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Long Time Behaviour of Solutions of Abstract Inequalities: Applications to Thermo-Hydraulic and Magnetohydrodynamic Equations

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We study some scalar inequalities of parabolic type and we give the leading term of an asymptotic expansion as $t \to \infty$ for solutions of thermo-hydraulic equations without external excitation. A phenomenon of resonance is pointed out. We also treat M. H. D. equations and Navier–Stokes equations on a Riemannian manifold.

INTRODUCTION

Due to dissipation effects, and in the absence of an external excitation, the solutions of thermo-hydraulic equations decay at least exponentialy when time goes to infinity. The aim of this paper is to give a qualitative description of this behaviour similar to that obtained by C. Foias and J. C. Saut [3, 4] for the Navier–Stokes equations. It is proved in [3, 4] that for these equations without exterior forces, the velocity decays exactly exponentialy with a rate which is an eigenvalue of Stokes operator.

Let us now describe our main results. For thermo-hydraulic equations, we show (see Theorem 2.1) that two different situations may occur. In the first one, tempeature and velocity decay exactly exponentialy. In the second situation, a phenomenon of resonance between Stokes operator and the heat operator appears. Although temperature decays exactly exponentialy, the velocity decays like the product of an integer power of time and a decreasing exponential whose rate is the same as that of the temperature. This is a case of resonance.

For Navier–Stokes equations on a Riemannian manifold, for magnetohydrodynamic equations, we prove that solutions of these

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equations decay exactly exponentialy, w.r.t. the L^2-Sobolev norm of order m for every $m \geqslant 0$. Finally we consider scalar inequalities of the type ($u(x, t) \in \mathbb{C}$, Ω bounded set in $\mathbb{R}^d$):

$$\left| \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) \right|_{L^2(\Omega)} \leqslant n \, |\nabla u|_{L^2(\Omega)^n}.$$

We show that under the assumptions that $a_{ij}(\cdot, t)$ goes to $a_{ij}^\infty(\cdot)$ and $n(t)$ goes to zero in a certain sense when t goes to infinity; there exists an eigenvalue Λ^∞ of the second order differential operator $A^\infty \equiv -\partial/\partial x_i$ $(a_{ij}^\infty(x)\, \partial/\partial x_j)$ associated to appropriate B.C., such that $e^{\Lambda^\infty t} u(\cdot, t)$ goes to some function $u^\infty(\cdot)$ eigenfunction of A^∞ associated with Λ^∞.

The paper is organized as follows: in Section 1 we present some abstract results on differential inequalities of the type ($v > 0$):

$$\left\| \frac{d\phi}{dt} + vA\phi \right\|_H \leqslant n \, \|\phi\|_{D(A^{1/2})},$$

where $\{A(t)\}$ is a family of self-adjoint unbounded operators on a Hilbert space H.

In Section 2 we make use of the results of Section 1 to study the long time behaviour of the equations previously mentioned: we consider successively the thermo-hydraulic equations (Sect. 2.1), the M.H.D. equations (Sect. 2.2) and scalar inequalities (Sect. 2.3).

1. Remarks and Complements on the Asymptotic Behaviour for Solutions of Abstract Inequalities

1.1. *Notations and Hypotheses*

Let V be a separable Hilbert space included with continuous and compact injection in a Hilbert space H; we denote by $(\cdot,\cdot)$ and $|\cdot|$ the scalar product and the norm on H and by $|\cdot|_V$ the norm on V.

We suppose that V is dense in H and thus, identifying H with its antidual, we have the usual injections

$$V \subset H \subset V', \tag{1.1}$$

we shall also denote by $(\cdot,\cdot)$ the duality between V and V'.

A bounded operator $\mathscr{A}$ from V into V' restricted to H defines an unbounded operator still denoted by $\mathscr{A}$ with domain

$$D(\mathscr{A}) = \{\phi \in V, \mathscr{A}\phi \in H\}. \tag{1.2}$$

Let $t \to A(t)$ be a measurable mapping from $\mathbb{R}_+ = [0, +\infty[$ into $\mathscr{L}(V, V')$; we assume that for almost every $t \in \mathbb{R}_+$ (in the sequel we shall simply write for a.e. $t \geqslant 0$) the operator $A(t)$ is self-adjoint.

ASSUMPTION 1.1. (i) *There exists* $\lambda \in \mathbb{R}$ *and* $\eta_1 > 0$ *such that*

$$(A(t)\,\phi, \phi) + \lambda\,|\phi|^2 \geqslant \eta_1\,|\phi|_V^2, \qquad \forall \phi \in V \text{ for a.e. } t \geqslant 0.$$

(ii) *There exists a function of* $L^1(\mathbb{R}_+; \mathscr{L}(V, V'))$ *denoted* dA/dt *such that*

$$\frac{d}{dt}(A(t)\,\phi, \psi) = \left(\frac{dA}{dt}(t)\,\phi, \psi\right), \qquad \forall \phi, \psi \in V \text{ for a.e. } t \geqslant 0.$$

The point (ii) clearly implies that $\{A(t)\}$ converges, when t goes to $+\infty$, to a limit denoted by A^∞ in the topology of $\mathscr{L}(V, V')$. This limit satisfies

$$(A^\infty \phi, \phi) + \lambda\,|\phi|^2 \geqslant \eta_1\,|\phi|_V^2, \qquad \forall \phi \in V,$$

and A^∞ is self-adjoint. The operator $A_\lambda^\infty \equiv A^\infty + \lambda$ is an isomorphism from V onto V', Thus $\|\phi\| \equiv (A_\lambda^\infty \phi, \phi)^{1/2}$ and $|\phi|_V$ are two equivalent norms on V. In the following we shall use the first one. We rewrite the point (i) of Assumption 1.1 as

$$\exists \eta > 0\ (A(t)\,\phi, \phi) + \lambda\,|\phi|^2 \geqslant \eta\,\|\phi\|^2, \qquad \forall \phi \in V \text{ for a.e. } t \geqslant 0. \tag{1.3}$$

Let $\alpha(t)$ be the function

$$\alpha(t) = \left\|\frac{dA}{dt}(t)\right\|_{\mathscr{L}(V,V')} \in L^1(0, +\infty), \tag{1.4}$$

and define

$$A^D(t) = -\int_t^{+\infty} \frac{dA}{ds}(s)\,ds = A(t) - A^\infty. \tag{1.5}$$

It follows from (ii) of Assumption 1.1 that $A^D(t)$ goes to zero and

$$\|A^D(t)\|_{\mathscr{L}(V,V')} \leqslant \int_t^{+\infty} \alpha(s)\,ds. \tag{1.6}$$

Diagonalisation of A^∞. The operator A_λ^∞ is an isomorphism from $D(A_\lambda^\infty)$ onto H and $(A_\lambda^\infty)^{-1}$ is a self-adjoint and compact operator from H into H. Thus there exists a complete family in H constituted by eigenfunctions of

A^∞. More precisely there exists $\{\xi_j\}_{j\geq 1}\subset H$ and a non-decreasing sequence $\{\lambda_j\}_{j\geq 1}$ of real numbers such that

$$\begin{aligned} &\mathrm{Sp}\{\xi_j\}_{j\geq 1} \quad \text{is dense in } H,\\ &(\xi_i, \xi_j)=\delta_{ij}, \\ &A^\infty \xi_i=\lambda_i \xi_i. \end{aligned} \tag{1.7}$$

We also denote by $\sigma(A^\infty)=\{\Lambda_j\}_{j\geq 1}$ the increasing sequence of distinct eigenvalues. If $\tilde{\Lambda}\in\sigma(A^\infty)$, we denote by $\Pi_{\tilde{\Lambda}}$ the projector onto $\mathrm{Ker}(A^\infty-\tilde{\Lambda})$ and by $P_{\tilde{\Lambda}}$ the projector $P_{\tilde{\Lambda}}=\sum_{\lambda\leqslant\tilde{\Lambda}}\Pi_\lambda$.

1.2. *Asymptotic Behaviour*

This Section is devoted to the study of the behaviour as $t\to+\infty$ of a function ϕ from $\mathbb{R}_+$ into V satisfying

$$\phi\in\mathscr{C}(\mathbb{R}_+;V)\cap L^2(0,T;D(A(t))), \qquad \forall T>0, \tag{1.8}$$

(the notation $\phi\in L^2(0,T;D(A(t))$ means that for almost every $t\in[0,T]$; $u(t)\in D(A(t))$ and $A(t)\,u(t)\in L^2(0,T;H))$.

$$\begin{aligned} &\frac{d\phi}{dt}+\nu A\phi\in H \qquad && \text{for a.e. } t\geqslant 0,\\ &\left|\frac{d\phi}{dt}+\nu A\phi\right|\leqslant n\,\|\phi\| && \text{for a.e. } t\geqslant 0, \end{aligned} \tag{1.9}$$

where $\nu>0$ and

$$n\in L^2(0,+\infty). \tag{1.10}$$

According to a backward uniqueness result for which the reader is refered to J. L. Lions and B. Malgrange [18], C. Bardos and L. Tartar [1], and J. M. Ghidaglia [9] we know that either ϕ is identically zero or ϕ never vanishes. Thus assuming that $\phi(0)\neq 0$, we have

$$|\phi(t)|>0, \qquad \forall t\geqslant 0. \tag{1.11}$$

In this last case, [9] Theorem 1.1 proves that the ratio $\|\phi\|^2/|\phi|^2$ converges when $t\to+\infty$ to some $\Lambda^\infty\in\sigma(A^\infty)$, and for every $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$|\phi(t)|\geqslant C_\varepsilon e^{-\nu(1+\varepsilon)\Lambda^\infty t}, \qquad \forall t\geqslant 0. \tag{1.12}$$

We shall show that, under stronger hypothesis on the convergence of

$A(t)$ to A^∞ (see (1.16)) and on the decay of n (see (1.15)), (1.12) can be improved. More precisely Theorem 1.1 (which is the main result of this paragraph) proves that there exists an eigenfunction $\phi^\infty \neq 0$ of A^∞ associated with Λ^∞ such that

$$\lim_{t \to +\infty} \| e^{\nu \Lambda^\infty t} \phi(t) - \phi^\infty \| = 0. \tag{1.13}$$

Let us note that (1.12) was proved without the hypothesis of compactness of the injection from V into H.

Before stating our result, let us introduce some notations when $\phi(0) \neq 0$ so that (1.11) holds:

$$\Lambda(t) = \frac{(A(t)\,\phi(t), \phi(t))}{|\phi|^2}, \; \Lambda^l(t) = \frac{(A^\infty \phi(t), \phi(t))}{|\phi|^2}, \; \Lambda^D = \Lambda - \Lambda^l. \tag{1.14}$$

THEOREM 1.1. *The hypotheses are those of Section* 1.1. *Let* ϕ *satisfy* (1.9) *and* $\phi(0) \neq 0$. *We make the following supplementary assumptions*

$$n \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \tag{1.15}$$

$$\int_0^{+\infty} \left(\int_t^{+\infty} \left\| \frac{dA}{dt} \right\|_{\mathscr{L}(V,V')} ds \right) dt < +\infty. \tag{1.16}$$

Then the ratio $\|\phi(t)\|^2/|\phi(t)|^2$ *converges to* $\Lambda^\infty \in \sigma(A^\infty)$ *when* $t \to +\infty$ *and there exists* $\phi^\infty \in D(A^\infty)$ *such that*

$$e^{\nu \Lambda^\infty t} \phi(t) \quad \text{converges to } \phi^\infty \text{ in the norm of } V, \tag{1.17}$$

$$A^\infty \phi^\infty = \Lambda^\infty \phi^\infty. \tag{1.18}$$

Remark 1.1. The hypothesis $n \in L^1(\mathbb{R}_+)$ and (1.16) are optimal in a certain sense. Take, for instance, $V = H = \mathbb{R}$, and

(i) $A = Id$, $n \geqslant 0$, $(d\phi/dt) + \phi = n\phi$ then $e^t \phi(t)$ converges iff $n \in L^1(\mathbb{R}_+)$,

(ii) $A(t) = [1 + a(t)]\, Id$, $a \geqslant 0$, $n = 0$, $(d\phi/dt) + (1 + a)\,\phi = 0$, $e^t \phi$ converges in $\mathbb{R}$ iff $a \in L^1(\mathbb{R}_+)$.

The proof of Theorem 1.1 consists in successive lemmas. In all of them, hypotheses of Section 1.1 are understood.

Let us remark that using $\phi(t)\, e^{-\nu \lambda t}$ and $A + \lambda$ instead of $\phi(t)$ and A amounts to the case $\lambda = 0$.

LEMMA 1.1. *Let* ϕ *satisfy* (1.9) *with* (1.10). *If* $\phi(0) \neq 0$ *then* $\Lambda(t)$ *and* $\Lambda^l(t)$ *converge to* $\Lambda^\infty \in \sigma(A^\infty)$ *when* $t \to +\infty$.

Proof. We denote by $L(\phi) = (d\phi/dt) + \nu A\phi$. We have the following equation for $\Lambda(t)$:

$$\frac{d\Lambda}{dt} = 2\mathrm{Re}\,\frac{(A\phi - \Lambda\phi, \phi')}{|\phi|^2} + \frac{(A'\phi, \phi)}{|\phi|^2}.$$

Since $(A\phi - \Lambda\phi, \phi) = 0$, it follows then that

$$\frac{1}{2}\frac{d\Lambda}{dt} + \nu\frac{|A\phi - \Lambda\phi|^2}{|\phi|^2} = \mathrm{Re}\,\frac{(A\phi - \Lambda\phi, L(\phi))}{|\phi|^2} + \frac{1}{2}\frac{(A'\phi, \phi)}{|\phi|^2}; \tag{1.19}$$

dashes denoting derivatives with respect to t. From (1.19), thanks to (1.3), (1.4), and (1.9) we have

$$\frac{d\Lambda}{dt} + \nu\frac{|A\phi - \Lambda\phi|^2}{|\phi|^2} \leqslant m\Lambda, \tag{1.20}$$

where

$$m(t) \equiv \frac{1}{\eta}\left(\alpha(t) + \frac{n^2(t)}{\nu}\right). \tag{1.21}$$

From (1.20), it follows that $d\Lambda/dt \leqslant m\Lambda$ and by integration from t_0 to $t_1 (t_1 \geqslant t_0)$ we find

$$\Lambda_1 \leqslant \Lambda(t_1) \leqslant \Lambda(t_0) \exp \int_{t_0}^{t_1} m(s)\, ds.$$

Using (1.4), (1.10), and (1.21) we deduce

$$\Lambda_1 \leqslant \limsup_{t_1 \to +\infty} \Lambda(t_1) \leqslant \Lambda(t_0) \exp \int_{t_0}^{+\infty} m(s)\, ds,$$

taking the lim inf when t_0 goes to $+\infty$ we have then

$$\Lambda_1 \leqslant \limsup_{t_1 \to +\infty} \Lambda(t_1) \leqslant \liminf_{t_0 \to +\infty} \Lambda(t_0).$$

It follows that $\Lambda(t)$ converges to a limit Λ^∞ when $t \to +\infty$. Using now (1.6) we estimate $\Lambda^D(t)$:

$$|\Lambda^D(t)| \leqslant \frac{\|u\|^2}{|u|^2} \int_t^{+\infty} \alpha(s)\, ds \leqslant \frac{\Lambda(t)}{\eta} \int_t^{+\infty} \alpha(s)\, ds$$

and thus $\lim_{t \to +\infty} \Lambda(t) = \lim_{t \to +\infty} \Lambda^l(t) = \Lambda^\infty$.

It remains to show that $\Lambda^\infty \in \sigma(A^\infty)$. Denoting by $\psi = \phi/|\phi|$ we deduce from (1.20) and from the fact that $\Lambda(t)$ is bounded for $t \in \mathbb{R}_+$ that $\psi \in L^2(\mathbb{R}_+, H)$. Hence there exists a sequence $t_j \to +\infty$ as $j \to +\infty$ such that $(A-\Lambda)\psi(t_j) \to 0$ in H. Then $r_j \equiv (A-\Lambda^\infty)\psi(t_j) \to 0$ in H. If $\Lambda^\infty \notin \sigma(A^\infty)$, $(A^\infty - \Lambda^\infty)^{-1} \in \mathscr{L}(V', V)$. Since $A^D\psi(t_j) + r_j \to 0$ in V', $\psi(t_j) = (A^\infty - \Lambda^\infty)^{-1}\{A^D\psi(t_j) + r_j\}$ goes to 0 in V. This conradicts the fact that $|\psi(t_j)| = 1$. This completes the proof of Lemma 1.1.

LEMMA 1.2. *Let ϕ satisfy* (1.9) *with* (1.10). *If $\phi(0) \neq 0$, under the assumption* (1.16), *we have*

$$\lim_{t \to +\infty} \left\| (I - \Pi_{\Lambda^\infty}) \frac{\phi}{|\phi|} \right\| = 0, \tag{1.22}$$

$$\int_0^{+\infty} |\Lambda(t) - \Lambda^\infty| \, dt < +\infty. \tag{1.23}$$

Proof. We have the following equation for $\psi = \phi/|\phi|$ (recall that $L(\phi) \equiv (d\phi/dt) + \nu A\phi$)

$$\frac{d\psi}{dt} + \nu(A - \Lambda)\psi = \frac{L(\phi)}{|\phi|} - \operatorname{Re}\frac{(L(\phi), \phi)}{|\phi|^2}\psi. \tag{1.24}$$

We shall prove that (1.22), (1.23) follow from (1.24), thanks to the following lemma (the proof of Lemma 1.3 is given hereafter).

LEMMA 1.3. *Let ψ be a function with values in V such that*

$$\psi \in \mathscr{C}(\mathbb{R}_+, V) \cap L^2_{\text{loc}}(\mathbb{R}_+; D(A(t))),$$

$$\left| \frac{d\psi}{dt} + \nu(A - \Lambda)\psi \right| \leqslant \rho, \ |\psi| = 1,$$

where

$$\Lambda(t) = \|\psi(t)\|^2 \text{ converges to } \Lambda^\infty \in \sigma(A^\infty) \text{ as } t \to +\infty, \tag{1.25}$$

$$\rho \in L^2(\mathbb{R}_+). \tag{1.26}$$

Then

$$\lim_{t \to +\infty} \|(I - \Pi_{\Lambda^\infty})\psi\| = 0, \tag{1.27}$$

and

$$\int_0^{+\infty} \|(I - \Pi_{\Lambda^\infty})\psi\|^2(s)\, ds < +\infty. \tag{1.28}$$

We return to the proof of Lemma 1.2. Denoting by $\rho = |(d\Psi/dt) + v(A - \Lambda)\psi|$, we infer from (1.9) and (1.10) that ρ satisfies (1.26). The convergence (1.25) results from Lemma 1.1 and thus applying Lemma 1.3, (1.22) follows. To prove (1.23) we write $\Lambda - \Lambda^\infty = \Lambda^l - \Lambda^\infty + \Lambda^D$; using (1.6) and (1.14) we have

$$|\Lambda - \Lambda^\infty| \leqslant |\Lambda - \Lambda^\infty| + \Lambda^l \int_t^{+\infty} \alpha(s)\, ds.$$

On the other hand one checks easily the estimate (expand in term of $\{\xi_j\}_{j \geqslant 1}$)

$$|\Lambda^l - \Lambda^\infty| \leqslant \left(1 + \frac{\Lambda^\infty}{\Lambda_1}\right) \|(I - \Pi_{\Lambda^\infty})\, \psi(s)\|^2.$$

Now (1.23) follows from (1.16), (1.28) and the two last estimates. The proof of Lemma 1.2 is complete.

We establish now a result which shows that $\|\phi(t)\|$ decays at least like $e^{-v\Lambda^\infty t}$.

LEMMA 1.4. *Under the assumptions of Theorem* 1.1, *there exists a positive constant C such that*

$$|\phi(t)| \leqslant Ce^{-v\Lambda^\infty t}, \qquad \forall t \geqslant 0, \tag{1.29}$$

$$\|\phi(t)\| \leqslant Ce^{-v\Lambda^\infty t}, \qquad \forall t \geqslant 0. \tag{1.30}$$

Proof. Since $\Lambda^l(t)$ is bounded, it suffices to prove (1.29). We have the following estimate:

$$\frac{1}{2}\frac{d}{dt}|\phi|^2 + v\Lambda\, |\phi|^2 = \operatorname{Re}(L(\phi), \phi) \leqslant n(\Lambda^l)^{1/2}\, |\phi|^2.$$

It results that

$$\frac{d}{dt} \log |\phi|^2\, e^{v\Lambda^\infty t} \leqslant 2n(\Lambda^l)^{1/2} + 2v(\Lambda - \Lambda^\infty).$$

Now from (1.15) and (1.23) the r.h.s. of this last inequality belongs to $L^1(\mathbb{R}_+)$ and thus (1.29) is satisfied: Lemma 1.4 is proved.

We conclude by giving

Proof of Theorem 1.1. We denote by $\Pi(t) = \Pi_{\Lambda^\infty}\phi(t)$, applying Π_{Λ^∞} to $L(\phi) = (d\phi/dt) + vA\phi$ it follows that

$$\frac{d}{dt}(e^{v\Lambda^\infty t}\, \Pi(t)) = e^{v\Lambda^\infty t}\, \Pi_{\Lambda^\infty}(L(\phi) - v\Lambda^D \phi).$$

By integration we deduce

$$e^{\nu\Lambda^\infty t}\Pi(t)-e^{\nu\Lambda^\infty s}\Pi(s)=\int_s^t e^{\nu\Lambda^\infty\sigma}\Pi_{\Lambda^\infty}(L(\phi)-\nu A^D\phi)\,d\sigma. \qquad (1.31)$$

From (1.6) and (1.9) we have

$$|\Pi_{\Lambda^\infty}(L(\phi)-\nu A^D\phi)|\leqslant\left[n+(\Lambda^\infty)^{1/2}\int_t^{+\infty}\alpha(s)\,ds\right]\|\phi\|.$$

Using (1.15), (1.16), and (1.30) we deduce that there exists a function $n_1\in L^1(\mathbb{R}_+)$ such that

$$e^{\nu\Lambda^\infty\sigma}|\Pi_{\Lambda^\infty}(L(\phi(\sigma)))-\nu A^D(\sigma)\,\phi(\sigma)|\leqslant n_1(\sigma). \qquad (1.32)$$

Hence the integral in the r.h.s. of (1.31) is convergent and therefore $e^{\nu\Lambda^\infty t}\Pi(t)$ converges to ϕ^∞ in the norm of H. If $\phi^\infty=0$, then

$$e^{\nu\Lambda^\infty t}\Pi(t)=\int_t^{+\infty}e^{\nu\Lambda^\infty\sigma}\Pi_{\Lambda^\infty}(L(\phi)-\nu A^D\phi)\,d\sigma$$

and according to (1.32)

$$|\Pi(t)|\leqslant e^{-\nu\Lambda^\infty t}\int_t^{+\infty}n_1(\sigma)\,d\sigma.$$

According to (1.22), there exists $C_1>0$ such that

$$|\phi(t)|\leqslant C_1 e^{-\nu\Lambda^\infty t}\int_t^{+\infty}n_1(\sigma)\,d\sigma. \qquad (1.33)$$

From

$$\frac{1}{2}\frac{d}{dt}|\phi|^2+\nu\Lambda\,|\phi|^2=\mathrm{Re}(\phi,L(\phi))\geqslant -n(\Lambda^l)^{1/2}\,|\phi|^2$$

we deduce that

$$|\phi(t)|\geqslant|\phi(0)|\exp-\int_0^t(\nu\Lambda+n(\Lambda^l)^{1/2})(\sigma)\,d\sigma.$$

By comparison with (1.33) we have

$$\int_0^t|\nu(\Lambda-\Lambda^\infty)+n(\Lambda^l)^{1/2}|\,d\sigma\geqslant\log\frac{|\phi(0)|}{C_1}-\log\int_t^{+\infty}n_1(\sigma)\,d\sigma.$$

According to (1.15) and (1.23) the l.h.s. of this last inequality is majored but since $n_1 \in L^1(\mathbb{R}_+)$ the r.h.s. goes to $+\infty$ when $t \to +\infty$. It is absurd and $\phi^\infty \neq 0$. We have proved that $e^{\nu \Lambda^\infty t} \Pi_{\Lambda^\infty} \phi(t)$ converges to ϕ^∞ w.r.t. the norm on H. It follows then from (1.22) and (1.29) that $e^{\nu \Lambda^\infty t} \phi(t)$ converges to ϕ^∞ w.r.t. the norm on V. Since Ker Π_{Λ^∞} is closed, (1.18) is satisfied.

Proof of Lemma 1.3. We denote by $q = (I - P_{\Lambda^\infty}) \psi$. Applying $I - P_{\Lambda^\infty}$ *to* $(d\psi/dt) + \nu(A - \Lambda) \psi$ and taking the scalar product in H with q we get

$$\frac{1}{2} \frac{d}{dt} |q|^2 + \nu(\|q\|^2 - \Lambda |q|^2) = \operatorname{Re}\left(q, \frac{d\psi}{dt} + \nu(A - \Lambda) \psi - \nu A^D \psi\right).$$

Now by (1.6) we obtain

$$\frac{1}{2} \frac{d}{dt} |q|^2 + \nu(\|q\|^2 - \Lambda |q|^2) \leqslant \rho |q| + \nu \|\psi\| \|q\| \int_t^{+\infty} \alpha(s) \, ds,$$

and if we denote by

$$m_2(t) = \frac{\rho}{\Lambda_1^{1/2}} + \nu(\Lambda')^{1/2} \int_t^{+\infty} \alpha(s) \, ds, \tag{1.34}$$

we have

$$\frac{1}{2} \frac{d}{dt} |q|^2 + \nu(\|q\|^2 - \Lambda |q|^2) \leqslant m_2 \|q\|. \tag{1.35}$$

It follows from (1.16), (1.26) that

$$m_2 \in L^2(\mathbb{R}_+). \tag{1.36}$$

Now, thanks to Schwarz inequality ($\varepsilon > 0$ will be chosen later) it comes from (1.35):

$$\frac{d}{dt} |q|^2 + 2\nu(\|q\|^2 - \Lambda |q|^2) \leqslant 2m_2 \|q\| \leqslant 2\nu\varepsilon \|q\|^2 + \frac{1}{2\nu\varepsilon} m_2^2.$$

Now, let $(1 + \delta_1) \Lambda^\infty$ be the first eigenvalue of A^∞ which is strictly greater than Λ^∞, then $|q|^2 \leqslant \|q\|^2/(1 + \delta_1) \Lambda^\infty$ and

$$\frac{d}{dt} |q|^2 + 2\nu\left\{(1 - \varepsilon) - \frac{\Lambda}{(1 + \delta_1) \Lambda^\infty}\right\} \|q\|^2 \leqslant \frac{1}{2\nu\varepsilon} m_2^2. \tag{1.37}$$

But since $(1 - \varepsilon) - (\Lambda/(1 + \delta_1) \Lambda^\infty)$ tends to $(\delta_1/1 + \delta_1) - \varepsilon$, choosing $\varepsilon = \delta_1/4(1 + \delta_1)$, there exists $t_0 \geqslant 0$ such that

$$1 - \varepsilon - \frac{\Lambda}{(1 + \delta_1) \Lambda^\infty} \geqslant \frac{1}{2} \frac{\delta_1}{1 + \delta_1}, \qquad \forall t \geqslant t_0. \tag{1.38}$$

Now from (1.37) it follows that

$$\frac{d}{dt}|q|^2 + \nu \frac{\delta_1}{1+\delta_1}\|q\|^2 \leqslant \frac{1}{2\nu\varepsilon} m_2^2. \tag{1.39}$$

From this last inequality, using $|q|^2 \leqslant \|q\|^2/(1+\delta_1)\Lambda^\infty$, one obtains by Gronwall's lemma for $t \geqslant t_0$,

$$|q(t)|^2 \leqslant |q(t_0)|^2 e^{-\nu\delta_1\Lambda^\infty t} + \frac{1}{2\nu\varepsilon}\int_{t_0}^{t} m_2^2(s)\, e^{-\nu\delta_1\Lambda^\infty(t-s)} ds. \tag{1.40}$$

Now from (1.36) and (1.40) it follows easily that

$$\lim_{t\to+\infty} |q(t)| = 0. \tag{1.41}$$

Returning to (1.39) we obtain that

$$\int_t^{+\infty} \|q(s)\|^2\, ds < +\infty. \tag{1.42}$$

Let us remark that if $\Lambda^\infty = \Lambda_1$, (1.41) can be written as

$$\lim_{t\to+\infty} \left|(I - \Pi_{\Lambda^\infty})\frac{\phi}{|\phi|}\right| = 0. \tag{1.43}$$

If $\Lambda^\infty > \Lambda_1$, let $(1-\delta_2)\Lambda^\infty$ be the first eigenvalue of A^∞ which is strictly smaller than Λ^∞. Denoting by $\theta = P_{(1-\delta_2)\Lambda^\infty}\psi$, applying $P_{(1-\delta_2)\Lambda^\infty}$ to $(d\psi/dt) + \nu(A - \Lambda)\psi$ and taking the scalar product in H with θ we get

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + \nu(\|\theta\|^2 - \Lambda|\theta|^2) \geqslant -\rho|\theta| - \nu\|v\|\|\theta\| \int_t^{+\infty} \alpha(s)\, ds.$$

Denoting by

$$m_3(t) = \rho + \nu(\Lambda^l \Lambda^\infty(1-\delta_2))^{1/2} \int_t^{+\infty} \alpha(s)\, ds,$$

we have

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + \nu(\|\theta\|^2 - \Lambda|\theta|^2) \geqslant -m_3|\theta|, \tag{1.44}$$

and from (1.16) and (1.26) it follows that

$$m_3 \in L^2(\mathbb{R}_+). \tag{1.45}$$

Since $\|\theta\|^2 \leqslant \Lambda^\infty(1-\delta_2)|\theta|^2$, there exists $t_1 \geqslant 0$ such that

$$\Lambda(t) - \Lambda^\infty(1-\delta_2) \geqslant \frac{3\delta_2}{4} \qquad \text{for } t \geqslant t_1. \tag{1.46}$$

Now from (1.46),

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 \geqslant \frac{3\nu\delta_2}{4}\Lambda^\infty |\theta|^2 - m_3|\theta|.$$

Thanks to Schwarz inequality we deduce

$$\frac{d}{dt}|\theta|^2 \geqslant \frac{\nu\delta_2\nu}{4}\Lambda^\infty|\theta|^2 - \frac{2}{\nu\delta_2\Lambda^\infty}m_3^2.$$

Let $T \geqslant t \geqslant t_1$, by Gronwall's lemma it follows that

$$|\theta(t)|^2 \leqslant e^{-\nu\delta_2\Lambda^\infty(T-t)}|\theta(T)|^2 + \int_t^T \frac{2m_3^2}{\nu\delta_2\Lambda^\infty} e^{-\nu\delta_2\Lambda^\infty(s-t)}\, ds.$$

Letting $T \to +\infty$, observing that $|\theta| \leqslant |\psi| = 1$, we obtain

$$|\theta(t)|^2 \leqslant \frac{2}{\nu\delta_2\Lambda^\infty}\int_t^{+\infty} m_3^2(s)\, e^{-\nu\delta_2\Lambda^\infty(s-t)}\, ds. \tag{1.47}$$

Using (1.45), this last inequality proves that $\lim_{t\to+\infty}|\theta(t)| = 0$. Now according to (1.41), (1.43) is also proved in the case where $\Lambda^\infty > \Lambda_1$. We are going to prove that (1.43) and $\lim_{t\to+\infty}\|\psi(t)\|^2 = \Lambda^\infty$ shows (1.27). We write $\|\psi\|^2 = \|(I-\Pi_{\Lambda^\infty})\psi\|^2 + \Lambda^\infty|\Pi_{\Lambda^\infty}\psi|^2$, then $\|(I-\Pi_{\Lambda^\infty})\psi\|^2 = \Lambda - \Lambda^\infty + \Lambda^\infty|(I-\Pi_{\Lambda^\infty})\psi|$ and this proves our claim.

Now according to (1.42), to prove (1.28) it suffices to show that

$$\int_{t_1}^{+\infty} \|\theta(s)\|^2\, ds < +\infty. \tag{1.48}$$

But (1.48) is a consequence of (1.47) since $\|\phi\|^2 \leqslant \Lambda^\infty|\theta|^2$ and

$$\int_0^{+\infty}\left(\int_t^{+\infty} m_3^2(s)\, e^{-\nu\delta_2\Lambda^\infty(s-t)}\, ds\right) dt = \int_0^{+\infty}\int_0^{+\infty} m_3^2(s+t)\, e^{-\nu\delta_2\Lambda^\infty s}\, ds\, dt$$

$$\leqslant \left(\int_0^{+\infty} m_3^2(t)\, dt\right)\left(\int_0^{+\infty} e^{-\nu\delta_2\Lambda^\infty s}\, ds\right)$$

$$< +\infty. \quad \blacksquare$$

1.3. *The Case of a Quadratic Nonlinearity*

Theorem 1.1 shows that solutions of (1.9) behave, w.r.t. the V-norm, like $e^{-\nu A^\infty t}\phi^\infty$ as $t \to +\infty$. In this paragraph we strengthen this result in the case where ϕ satisfies a nonlinear equation (see (1.50)). We prove in particular that for every $m \in \mathbb{N}$, $d^m\phi/dt^m$ behaves, w.r.t. the $D(A)$-norm, like $(-\nu A^\infty)^m e^{-\nu A^\infty t}\phi^\infty$ as $t \to +\infty$. Throughout this paragraph we assume that the family $\{A(t)\}_{t \geqslant 0}$ does not depend on t, i.e., $A(t) \equiv A$ and we make the assumption that A is positive (i.e., point (i) of assumption 1.1 is satisfied by $\lambda = 0$).

This section is motivated by the case where A is the Stokes operator. Due to the classical regularity properties of this operator, the Theorem 1.3 established at the end of this paraıgraph leads, in this particular case, to decay results w.r.t. the Sobolev's norm H^m for every $m \geqslant 0$ (see Remark 2.2).

1.3.1. *Behaviour in* $D(A)$. Let B a bilinear continuous operator from $V \times V$ into V', from $D(A) \times V$ and $V \times D(A)$ into H, such that $(K \geqslant 0)$

$$|B(\phi, \psi)| + |B(\psi, \phi)| \leqslant K\,|A\phi|\,\|\psi\|, \qquad \forall \phi, \psi, \in D(A). \tag{1.49}$$

We study the behaviour or $\phi(t)$ solution of the differential equation

$$\phi'(t) + \nu A\phi(t) + B(\phi(t), \phi(t)) = 0. \tag{1.50}$$

Let us remark that we do not ask B to satisfy the identity $(B(\phi, \phi), \phi) = 0$. But we are interested in solutions of (1.50) which satisfy

$$\phi \in \mathscr{C}(\mathbb{R}_+, D(A)), \qquad \phi' \in \mathscr{C}(\mathbb{R}_+, H), \tag{1.51}$$

$$\int_0^{+\infty} (|A\phi| + |A\phi|^2)\, ds < +\infty. \tag{1.52}$$

Remark 1.2. If ϕ satisfies (1.50) and (1.51) then $\phi \in \mathscr{C}^\infty(]0, +\infty[; D(A))$ (see J. M. Ghidaglia [6]).

Let us notice that denoting by $n(t) = \kappa\,|A\phi(t)|$ it follows from (1.49) and (1.50) that

$$|\phi' + \nu A\phi| \leqslant n\,\|\phi\|. \tag{1.53}$$

As mentioned before according to (1.51)-(1.53), the following alternative holds: either ϕ is identically zero or ϕ never vanishes. We shall restrict ourselves to the latter case throughout this paragraph. Henceforth

$$|\phi(t)| > 0, \forall t \geqslant 0. \tag{1.54}$$

We state now a result which makes more precise the behaviour of $\phi(t)$ with respect to the $D(A)$-norm. It shows that under the previous hypotheses the behaviour of $\phi(t)$ when t goes to $+\infty$ is exactly exponential.

THEOREM 1.2. *Let ϕ satisfying (1.50) to (1.52). Then there exists an eigenvector ϕ^∞ of A associated with the eigenvalue Λ^∞ such that for every $j \in \mathbb{N}$*

$$\phi_j(t) \equiv \frac{d^j\phi}{dt^j} \sim (-\nu\Lambda^\infty)^j \phi^\infty e^{-\nu\Lambda^\infty t} \qquad \text{in } D(A) \text{ when } t \to +\infty. \quad (1.55)$$

Proof. We only sketch the proof, for further details the reader is refered to J. M. Ghidaglia [7]. According to (1.51), (1.52), and (1.53), Theorem 1.1 applies hence

$$\phi(t) \sim \phi^\infty e^{-\nu\Lambda^\infty t} \qquad \text{in } V.$$

Equation (1.50) differentiated with respect to t reads:

$$\phi_1' + \nu A\phi_1 + B(\phi, \phi_1) + B(\phi_1, \phi) = 0. \qquad (1.56)$$

Using (1.49), and applying Theorem 1.1, one finds that

$$\phi \sim \phi^\infty e^{-\nu\Lambda^\infty t} \qquad \text{in } D(A),$$

$$\phi_1 \sim -\nu\Lambda^\infty e^{-\nu\Lambda^\infty t} \qquad \text{in } V.$$

Equation (1.50) differentiated twice w.r.t. t reads

$$\phi_2^1 + \nu A\phi_2 = -2B(\phi_1, \phi_1) - B(\phi_2, \phi) - B(\phi, \phi_2). \qquad (1.57)$$

From (1.56) one derives that

$$\int_0^{+\infty} (|A\phi_1| + |A\phi_1|^2)\, dt < +\infty. \qquad (1.58)$$

Now if we introduce $\boldsymbol{\phi} = (\phi_1, \phi_2)$, it follows from (1.52) and (1.56)–(1.58) that Theorem 1.1 applies to $\boldsymbol{\phi}$. Hence we deduce that

$$\phi_1 \sim -\nu\Lambda^\infty e^{-\nu\Lambda^\infty t} \qquad \text{in } D(A),$$

$$\phi_2 \sim (-\nu\Lambda^\infty)\, e^{-\nu\Lambda^\infty t} \qquad \text{in } V.$$

The proof proceeds by induction on $l \geq 2$. Let $(\mathscr{H})_l$ be

$$\left.\begin{array}{ll} \phi_j \sim (-\nu\Lambda^\infty)^j e^{-\nu\Lambda^\infty t} & \text{in } D(A) \text{ for } j = 1, -, l-1, \\ \phi_l \sim (-\nu\Lambda^\infty)^l e^{-\nu\Lambda^\infty t} & \text{in } V. \end{array}\right\} (\mathscr{H})_l$$

We have just proved $(\mathscr{H})_2$. Assuming $(\mathscr{H})_l$, $l \geqslant 2$ we differentiate (1.50) l times w.r.t. t,

$$\frac{d\phi_l}{dt} + \nu A\phi_l = -\sum_{k=0}^{l} \binom{l}{k} B(\phi_k, \phi_{l-k}).$$

Denoting $\boldsymbol{\phi} = (\phi_1, -, \phi_{l+1})$ we find that, according to $(\mathscr{H})_l$, Theorem 1.1 applies to $\boldsymbol{\phi}$ and $(\mathscr{H})_{l+1}$ follows. ▌

Remark 1.3. We have just proved that $e^{\nu \Lambda^\infty t}\phi(t)$ converges to an element ϕ^∞ of $\mathrm{Ker}(A - \Lambda^\infty)$ when $t \to +\infty$. The following result will make more precise this convergence (this type of result has been first derived for the Navier–Stokes equations by C. Foias and J. C. Saut [3, 4]).

Under the hypotheses of Theorem 1.2, let λ be an eigenvalue of A, there exists $\alpha_\lambda > 0$ such that for $\Lambda_1 \leqslant \lambda < \Lambda^\infty$,

$$\Pi_\lambda \phi(t) = \frac{e^{-2\nu\Lambda^\infty t}}{\nu(2\Lambda^\infty - \lambda)} \Pi_\lambda B(\phi^\infty, \phi^\infty) + O(e^{-(2\nu\Lambda^\infty + \alpha_\lambda)t}),$$

for $\Lambda^\infty \leqslant \lambda < 2\Lambda^\infty$,

$$\Pi_\lambda \phi(t) = e^{-\nu\lambda t}\phi^\infty + \frac{e^{-2\nu\Lambda^\infty t}}{\nu(\Lambda^\infty - \lambda)} \Pi_\lambda B(\phi^\infty, \phi^\infty) + O(e^{-(2\nu\Lambda^\infty + \alpha_\lambda)t}),$$

for $2\Lambda^\infty = \lambda$,

$$\Pi_\lambda \phi(t) = -te^{-\nu\lambda t}\Pi_\lambda B(\phi^\infty, \phi^\infty) + O(e^{-2\nu\Lambda^\infty t}),$$

for $\lambda > 2\Lambda^\infty$,

$$\Pi_\lambda \phi(t) = -\frac{e^{-\nu\Lambda^\infty t}}{\nu(\lambda - 2\Lambda^\infty)} \Pi_\lambda B(\phi^\infty, \phi^\infty) + O(e^{-(\nu\lambda^\infty + \alpha_\lambda)t}).$$

(The $O(\ )$ are understood w.r.t. the $D(A)$-norm). For the proof, the reader is refered to J. M. Ghidaglia [7]. ▌

1.3.2. *Behaviour in* E_m. We denote by V_m the scale of Hilbert spaces endowed with the norm $\|\phi\|_m \equiv |A^{m/2}\phi|$.

We introduce a family of Hilbert spaces E_m, $m \in \mathbb{N}$, with

$$E_{m+1} \subset E_m, \qquad \forall m \in \mathbb{N} \text{ the injection being continuous,} \tag{1.59}$$

V_m is a closed subspace of E_m, $\forall m \in \mathbb{N}$ the norm induced by

$$E_m \text{ on } V_n \text{ being equivalent to } \|\cdot\|_m. \tag{1.60}$$

We assume that

$$A^{-1} \quad \text{is continuous from } E_m \text{ into } E_{m+2} \cap V, \forall m \geq 0, \tag{1.61}$$

$$E_0 = H,\ E_2 = D(A). \tag{1.62}$$

We make the following assumptions on B:

$$\begin{aligned}&B \text{ is continuous from } E_1 \times E_2 \text{ and } E_2 \times E_1 \text{ into } E_0, \text{ from}\\ &E_2 \times E_2 \text{ into } E_1 \text{ and from } E_m \times E_{m+1} \text{ into } E_m, \quad \forall m \geq 2.\end{aligned} \tag{1.63}$$

Theorem 1.2 gives the behaviour of $\phi(t)$ in $D(A)$. With the previous assumptions we are able to state

THEOREM 1.3. *The hypotheses are the same as in Theorem* 1.2. *For every* $(j, m) \in \mathbb{N} \times \mathbb{N}$ *we have*

$$\frac{d^j\phi}{dt^j} = \phi_j(t) \sim (-\nu A^\infty)^j \phi^\infty e^{-\nu A^\infty t} \qquad \textit{in } E_m.$$

Proof. We prove this theorem by induction. Let us denote by

$$\phi_j \sim (-\nu A^\infty)^j \phi^\infty e^{-\nu A^\infty t} \qquad \text{in } E_k \qquad (\mathscr{H})_k$$

According to Theorem 1.2, $(\mathscr{H})_2$ is proved. Let us suppose that $(\mathscr{H})_k$ is established with $k \geq 2$. We have

$$-\nu A\phi_j = \phi_{j+1} + \sum_{l=0}^{j} \binom{j}{l} B(\phi_l, \phi_{j-l}).$$

From (1.63) and $(\mathscr{H})_k$ we find that

$$B(\phi_l, \phi_{j-l}) = O(e^{-2\nu A^\infty t}) \text{ in } E_{k-1},$$

$$\phi_{l+1} \sim (-\nu A^\infty)^{j+1} \phi^\infty e^{-\nu A^\infty t} \text{ in } E_{k-1}.$$

Now

$$-\nu A\phi_j \sim (-\nu A^\infty)^{j+1} \phi^\infty e^{-\nu A^\infty t} \text{ in } E_{k-1},$$

and according to (1.61), $(\mathscr{H})_{k+1}$ is proved. ∎

2. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THERMO-HYDRAULIC AND M.H.D. EQUATIONS

2.1. *Thermo-Hydraulic Equations*

We consider the motion of a viscous fluid, subjected to thermal effects, which fills some bounded region Ω. In the Boussinesq approximation the velocity $u(x, t)$, pressure $p(x, t)$ and temperature $\theta(x, t)$ are determined, in case of homogeneous boundary conditions, by the equations (see Chandrasekhar [2]):

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \sigma\theta + \nabla p = 0, \\
&\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\, \theta - \kappa \Delta \theta = 0, \\
&\operatorname{div} u = 0, \\
&u(x, t) = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}_+, \\
&\theta(x, t) = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}_+.
\end{aligned} \tag{2.1}$$

Where κ denotes Fourier's coefficient of the fluid and σ is a fixed vector in $\mathbb{R}^d$, parallel to the descending vertical.

We assume that Ω is an open connected bounded set in $\mathbb{R}^d$ ($d = 2$ or 3) and let Γ be its boundary:

Γ is a manifold of class C^∞ of dimension $d-1$ and Ω is locally located on one side of Γ. (2.2)

Let H and V be the closures in $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively, of

$$\mathscr{V} = \{v \in \mathscr{D}(\Omega)^d, \operatorname{div} v = 0\},$$

where $H_0^m(\Omega)$ denotes the closure of $\mathscr{D}(\Omega)$ (the space of $\mathscr{C}^\infty$ functions with compact support in Ω) in the Sobolev L^2-space $H^m(\Omega)$ of order m. The spaces H and V are endowed with the scalar products

$$(u, v) = \int_\Omega u_i v_i \, dx$$

and

$$((u, v)) = \int_\Omega D_i u_j D_i v_j \, dx,$$

respectively. We set also $|u| = (u, u)^{1/2}$ and $\|u\| = ((u, u))^{1/2}$. The injection from V into H is dense, continuous and compact (thanks to Rellich's lemma). Let A be the operator from V into V' defined by

$$\langle Au, v \rangle = ((u, v)), \qquad \forall u, v \in V. \tag{2.3}$$

For u, v, w in $H^1(\Omega)^d$ we set

$$\langle B(u, v), w \rangle = \int_\Omega u_j (D_j v_h)\, w_k \, dx.$$

Since no confusion is possible, we also denote by $|\cdot|$ the norm on $L^2(\Omega)$ and $\|\theta\| = |\nabla\theta|$ on $H_0^1(\Omega)$,

Let A_1 be the operator $-\Delta$ on $H_0^1(\Omega)$ and $B_1 \in \mathscr{L}(V \times H_0^1(\Omega), H^{-1}(\Omega))$:

$$\langle B_1(u, \theta), \eta \rangle = \int_\Omega u_i \frac{\partial \theta}{\partial x_i} \eta \, dx.$$

Let P be the orthogonal projector in $L^2(\Omega)^d$ onto H, we introduce the operator $\sum$ defined by

$$\sum \theta = P(\sigma\theta), \qquad \theta \in H_0^1(\Omega). \tag{2.4}$$

With the previous notations Eqs. (2.1) reads

$$\frac{du}{dt} + vAu + B(u, u) + \sum \theta = 0, \tag{2.5}$$

$$\frac{d\theta}{dt} + \kappa A_1 \theta + B_1(u, \theta) = 0. \tag{2.6}$$

We study now the Cauchy Problem for (2.5), (2.6) and set

$$\theta(0) = \theta^0 \in H_0^1(\Omega), \qquad u(0) = u^0 \in V. \tag{2.7}$$

Equations (2.5)–(2.7) possess at least a weak solution which satisfies the inequality of energy. Moreover, after some transient period, say $t \geqslant t_0$, u and θ become regular (i.e., $(u, \theta) \in C([t_0, +\infty[; H^1(\Omega)^{d+1}))$. Since Eqs. (2.5), (2.6) are autonomous, we can take the origin of time to be t_0 and then (u, θ) lies in $C^\infty(]0, +\infty[, H^2(\Omega)^{d+1})$ and

$$\int_0^{+\infty} (|Au| + |Au|^2 + |A_1\theta| + |A_1\theta|^2) \, ds < +\infty. \tag{2.8}$$

These results are well known in the case of Navier–Stokes equations (J. Leray [15, 16], J. L. Lions [17], R. Temam [21],...); the details of the similar results for the thermo-hydraulic equations are given in [6, 7].

Equations (2.5), (2.6) can also be written as a single equation but the operator $\mathscr{A} = \begin{pmatrix} vA & \Sigma \\ 0 & \kappa A_1 \end{pmatrix}$ which occurs, is not self-adjoint unless $\sigma = 0$. Thus the results of paragraph 1 seem not to apply to this situation. We shall show that we can, however, obtain the exact asymptotic behaviour of u and θ by proceeding directly.

THEOREM 2.1. *For every* $(u^0, \theta^0) \in V \times H_0^1(\Omega)$ *with* $\theta^0 \neq 0$, *the ratios* $(\log |u(t)|)/t$ *and* $(\log |\theta(t)|)/t$ *converge, respectively, to some limit denoted*

$-\nu\Lambda^{\infty}$ and $-\kappa\omega^{\infty}$, where either $\Lambda^{\infty} \geqslant \kappa\omega^{\infty}/\nu$, or Λ^{∞} is an eigenvalue of A smaller than $\kappa\omega^{\infty}/\nu$. Furthermore there exists $u^{\infty} \neq 0$ and $\theta^{\infty} \neq 0$ such that

$$\theta(t) \sim \theta^{\infty} e^{-\kappa\omega^{\infty}t} \qquad \text{in } H^2(\Omega),\ -\Delta\theta^{\infty} = \omega^{\infty}\theta^{\infty},\ \theta^{\infty} \in H_0^1(\Omega) \cap H^2(\Omega);$$

(i) *if $\nu\Lambda^{\infty} < k\omega^{\infty}$*

$$u(t) \sim u^{\infty} e^{-\nu\Lambda^{\infty}t} \qquad \text{in } H^2(\Omega)^d,$$

(ii) *if $\nu\Lambda^{\infty} \geqslant k\omega^{\infty}$ there exists some $p \in \mathbb{N}$ such that*

$$u(t) \sim -t^p u^{\infty} e^{-\nu\Lambda^{\infty}t} \qquad \text{in } H^2(\Omega)^d.$$

Remark 2.1. (i) u^{∞} is an eigenfunction of A associated with Λ^{∞} when $\Lambda^{\infty} < (\kappa/\nu)\,\omega^{\infty}$.

(ii) The equivalences obtained in Theorem 2.1 are also valid in $H^m(\Omega)$, $\forall m \geqslant 0$.

Proof. We begin by making two remarks. First, when $\sigma \neq 0$, the operator $\sum$ is injective and

$$\sum \theta = 0 \Rightarrow \theta = 0. \tag{2.9}$$

Second, the operator $\sum$ and A satisfy:

$$\sum A_1\theta = A \sum \theta, \qquad \forall\theta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.10}$$

(1) *The case $\nu \neq \kappa$.*

We set $S = A^{-1}\sum$ and we shall use the new function:

$$v(t) = u(t) - \frac{S\theta(t)}{\kappa - \nu}.$$

According to (2.5), (2.6) we have

$$\frac{dv}{dt} + \nu A v + B\left(v + \frac{S\theta}{\kappa - \nu}, v + \frac{S\theta}{\kappa - \nu}\right) - SB_1\left(v + \frac{S\theta}{\kappa - \nu}, \theta\right)$$
$$= ((\nu AS - \kappa SA_1)/\kappa - \nu + \textstyle\sum)\theta. \tag{2.11}$$

And now by (2.10), the r.h.s. of (2.11) vanishes.

Let us denote by $\phi = (v, \theta)$, $\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & (\kappa/\nu)A_1 \end{pmatrix}$,

$$\mathbf{B}(\phi_1, \phi_2) = \begin{bmatrix} B\left(v_1 + \frac{S\theta_1}{\kappa - \nu}, v_2 + \frac{S\theta_2}{\kappa - \nu}\right) - SB_1\left(v_1 + \frac{S\theta_1}{\kappa - \nu}, \theta_2\right) \\ B_1\left(v_1 + \frac{S\theta_1}{\kappa - \nu}, \theta_2\right) \end{bmatrix} \tag{2.12}$$

Equations (2.6) and (2.11) read

$$\frac{d\phi}{dt} + \nu \mathbf{A}\phi + \mathbf{B}(\phi, \phi) = 0. \tag{2.13}$$

LEMMA 2.1. *Under the assumptions of Theorem 2.1, there exists $(\tilde{\Lambda}, \tilde{\phi})$ such that $\tilde{\phi} \neq 0$ and*

$$e^{\nu\tilde{\Lambda}t}\phi(t) \qquad \text{converges to } (\tilde{v}, \tilde{\theta}) \equiv \tilde{\phi} \text{ in } D(A) \times D(A_1),\ A\tilde{v} = \tilde{\Lambda}\tilde{v},\ \frac{\kappa}{\nu} A_1\tilde{\theta} = \tilde{\Lambda}\tilde{\theta}.$$

Proof of Lemma 2.1. We denote by $\mathbf{V} = V \times H_0^1(\Omega)$, $\mathbf{H} = H \times L^2(\Omega)$. By classical estimates (see R. Temam [23] for instance) we have

$$|B(u_1, u_2)| + |B(u_2, u_1)| \leqslant C_1 |Au_1| \|u_2\|, \qquad \forall u_i \in D(A),$$
$$|B_1(u, \theta)| \leqslant C_2 |Au| \|\theta\|, \qquad \forall u \in D(A),\ \forall \theta \in H_0^1(\Omega),$$
$$|AS\theta| \leqslant C_3 |\theta|, \qquad \forall \theta \in H_0^1(\Omega).$$

It follows then from (2.8) that Theorem 1.2 applies and Lemma 2.1 is proved. ∎

By the same procedure it follows also from (2.8) that Theorem 1.1 applies to Eq. (2.6) and thus there exists $(\omega^\infty, \theta^\infty)$ such that

$$e^{\kappa\omega^\infty t}\theta(t) \qquad \text{converges to } \theta^\infty \text{ in } H_0^1(\Omega), \tag{2.14}$$

$$A_1\theta^\infty = \omega^\infty\theta^\infty \neq 0. \tag{2.15}$$

From (2.14) and Lemma 2.1 we deduce that

$$\begin{aligned} u(t) &= \tilde{v}e^{-\nu\tilde{\Lambda}t} + \frac{S\tilde{\theta}}{\kappa - \nu} e^{-\nu\tilde{\Lambda}t} + o(e^{-\nu\tilde{\Lambda}t}), \\ \theta(t) &= \tilde{\theta}e^{-\nu\tilde{\Lambda}t} + o(e^{-\nu\tilde{\Lambda}t}) = \theta^\infty e^{-\kappa\omega^\infty t} + o(e^{-\kappa\omega^\infty t}). \end{aligned} \tag{2.16}$$

(i) If $\nu\tilde{\Lambda} \neq \kappa\omega^\infty$, from (2.16) and $\theta^\infty \neq 0$ it results that $\tilde{\theta} = 0$. Then $\tilde{v} \neq 0$ and $\nu\tilde{\Lambda} < \kappa\omega^\infty$, $A\tilde{v} = \tilde{\Lambda}\tilde{v}$: in this case point (i) of Theorem 2.1 is proved with $\tilde{\Lambda} \equiv \tilde{\Lambda}$, $\tilde{v} \equiv v^\infty$.

(ii) If $\nu\tilde{\Lambda}=\kappa\omega^{\infty}$, from $(2.16)_2$ $\tilde{\theta}=\theta^{\infty}$ and from (2.16):

$$u(t)=\left(\tilde{v}+\frac{S\theta^{\infty}}{\kappa-\nu}\right)e^{-\kappa\omega^{\infty}t}+O(e^{-\kappa\omega^{\infty}t}).$$

When $\tilde{v}+(S\theta^{\infty}/\kappa_{-}\nu)\equiv u^{\infty}\neq 0$, point (ii) of Therem 2.1 is proved with $\tilde{\Lambda}\equiv\Lambda^{\infty}$ and $p=0$. When $\tilde{v}+(S\theta^{\infty}/\kappa-\nu)=0$, we need the subsequent terms in the expansion of $u(t)$. We introduce with C. Foias and J. C. Saut [5] the additive semi-group generated by the $\{\Lambda_j\}_{j\geqslant 1}$ (eigenvalues of $\mathbf{A}$): $\{\mu_j\}_{j\geqslant 1}$.

According to [5], there exist polynomes $\{\phi_j(t)\}$ of degree $j-1$ with values in $D(A)$ such that for every $N\geqslant 1$,

$$\phi(t)=\sum_{j=1}^{N}\phi_j(t)\,e^{-\mu_j t}+O(e^{-(\mu_N+\varepsilon_N)t}),\tag{2.17}$$

where $\varepsilon_N>0$ and $\mu_{j+1}>\mu_j$.

Henceforth, denoting $\phi_j=(v_j,\ \theta_j)$, $u(t)$ admits a similar asymptotic expansion with

$$u_j(t)=v_j(t)+\frac{S\theta_j(t)}{\kappa-\nu}.$$

If we show that there exists some integer j such that $u_j(t)$ is not identically zero then the point (ii) of Theorem 2.1 will be proved. We argue by contradiction, if

$$\forall\mu\in\mathbb{R}_+,\qquad u(t)=O(e^{-\mu t})\text{ in }D(A),\tag{2.18}$$

then from (2.5) and (2.14) we deduce that

$$\frac{du}{dt}\sim-\sum\theta^{\infty}e^{-\kappa\omega^{\infty}t}\text{ in }H.$$

In particular, $(d/dt)(u,\sum\theta^{\infty})\sim|\sum\theta^{\infty}|^2e^{-\kappa\tilde{\omega}t}$ then for t sufficently large (say $t\geqslant T$)

$$\frac{d}{dt}\left(u,\sum\theta^{\infty}\right)\leqslant-\frac{1}{2}\left|\sum\theta^{\infty}\right|^2e^{-\kappa\omega^{\infty}t},$$

by integration from t to $+\infty$ $(t\geqslant T)$,

$$\left(u(t),\sum\tilde{\theta}\right)=-\int_t^{+\infty}\frac{d}{ds}\left(u(s),\sum\theta^{\infty}(1)\right)ds\geqslant\frac{1}{2}\frac{|\sum\theta^{\infty}|}{\kappa\omega^{\infty}}e^{-\kappa\omega^{\infty}t},$$

which contradicts (2.18) and prove our claim.

(2) *The case* $\nu = \kappa$.

We now set $v(t) = u(t) + t \sum \theta(t)$.

As in the previous case we deduce point (ii) of Theorem 2.1 (for further details, the reader is refered to J. M. Ghidaglia [7]. Note that (2.9) is useful to prove that $\sum \theta^\infty \neq 0$ because $\theta^\infty \neq 0$).

In all cases, thanks to the asymptotic expansions we obtained, we deduce that the ratios $(\log |u(t)|)/t$ and $(\log |\theta(t)|)/t$ converge, respectively, to $-\nu\Lambda^\infty$ and $-\kappa\omega^\infty$. ∎

The space periodic case. Instead of Dirichlet boundary conditions, let us consider another boundary condition, namely the space periodic one:

$$u(x + L_k e_k) = u(x, t), \qquad \theta(x + L_k e_k, t) = \theta(x, t), \quad \forall x, t,$$

where e_1, –, e_d is a basis of $\mathbb{R}^d$ and L_k the period in the kth direction. In this case we obtain the same result (Theorem 2.1) with the only difference that if σ is parallel to one of the e_k's, $\sum$ is not injective.

Remark 2.2. If $\theta^0 = 0$, then by uniqueness of strong solutions, $\theta(t) = 0$ for every $t \geqslant 0$. Equations (2.5), (2.6) reduce to Navier–Stokes equations:

$$\frac{du}{dt} + \nu A u + B(u, u) = 0. \tag{2.19}$$

It is known (see C. Foias and J. C. Saut [3, 4]), that $u(t)$ behaves exactly like $u^\infty e^{-\nu\Lambda^\infty t}$ w.r.t. the H^1-norm where $A^\infty u^\infty = \Lambda^\infty u^\infty$. C. Guillopé [11] proved that this behaviour is also valid w.r.t. the H^m-norm for every m. Taking $E_m = H^m(\Omega)^d \cap V$, and applying Theorem 1.3 we recover these results by a slightly less technical proof.

Remark 2.3. The result mentioned in the previous remark extends to Navier–Stokes equations on a compact Riemannian manifold without boundary (such a situation occurs in Meteorology). In this case we can take $E_m = V_m = D(A^{m/2})$ and the proof can be done directly from Theorem 1.1 without using the results of section 1.3 (see J. M. Ghidaglia [7]).

2.2. *Magnetohydrodynamic Equations*

We consider the motion of a viscous incompressible and resistive fluid. The velocity $u(x, t)$, pressure $p(x, t)$ and the magnetic field $B(x, t)$ are determined by the equations (see L. Landau and E. Lifchitz [12])

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)\,u-\frac{1}{Re}\,\Delta u-(B\cdot\nabla)\,B+\nabla\left(p+\frac{B^2}{2}\right)=0,$$

$$\frac{\partial B}{\partial t}+(u\cdot\nabla)\,B+\frac{1}{Rm}\operatorname{curl}(\operatorname{curl} B)-(B\cdot\nabla)\,u=0,$$

$$\operatorname{div} u=0, \qquad \operatorname{div} B=0, \tag{2.20}$$

$$u=0, \qquad B\cdot n, \qquad \text{and} \qquad (\operatorname{curl} B)\times n=0 \text{ on } \Gamma,$$

where n is the unit outward normal on Γ.

The two positive numbers Re and Rm appearing in (2.20) are, respectively, the Reynolds number and the magnetic Reynolds number.

We assume that Ω is simply connected (this assumption is not essential) and we introduce the space

$$\tilde{V}=\{B\in H^1(\Omega)^d, \operatorname{div} B=0, B\cdot n=0\}.$$

If we denote by ϕ the pair $\{u, B\}$ and introduce the oeprators

$$\langle \mathscr{A}\phi_1, \phi_2\rangle=\frac{1}{Re}\int_\Omega \nabla u_1\cdot\nabla u_2+\frac{1}{Rm}\int_\Omega \operatorname{curl} B_1\cdot\operatorname{curl} B_2\,dx,$$

$$\langle \mathscr{B}(\phi_1, \phi_2), \phi_3\rangle=b(u_1, u_2, u_3)-b(B_1, B_2, u_3)$$
$$+b(u_1, B_2, B_3)-b(B_1, u_2, B_3).$$

Then $\mathscr{A}\in\mathscr{L}(V\times\tilde{V}, V'\times\tilde{V}')$ and $B\in\mathscr{L}((V\times\tilde{V})^2, V'\times\tilde{V}')$. Equations (2.20) can be written with these notations in the form (see J. M. Ghidaglia [6])

$$\frac{d\phi}{dt}+\mathscr{A}\phi+\mathscr{B}(\phi, \phi)=0. \tag{2.21}$$

The study of the Cauchy's problem for (2.21) leads to

$$\phi(0)=\phi^0\in V\times\tilde{V}. \tag{2.22}$$

Arguying as for thermo-hydraulic equations, we know that (2.21), (2.22) possess at least a weak solution which satisfies the inequality of Energy. We change the origin of time so that $\phi\in C^\infty(]0,+\infty[; H^2(\Omega)^{2d})$ and

$$\int_0^{+\infty}(|\mathscr{A}\phi|+|\mathscr{A}\phi|^2)\,dt<+\infty, \tag{2.23}$$

where $|\cdot|$ is the norm on $\mathscr{H}=H\times H$.

To apply the results of paragraph 1.3 we must check (1.59) to (1.63). These properties follow from regularity results of Stokes' operator and

Maxwell's operator (see R. Temam [21] and J. M. Ghidaglia [8]) and from various estimates on b (see R. Temam [23]). Now applying Theorem 1.3 it follows

THEOREM 2.2. *There exists an eigenfunction* (u^∞, B^∞) *of* $\mathscr{A}$ *associated with the eigenvalue* Λ^∞ *such that for every* $j \in \mathbb{N}$ *and* $m \in \mathbb{N}$,

$$\left(\frac{d^j u}{dt^j}, \frac{d^j B}{dt^j}\right) \sim (-\nu\Lambda^\infty)^j e^{-\nu\Lambda^\infty t} (u^\infty, B^\infty) \qquad \text{in } H^m(\Omega)^{2d}.$$

Remark 2.4. When $B^0 \neq 0$, applying Theorem 1.1 to the equation satisfied by B, we have the existence of $(\Lambda_B, \tilde{B})$ such that

$$B \sim e^{-\nu\Lambda_B t}\tilde{B} \qquad \text{in } H^1(\Omega)^d.$$

(i) If $u^\infty \neq 0$, then $\Lambda^\infty \leqslant \Lambda_B$ and $u \sim u^\infty e^{-\nu\Lambda^\infty t}$ in $H^1(\Omega)^d$.

(ii) If $u^\infty = 0$, then $B^\infty \neq 0$ and $\Lambda^\infty = \Lambda_B$.

If $(B^\infty \cdot \nabla) B^\infty$ is *not* a gradient, then using the expansion (2.17) we can argue as in the proof of Theorem 2.1 and conclude that there exist $p \in \mathbb{N}$ and $\tilde{u} \neq 0$ such that

$$u(t) \sim t^p \tilde{u} e^{-\nu\Lambda^\infty t} \qquad \text{in } H^1(\Omega)^d.$$

If $(B^\infty \cdot \nabla) B^\infty$ is a gradient, one needs to consider further terms in the expansion of B in (2.17).

2.3. *Applications to Scalar Inequalities*

Let Ω be a bounded open set in $\mathbb{R}^d$ with smooth boundary Γ. Let also be given a family of hermitian $n \times n$ tensors $a_{ij}(x, t)$, $(x, t) \in \bar{\Omega} \times \mathbb{R}_+$ such that

$$\forall i, j, \qquad a_{ij} \in W^{1,1}(\mathbb{R}_+ ; L^\infty(\Omega)), \tag{2.24}$$

and such that there exists $\eta > 0$,

$$a_{ij}(x, t) \geqslant \eta\delta_{ij} \qquad \text{uniformly w.r.t. } (x, t) \in \bar{\Omega} \times \mathbb{R}_+ .$$

We denote by $\alpha(t)$ the function of $L^1(\mathbb{R}_+)$:

$$\alpha(t) = \sup_{1 \leqslant i \leqslant j \leqslant n} \left| \frac{d}{dt} a_{ij}(\cdot, t) \right|_{L^\infty(\Omega)}, \tag{2.25}$$

by $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and finally $A(t)$ is defined by

$$\langle A(t)\, u, v \rangle_{H_0^1 H^{-1}} \equiv \int_\Omega a_{ij}(x, t) \frac{\partial u}{\partial x_j} \frac{\partial v}{dx_j}\, dx, \qquad \forall u, v \in H_0^1(\Omega).$$

Thanks to Poincaré's inequality, $H_0^1(\Omega)$ can be endowed with the norm $(\int_\Omega |\nabla u|^2\, dx)^{1/2}$ and it is classical that

$$D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega).$$

Applying Theorem 1.1 to this situation one obtains

THEOREM 2.3. *Let $u(x, t)$ be a solution of*

$$\int_\Omega \left| \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right)\right|^2 dx \leqslant n^2(t) \int_\Omega \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_k}\, dx$$

with $u \in \mathscr{C}(\mathbb{R}_+; H_0^1(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^2(\Omega))$, $(du(t)/dt) + A(t)\, u(t) \in L^2(\Omega)$ *for a.e.* $t \geqslant 0$, *where* $n \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$.

If α *(see (2.25)) satisfies* $\int_0^{+\infty} (\int_t^{+\infty} \alpha(s)\, ds)\, dt < +\infty$, *there exists* $u^\infty \in H^2(\Omega)$ *and* $\Lambda^\infty > 0$ *such that*

$$\lim_{t \to +\infty} \| e^{\nu \Lambda^\infty t} u(\cdot, t) - u^\infty(\cdot)\|_{H^1(\Omega)} = 0. \quad \blacksquare \tag{2.26}$$

This theorem improves, as far as we know, previous results (see P. D. Lax [13], M. H. Protter [20], M. Lees [14],...) where only lower bounds for $\int_\Omega |u|^2\, dx$ were derived. Let us also note that our assumptions on differentiability with respect to t on the coefficients $a_{ij}(\cdot, \cdot)$ are weaker than those used in these references.

Remark 2.5. According to (2.25), the family $\{A(t)\}_{t \geqslant 0}$ converges as $t \to +\infty$ with respect to the $\mathscr{L}(V, V')$-norm to some elliptic operator A^∞. Theorem 1.1 shows that $A^\infty u^\infty = \Lambda^\infty u^\infty$.

Remark 2.6. (Non self-adjoint case). Let $b_{ij}(x, t)$ satisfying $b_{ij} - b_{ji} \in \mathbb{R}$. Define $\beta(t) = \sup_{1 \leqslant i \leqslant n} |(\partial a_{ij}/\partial x_j)(\cdot, t)|_{L^\infty(\Omega)}$ with $a_{ij} \equiv (b_{ij} + b_{ji})/2$. Then if $u(x, t)$ is solution of

$$\int_\Omega \left| \frac{\partial u}{\partial t} - b_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}\right|^2 dx \leqslant m^2(t) \int_\Omega \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_h}\, dx$$

under the same hypothesis on a_{ij} and $n \equiv m + \beta$ the conclusion (2.26) of Theorem 2.3 holds.

Remark 2.7. (i) We have presented the case of homogeneous Dirichlet boundary condition. Theorem 2.3 is also applicable to the homogeneous Neumann boundary condition and to the more general case of oblique derivative problem (with suitable restrictions). Note that in this last case $D(A(t))$ could depend on t.

(ii) In this section we only dealt with second order operators. The results of the first part apply to higher order operators. As an example we can consider the Kurarmoto–Sivashinsky equation (see B. Nicolaenko, B. Scheurer, and R. Temam [19]):

$$\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2=0.$$

The boundary condition is periodicity w.r.t. $x \in]0, L[$. In the case: $0 < L < 2\pi$ it can be shown that there exists $a^\infty \in \mathbb{C}$, $k^\infty \in \mathbb{Z}$ such that

$$u(x, t) \sim \operatorname{Re}\left[a^\infty e^{[((2\Pi/L)k^\infty)^2 - ((2\Pi/L)k^\infty)^4]t + (2i\Pi k^\infty/L)x}\right]$$

with respect to the H^m-norm for every $m \geqslant 0$.

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