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Limiting phase trajectories as an alternative to nonlinear normal modes

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Abstract

We discuss a recently developed concept of limiting phase trajectories (LPTs) allowing a unified description of resonance, highly non-stationary processes for a wide range of classical and quantum dynamical systems with constant and varying parameters. This concept provides a far going extension and adequate mathematical description of the well-known linear beating phenomenon to a diverse variety of nonlinear systems ranging from classical multi-particle models to nonlinear quantum tunneling. While stationary (and non-stationary, but non-resonant) oscllations can be described in the framework of non-linear normal modes (NNMs) concept, it is not so in the considered case of resonant non-stationary processes. In the latter case which is characterized by intense energy exchange between different parts of a system, an additiional slow time scale appears. The energy exchange proceeds in this time scale and can be identified as strong modulation of the fast oscillations. The aforementioned resonant non-staionary prcesses include, e.g., targeted energy transfer, non-stationary vibrations of carbon nanotubes, quantum tunneling, auto-resonance and non-conventional synchronization. Besides the non-linear beating, the LPT concept allows one to find the conditions of transition from intense energy exchange to strongly localized (e.g. breather-like) excitations. A special mathematical technique based on the non-smooth temporal transformations leads to the clear and simple description of strongly modulated regimes. The role of LPTs in the theory of resonance non-stationary processes turns out to be similar to that of NNMs in stationary case.

As an example we present results of analytical and numerical study of planar dynamics of a string with uniformly distributed discrete masses without a preliminary stretching. Each mass is also affected by grounding support with cubic characteristic (which is equivalent to transversal unstretched string). We consider the most important case of low-energy transversal dynamics. This example is especially instructive because the considered system cannot be linearized. Adequate analytical description of resonance non-stationary processes which correspond to intensive energy exchange between different parts of the system (clusters) in low frequency domain was obtained in terms of LPTs. We have revealed also in these terms the conditions of energy localization on the initially excited cluster. Analytical results are in agreement with the results of numerical simulations. It is shown that the considered system can be used as an efficient energy sink.

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Keywords: Weakly Coupled Nonlinear Oscillators; Nonlinear Normal Mode; Energy Transfer; Energy Localization; Limiting Phase Trajectory; un-stretched string; acoustic vacuum; energy sink; Poincaré sections; transition to chaotic dynamics;

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1. Introduction

The accepted classification of the problems of mathematical physics (in application to models of the oscillation and wave theory) draws first of all a sharp distinction between linear and nonlinear model[1,2,3,4,5]. Such a distinction is caused by understandable mathematical reasons including absence of superposition principle in the nonlinear case. However it was recently shown[6,7,8,9,10,11,12,13] that in-depth physical analysis allows us to introduce other basis for classification of the oscillation problems, focusing on the difference between stationary (or non-stationary, but non-resonance) and resonance non-stationary processes. In the latter case a discrimination of linear and nonlinear problems is not fundamental if we deal with regular (non-chaotic) motions, and a specific technique has been developed which is efficient in the same degree for description of both linear and nonlinear resonance non-stationary processes. The existence of two alternative approaches in the framework of linear theory seems unexpected. Really, the superposition principle allows us to find a solution describing arbitrary non-stationary oscillations as a combination of linear normal modes which correspond to stationary processes. However, in the systems of weakly coupled oscillators, in which resonance non-stationary vibrations can occur, other type of fundamental solution exists. It describes strongly modulated non-stationary oscillations characterized by the maximum possible energy exchange between the oscillators or the clusters of the oscillators (effective particles). Such solutions are referred to as Limiting Phase Trajectories (LPTs). It was demonstrated that the LPT concept suggests a unified approach to the study of highly non-stationary processes in a wide range of classical and quantum dynamical systems with constant and time-varying parameters[12]. The development and use of analytical tool based on the LPT concept is motivated by the fact that resonance non-stationary processes occurring in a broad variety of finite dimensional physical models are beyond the well-known paradigm of nonlinear normal modes (NNMs), fully justified only for quasi-stationary and non-stationary, but non-resonance processes. While the NNMs approach has been proved to be an effective tool for the analysis of stationary regimes, their instability and bifurcations (see, e.g.,[2,3]), the use of the LPTs concept provides the adequate procedures for studying strongly non-stationary regimes as well as the transitions between different types of non-stationary motions, including propagation of localized excitations[7,8]. It makes possible, at the first time, to extend the notion of beating phenomenon to the systems with many degrees of freedom. Moreover, the concept of the limiting phase trajectories allows the prediction of the new type of synchronization (LPT-synchronization) in the system of weakly coupled autogenerators[9] and this is in contrast to the conventional NNM-synchronization[14]. Note that, along with the well-known asymptotic methods, the investigation of the phenomena under discussion has required the application of the special technique of non-smooth temporal transformations providing a simple description of strongly modulated and transient regimes. This technique was initially elaborated for description of vibro-impact (or close to them) processes[15].

In this paper we demonstrate the role of the LPT concept in Non-stationary Resonance Dynamics and its relation to the NNM concept on the example of unstretched string with grounding cubic supports undergoing predominant transversal motion. It was shown recently[16] that in the limit of low energy a fixed-fixed chain of linearly coupled particles performing in-plane transverse oscillations possesses strongly nonlinear dynamics and acoustics due to geometric nonlinearity, forming a nonlinear acoustic vacuum. This designation denotes the fact that the speed of sound as defined in the sense of classical acoustic theory is zero in that medium, so the resulting equations of motion lack any linear stiffness components. A significant feature of that system was the presence of strongly non-local terms in the governing equations of motion (in the sense that each equation directly involves all particle displacements), in-spite of the fact that the physical spring-mass chain has only local (nearest-neighbor) interactions between particles. These non-local terms constitute a time-dependent ‘effective speed of sound’ for this medium, which is completely tunable with energy. A rich structure of resonance manifolds of varying dimensions were identified in the nonlinear sonic vacuum, and 1:1 resonance interactions are studied asymptotically to prove the possibility of strong energy exchanges between nonlinear modes.

One of the distinctive features of a chain without grounding support was that its nonlinear normal modes – NNMs[3] could be exactly determined. Moreover, the analysis has shown that the number of NNMs in the sonic vacuum is equal to the dimensionality of the configuration space and that no NNM bifurcations are possible. In addition, the most intensive 1:1 resonance intermodal interaction was the one realized by the two NNMs with the highest wave numbers. However, the unstretched string model considered in[16] is in some sense a special case, since one of the most significant features of dynamical systems with homogeneous potentials is that the number of NNMs may exceed the number of degrees of freedom due to mode bifurcations[1]. One can expect that such NNM bifurcations will also lead to drastic modification of the non-stationary resonance dynamics of the sonic vacuum described by LPTs. Thus it is of great interest to consider an extension of the nonlinear sonic vacuum developed in[16] so that the modified system has the capacity to undergo NNM bifurcations. Such a study can provide us with the opportunity to investigate how these bifurcations can affect the non-stationary resonant dynamics corresponding to resonant energy exchange and localization.

These questions were discussed in our paper devoted to unstretched string with grounding support but carrying only two discrete masses[17]. Here we present an extension to a more complicated system with an arbitrary but finite number of discrete masses.

2. The model and equations of motion

Let's consider an unstretched string with uniformly distributed equal masses and returning forces, proportional to cubes of deformations (see Fig. 1). The equations of motion are

$$\begin{aligned} m\frac{d^2U_j}{dt^2} + T_j\cos\theta_j - T_{j+1}\cos\theta_{j+1} &= 0; j = 1,\ldots,N, \\ m\frac{d^2V_j}{dt^2} + cV_j^3 + T_j\sin\theta_j - T_{j+1}\sin\theta_{j+1} &= 0; j = 1,\ldots,N, \end{aligned} \tag{1}$$

with U_j, V_j being the longitudinal and transversal displacements of j-th mass respectively; θ_j is angle between j-th segment and its equilibrium position. Tensile forces are proportional to deformations and may be written as

$$T_j = K\frac{1}{l}\left[(U_j - U_{j-1}) + \frac{1}{2l}(V_j - V_{j-1})^2\right],$$

with l being equilibrium length of one segment and K being stiffness coefficient.

The mechanism of nonlocal force formation was discussed in the paper[16]. According to this mechanism, the tensile forces in all segments are approximately equal to their mean value:

$$T =< T_j >= \frac{1}{N+1}K\frac{1}{2l^2}\sum_{s=0}^{N}(V_{s+1} - V_s)^2$$

Introducing the "slow" time scale $\tau_0 = \varepsilon t$, where small parameter ε describes the relative smallness of tranversal frequencies ($\varepsilon = a/l$, with a being an amplitude of transversal oscillations), we obtain the following equation system for transversal motion (parameter $\mu = \frac{K}{Cl^3}$ describes relation between contributions of string itself and grounding supports):

$$\frac{d^2v_j}{d\tau_0^2} + \frac{1}{\mu}v_j^3 + \frac{1}{2(N+1)}\sum_{s=0}^{N}(v_{s+1} - v_s)^2\left(2v_j - v_{j+1} - v_{j-1}\right) = 0; j = 1,\ldots,N, v_0(0) = v_{N+1}(0) = 0, \tag{2}$$

where $V_j = \varepsilon v_j$, v_j are normalized displacements, and $\omega_0 = \sqrt{\frac{K}{lm}}$.

3. Continuum limit

If we deal with the case $N >> 1$, the continuum limit is an adequate approximation. There are two possibilities. In the case of long-wave length dynamics we introduce continuous length parameter ξ instead of discrete index parameter j and continuous function $v(t,\xi)$ (instead of $v_j(t)$). Then the equations of motion turn into following PDE (L is dimensionless length, $L = N + 1$):

$$\frac{\partial^2 v}{\partial\tau_0^2} + \frac{1}{\mu}v^3 - \frac{1}{2L}\left[\int_0^L\left(\frac{\partial v}{\partial x}\right)^2 dx\right]\cdot\frac{\partial^2 v}{\partial\xi^2} = 0; v(t,0) = v(t,L) = 0. \tag{3}$$

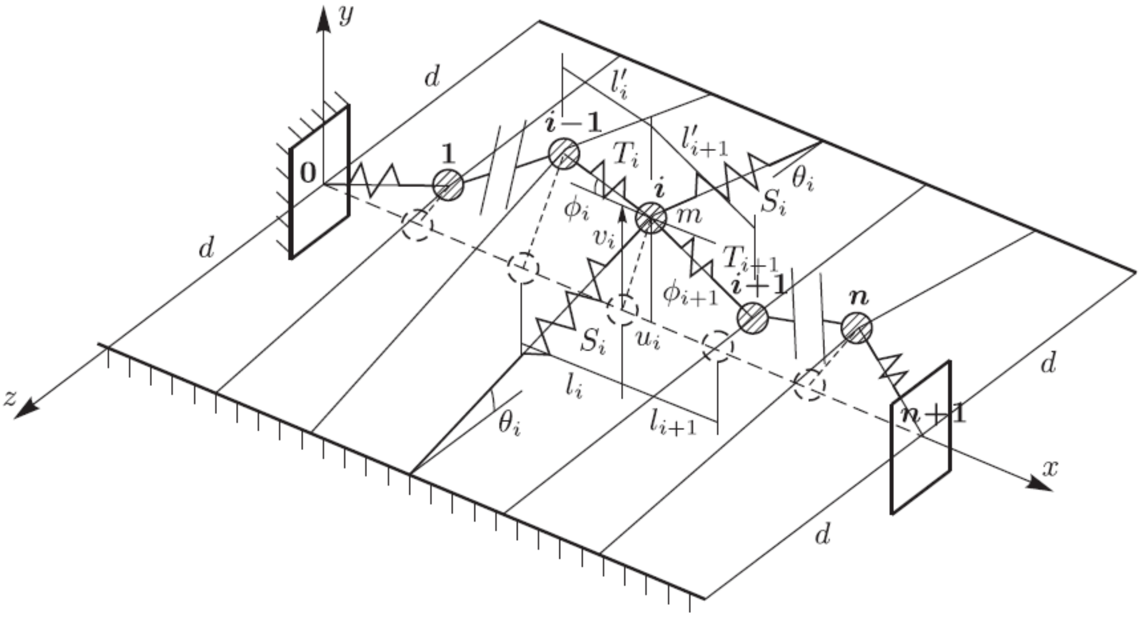

Fig. 1. Oscillator chain with elastic support

However, for short-wave oscillations the value $v_j(\tau_0)$ changes quickly as a function of j, and the continuum description of displacements is not more valid. If we introduce a change such as $v_j = (-1)^j w_j$ (w_j are "invert variables "), continuum description is again possible. The equations of motion transform in following manner:

$$\frac{d^2 w_j}{d\tau_0^2} + \frac{1}{\mu} w_j^3 + \frac{1}{2(N+1)} \sum_{s=0}^{N} (w_{s+1} + w_s)^2 \left(2w_j + w_{j+1} + w_{j-1}\right) = 0; w_0 = w_{N+1} = 0. \tag{4}$$

These equations describe a modulation of quickly changing (along the length of the string) displacements. A continuum version of the equations of motions for invert variables is as follows:

$$\frac{\partial^2 w}{\partial \tau_0^2} + \frac{1}{\mu} w^3 + \frac{1}{2L} \int_0^L \left(2w(x, \tau_0) + \frac{\partial w(x, \tau_0)}{\partial x}\right)^2 dx \left(\frac{\partial^2 w}{\partial \xi^2} + 4w\right) = 0; w_0 = w_L = 0. \tag{5}$$

4. Two-mode approximation (case of short waves)

We consider the following two-mode approximation

$$w = a_m(\tau_0) \sin \frac{m\pi\xi}{L} + a_n(\tau_0) \sin \frac{n\pi\xi}{L}; m = 2, n = 1,$$

which correspond to interaction of only the two highest NNMs. We denote for convenience $n_L = \frac{\pi n}{N+1}$ and $m_L = \frac{\pi m}{N+1}$.

The equations obtained by projecting onto these modes are

$$\frac{d^2a_m}{d\tau_0^2}+\frac{3}{4\mu}\left(a_m^3+2a_ma_n^2\right)+\frac{1}{4}\left[\left(4+m_L^2\right)a_m^2+\left(4+n_L^2\right)a_n^2\right]\left(4-m_L^2\right)a_m=0,$$
$$\frac{d^2a_n}{d\tau_0^2}+\frac{3}{4\mu}\left(a_n^3+2a_na_m^2\right)+\frac{1}{4}\left[\left(4+m_L^2\right)a_m^2+\left(4+n_L^2\right)a_n^2\right]\left(4-n_L^2\right)a_n=0. \qquad (6)$$

Because the considered NNMs interact strongly, it is necessary to introduce "cluster" variables $Y_1=\frac{a_m+a_n}{2}, Y_2=\frac{a_m-a_n}{2}$, characterizing the dynamics of weakly interacting domains of the system ("clusters").

In these new variables we deal with the system:

$$\frac{d^2Y_1}{d\tau_0^2}+MY_1\left(3Y_1^2+Y_2^2\right)+\frac{A}{8}Y_1^3+\frac{3C}{8}Y_1^2Y_2+\frac{B}{8}Y_1Y_2^2+\frac{C}{8}Y_2^3=0,$$
$$\frac{d^2Y_2}{d\tau_0^2}+MY_2\left(3Y_2^2+Y_1^2\right)+\frac{A}{8}Y_2^3+\frac{3C}{8}Y_2^2Y_1+\frac{B}{8}Y_2Y_1^2+\frac{C}{8}Y_2^3=0. \qquad (7)$$

We consider 1:1 resonance on the frequency ω and re-write the system (7) in the following form:

$$\frac{d^2Y_1}{d\tau_0^2}+\omega^2Y_1=-\varepsilon_1\gamma\left(MY_1\left(3Y_1^2+Y_2^2\right)+\frac{A}{8}Y_1^3+\frac{3C}{8}Y_1^2Y_2+\frac{B}{8}Y_1Y_2^2+\frac{C}{8}Y_2^3-\omega^2Y_1\right),$$
$$\frac{d^2Y_2}{d\tau_0^2}+\omega^2Y_2=-\varepsilon_1\gamma\left(MY_2\left(3Y_2^2+Y_1^2\right)+\frac{A}{8}Y_2^3+\frac{3C}{8}Y_2^2Y_1+\frac{B}{8}Y_2Y_1^2+\frac{C}{8}Y_2^3-\omega^2Y_2\right). \qquad (8)$$

Combination in the right hand side should be small (since we consider a system near resonance). It is achieved by introducing the small parameter $\varepsilon_1 \ll 1$. We introduce a parameter $\gamma=\varepsilon_1^{-1}$ to provide an equivalence of systems (7) and (8). We introduce also complex variables: $\psi_j=dY_j/d\tau_0+i\omega Y_j$ (and $\psi_j^*=dY_j/d\tau_0-i\omega Y_j$), $j=1,2$. Then

$$Y_j=\frac{\psi_j-\psi_j^*}{2i\omega},\ \frac{dY_j}{d\tau_0}=\frac{\psi_j+\psi_j^*}{2},\ j=1,2. \qquad (9)$$

Applying a procedure of multiscale expansion we introduce a super-slow time scale $\tau_1=\varepsilon_1\tau_0$. Taking into account that $\frac{d}{d\tau_0}=\frac{\partial}{\partial\tau_0}+\varepsilon_1\frac{\partial}{\partial\tau_1}+\ldots$, we are looking for a solution in the following form: $\psi_j=\psi_{j0}+\varepsilon_1\psi_{j1}+\ldots$, $j=1,2$. We substitute this expansion into the system (8), keeping in mind (9) and equating the terms of each order by parameter ε_1 to zero. In the first approximation we get: $\psi_{j0}=e^{i\omega\tau_0}\varphi_j(\tau_1)$, $j=1,2$. We substitute this expression into the equation for complex variables and consider next order of smallness. To avoid appearance of secular terms while integrating over time τ_0, coefficient before $e^{i\omega\tau_0}$ should be zero. Thus we obtain the system which determines the "amplitude" functions $\varphi_j(\tau_1)$, $j=1,2$ in super-slow time τ_1:

$$\frac{d\varphi_j}{d\tau_1}=\gamma\left(3\frac{Mi}{8\omega^3}\left(9\varphi_j|\varphi_j|^2+\varphi_j^*\varphi_{3-j}^2+2\varphi_j|\varphi_{3-j}|^2\right)+\right.$$
$$\left.+\frac{1}{64}\left(3A\varphi_j|\varphi_j|^2+3C\varphi_{3-j}^*\varphi_j^2+3C\varphi_{3-j}|\varphi_{3-j}|^2+6C\varphi_{3-j}|\varphi_j|^2+2B\varphi_j|\varphi_{3-j}|^2\right)-\frac{i\omega}{2}\varphi_j\right),\ j=1,2.$$

This techniques is described in details in the paper[18].

The obtained system is integrable because besides the integral of energy it possesses a second integral

$$N=|\varphi_1|^2+|\varphi_2|^2, \qquad (10)$$

what can be verified directly. Due to existence of second integral it is possible to introduce angular variables:

$$\varphi_1=\sqrt{N}\cos\theta e^{i\delta_1};\ \varphi_2=\sqrt{N}\sin\theta e^{i\delta_2}.$$

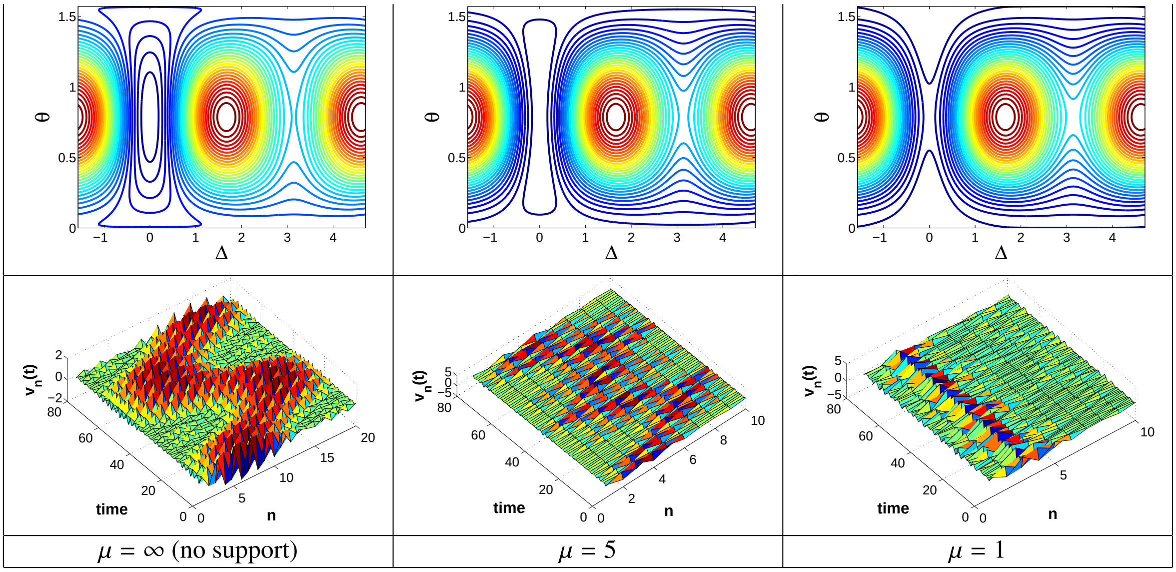

Fig. 2. Energy exchange and energy localization in initial variables and corresponding phase plane in angular variables.

Here θ and $\Delta = \delta_1 - \delta_2$ characterize relationship between amplitudes of two clusters and the phase shift between them.

For these variables, we obtain following system:

$$\begin{aligned} \frac{1}{2}\sin 2\theta\dot{\Delta} &= M\left(-\frac{7}{4}\sin 4\theta + \frac{1}{4}\sin 4\theta\cos 2\Delta\right) - \frac{1}{2}\left(\frac{3A}{4}\sin 4\theta - \frac{B}{4}\sin 4\theta(\cos 2\Delta + 2) - 3C\cos 2\theta\cos\Delta\right) \\ \dot{\theta} &= \frac{3}{2\mu}\frac{4 - n_L^2}{4 + n_L^2}\sin 2\theta\sin 2\Delta + \frac{1}{2}\left(B\sin\theta\cos\theta\sin 2\Delta + 3C\sin\Delta\right). \end{aligned} \tag{11}$$

Where overdot denotes differentiation with respect to normalized (for convenience) time $\tau_1^* = \frac{\gamma N}{32\omega^3}\tau_1$.

This first-order system of real equations possesses the energy integral:

$$\begin{aligned} H = -M\left(\frac{9}{2}\sin^4\theta + \frac{9}{2}\cos^4\theta + \frac{1}{4}\sin^2 2\theta(\cos 2\Delta + 2)\right) + \\ \frac{1}{2}\left(-\frac{3A}{2}(\sin^4\theta + \cos^4\theta) - 3C\sin 2\theta\cos\Delta - B\sin^2\theta\cos^2\theta(\cos 2\Delta + 2)\right), \end{aligned} \tag{12}$$

hence it is integrable. In angular variables the stationary (equilibrium) points correspond to NNMs of initial system.

5. Phase plane

Due to existence of the integral of motion the simplest way of investigation is to study a topology of phase plane. By comparing phase planes for different values of parameter μ we reveal two dynamical transitions, which are reflected in the phase plane topology. The first one is caused by instability and bifurcation of the highest NNMs. When $\mu > \mu_{cr1}$ (as in a particular case when $1/\mu = 0$, that is there is no grounding supports), there are four critical points. When $\mu < \mu_{cr1}$, a bifurcation is observed: the point ($\theta = \pi/4, \Delta = 0$) (corresponding to in-phase motion of clusters) becomes unstable and two additional equilibrium points appear. The first topological transition caused by bifurcation of the NNM and appearance of new NNMs, is a significant stage of the system evolution (in parametric space). This stage precedes to second topological transition which leads to spontaneous energy localization on initially excited cluster, when $\mu < \mu_{cr2}$ (complete energy exchange becomes impossible). It is possible to find a critical value μ_{cr2} analytically

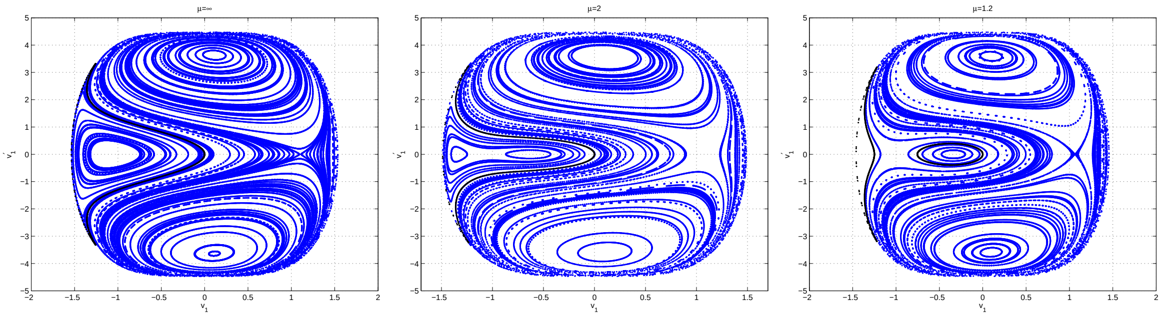

Fig. 3. Poincares' section for the case of (a)no support ($\mu = \infty$); (b) $\mu = 2$; (c) $\mu = 1.2$.

from the condition of coincidence of separatrix and LPT: $H(\pi/4, 0) = H(0, \pi)$. Hence

$$\mu_{cr2} = \frac{6\left(n_L^4 - 4\right)}{(n_L^2 + 4)(16m_L^2 - 3m_L^4 - 16n_L^2 + 2m_L^2 n_L^2 + n_L^4)}.$$

For two highest NNMs $\mu_{cr2} = 1.35$ if $N = 10$.

The obtained results are confirmed by numerical integration of the initial system (2) with initial conditions corresponding to excitation of one cluster which is formed by resonance interaction of the two highest modes ($v_j = \sin\frac{\pi j(N-1)}{N+1} + \sin\frac{\pi jN}{N+1}$). When $\mu < \mu_{cr2}$ the energy localization is realized; when $\mu > \mu_{cr2}$ we observe complete energy exchange.

6. Poincaré section

Since the initial system after projection onto the two highest NNMs remains non-integrable, it is of great interest to investigate Poncaré sections corresponding to (7).

Consider the set of trajectories with same value of energy. The section plane is taken as $\dot{v}_2 = 0$. The points of intersection of trajectories and section plane are projected onto the plane $(v_1, \dot{v}_1)$. The LPT can be obtained from trajectories corresponding to excitation of one cluster.

The obtained sections are shown in the Fig. 3. One can note that these sections correspond to phase portraits presented above. It's very important that topological transitions predicted in the course of asymtotical analysis, are also observed in the initial non-integrable system, and that their appearance can be predicted analytically. It's rather unexpected that there is no chaotic behaviour for any value of μ. This can be explained by closeness to the degenerate system[16].

7. Conclusions

Adequate analysis of strongly modulated processes in nonlinear dynamics goes out of framework of the existing paradigm. The concept of Limiting Phase Trajectories which turns out to be an alternative to the Nonlinear Normal Modes concept gives an efficient tool for such analysis. The mathematical content of this concept is closely connected with non-smooth transformations which were used earlier in the study of vibro-impact processes.

In particular, we reveal that for a string with arbitrary number of discrete masses in conditions of acoustic vacuum, there exists a regular regime of complete energy exchange between different domains of the string (clusters) and nonstationary energy localization on the excited cluster, alongside NNMs and stationary energy localization. These regimes have been described analytically, and corresponding thresholds in parametric space were defined. Possibility of existence of different regimes in the same system is due to nonlinear grounding support, which also enables widening of the resonance domain. Therefore, the considered string can be used as an efficient energy sink.

Finally, we conclude with a tabulated comparison of two basic concepts of finite dimensional nonlinear dynamics.

Table 1. Comparison of two nonlinear dynamics basic concepts.

Nonlinear Normal Mode	Limiting Phase Trajectories
elementary stationary process	elementary strongly non-stationary process
is not involved into processes of energy exchange	describes maximum possible (under given conditions) energy exchange between different parts of the system
can be localized (localized NNM – stationary localization)	can be localized (localized LPT – non-stationary localization)
can bifurcate (transformation to localized NNMs)	can bifurcate (transformation to localized LPT)
in the presence of forcing is transformed into steady-state oscillations	in the presence of forcing is transformed into LPT describing maximum possible taking away from the energy source
can be attractor in active system	can be attractor in active system
can be presented by a sine-like basic functions	can be presented by non-smooth basic functions

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