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LINEAR ALGEBRA AND ITS APPLICATIONS

Inversion formulas and fast algorithms for Löwner–Vandermonde matrices

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Abstract

Matrices consisting of two parts one of Vandermonde and the other of Löwner type are considered. Inversion formulas (also for one-sided inverses) are presented. Moreover, in the nonsingular case the parameters of the inversion formulas are described in terms of the evaluation of numerator and denominator polynomials which constitute the solution of associated rational interpolation problems. These results can be used to establish an algorithm for the solution of Löwner–Vandermonde systems of equations with $O(n \log^2 n)$ complexity. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

To introduce the matrix class under consideration we associate with given (complex) vectors

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$$w^+ := [w_i]_{i=1}^n, \qquad w^- := [w_{-i}]_{i=1}^k,$$
$$z^+ := [z_i]_{i=1}^n, \qquad z^- := [z_{-i}]_{i=1}^k \quad (z_i \neq z_j \text{ for all } i \neq j),$$

the $k \times n$ Löwner matrix

$$L_{kn}(w,z) = \left[\frac{w_{-i} - w_j}{z_{-i} - z_j}\right]_{i=1\ j=1}^{k\quad n}, \quad w := \begin{bmatrix} w^- \\ w^+ \end{bmatrix}, \ z := \begin{bmatrix} z^- \\ z^+ \end{bmatrix},$$

as well as the $l \times n$ Vandermonde matrix

$$V_{ln}(z^+) = [z_j^i]_{i=0\ j=1}^{l-1\ n}.$$

Composed matrices of the form

$$A_n^{lk} := \begin{bmatrix} V_{ln}(z^+) \\ L_{kn}(w,z) \end{bmatrix} \tag{1.1}$$

(as well as their transposes) are called Löwner–Vandermonde matrices (with respect to z and w).

Clearly, in case $l = 0$ ($k = 0$) we have pure Löwner (Vandermonde) matrices. Putting

$$w_i = \begin{cases} 0 & \text{for } i > 0 \\ 1 & \text{for } i < 0 \end{cases},$$

the Löwner matrix has the form of a Cauchy matrix,

$$L_{kn}(w,z) = C_{kn}(z) := \left[\frac{1}{z_{-i} - z_j}\right]_{i=1\ j=1}^{k\quad n}.$$

In this way, connections with the results concerning Cauchy–Vandermonde matrices [1,2] are established.

Before we formulate our main result let us introduce the following vectors belonging to $\mathbb{C}^n$

$$\mathbf{1}_n := [1, 1, \dots, 1]^{\mathrm{T}}, \qquad \mathbf{O}_n = [0, 0, \dots, 0]^{\mathrm{T}}, \qquad e_l^n = [\delta_{il}]_{i=1}^n,$$

where δ_{il} denotes the Kronecker delta as well as the following $n \times n$ matrices:

the backward shift matrix $S_n := [\delta_{i,j-1}]_{i,j=1}^n$,
the counteridentity $J_n := [\delta_{i,n+1-j}]_{i,j=1}^n$,
the diagonal matrix $\mathrm{diag}(z^+) := [z_i\delta_{ij}]_{i,j=1}^n$ and
the triangular Hankel matrix

$$H_n(z^+) = \begin{bmatrix} z_1 & z_2 & z_3 \cdots & z_n \\ z_2 & z_3 & & \\ z_3 & & \iddots & \\ \vdots & \iddots & & \\ z_n & & & 0 \end{bmatrix}.$$

Theorem 1.1. *Let $k+l=n, l\neq 0$, and let the equations*

$$(A_n^{lk})^{\mathrm{T}}y_1 = w^+, \qquad (A_n^{lk})^{\mathrm{T}}y_2 = [z_i^l]_{i=1}^n \tag{1.2}$$

be solvable. Then A_n^{lk} is nonsingular, and its inverse is completely determined by these solutions $y_i := [y_{ij}]_{j=1}^n$ and by the solutions x_i $(i=1,2)$ of

$$A_n^{lk}x_1 = \begin{bmatrix} \mathbf{O}_l \\ \mathbf{1}_k \end{bmatrix}, \qquad A_n^{lk}x_2 = e_l^n \tag{1.3}$$

via the formula

$$(A_n^{lk})^{-1} = \sum_{i=1}^{2} \operatorname{diag}(x_i)\ [\, V_{ln}^{\mathrm{T}}(z^+)\ C_{kn}^{\mathrm{T}}(z)\,] \begin{bmatrix} H_l(\hat{y}_i) & \mathrm{O} \\ \mathrm{O} & \operatorname{diag}(\check{y}_i) \end{bmatrix}. \tag{1.4}$$

Here

$$\hat{y}_1 := -[\, y_{12},\ldots,y_{1l},\ 0\,]^{\mathrm{T}}, \qquad \hat{y}_2 := -[\, y_{22},\ldots,y_{2l},\ -1\,]^{\mathrm{T}},$$
$$\check{y}_i := -\,[y_{i,l+1},\ y_{i,l+2},\ldots,y_{in}\,]^{\mathrm{T}} \quad (i=1,2).$$

If the solutions x_i, y_i $(i=1,2)$ are given the formula (1.4) can be used to solve linear systems of equations with Löwner–Vandermonde coefficient matrices. The multiplication of $(A_n^{lk})^{-1}$ in the form (1.4) by a vector consists of several steps and the complexity of each of them is $\mathrm{O}(n\log^2 n)$ on a sequential computer and $\mathrm{O}(n)$ on an n-processor parallel computer (comp. [1]).

For the evaluation of the special solutions x_i, y_i $(i=1,2)$ of (1.2) and (1.3) we can utilize the three-term recursion presented in [3–5]. But these "Levinson - type" algorithms have complexity $\mathrm{O}(n^2)$.

Section 5 is dedicated to the connection of the solutions of (1.2), (1.3) and the solutions of certain interpolation problems. Using these results we come to an algorithm of $\mathrm{O}(n\log^2 n)$ complexity also for the determination of the solutions of (1.2) and (1.3). This leads finally to an $\mathrm{O}(n\log^2 n)$ solution algorithm for Löwner–Vandermonde systems of equations.

Note that the study of connections of Löwner matrices and interpolation has a long history which goes back to the beautiful paper of Löwner [6]. Several authors continued and generalized Löwner's results; see [7–12]. In principle, there is the following connection: The parameters of the Löwner matrix determine interpolation data (z_i are the nodes and w_i the prescribed values) and the numerator or denominator of the rational function (with properly given bounds on degrees) are connected with solutions of homogeneous systems or other special systems of equations with the given Löwner matrix as coefficient matrix. Besides that, solvability of interpolation is connected with rank properties of the matrix.

While Löwner matrices require the same degree bound for the numerator and denominator, in the present paper we show that similar connections hold for Löwner–Vandermonde matrices but the degree bounds have difference l (which equals the number of Vandermonde rows), see [3,4]. The formulas we give here for x_i and y_i, $i = 1, 2$, have a similar form as those occuring in [13].

In Section 3 we give criteria for A_n^{lk} be of full rank and formulas for one-sided inverses of A_n^{lk}. The proof of Theorem 1.1 is presented in Section 4.

Finally, in Section 6 we shortly discuss the special cases of Löwner, Vandermonde, Cauchy and Cauchy–Vandermonde matrices.

2. Preliminaries

One of our main tools used here is (in the terminology of [14–17]) the UV-displacement rank of $A := A_n^{lk}$ for

$$U = \operatorname{diag}(z^+), \qquad V = \begin{bmatrix} S_l & 0 \\ 0 & \operatorname{diag}(z^-) \end{bmatrix}. \tag{2.1}$$

Clearly U is of order n and V of order $m := l + k$. A straightforward calculation shows the following.

Lemma 2.1. *Let $A = A_n^{lk}$ be the $m \times n$ Löwner–Vandermonde matrix (1.1), $k + l = m$, and U, V be defined in (2.1).*

Then

$$AU - VA = \sum_{i=1}^{3} g_i f_i^{\mathrm{T}}, \tag{2.2}$$

where

$$g_1 = \begin{bmatrix} \mathbf{O}_l \\ \mathbf{1}_k \end{bmatrix}, \qquad g_2 = e_l^n, \qquad g_3 = \begin{bmatrix} \mathbf{O}_l \\ -w^- \end{bmatrix}, \tag{2.3}$$

$$f_1 = w^+, \qquad f_2 = [\, z_1^l, \, z_2^l, \ldots, z_n^l \,]^{\mathrm{T}}, \qquad f_3 = \mathbf{1}_n. \tag{2.4}$$

(In case $l = 0$ we have to leave out the vectors g_2 and f_2, whereas in case $k = 0$ we leave out g_1, g_3, f_1, f_3.)

If rank $A = m \leqslant n$ the matrix A is right invertible, that means there exists a matrix A_{R} such that $A \cdot A_{\mathrm{R}} = I_m$. To construct a formula for A_{R} we consider any solution x_i of

$$Ax_i = g_i \quad (i = 1, 2, 3) \tag{2.5}$$

and denote by X the $n \times 3$ matrix

$$X := \operatorname{row}(x_i)_1^3.$$

In case rank $A = n \leqslant m$ the matrix A is left invertible, that is there exists a matrix A_{L} such that $A_{\mathrm{L}} \cdot A = I_n$. To construct a formula for A_{L} we consider here any solution y_i of

$$A^{\mathrm{T}} y_i = f_i \quad (i = 1, 2, 3) \tag{2.6}$$

and designate

$$Y := \mathrm{row}(y_i)_1^3. \tag{2.7}$$

Clearly, in case $l > 0$, we may choose

$$y_3 = e_1^n. \tag{2.8}$$

In the case $m = n$ Eqs. (1.2) and (1.3) are just Eqs. (2.6) and (2.5) for $i = 1, 2$. Moreover,

$$G := \mathrm{row}(g_i)_1^3, \qquad F := \mathrm{row}(f_i)_1^3.$$

In Theorem 3.2 below we assume $w_i \neq 0$ for all $1 \leqslant i \leqslant n$ which is no loss of generality. (Otherwise we consider instead of w the vector

$$w_c := \begin{bmatrix} w_c^- \\ w_c^+ \end{bmatrix}, \qquad w_c^- := [w_- + c]_{i=1}^k, \qquad w_c^+ := [w_i + c]_{i=1}^n$$

defining the same Löwner-matrix $L_{kn}(w, z) = L_{kn}(w_c, z)$, but we choose the constant c such that $w_i + c \neq 0$ for all $1 \leqslant i \leqslant n$.)

3. The full rank cases – construction of one-sided inverses

In formulas for one-sided inverses, we need the following notion:

Given a vector $b \in \mathbb{C}^n$ and an $n \times n$ matrix A by the *Krylov matrix* $\mathscr{K}(A, b, m)$ is meant the $n \times m$ matrix

$$\mathscr{K}(A, b, m) := [\, b,\ Ab,\ A^2 b, \ldots, A^{m-1} b \,].$$

Theorem 3.1. *Let $A = A_n^{lk}$ be the $m \times n$ Löwner–Vandermonde matrix (1.1), $k + l = m$. Assume that Eq. (2.5) are solvable. Then A has full (row) rank $m \leqslant n$, and the matrix*

$$A_{\mathrm{R}} = [\, K_2 J_l \; K_1 V_{kk}^{-\mathrm{T}}(z^-)] \tag{3.1}$$

is a right inverse to A, where

$$K_1 := \mathscr{K}((U - XF^{\mathrm{T}}), x_1, k) \quad \textit{and} \quad K_2 := \mathscr{K}(U - XF^{\mathrm{T}}), x_2, l).$$

Proof. From equality (2.2) it becomes clear that

$$VA = A(U - XF^{\mathrm{T}}),$$

and, consequently, we get for $j = 1, 2, \ldots$

$$V^j A = A(U - XF^{\mathrm{T}})^j.$$

In particular, we observe that

$$V^j g_1 = A(U - XF^{\mathrm{T}})^j x_1 \quad \text{for } j = 0, 1, \ldots, k-1$$

and

$$V^j g_2 = A(U - XF^{\mathrm{T}})^j x_2 \quad \text{for } j = 0, 1, \ldots, l-1,$$

which leads to

$$A\,[K_1 K_2] = \begin{bmatrix} 0 & J_l \\ V_{kk}^{\mathrm{T}}(z^-) & 0 \end{bmatrix}. \tag{3.2}$$

Since the matrix on the right-hand side of (3.2) is nonsingular A has to be of full (row) rank, which means A is right invertible. Postmultiplying (3.2) from the right by

$$\begin{bmatrix} 0 & V_{kk}^{-\mathrm{T}}(z^-) \\ J_l & 0 \end{bmatrix}$$

formula (3.1) is obtained. This completes the proof. □

Theorem 3.2. *Let $A = A_n^{lk}$ be the $m \times n$ Löwner–Vandermonde matrix (1.1). Assume that Eq. (2.6) for $i = 1, 2$ are solvable. Then the following matrix A_{L} is a left inverse to A.*

$$A_{\mathrm{L}} = \operatorname{diag}\left(\frac{1}{w^+}\right) \cdot V_{nn}^{-1}(z^+) \cdot K^{\mathrm{T}}, \tag{3.3}$$

where

$$\frac{1}{w^+} := \left[\frac{1}{w_i}\right]_{i=1}^{n},$$

Y is defined in Eq. (2.7), and

$$K := \mathscr{K}((V^{\mathrm{T}} + YG^{\mathrm{T}}), y_1, n).$$

Proof. According to Eq. (2.2) we have

$$AU^j = (V + GY^{\mathrm{T}})^j A \quad (j = 1, 2, \ldots).$$

Consequently, we observe that for $j = 0, 1, \ldots, n-1$

$$y_1^{\mathrm{T}} A U^j = f_1^{\mathrm{T}} U^j = y_1^{\mathrm{T}}(V + GY^{\mathrm{T}})^j A. \tag{3.4}$$

Taking into account that

$$f_1^{\mathrm{T}} U^j = [\, w_1 z_1^j, w_2 z_2^j, \dots, w_n z_n^j \,] \tag{3.5}$$

the equality (3.4) leads to

$$V_{nn}(z^+)\ \mathrm{diag}(w^+) = K^{\mathrm{T}} A\ .$$

Postmultiplying the latter equality from the left-hand side by

$$(V_{nn}(z^+) \cdot \mathrm{diag}(w^+))^{-1}$$

we get (3.3), and the proof is complete. □

Remark 3.1. In the square case $m = n$ the above theorems contain invertibility criteria. In particular, in Theorem 3.2 we have already proved the first part of Theorem 1.1.

Remark 3.2. In case of a regular Löwner–Vandermonde matrix A the formulas Eqs. (3.1) and (3.3) are formulas for the inverse of A.

4. Proof of Theorem 1.1

Taking Remark 3.1 into account it remains to prove the inversion formula (1.4). We shall again write shortly A instead of A_n^{lk} and use Lemma 2.1. Multiplying (2.2) from both sides by A^{-1} we obtain

$$A^{-1} V - U A^{-1} = -\sum_{i=1}^{3} x_i y_i^{\mathrm{T}},$$

where x_i, y_i are the solutions of (2.5) and (2.6). Putting $e_0^n := \mathbf{O}_n$ and applying the matrices on both sides of this equality to e_{j+1}^n for $j = 0, 1, \dots, l-1$ and to e_j^n for $j = l+1, l+2, \dots, n$ we get

$$A^{-1} e_j^n - U A^{-1} e_{j+1}^n = -\sum_{i=1}^{3} y_{i,j+1} x_i \quad (j = 0, 1, \dots, l-1),$$

$$(z_{l-j} I_n - U) A^{-1} e_j^n = -\sum_{i=1}^{3} y_{ij} x_i \quad (j = l+1, l+2, \dots, n).$$

Taking into account that $y_3 = e_1^n$ and that $A^{-1} e_l^n = x_2$ (comp. (2.8) and (1.3)) we get the following recurrence formula for the columns $s_j := A^{-1} e_j^n$ $(j = 1, 2, \dots n)$ of A^{-1}

$$s_j = \begin{cases} Us_{j+1} - \sum_{i=1}^{2} y_{i,j+1} x_i & \text{for } j = 1, \ldots, l-1, \\ x_2 & \text{for } j = l, \\ (U - z_{l-j} I_n)^{-1} \sum_{i=1}^{2} y_{ij} x_i & \text{for } j = l+1, \ldots, n. \end{cases} \tag{4.1}$$

The equality $s_l = x_2$ together with

$$s_j - Us_{j+1} = -\sum_{i=1}^{2} y_{i,j+1} x_i, \quad j = l-1, \ldots, 1 \tag{4.2}$$

determine the vectors $s_l, s_{l-1}, \ldots, s_1$ uniquely.

We denote the jth column of the matrix on the right-hand side of (1.4) by r_j and show that $r_j = s_j$ for $j = 1, 2, \ldots, n$. Indeed, for $j = 1, \ldots, l$ we have

$$r_j = -\sum_{i=1}^{2} \operatorname{diag}(x_i) V_{ln}^{\mathrm{T}}(z^+) \left[y_{i,j+1}, y_{i,j+2}, \ldots, y_{i,l+1}, 0, \ldots, 0\right]^{\mathrm{T}}.$$

Since $y_{1,l+1} = 0$ and $y_{2,l+1} = 1$, it follows that $r_l = x_2$. Note also that

$$Ur_{j+1} = -\sum_{i=1}^{2} \operatorname{diag}(x_i z_i^+) V_{ln}^{\mathrm{T}}(z^+) \left[y_{i,j+2}, y_{i,j+3}, \ldots, y_{i,l+1}, 0, \ldots, 0\right]^{\mathrm{T}}$$

and, consequently,

$$r_j - Ur_{j+1} = -\sum_{i=1}^{2} y_{i,j+1} x_i. \tag{4.3}$$

Comparing (4.2) and (4.3), we conclude that $s_j = r_j$ for $j = l, l-1, \ldots, 1$. By (1.4), we can write for $j = l+1, \ldots, n$:

$$r_j = -\sum_{i=1}^{2} y_{i,j} \operatorname{diag}(x_i) \operatorname{col}\left(\frac{1}{z_i - z_{l-j}}\right) = -\sum_{i=1}^{2} y_{i,j} \operatorname{diag}\left(\frac{1}{z_i - z_{l-j}}\right) x_i,$$

which is equal to s_j for $j = l+1, \ldots, n$. This completes the proof. □

5. Connection of the special solutions and rational interpolants. A superfast algorithm

In this section we are going to discuss how the solutions

$$x_i = [x_{ij}]_{j=1}^{n}, \qquad y_i = [y_{ij}]_{j=1}^{n} \tag{5.1}$$

of Eqs. (1.2) and (1.3) needed for the representation of the inverse of $A = A_n^{lk}, n = l + k$, in the form (1.4) are connected with solutions of the following interpolation problems:

Problem I1. Find a nonzero couple of polynomials (p, q), $\deg p = n$, $\deg q \leqslant k - 1$, such that

$$p(z_i) - w_i q(z_i) = 0 \quad \text{for} \quad i = -k, \dots, -1 \quad \text{and} \quad i = 1, \dots, n. \tag{5.2}$$

Problem I2. Find a nonzero couple of polynomials (p, q), $\deg p \leqslant n - 1$, $\deg q = k$ such that (5.2) is satisfied.

We designate any solution of Problem I1 and of Problem I2 by (p_{I1}, q_{I1}) and (p_{I2}, q_{I2}), respectively. Denoted by $g^{(n)}(\lambda), g^{(-k)}(\lambda)$ the polynomials

$$g^{(n)}(\lambda) := \prod_{i=1}^{n}(\lambda - z_i), \qquad g^{(-k)}(\lambda) := \prod_{i=1}^{k}(\lambda - z_{-i}),$$

we consider the following basis systems in the linear space $\mathscr{P}_n$ of all (complex) polynomials of degree not greater than n:

$$\left\{ g_i^{(n)}(\lambda) := \frac{g^{(n)}(\lambda)}{\lambda - z_i} \right\}_{i=1}^{n} \bigcup g^{(n)}(\lambda),$$

$$\{\lambda^i g^{(-k)}(\lambda)\}_{i=0}^{l} \bigcup \left\{ g_{-i}^{(-k)}(\lambda) = \frac{g^{(-k)}(\lambda)}{\lambda - z_{-i}} \right\}_{i=1}^{k}.$$

Assume A is nonsingular and introduce the following polynomials belonging to $\mathscr{P}_n$ *for* $j = 1, 2$:

$$a_j(\lambda) := \alpha_j g^{(n)}(\lambda) + \sum_{i=1}^{n} w_i x_{ji} g_i^{(n)}(\lambda),$$

$$b_j(\lambda) := \sum_{i=1}^{n} x_{ji} g_i^{(n)}(\lambda),$$

$$c_j(\lambda) := (1 - \alpha_j)\lambda^l g^{(-k)}(\lambda) - \left(\sum_{i=0}^{l-1} y_{j,i+1}\lambda^i \right) g^{(-k)}(\lambda) + \sum_{i=1}^{k} w_{-i} y_{j,l+i} g_{-i}^{(-k)}(\lambda),$$

$$d_j(\lambda) := -\alpha_j g^{(-k)}(\lambda) + \sum_{i=1}^{k} y_{j,l+i} g_{-i}^{(-k)}(\lambda),$$

where

$$\alpha_j = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j = 2. \end{cases}$$

Thus $a_1, c_2 \in \mathscr{P}_n$, $a_2, c_1 \in \mathscr{P}_{n-1}$, $d_1 \in \mathscr{P}_k$, $d_2 \in \mathscr{P}_{k-1}$.

Theorem 5.1. *If A is nonsingular then the Problems I1, I2 have (up to a multiplicative constant) the unique solutions*

$$p_{l1} = a_1 = c_2, \qquad q_{l1} = b_1 = d_2,$$
$$p_{l2} = a_2 = c_1, \qquad q_{l2} = b_2 = -d_1.$$

Proof. Let us start with proving the bounds for the degrees of b_1 and b_2. Since

$$\frac{b_1(\lambda)}{g^{(n)}(\lambda)} = \sum_{i=1}^{n} \frac{x_{1i}}{\lambda - z_i} = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} x_{1i} z_i^j \right) \lambda^{-j-1} \tag{5.3}$$

and since $V_{ln}(z^+)x_1 = \mathbf{O}_l$ we conclude $b_1(\lambda)/g^{(n)}(\lambda) = 0(\lambda^{-l-1})$ and $\deg b_1(\lambda) \leqslant k - 1$. Similarly, in view of

$$V_{ln}(z^+)x_2 = [0, \dots, 0, 1]^{\mathrm{T}},$$

we get $\deg b_2 = k$.

Now let us show that the interpolation conditions (5.2) are satisfied. If we start with considering the couple (a_j, b_j), $j = 1, 2$, it is easy to verify that (5.2) holds true for $i = 1, \dots, n$. For $i = 1, \dots, k$ we rewrite conditions (5.2)

$$a_j(z_{-i}) - w_{-i} b_j(z_{-i}) = 0$$

as

$$w_{-i} \frac{b_j(z_{-i})}{g^{(n)}(z_{-i})} - \frac{a_j(z_{-j})}{g^{(n)}(z_{-i})} = 0.$$

Then, by definition of a_j and b_j, we get for $r = 1, \dots, k$

$$\sum_{i=1}^{n} \frac{w_{-r} - w_i}{z_{-r} - z_i} x_{ji} = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } j = 2, \end{cases}$$

which is, due to (1.3), satisfied.

Moreover, it is obvious that the couples $(c_j, d_j), j = 1, 2$, satisfy (5.2) for $i = -k, \dots, -1$. As above let us rewrite the conditions (5.2) for $i = 1, \dots, n$ in the form

$$w_i \frac{d_j(z_i)}{g^{(-k)}(z_i)} - \frac{c_j(z_i)}{g^{(-k)}(z_i)} = 0,$$

which means

$$-\alpha_j w_i + \sum_{r=1}^{k} \frac{w_i - w_{-r}}{z_i - z_{-}} y_{j,l+r} + \sum_{r=0}^{l-1} y_{j,r+1} z_i^r - (1 - \alpha_j) z_i^l = 0$$

and is equivalent to (1.2).

It remains to prove the uniqueness of the solutions. Assume (p, q) as well as $(\tilde{p}, \tilde{q})$ are solutions of Problem I1. First note that $q \neq 0$ since, otherwise, also p would be a zero polynomial but then (p, q) would not be a solution of Problem I1. The same reasoning applies to $(\tilde{p}, \tilde{q})$. The interpolation conditions imply that $p\tilde{q} - \tilde{p}q$ is divisible by $g^{(n)}g^{(-k)}$. Taking into account the bounds of the degrees, we state that

$$p\tilde{q} - \tilde{p}q = 0 \quad \text{or, in other words,} \quad \frac{p}{q} = \frac{\tilde{p}}{\tilde{q}}.$$

Denoting by p_0, q_0 the greatest common divisor of $p, \tilde{p}$, respectively $q, \tilde{q}$ we establish that (p_0, q_0) satisfies the interpolation conditions for all i. If $\deg p_0 < n$, the following expansions are possible

$$p_0(\lambda) = \sum_{i=1}^{n} w_i p_{0i} g_i^{(n)}(\lambda), \quad q_0(\lambda) = \sum_{i=1}^{n} p_{0i} g_i^{(n)}(\lambda).$$

If we expand the function $q_0/g^{(n)}$ like in (5.3), we find that the condition

$$\deg q_0 \leqslant k - 1$$

is equivalent to $V_{ln}(z^+)\text{col}\,(p_{0i})_1^n = 0$. Also the conditions

$$p_0(z_{-i}) - w_{-i} q_0(z_{-i}) = 0$$

are equivalent to $L_{kn}(w, z)\text{col}\,(p_{0i})_1^n = 0$. In this way, we get that $\text{col}\,(p_{0i})_1^n$ is a nontrivial element of the kernel of A, which contradicts the assumption $\det A \neq 0$. Thus $\deg p_0 = n$. Consequently, $(p, q) = c(\tilde{p}, \tilde{q})$ with $c = \text{const}$.

Analogously the uniqueness of the solution of Problem I2 can be proved.

Finally, taking into account that the polynomials a_1, b_2, c_2, $-d_1$ are monic, we conclude $a_1 = c_2$, $b_1 = d_2$, $a_2 = c_1$, $b_2 = -d_1$, and the proof is complete. □

The preceding theorem proves that an alternative way exists to compute the parameters (5.1) needed in the inversion formula (1.4) for $A = A_n^{kl}$. It consists of the following steps:

- Solving the rational interpolation Problems I1 and I2.
- Evaluating, for $j = 1, 2$,

$$x_{ji} = \frac{q_{lj}(z_i)}{g_i^{(n)}(z_i)}, \quad i = 1, \ldots, n,$$

$$y_{j,l+i} = (-1)^{\alpha_j} \frac{q_{lj}(z_{-i})}{g_{-i}^{(-k)}(z_{-i})}, \quad i = 1, \ldots, k.$$

- Computing y_{1i}, resp. y_{2i} $(i = 1, \ldots, l)$ as the coefficients at the powers λ^{i-1} of the quotient in the division $p_{l2} : g^{(-k)}$, resp. $p_{l1} : g^{(-k)}$.

It is known that all these steps can be realized superfast, i.e. with complexity $O(n\log^2 n)$. For details, see e.g. [18], Section 4.4, 4.5. Consequently, taking inversion formula (1.4) into account we have proved the following theorem.

Theorem 5.2. *Linear systems of equations with nonsingular Löwner–Vandermonde coefficient matrices can be solved with $O(n\log^2 n)$ complexity on a sequential computer and with $O(n)$ complexity on an n-processor parallel computer.*

6. Special cases

Let us shortly discuss formulas for the inverse of Vandermonde, Löwner, Cauchy, and Cauchy–Vandermonde matrices. Since these special cases are widely investigated and formulas of this type have been already known (see e.g. [15, 19]) we restrict ourselves to a specification of the inversion formula (1.4) basing on a specification of Eqs. (2.5) and (2.6). Hereby we do not change the subscripts. Clearly, in the Löwner case $A_n^{0n} = \mathscr{L}_{nn}(w,z)$ (2.8) fails to be true and $\check{y}_i = y_i$. Thus we have to determine the solutions of

$$L_{nn}(w,z)x_1 = \mathbf{1}_n, \qquad L_{nn}(w,z)x_3 = -w^-,$$

$$L_{nn}^{\mathrm{T}}(w,z)y_1 = w^+, \qquad L_{nn}^{\mathrm{T}}(w,z)y_3 = \mathbf{1}_n$$

and get

$$L_{nn}^{-1}(w,z) = \mathrm{diag}(x_1)C_{nn}^{\mathrm{T}}(z)\mathrm{diag}(y_1) + \mathrm{diag}(x_3)C_{nn}^{\mathrm{T}}(z)\mathrm{diag}(y_3).$$

In case that

$$w^- = \mathbf{1}_n \quad \text{and} \quad w^+ = \mathbf{O}_n, \tag{6.1}$$

which means $L_{nn}(w,z)$ is just the Cauchy matrix $C_{nn}(z)$ we obtain

$$C_{nn}^{-1}(z) = \mathrm{diag}\,(x_3)C_{nn}^{\mathrm{T}}(z)\,\mathrm{diag}\,(y_3).$$

The case of a pure Vandermonde matrix $A_n^{n0} = V_{nn}(z^+)$ leads to

$$V_{nn}^{-1}(z^+) = \mathrm{diag}(x_2)V_{nn}^{\mathrm{T}}(z^+)H_n(\hat{y}_2),$$

where x_2, y_2 are solutions of

$$V_{nn}(z^+)x_2 = e_n^l, \qquad V_{nn}^{\mathrm{T}}(z^+)y_2 = [z_i^n]_{i=1}^n.$$

Finally we consider the case of a Cauchy–Vandermonde matrix

$$W = \begin{bmatrix} V_{ln}(z^+) \\ C_{kn}(z) \end{bmatrix} \quad (l \neq 0, k \neq 0)$$

choosing again w due to (6.1). In this case we obtain the inverse of W from the solutions of

$$Wx_2 = e_l^n, \qquad W^{\mathrm{T}}y_2 = [z_i^l]_{i=1}^n$$

via the formula

$$W^{-1} = \mathrm{diag}(x_2)\; W^{\mathrm{T}} \begin{bmatrix} H_l(\hat{y}_2) & 0 \\ 0 & \mathrm{diag}(\check{y}_2) \end{bmatrix}$$

(compare [1,2]).

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