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# A new butterfly-shaped chaotic attractor

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ABSTRACT

In this paper, a new chaotic system is proposed that consists of six terms including one multiplier and one quadratic term. The characteristics of this system are examined by theoretical and numerical analysis, such as equilibria, their stabilities, Lyapunov exponents and Lyapunov dimension, dissipativity, as well as, Poincaré maps, bifurcations, waveforms, power spectrums are performed. In addition, the forming mechanisms of compound structures of the new chaotic attractor are investigated.



## 1. Introduction

In this paper, a new chaotic system is proposed that consists of three first-order autonomous ordinary differential equations having *six terms* on their right-hand side that contain one multiplier and one quadratic term. This system is characterized with a butterfly-shaped attractor.

For this system, the dynamic properties of this attractor are examined in detail theoretically and numerically. These dynamic properties include equilibria with their stabilities, Lyapunov exponents, Lyapunov dimension, Poincaré maps, and bifurcation diagram. In addition, we also present the forming mechanisms of its compound structures obtained by merging together two simple attractors after performing one mirror operation.

## 2. Theoretical analysis of the new chaotic attractor

The proposed chaotic system has the governing equations as the following:

$$\begin{aligned}\dot{x} &= a(y - x)\\ \dot{y} &= xz + by\\ \dot{z} &= -x^2 - cz\end{aligned} \tag{1}$$

where $\mathrm{x} = (x, y, z)^T \in R^3$ denotes the state variables of the system, with $a$, $b$ and $c$ representing real constants.

It has six terms on the right-hand side of the governing equations, and relies on one multiplier ($xz$) and one quadratic term ($x^2$) to introduce the nonlinearity necessary for folding trajectories.

### 2.1. Symmetry and invariance

The system (1) is symmetrical on the $z$-axis as it has the transformation:

$$(x,\ y,\ z)\ \rightarrow\ (-x,\ -y,\ z)$$

It implies that this system is invariant for all values of the parameters $a$, $b$, $c$ and that the $z$-axis itself is an orbit that goes toward the origin at $t \rightarrow 0$.

### 2.2. Equilibria and stability

The equilibria of the system (1) are found by using the following equations:

$$\begin{aligned}a(y - x) &= 0,\\ xz + by &= 0,\\ -x^2 - cz &= 0.\end{aligned} \tag{2}$$

Equations (2) lead to three equilibrium points,

$$\begin{aligned}&O(0, 0, 0)\\ &P_+(\sqrt{bc}, \sqrt{bc}, -b)\\ &P_-(-\sqrt{bc}, -\sqrt{bc}, -b)\end{aligned} \tag{3}$$

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For these equilibrium points, we investigate stability of the system (1). To this end, we use the method of linearization (see for instance [1,2]). The system (1) is linearized at the equilibrium $O(0,0,0)$to obtain the Jacobian matrix as follows:

$$J_0 = \begin{bmatrix} -a & a & 0 \\ Z & b & x \\ -2x & 0 & -c \end{bmatrix}_0 = \begin{bmatrix} -a & a & 0 \\ 0 & b & 0 \\ 0 & 0 & -c \end{bmatrix} \tag{4}$$

By letting $|\lambda I - J_0| = 0$, the characteristic equation of $J_0$ is obtained as the following:

$$(\lambda + a)(\lambda - b)(\lambda + c) = 0,$$

which leads to the eigenvalues of $J_0$

$$\lambda_1 = -a, \quad \lambda_2 = b, \quad \lambda_3 = -c \tag{5}$$

For $a, c > 0$ and $b < 0$, the equilibrium $O(0,0,0)$ is stable; whereas, for either $a < 0$ or $b > 0$ or $c < 0$, the equilibrium $O(0,0,0)$ is unstable. In the case that $\mathrm{Re}(\lambda_i)$ is zero, the stability could further be analyzed by other methods such as the center manifold theory and Lyapunov stability theory (see for instance [1,2]).

In the same way, the stability at $P_+$ and $P_-$ can also be examined. Now that the invariance of this system under the transformation $(x, y, z) \to (-x, -y, z)$, that is, the stabilities of both are alike, we have only to consider the stability at $P_+$. The Jacobian matrix of the system (1) at $P_+(\sqrt{bc}, \sqrt{bc}, -b)$

$$J_{P_+} = \begin{bmatrix} -a & a & 0 \\ z & b & x \\ -2x & 0 & -c \end{bmatrix}_{P_+} = \begin{bmatrix} -a & a & 0 \\ -b & b & \sqrt{bc} \\ -2\sqrt{bc} & 0 & -c \end{bmatrix} \tag{6}$$

Letting $|\lambda I - J_{p_+}| = 0$, one can obtain the characteristic equation of $J_{p_+}$ as follows:

$$(\lambda^2 + (a + c)\lambda + ac - 2a\sqrt{bc})(\lambda - b) = 0$$

The eigenvalues corresponding to equilibrium $P_+(\sqrt{bc}, \sqrt{bc}, -b)$ are

$$\lambda_{1,2} = \frac{-(a + b) \pm \sqrt{(a + b)^2 + 4(ac - 2a\sqrt{bc})}}{2}, \quad \lambda_3 = b \tag{7}$$

For $(a + b) > 0$ and $b < 0$, the equilibrium $P_+$ is stable; whereas for $(a + b) < 0$ or $b > 0$, the equilibrium $P_+$ is unstable. But, if $a + b = 0$ or $b = 0$, that is $\mathrm{Re}(\lambda_i)$ is zero, the stability cannot be analyzed by linearization. For further analysis in this case, the center manifold theory or Lyapunov stability theory could be employed.

### 2.3. Dissipativity and the existence of attractor

In order to prove that the system (1) is a dissipative system, the divergence of flow of the system is examined as:

$$\frac{1}{V}\frac{dV}{dt} = div V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + b - c \leqslant 0 \tag{8}$$

$$V(t) = V(0)e^{(-a+b-c)t} \tag{9}$$

Hence, the system (1) is dissipative if and only if $(-a + b-c) < 0$ with an exponential rate of contraction as

$$\frac{dV}{dt} = e^{-a+b-c} \tag{10}$$

An initial volume element $V(0)$ shrinks exponentially by the flow to a volume element $V(0)e^{(-a+b-c)t}$ as time goes. That is, each volume containing the system trajectory becomes zero as $t \to \infty$. Every trajectory is eventually confined to a specific zero-volume limit set and the asymptotic motion settles onto an attractor of the system (1). This means that the dynamics go toward the attractor as $t \to \infty$.

## 3. Numerical analysis of the new chaotic attractor

### 3.1. Phase portraits

With the parameters $a = 30$, $b = 15$, $c = 11$, the system (1) is dissipative and its three equilibrium points are all unstable. With

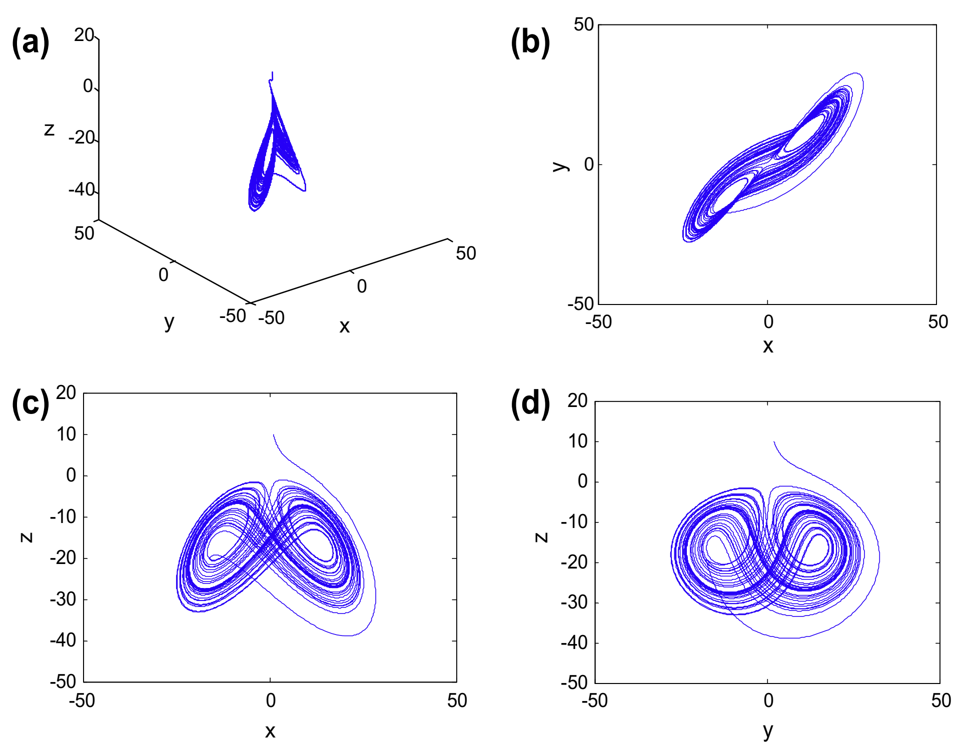


**Fig. 1.** A new chaotic attractor. (a) three-dimensional view. (b) $x$–$y$ plane. (c) $x$–$z$ plane. (d) $y$–$z$ plane.

initial conditions $(x_0, y_0, z_0) = (1, 2, 10)$, this system has a single 2-scroll chaotic attractor that exhibits abundant complex behaviors of chaotic dynamics. Fig. 1(a) shows the trajectory of the system (1) plotted for a three-dimensional view, whereas Fig. 1(b–d) show its projections onto an $x$–$y$ plane, an $x$–$z$ plane, $y$–$z$ plane, respectively. A notable feature is that the attractor has the butterfly shape, but as shown in Fig. 1(c) it has an upside-down shape of that of the family of the Lorenz-like systems.

### 3.2. Lyapunov exponents and Lyapunov dimension

Lyapunov exponent is a quantity that characterizes the rate of separation of infinitesimally close trajectories. In other words, it indicates the average exponential rates of divergence or convergence of adjacent trajectories in phase space. An attractor with one or more positive Lyapunov exponents is said to be chaotic [3].

By using the Wolf algorithm proposed in [3], Lyapunov exponents of the system (1) are calculated after $10^5$ iterations with a normalized step-sized 0.05 as follows:

$$\lambda_{L1} = 0.9727, \quad \lambda_{L2} = -0.0720, \quad \lambda_{L3} = -26.9007$$

Since $\lambda_{L1}$ is positive, the system (1) possesses expanding nature of different directions in phase space; whereas negative $\lambda_{L2}, \lambda_{L3}$ means the contracting nature of different direction in phase space.

The Lyapunov dimension of the system (1), a quantity of fractal dimension of an attractor [4], is described as

$$D_L = k + \frac{1}{|\lambda_{k+1}|}\sum_{i=1}^{k}\lambda_{Li} = 2 + \frac{\lambda_{L1} + \lambda_{L2}}{|\lambda_3|} = 2.0335$$

where $k$ is the maximum value of $i$ such that $\xi_i = \lambda_{L1} + \cdots + \lambda_{Li} > 0$.

As a chaotic attractor always has non-integer dimension, namely, fraction dimension, the attractor of the system (1) is of fraction dimension.

### 3.3. Bifurcation analysis

A bifurcation is a change in qualitative behavior of a system as a parameter varies [5]. The bifurcation diagrams of the system (1) can be obtained by examining the peak of state $x$ versus each of the parameters $a$, $b$, $c$ respectively while the others are fixed.

First, the chaotic dynamics of the system versus varying parameter $b$ is investigated. Figs. 2(a–b) shows the bifurcation diagram of state $x$ and the Lyapunov-exponent spectra versus increasing parameter $b$, respectively, with fixed values of $a = 30$, $c = 11$. It is observed that the bifurcation diagram well coincides with the spectra of the Lyapunov exponents.

The representative dynamical routes are summarized as follows:

(1) $0 < b < 12.5$: $\lambda_{L1,2,3} < 0$, the system (1) is stable.
(2) $12.5 \leqslant$; $b < 17.4$: $\lambda_{L1} > 0$, $\lambda_{L2} \approx 0$, $\lambda_{L3} < 0$, the system (1) is chaotic. But there are some periodic windows in the chaotic band.
(3) $17.4 \leqslant b < 25$: $\lambda_{L1} = 0$, $\lambda_{L2} < 0$, $\lambda_{L3} < 0$, there is a very long period-doubling bifurcation window.

Next, the chaotic dynamics is investigated as $c$ varies, with $a = 30$, $b = 11$. Fig. 3 displays the bifurcation diagram of state $x$ and the Lyapunov-exponent spectra. In this case, too, the result shows that the bifurcation diagram coincides well with the spectra of the Lyapunov exponents

The representative dynamical routes are summarized as the following:

(1) $0 < c < 0.3$: $\lambda_{L1,2,3} < 0$, the system (1) is stable.
(2) $0.3 \leqslant c < 13.7$: $\lambda_{L1} > 0$, $\lambda_{L2} \approx 0$, $\lambda_{L3} < 0$, the system (1) is chaotic. But there are some periodic windows in the chaotic band.

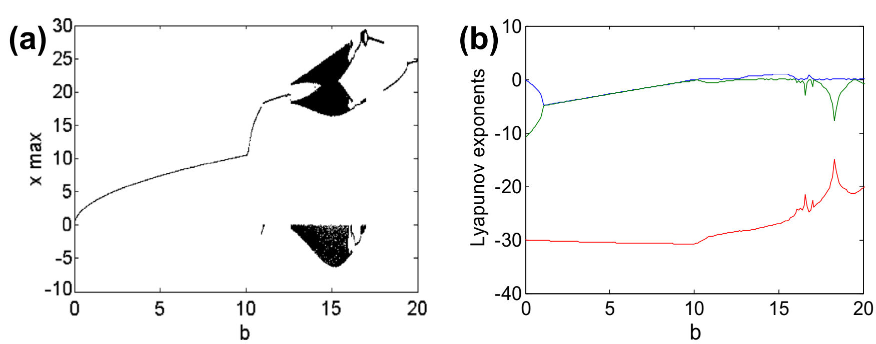


**Fig. 2.** (a) Bifurcation diagram of ‘x max’, (b) Lyapunov exponent spectra; versus the parameter $b$ with $a = 30$, $c = 11$.

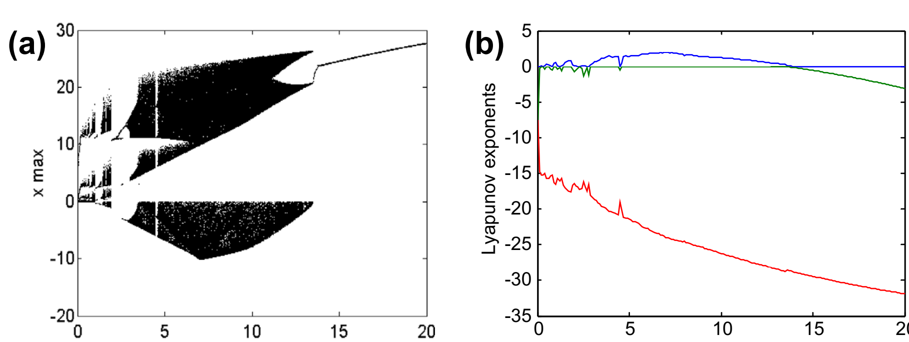


**Fig. 3.** (a) Bifurcation diagram of ‘x max’, (b) Lyapunov exponent spectra; versus the parameter $c$ with $a = 30$, $b = 15$.

(3) $13.7 \leqslant c < 25$: $\lambda_{L1} = 0$, $\lambda_{L2} < 0$, $\lambda_{L3} < 0$, there is a very long period-doubling bifurcation window.

We omit the investigation of the chaotic dynamics of the system versus varying parameter *a.*

### 3.4. Poincaré maps, wave forms and spectrum

The dynamical behaviors of the system (1) are further investigated by means of Poincaré maps, wave forms and spectrum. Fig. 4(a–c) shows the Poincaré maps in the planes where $x = 0$, $y = 0$ and $z = 0$, respectively. It is observed that several sheets of the attractors are separated symmetrically and folded. It is noticed that the Poincaré map in plane where $z = 0$ shows a diagonal distribution. We easily find out that the reason for it comes from the fact that the wave forms of $x(t)$ and $y(t)$ have almost the same behavior as shown in Fig. 5

Fig. 6(a) shows the power spectrum of the signal $x(t)$ of the system (1). In this system $x$ has a bandwidth between about 0–8 Hz, which is a little broader than that of the original Lorenz system $\dot{x} = 10(y - x)$, $\dot{y} = 28x - y - xz$, $\dot{z} = xy - (8/3)z$ as shown in Fig. (b).

## 4. Forming mechanisms of the new chaotic attractor

Compound structures of an attractor can be obtained by merging together two simple attractors after performing one mirror operation, namely, a half-image operation to obtain only the left or the right half-image attractors.

To perform one mirror operation, a controlled system inherited from the system (1) is proposed. This controlled system equals to

$$\dot{x} = a(y - x)\dot{y} = xz + by + u\dot{z} = -x^2 - cz \tag{11}$$

where $u$ is a control parameter.

Fig. 7(a) shows the bifurcation diagram versus varying $u$ of the controlled system. This diagram indicates that for about $u < 42$, the controlled system maintains the two-lobed orbits of the original system (1), whereas for about $u \geqslant 42$, the controlled system has a period doubling bifurcation.

As shown in Fig. 7(b), there are some periodic windows, which play a role

In the evolution of the complex dynamics for the controlled system, near at $12.1 < u < 13$, $u = 14.7$ and $u = 15.3$. Maintaining that $a = 30, b = 15, c = 11$, $(x_0, y_0, z_0) = (1, 2, 10)$, we plot trajectories on the *x-z* plane at specific values of the parameter $u$ as follows:

- When $u = 50$, the attractor evolves into a period doubling bifurcation, which is shown in Fig. 8(a).
- When $u = 42$, the attractor of the controlled system is evolved into the single right scroll attractor; it is only one half the original system (1) as shown in Fig. 8(b).
- When $u = -42$, on the other hand, the attractor of the controlled system is evolved into the single left scroll attractor; it is only one half the original system (1) as shown in Fig. 8(c).
- When $u = -50$, the attractor evolves into a period doubling bifurcation, as is shown in Fig. 8(d).

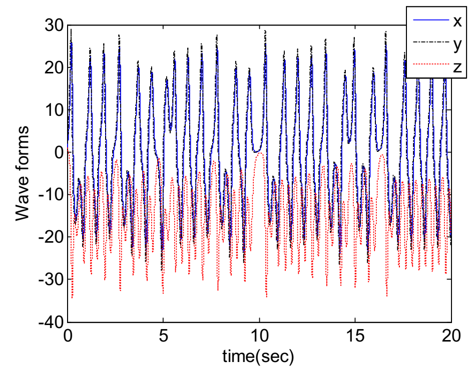


**Fig. 5.** Wave forms of the signals of the system (1).

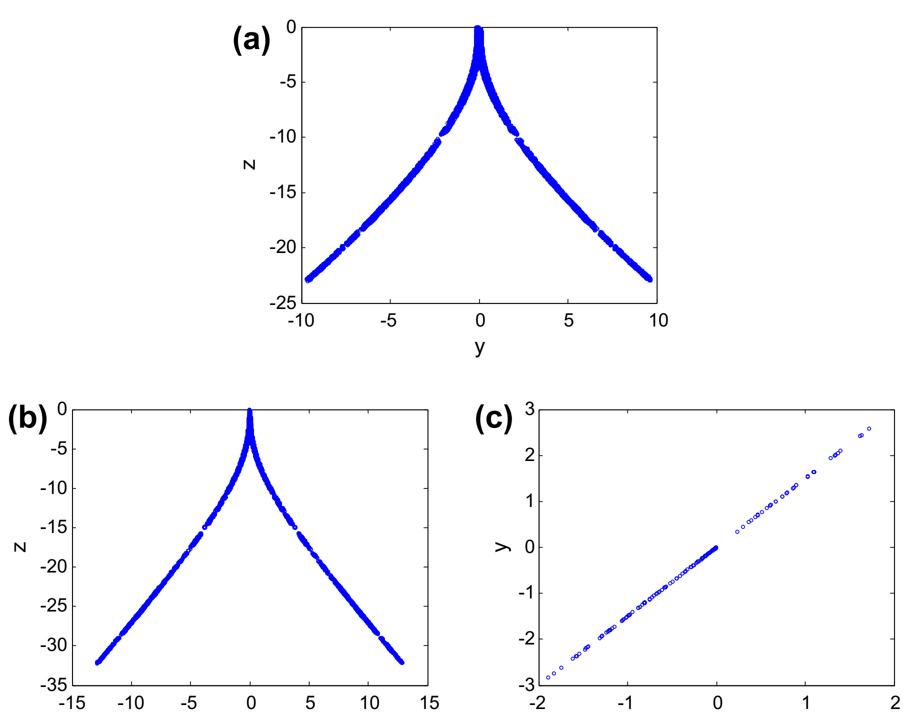


**Fig. 4.** Poincaré map in plane where (a) $x = 0$, (b) $y = 0$, (c) $z = 0$.

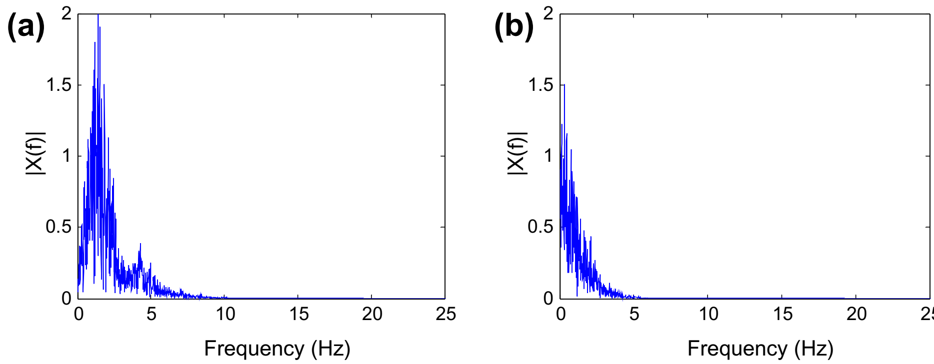


**Fig. 6.** Power spectrum of the signal $x(t)$ of (a) the system (1), (b) Lorenz system.

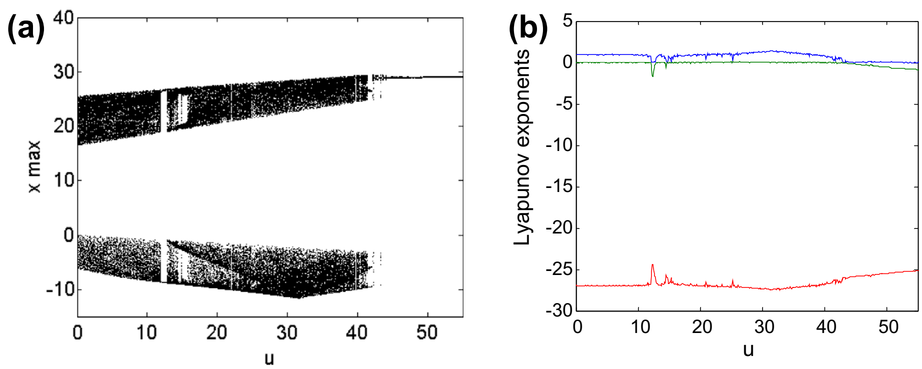


**Fig. 7.** (a) Bifurcation diagram of '$x$ max', (b) Lyapunov exponent spectra; versus the control parameter $c$.

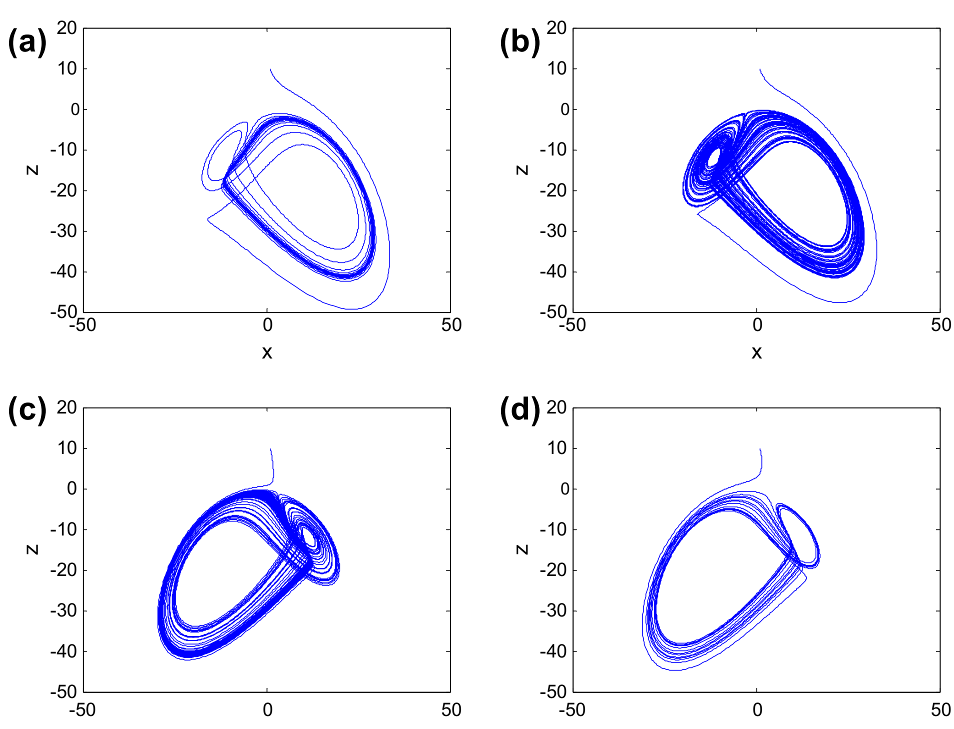


**Fig. 8.** $x$–$z$ phase planes of the system (1) at (a) $u = 50$, (b) $u = 42$, (c) $u = -42$, (d) $u = -50$.

It means that the reversed butterfly-shaped attractor of the system (1) in Fig. 1(c) is a compound structure obtained by merging together two simple attractors after performing one mirror operation.

## 5. Conclusions

In this paper, we have proposed a new butterfly-shaped chaotic attractor. The new attractor consists of six terms in three first-order autonomous ODEs with one multiplier ($xz$) and one quadratic term ($x^2$). The new system has been analyzed both theoretically and numerically by examining equilibria, stability, Lyapunov exponents and Lyapunov dimension, dissipativity, as well as waveform, power spectrum, Poincaré maps, bifurcations are studied. In addition, we have investigated the forming mechanisms of compound structures of the new chaotic attractor.

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