Note on the Generalization of the Godunova–Levin–Opial Inequality in Several Independent Variables

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1. INTRODUCTION

The aim of this note is the generalization of Opial-type inequalities for the case of many independent variables. First, we are going to recall some important results.

(a) Inequalities in One Variable

First, in 1960, Z. Opial [5] proved the following inequality:

**Theorem A.** Let $f'$ be a continuous function on $[0, h]$, with $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$. Then the integral inequality

$$
\int_0^h |f(x)f'(x)| \, dx \leq \frac{h}{4} \int_0^h \left(f'(x)\right)^2 \, dx
$$

holds.
In 1967, E. K. Godunova and V. I. Levin [3] proved the following theorem:

**Theorem B.** Let $f$ be a real-valued absolutely continuous function defined on $[a, b]$ with $f(a) = 0$. Let $F$ be a convex, increasing function on $[0, \infty)$ with $F(0) = 0$. Then the integral inequality

$$\int_a^b F'(|f(t)|) |f'(t)| \, dt \leq F\left( \int_a^b |f'(t)| \, dt \right)$$

holds.

The multidimensional generalization is given in [11] by J. Pečarić:

**Theorem C.** Let $f_i, i = 1, \ldots, n$, be real-valued absolutely continuous functions defined on $[a, b]$ with $f_i(a) = 0, i = 1, \ldots, n$. Let $F$ be a nondecreasing function on $[0, \infty)^n$ with $F(0, \ldots, 0) = 0$ such that all its first partial derivatives $D_iF, i = 1, \ldots, n$, are nondecreasing functions. Let $p_i, i = 1, \ldots, n$, be positive functions defined on $[a, b]$ with $\int_a^b p_i(t) \, dt = 1, i = 1, \ldots, n$. Let $h_i, i = 1, \ldots, n$, be positive convex and increasing functions on $(0, \infty)$. Then the integral inequality

$$\int_a^b \left( \sum_{i=1}^n D_iF(|f_1(t)|, \ldots, |f_n(t)|, |f'_1(t)|, \ldots, |f'_n(t)|) \right) \, dt$$

$$\leq F\left( \int_a^b p_1(t) h_1 \left( \frac{|f_1(t)|}{p_1(t)} \right) \, dt \right) \cdot \ldots \cdot \int_a^b p_n(t) h_n \left( \frac{|f_n(t)|}{p_n(t)} \right) \, dt$$

holds.

The most interesting special case of this theorem is for $h_i(x) = x, i = 1, \ldots, n$.

The special case obtained for $F(x_1, \ldots, x_n) = \prod_{i=1}^n F(x_i), h_i = h$, for each $i = 1, \ldots, n$ is proved by B. G. Pachpatte in [9].

An interesting generalization of Theorem B was given in 1972 by G. I. Rozanova [12]:

**Theorem D.** Let $f, F$ be defined as in Theorem B. Let $\phi$ be a convex increasing function on $(0, \infty)$. Let $r(x) \geq 0, r'(x) > 0$, for $x \in [a, b]$ and $r(a) = 0$. Then the integral inequality

$$\int_a^b F\left( r(t) \phi \left( \frac{|f(t)|}{r(t)} \right) \right) r'(t) \phi \left( \frac{|f'(t)|}{r'(t)} \right) \, dt \leq F\left( \int_a^b r'(t) \phi \left( \frac{|f'(t)|}{r'(t)} \right) \, dt \right)$$

holds.
Generalization of this result taking the product of two functions is done by B. G. Pachpatte in [10].

(b) The Inequalities in Several Independent Variables

First, we will introduce notation we will use throughout the paper.

Let \( \Omega = \prod_{i=1}^{m} [a_i, b_i] \) be a general point in \( \Omega \), \( \Omega_i = \prod_{i=1}^{m} [a_i, b_i] \) and \( dt \) the volume form \( dt_1 \ldots dt_m \). Further, let \( D h(u) \) stand for \( d h(u)/d u \), \( D_k h(t_1, \ldots, t_m) \) stand for \( \partial^k / \partial t_1 \ldots \partial t_k h(t_1, \ldots, t_m) \), \( 1 \leq k \leq m \), and \( D^k h(t_1, \ldots, t_m) \) stand for \( \partial^k / \partial t_1 \ldots \partial t_k h(t_1, \ldots, t_m) \), \( 1 \leq k \leq m \). Following this notation we have \( D h = D^1 h \).

The generalization of Opial’s inequality for the case of two independent variables was done by G. S. Yang [13] in 1982:

**Theorem E.** Let \( f, D^1 f, D^2 f \) be real-valued continuous functions on \( [a_1, b_1] \times [a_2, b_2] \) with \( f(a_1, t_2) = f(b_1, t_2) = D^1 f(t_1, a_2) = D^2 f(t_1, b_2) = 0 \) for each \( t_i \in [a_i, b_i], \ i = 1, 2. \) Then the following integral inequality holds

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} |f(t_1, t_2)| D^2 f(t_1, t_2) dt_1 dt_2 
\leq \frac{(b_1 - a_1)(b_2 - a_2)}{8} \int_{a_1}^{b_1} \int_{a_2}^{b_2} (D^2 f(t_1, t_2))^2 dt_1 dt_2
\]

One of the generalizations of the Yang inequality was given by W. S. Cheung in [2]:

**Theorem F.** Let \( m \geq 2 \) and let \( f_i, D^1 f_i, \ldots, D^m f_i, \ i = 1, \ldots, n \) be real-valued continuous functions on \( \Omega \) with

\[
f_i(t) \big|_{t_i = a_i} = f_i(t) \big|_{t_i = b_i} = D^1 f_i(t) \big|_{t_i = a_i} = D^1 f_i(t) \big|_{t_i = b_i} \\
\ldots = D^m f_i(t) \big|_{t_m = a_m} = D^m f_i(t) \big|_{t_m = b_m} = 0
\] (1)

for each \( i = 1, \ldots, n \). Let \( F_i, \ i = 1, \ldots, n, \) be nonnegative and differentiable functions on \([0, \infty)\) with \( F_i(0) = 0 \) such that \( DF_i, \ i = 1, \ldots, n \) are nonnegative, continuous, and nondecreasing on \([0, \infty)\). Then the integral inequality

\[
\int_{\Omega} \sum_{i=1}^{n} \left( \prod_{j=1, j \neq i}^{n} F_j(|f_j(t)|) \right) |D F_i(|f_i(t)|)| D^m f_i(t) dt
\leq \sum_{k=1}^{2^m} \prod_{l=1}^{n} F_k \left( \int_{\Omega_k} |D^m f_l(t)| dt \right)
\]
holds, where $\Omega_k$, $k = 1, \ldots, 2^m$ are rectangular subregions of $\Omega$ bounded by the hyperplanes $t_i = a_i$, $i = c_i$, and $t_i = b_i$, where, for each $i \in \{1, \ldots, m\}$, $c_i \in [a_i, b_i]$ is arbitrary.

*Remark.* Instead of condition (1) we can suppose that for each $i = 1, \ldots, n$ and each $j = 1, \ldots, m$,

$$f_i(t)|_{t_j = a_j} = f_i(t)|_{t_j = b_j} = 0 \quad (2)$$

is valid (see [1]).

A special case of Theorem F for $m = 2$ was proved by B. G. Pachpatte in [8].

Our aim is to generalize the result of Theorem F following the idea of Theorem C and generalizing it for the case of several variables. Also we are going to obtain more general results using the idea of Theorem D.

### 2. THE MAIN RESULTS

First, we are going to generalize Theorem F taking the function of $m$ independent variables instead of the product of $m$ functions of one variable:

**Theorem 1.** Let $m \geq 2$ and let $f_i$, $D^1 f_i, \ldots, D^m f_i$, $i = 1, \ldots, n$ be real-valued continuous functions on $\Omega$ with

$$f_i(t)|_{t_j = a_j} = 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \quad (3)$$

or

$$f_i(t)|_{t_1 = a_1} = D^1 f_i(t)|_{t_2 = a_2} = \ldots = D^{m-1} f_i(t)|_{t_m = a_m} = 0, \quad i = 1, \ldots, n. \quad (4)$$

Let $F$ be a nonnegative and differentiable function on $[0, \infty)^n$ with $F(0, \ldots, 0) = 0$ such that $D_i F$, $i = 1, \ldots, n$, are nonnegative, continuous, and nondecreasing on $[0, \infty)^n$. Then the integral inequality

$$\int_\Omega \left( \sum_{i=1}^n D_i F(|f_i(t)|, \ldots, |f_n(t)|)|D^m f_i(t)| \right) dt$$

$$\leq F \left( \int_\Omega |D^m f_1(t)| dt, \ldots, \int_\Omega |D^m f_n(t)| dt \right) \quad (5)$$

holds.
Proof. First we define

\[ z_i(t) := \int_{\Omega_i} |D^m f_i(u)| du, \]

for any \( t = (t_1, \ldots, t_m) \in \Omega \) and \( i = 1, \ldots, n. \)

It is easy to show

\[ |f_i(t)| \leq z_i(t), \quad i = 1, \ldots, n. \]

Namely, in the case that condition (3) is fulfilled, we have

\[ |f_i(t)| = \left| \int_{\Omega_i} D^m f_i(u) du \right| \leq \int_{\Omega_i} |D^m f_i(u)| du = z_i(t), \quad i = 1, \ldots, n, \]

and, in the case that condition (4) is fulfilled, we have

\[ |f_i(t)| \leq \int_{a_1}^{b_1} |D^m f_i(u)| du = z_i(t), \quad i = 1, \ldots, n. \]

Let \( \Omega' = \prod_{i=2}^m [a_i, b_i] \). Now, using (7), the fact that functions \( z_i, i = 1, \ldots, n, \) are nondecreasing in each variable, and the hypotheses of Theorem 1, we have

\[
\int_{\Omega} \left( \sum_{i=1}^n D_i \left( |f_1(t)|, \ldots, |f_n(t)| \right) |D^m f_i(t)| \right) dt \\
\leq \int_{\Omega} \left( \sum_{i=1}^n D_i \left( z_1(t), \ldots, z_n(t) \right) |D^m f_i(t)| \right) dt \\
\leq \int_{a_1}^{b_1} \left( \sum_{i=1}^n D_i \left( z_1(t_1, b_2, \ldots, b_m), \ldots, z_n(t_1, b_2, \ldots, b_m) \right) \right) \\
\times \int_{\Omega'} |D^m f_i(t)| dt_2 \ldots dt_m dt_1
\]
\[
\leq \int_{a_1}^{b_1} \left( \sum_{i=1}^{n} D_i F(z_1(t_1, b_2, \ldots, b_m), \ldots, z_n(t_1, b_2, \ldots, b_m)) \times D^i z_i(t_1, b_2, \ldots, b_m) \right) dt_1
\]

\[
= \int_{a_1}^{b_1} \frac{d}{dt_1} \left( F(z_1(t_1, b_2, \ldots, b_m), \ldots, z_n(t_1, b_2, \ldots, b_m)) \right) dt_1
\]

\[
= F(z_1(b_1, \ldots, b_m), \ldots, z_n(b_1, \ldots, b_m))
\]

\[
= F\left( \int_{\Omega} |D^m f_1(t)| dt, \ldots, \int_{\Omega} |D^m f_n(t)| dt \right)
\]

which completes the proof.

Then the more general theorem follows easily:

**Theorem 2.** Let \( m \geq 2 \) and \( f, D^i f_1, \ldots, D^m f_i, i = 1, \ldots, n, \) be real-valued continuous functions on \( \Omega \) and let the condition (1) or (2) be satisfied. Let \( F \) be as in Theorem 1. Then the integral inequality

\[
\int_{\Omega} \left( \sum_{i=1}^{n} D_i F(|f_1(t)|, \ldots, |f_n(t)|) |D^m f_i(t)| \right) dt
\]

\[
\leq \sum_{k=1}^{2^m} \left( \int_{\Omega_k} |D^m f_1(t)| dt, \ldots, \int_{\Omega_k} |D^m f_n(t)| dt \right) \quad (8)
\]

holds, where \( \Omega_k, k = 1, \ldots, 2^m, \) are defined as in Theorem F.

**Proof.** It is clear that we can obtain analogies of Theorem 1 (by redefining the functions \( z_i \)) so that inequality (5) holds for every \( \Omega_k \). Now we obtain the required inequality (8) by adding these \( 2^m \) inequalities.

Now we are going to generalize the result of Theorem 1 following the ideas given in Theorem C and Theorem D:

**Theorem 3.** Let \( m \geq 2 \) and let \( f, i = 1, \ldots, n, \) and \( F \) be defined as in Theorem 1. Let \( r_i, i = 1, \ldots, n, \) be continuous nonnegative functions such that \( D^1 r_i, \ldots, D^m r_i, i = 1, \ldots, n, \) are continuous positive functions and let the condition (3) be satisfied for functions \( r_i \). Let \( \phi_i, i = 1, \ldots, n, \) be positive, convex, and increasing functions on \((0, \infty)\). Let \( p_i, i = 1, \ldots, n, \) be positive functions defined on \( \Omega \) and \( \int_{\Omega} p_i(t) dt = 1, i = 1, \ldots, n. \) Let \( h_i, i = 1, \ldots, n, \) be positive convex and increasing functions on \((0, \infty)\). Then the integral
inequality

\[
\int_\Omega \left( \sum_{i=1}^n D_i F \left( r_i(t) \phi_i \left( \frac{|f_i(t)|}{r_i(t)} \right) \right) \right) du \times D^{m} r_i(t) \phi_i \left( \frac{|D^m f_i(t)|}{D^m r_i(t)} \right) dt \\
\leq F \left( h^{-1}_n \left( \int_\Omega p_n(t) \left( D^{m} r_i(t) \phi_i \left( \frac{|D^m f_i(t)|}{D^m r_i(t)} \right) \right) dt \right) \right) 
\]

holds.

Proof. In the same way as it is shown in the proof of Theorem 1, we can obtain

\[
|f_i(t)| \leq \int_\Omega |D^m f_i(u)| du, \quad t \in \Omega, \ i = 1, \ldots, n \quad (10)
\]

and

\[
r_i(t) = \int_\Omega D^{m} r_i(u) du, \quad t \in \Omega, \ i = 1, \ldots, n. \quad (11)
\]

Since \( \phi_i, \ i = 1, \ldots, n, \) are increasing on \((0, \infty), \) from (10) and (11) and using Jensen’s inequality, we have

\[
\phi_i \left( \frac{|f_i(t)|}{r_i(t)} \right) \leq \phi_i \left( \frac{1}{\int_\Omega D^{m} r_i(u) du} \int_\Omega \frac{|D^m f_i(u)|}{D^m r_i(u)} du \right) \\
\leq \frac{1}{r_i(t)} \int_\Omega D^{m} r_i(u) \phi_i \left( \frac{|D^m f_i(u)|}{D^m r_i(u)} \right) du \quad (12)
\]

for each \( i = 1, \ldots, n. \)

Now, be defining

\[
z_i(t) := \int_\Omega D^{m} r_i(u) \phi_i \left( \frac{|D^m f_i(u)|}{D^m r_i(u)} \right) du, \quad i = 1, \ldots, n,
\]
using (12), and following the same arguments as in the proof of Theorem 1, we will obtain
\[
\int_{\Omega} \left( \sum_{i=1}^{n} D_{i} F\left(r_{1}(t) \phi_{1}\left(\frac{|f_{1}(t)|}{r_{1}(t)}\right), \ldots, r_{n}(t) \phi_{n}\left(\frac{|f_{n}(t)|}{r_{n}(t)}\right)\right) \times D^{m} r_{1}(t) \phi_{1}\left(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)}\right) dt \right.
\]
\[
\leq F\left(\int_{\Omega} D^{m} r_{1}(t) \phi_{1}\left(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)}\right) dt, \ldots, \int_{\Omega} D^{m} r_{n}(t) \phi_{n}\left(\frac{|D^{m} f_{n}(t)|}{D^{m} r_{n}(t)}\right) dt \right).
\]
To complete the proof we observe that, from equality
\[
\int_{\Omega} D^{m} r_{1}(t) \phi_{1}\left(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)}\right) dt = \frac{\int_{\Omega} (p_{1}(t) D^{m} r_{1}(t) \phi_{1}(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)})/p_{1}(t)) dt}{\int_{\Omega} p_{1}(t) dt},
\]
i = 1, \ldots, n,
and using Jensen’s inequality, we can obtain
\[
h_{1}\left(\int_{\Omega} D^{m} r_{1}(t) \phi_{1}\left(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)}\right) dt \right)
\]
\[
\leq \int_{\Omega} p_{1}(t) h_{1}\left(D^{m} r_{1}(t) \phi_{1}(\frac{|D^{m} f_{1}(t)|}{D^{m} r_{1}(t)})/p_{1}(t)\right) dt,
\]
i = 1, \ldots, n,
since \(h_{i}, i = 1, \ldots, n,\) are convex.

Now the inequality (9) follows from the facts that the \(h_{i}, i = 1, \ldots, n,\)
are increasing and \(F\) is nondecreasing in each variable.

REFERENCES


