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Faithful tropicalisation and torus actions

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Abstract. For any affine variety equipped with coordinates, there is a surjective, continuous map from its Berkovich space to its tropicalisation. Exploiting torus actions, we develop techniques for finding an explicit, continuous section of this map. In particular, we prove that such a section exists for linear spaces, Grassmannians of planes (reproving a result due to Cueto, Häbich, and Werner), matrix varieties defined by the vanishing of 3×3-minors, and for the hypersurface defined by Cayley's hyperdeterminant.

1. Introduction

Let K be a field with a non-Archimedean valuation $v : K \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, let $\mathbb{A}^n \supseteq \mathbb{G}_\mathrm{m}^n$ be the n-dimensional affine space over K and the n-dimensional torus with coordinates $x_1, \ldots, x_n$, respectively, and let $\mathbb{P}^{n-1}$ be the $(n-1)$-dimensional projective space over K with homogeneous coordinates $x_1, \ldots, x_n$. For a closed subvariety X of $\mathbb{G}_\mathrm{m}^n$ or $\mathbb{A}^n$ or $\mathbb{P}^{n-1}$, defined over K, we write $\mathrm{Trop}(X)$ for the tropicalisation of X sitting inside $\mathbb{R}^n$ or $\mathbb{R}_\infty^n$ or $(\mathbb{R}_\infty^n \setminus \{(\infty, \ldots, \infty)\})/\mathbb{R}(1, \ldots, 1)$, respectively.

Write X^{an} for the analytification of X in Berkovich's sense [2, Chapter 1]. We work with the negative logarithms of multiplicative seminorms, so in the affine case X^{an} is the set of all ring valuations $K[X] \to \mathbb{R}_\infty$ extending v, equipped with the topology of pointwise convergence. In particular, X^{an} is a Hausdorff topological space, and a sequence $w_1, w_2, \ldots$ in X^{an} converges if and only if the sequence $w_1(f), w_2(f), \ldots$ converges in $\mathbb{R}_\infty$ (with the topology of a half-open interval)

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for each $f \in K[X]$. Write ∞ for the valuation of $K[\mathbb{A}^n] = K[x_1, \dots, x_n]$ that maps a polynomial to the valuation of its constant term. In the projective case, let $\widehat{X} \subseteq \mathbb{A}^n$ be the affine cone over X. Then, as a topological space, X^{an} equals $\widehat{X}^{\mathrm{an}} \setminus \{\infty\}$ modulo the equivalence relation under which w_1 and w_2 are equivalent if and only if there exists a constant $C \in \mathbb{R}$ such that for each degree-d homogeneous polynomial f in the graded coordinate ring $K[\widehat{X}]$ we have $w_1(f) = dC + w_2(f)$ (see [6, Chapter 2] for the case of the projective line).

There is a continuous surjection

$$\mathrm{trop}: X^{\mathrm{an}} \to \mathrm{Trop}(X), \;\; w \mapsto (w(x_1), \dots, w(x_n)).$$

This can be taken either as a definition of $\mathrm{Trop}(X)$ or as a theorem when other definitions are chosen [9,11,19,20]. Indeed, in [19] it is proved that X^{an} is the projective limit of the tropicalisations $\mathrm{Trop}(X)$ for all choices of coordinates. The tropicalisation is the support of a finite polyhedral complex by [3].

In this paper we discuss a number of high-dimensional examples where trop has a continuous section. The results are motivated by exciting recent work for Grassmannians of planes [7] and for curves [5]. In particular, we will give another, more geometric proof of the main result of [7] that Grassmannians of planes admit such a section. In the recent paper [13] (written concurrently with our paper) it is proved that, if X is a subvariety of $\mathbb{G}_{\mathrm{m}}^n$, then a section exists on the locus in $\mathrm{Trop}(X)$ where the tropical multiplicity equals one [13]. This beautiful general theorem implies parts of our results, e.g. for linear spaces. The emphasis in our paper, however, is on explicit sections in concrete examples, and in several of these we also extend the section to the part of $\mathrm{Trop}(X)$ outside $\mathbb{R}^n$.

Throughout, we will assume that the valuation $K \to \mathbb{R}_\infty$ is surjective. This is no restriction for our purposes. Indeed, (K, v) always embeds into a valued field (L, v_L) with v_L surjective. This does not change $\mathrm{Trop}(X)$, and a suitable section $\mathrm{Trop}(X) \to X_L^{\mathrm{an}}$ can be composed with the restriction map $X_L^{\mathrm{an}} \to X^{\mathrm{an}}$ to obtain a section $\mathrm{Trop}(X) \to X^{\mathrm{an}}$.

We will use the following notation and facts. Given a point $\xi \in \mathbb{R}_\infty^n$ we write

$$K[x]_\xi := \left\{ \sum_{\alpha \in A} c_\alpha x^\alpha \mid A \subseteq \mathbb{N}^n \text{ finite and } v(c_\alpha) + \alpha \cdot \xi \geq 0 \text{ for all } \alpha \in A \right\}$$

for the *tilted group ring* [20]. This ring contains the valuation ring of K and it has an ideal with the same definition but with $\geq$ replaced by $>$. The quotient by this ideal is an algebra over the residue field k of K. By surjectivity of the valuation, this algebra is in fact a polynomial ring over k in at most n variables—generators can be obtained as the images y_i of $c_i x_i$ where i ranges through the set where $\xi_i \neq \infty$ and where the coefficients $c_i \in K$ are chosen such that $v(c_i) + \xi_i = 0$. Let $I(X)$ be the ideal of X in $K[x]$. The image of $I(X) \cap K[x]_\xi$ in the polynomial ring $k[y_i \mid i : \xi_i \neq \infty]$ is called the *initial ideal* $\mathrm{in}_\xi I(X)$ of $I(X)$ relative to ξ, and the scheme $\mathrm{in}_\xi X$ defined by it is called the *initial degeneration* of X. The point ξ lies in the tropical variety if and only if $\mathrm{in}_\xi I(X)$ does not contain monomials [18, Chapter 3].

The remainder of this paper is organised as follows. In Sect. 2 we prove that if $Y \subseteq \mathbb{A}^n$ is a linear space, then the surjection $Y^{\mathrm{an}} \to \mathrm{Trop}(Y)$ has a continuous section. In Sect. 3, given an action of an m-dimensional subtorus of $\mathbb{G}_{\mathrm{m}}^n$ on a subvariety $X \subseteq \mathbb{A}^n$, we construct an action of $\mathbb{R}^m$ on a retract $Z \subseteq X^{\mathrm{an}}$, which maps surjectively and $\mathbb{R}^m$-equivariantly onto $\mathrm{Trop}(X)$. In Sect. 4 we introduce techniques for finding sections $\mathrm{Trop}(X) \to Z$ when X is obtained by smearing around a linear space Y with a torus action. As an example, we treat the variety in $\mathbb{G}_{\mathrm{m}}^{m \times p}$ of matrices of less than full rank, where we show that a continuous section exists at least over a large open subset of the tropicalisation. In Sects. 5 and 6 we apply our techniques to Grassmannians of two-spaces and to matrices of rank two, respectively. We conclude with a brief discussion of A-discriminants in Sect. 7.

2. Linear spaces

In this section we assume that Y is a linear subspace through the origin $0 \in \mathbb{A}^n$. Tropical linear spaces are well-understood through their circuits and cocircuits [1,24], and the proof of the following theorem is very natural from that perspective.

Theorem 2.1. *For any linear subspace $Y \subseteq \mathbb{A}^n$ the projection* $\mathrm{trop}_Y : Y^{\mathrm{an}} \to \mathrm{Trop}(Y)$ *has a continuous section.*

Without loss of generality, we may restrict to the case where Y is not contained in any coordinate hyperplane, so that $\mathrm{Trop}(Y)$ is the closure of $\mathrm{Trop}(Y) \cap \mathbb{R}^n$. Nevertheless, we will need to check carefully that the section we construct is also continuous on $\mathrm{Trop}(Y) \setminus \mathbb{R}^n$. We will use that for $\eta \in \mathrm{Trop}(Y) \cap \mathbb{R}^n$ the initial degeneration $\mathrm{in}_\eta Y$ is a linear subspace of $\mathbb{A}_k^n$ of the same dimension as Y. For general $\eta \in \mathrm{Trop}(Y)$ we have $\mathrm{in}_\eta Y = \mathrm{in}_\eta Y'$, where Y' is the subspace of Y consisting of all y with $x_i(y) = 0$ for all i with $\eta_i = \infty$.

The K-space Y' defines a matroid on $[n]$ by declaring a subset $J \subseteq [n]$ independent if the restrictions $(x_j)|_{Y'},\ j \in J$ are K-linearly independent. Similarly, the k-space $\mathrm{in}_\eta Y'$ also defines a matroid on $[n]$, by declaring J independent if the restrictions of the y_j, $j \in J$ (from the definition of the tilted polynomial ring) to $\mathrm{in}_\eta Y'$ are k-linearly independent. The two matroids have the same rank, and any basis of the latter matroid is also a basis of the former matroid. Throughout the paper, these distinguished bases of the former matroid will be called *compatible with* η (and, conversely, η with those bases).

Proof of Theorem 2.1. We define the section $\sigma\colon \mathrm{Trop}(Y) \to Y^{\mathrm{an}}$ as follows. Pick $\eta \in \mathrm{Trop}(Y)$, set $S := \{i \in [n] \mid \eta_i = \infty\}$, and let $Y' \subseteq Y$ be the subspace of all $y \in Y$ with $x_i(y) = 0$ for all $i \in S$. Let J be a basis of the matroid defined by Y' that is compatible with η. In particular, J is disjoint from S. The inclusion $K[x_j \mid j \in J] \to K[Y']$ is an isomorphism, so for $f \in K[Y]$ we can uniquely write $f|_{Y'} = \sum_\alpha c_\alpha x^\alpha$ where the α run through $\mathbb{N}^J$. We set

$$\sigma(\eta)(f) := \min_\alpha (v(c_\alpha) + \alpha \cdot \eta).$$

This is clearly a valuation that maps x_j, $j \in J$ to η_j and that maps the x_i with $i \in S$ to ∞. What about x_i with $i \notin J \cup S$? Up to a scalar factor, there exists a unique non-zero linear relation

$$\sum_{j \in J \cup \{i\}} d_j x_j \in I(Y').$$

After scaling we may assume that $v(d_j) + \eta_j \geq 0$ for all $j \in J \cup \{i\}$ and that equality holds for at least one j. Then this element lies in $I(Y') \cap K[x_j \mid j \notin S]_\eta$. If $v(d_i) + \eta_i$ were strictly positive, then projecting down into $k[y_j \mid j \notin S]$ would yield a relation among the y_j with $j \in J$, a contradiction to the choice of J. Hence $v(d_i)+\eta_i = 0$. If $v(d_j)+\eta_j$ were strictly positive for all $j \in J$, then projecting down would yield $y_i \in \mathrm{in}_\eta I(Y')$, which contradicts $\eta \in \mathrm{Trop}(Y')$. Hence $v(d_j)+\eta_j = 0$ for some $j \in J$. This shows that

$$\sigma(\eta)(x_i) = \min_{j \in J}(v(-d_j/d_i) + \eta_j) = \eta_i,$$

as required. So $\sigma(\eta) \in Y^{\mathrm{an}}$ is a point in the fibre of trop_Y above η.

To define $\sigma(\eta)$, we have made the choice of a basis J in the matroid defined by $\mathrm{in}_\eta I(Y')$. But in fact, this choice does not influence the outcome. Indeed, *any* valuation $w \in Y^{\mathrm{an}}$ with $\mathrm{trop}_Y(w) = \eta$ must satisfy $w(f) \geq \sigma(\eta)(f)$ for all $f \in K[Y]$.[1] In particular, this must hold for all valuations constructed from other bases of the matroid. This shows that σ is well-defined on all of $\mathrm{Trop}(Y)$.

It remains to show that σ is continuous. This is immediate from the formula for $\sigma(\eta)$ on a subset of $\mathrm{Trop}(Y)$ where S and J compatible with η are fixed. Let Y' be as above. Suppose that a sequence $\eta^{(l)}, l = 1, 2, \dots$ in this set converges to a point $\eta \in \mathrm{Trop}(Y)$. Note that the set of i with $\eta_i = \infty$ contains S but may be strictly larger, and may even contain elements of J. Even so, for every non-zero relation $\sum_{j \in J \cup \{i\}} d_j x_j \in I(Y')$ for $i \notin J \cup S$ we have $\min_{j \in J}(v(d_j)+\eta_j^{(l)}) = v(d_i)+\eta_i^{(l)}$. This closed condition then also holds in the limit:

$$\min_{j \in J}(v(d_j) + \eta_j) = v(d_i) + \eta_i. \tag{1}$$

Let w be the valuation of $K[Y]$ defined by mapping $f \in K[Y]$ with $f|_{Y'} = \sum_{\alpha \in \mathbb{N}^J} c_\alpha x^\alpha$ to $\min_{\alpha \in \mathbb{N}^J}(v(c_\alpha) + \alpha \cdot \eta)$. Then w is, indeed, a valuation of $K[Y]$, which maps x_j to η_j for $j \in J$ (because $x_j|_{Y'}$ is a single term) and for $j \in S$ (because $x_j|_{Y'}$ has no terms) and for $j \notin J \cup S$ [by (1)]. Moreover, $w(f)$ is minimal among all such valuations, so $w(f) = \sigma(\eta)(f)$. This shows that σ is continuous on the closure of the set of all η compatible with a given S and J. These closures form a finite closed cover of $\mathrm{Trop}(Y)$, hence σ is continuous everywhere. □

Remark 2.2. In the *constant coefficient case*, where Y is a linear space defined over a subfield of K on which the valuation is trivial, the choice of J above can be made more constructive, as follows. Given $\eta \in \mathrm{Trop}(Y) \cap \mathbb{R}^n$, take a permutation $\pi \in S_n$ such that $\eta_{\pi(1)} \geq \dots \geq \eta_{\pi(n)}$. Then construct J by setting $J_0 := \emptyset$ and

[1] So $\sigma(\eta)$ is the *Shilov boundary point* [2, Chapter 2] in the fibre in Y^{an} above η.

$$J_i := \begin{cases} J_{i-1} \cup \{i\} & \text{if } x_i|_Y \text{ linearly independent of } \langle x_j|_Y \mid j \in J_{i-1}\rangle, \text{ and} \\ J_{i-1} & \text{otherwise.} \end{cases}$$

Then $J := J_n$ is a basis of the matroid defined by Y compatible with η. This is the greedy algorithm for finding a maximal-weight basis in a matroid [22, Chapter 40].

Conversely, given a basis J of that matroid, we can construct all $\eta \in \mathrm{Trop}(Y) \cap \mathbb{R}^n$ compatible with it by choosing the η_j with $j \in J$ arbitrarily and setting η_i for $i \notin J$ equal to the minimal value η_j for j in the unique circuit contained in $J \cup \{i\}$. We will use this explicit construction in Sects. 5 and 6. These remarks apply, *mutatis mutandis*, also to $\eta \in \mathrm{Trop}(Y) \setminus \mathbb{R}^n$.

3. Torus actions

Let $\varphi : \mathbb{G}_\mathrm{m}^m \to \mathbb{G}_\mathrm{m}^n$ be a homomorphism of tori. This is of the form $\varphi(t_1, \dots, t_m) = (t^{a_1}, \dots, t^{a_n})$ where $a_1, \dots, a_n \in \mathbb{Z}^m$. Let $A \in \mathbb{Z}^{n\times m}$ be the matrix with rows $a_1, \dots, a_m$. Let $X \subseteq \mathbb{A}^n$ or $X \subseteq \mathbb{G}_\mathrm{m}^n$ be a closed affine subvariety stable under the $\mathbb{G}_\mathrm{m}^m$-action on $\mathbb{A}^n$ (or $\mathbb{G}_\mathrm{m}^n$) given by φ. Then $\mathbb{R}^m$ has a continuous action on $\mathrm{Trop}(X)$ given by

$$(\tau, \xi) \mapsto A\tau + \xi.$$

The column space of A is contained in the *lineality space* of $\mathrm{Trop}(X)$. In this section we investigate to what extent this action can be lifted to X^{an}. For this, we denote by $\lambda : \mathbb{G}_\mathrm{m}^m \times X \to X, (t, x) \mapsto \varphi(t)x$ the action of $\mathbb{G}_\mathrm{m}^m$ on X and by $\lambda^* : K[X] \to K[\mathbb{G}_\mathrm{m}^m \times X]$ its comorphism.

Lemma 3.1. *There exists a commutative diagram of continuous maps:*

$$\begin{tikzcd} \mathbb{R}^m \times X^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{id} \times \mathrm{trop}"] & (\mathbb{G}_\mathrm{m}^m \times X)^{\mathrm{an}} \arrow[r, "w \mapsto w \circ \lambda^*"'] & X^{\mathrm{an}} \arrow[d, "\mathrm{trop}"'] \\ \mathbb{R}^m \times \mathrm{Trop}(X) \arrow[rr, "(\tau,\xi) \mapsto A\tau + \xi"] & & \mathrm{Trop}(X). \end{tikzcd}$$

Proof. The right-most map in the top row of the diagram is the analytification of the torus action, hence in particular continuous. The only map that needs a definition is the left-most map in that row. It sends (τ, w) to the valuation of $K[\mathbb{G}_\mathrm{m}^m \times X]$ defined by

$$\sum_{\beta \in \mathbb{Z}^m} f_\beta t^\beta \mapsto \min_\beta (w(f_\beta) + \beta \cdot \tau).$$

For each fixed element $\sum_{\beta \in \mathbb{Z}^m} f_\beta t^\beta$ of $K[\mathbb{G}_\mathrm{m}^m \times X]$ the right-hand side is continuous in (τ, w) (this uses the definition of the topology of X^{an} and the fact that a point-wise minimum of continuous functions is continuous). By definition of the topology of $(\mathbb{G}_\mathrm{m}^m \times X)^{\mathrm{an}}$, this implies that the map is continuous.

To see that the diagram commutes, pick $(\tau, w) \in \mathbb{R}^m \times X^{\mathrm{an}}$ and let $w' \in X^{\mathrm{an}}$ be the image of that pair along the top row. We have $\lambda^* x_i = t^{a_i} x_i$, and hence

$$w'(x_i) = w(x_i) + a_i \cdot \tau.$$

This implies that $\mathrm{trop}(w') = \mathrm{trop}(w) + A\tau$, as claimed. □

Let μ denote the composition of the two maps in the first row. Unwinding the definitions, we find that μ sends (τ, w) to a valuation on $K[X]$ defined as follows. Pick $f \in K[X]$ and decompose $f = \sum_{\beta \in \mathbb{Z}^m} f_\beta$ into $\mathbb{G}_\mathrm{m}^m$-weight vectors, i.e., with $\lambda^* f_\beta = t^\beta f_\beta$. Then

$$\mu(\tau, w)(f) = \min_\beta (w(f_\beta) + \beta \cdot \tau).$$

We remark that if $A\tau = 0$, then $\mu(\tau, w) = \mu(0, w)$. Indeed, τ is perpendicular to the rows of A, hence to any $\mathbb{Z}$-linear combination of these, and the β for which there exist non-zero $f_\beta \in K[X]$ of weight β are such linear combinations.

In general, μ is not an action of $\mathbb{R}^m$ on X^{an}. Indeed, while the valuations $\mu(0, w)$ and w do agree on monomials, they do not need to agree on other functions. For an explicit example, set $X = \mathbb{A}^2$ with coordinate ring $K[x_1, x_2]$, let $m = 2$, and let φ be the identity. Define $w \in X^{\mathrm{an}}$ by $w(f) := v(f(1, 1))$, so that $w(x_1) = w(x_2) = 0$. Then the image of $(0, w)$ along the first row equals the "Gauss point" w' of $K[x_1, x_2]$ defined by

$$\sum_{i,j} c_{ij} x_1^i x_2^j \mapsto \min_{i,j} v(c_{ij}).$$

Then we have $w(x_1 - 1) = \infty \neq 0 = w'(x_1 - 1)$. However, the following lemma shows that $\mu(0, w) \neq w$ is the *only* obstacle to μ being an action.

Lemma 3.2. *Define Z as the image of μ. Then Z is a closed subset of X^{an} and the restriction of μ to $\mathbb{R}^m \times Z$ defines a continuous action of $\mathbb{R}^m$ on Z. Moreover, the map $w \mapsto \mu(0, w)$ defines a continuous retraction from X^{an} to Z.*

Proof. First, for $\tau_1, \tau_2 \in \mathbb{R}^m$ and $f \in K[X]$ with $\mathbb{G}_\mathrm{m}^m$-weight decomposition $f = \sum_{\beta \in \mathbb{Z}^m} f_\beta \in K[X]$ and $w \in X^{\mathrm{an}}$ we compute

$$\mu(\tau_1, \mu(\tau_2, w))(f) = \min_{\beta \in \mathbb{Z}^m} (\beta \cdot \tau_1 + \mu(\tau_2, w)(f_\beta))$$
$$= \min_{\beta \in \mathbb{Z}^m} (\beta \cdot \tau_1 + \beta \cdot \tau_2 + w(f_\beta)) = \mu(\tau_1 + \tau_2, w)(f).$$

This implies that $\mu(0, \mu(\tau, w)) = \mu(\tau, w)$, so that 0 acts as the identity on Z. Hence μ is an action on Z. Furthermore, Z can be characterised as the pre-image of the diagonal in $X^{\mathrm{an}} \times X^{\mathrm{an}}$ under the continuous map $X^{\mathrm{an}} \to X^{\mathrm{an}} \times X^{\mathrm{an}}$, $w \mapsto (w, \mu(0, w))$. Since X^{an} is Hausdorff, the diagonal is closed, hence so is Z. The last statement is immediate. □

The following refinement of the statement that Z is a retract of X^{an} was pointed out to us by Joe Rabinoff.

Proposition 3.3. *In the setting above, Z is a strong deformation retract of X^{an}.*

Proof. This can be derived using the general techniques of [2, Chapter 6]; here is a shortcut in our language. For $r \in [0, \infty]$ and $w \in X^{\mathrm{an}}$ let w_r be the function $K[X] \to \mathbb{R}_\infty$ defined as follows. Take $f \in K[X]$, expand $f(\varphi(t)x) := \sum_{\beta \in \mathbb{Z}^m} f_\beta t^\beta$, and rewrite this Laurent series with $K[X]$-coefficients as a formal power series

$$\sum_{\beta \in \mathbb{Z}^m} f_\beta t^\beta = \sum_{\gamma \in (\mathbb{Z}_{\geq 0})^m} g_\gamma (t-1)^\gamma$$

around the identity element $1 = (1, \dots, 1)$ of $\mathbb{G}_{\mathrm{m}}^m$. Set

$$w_r(f) := \min_\gamma (w(g_\gamma) + |\gamma| r), \text{ where } |\gamma| := \gamma_1 + \cdots + \gamma_m.$$

We argue that this minimum is attained, and that it can be replaced by a minimum over a finite set of γs that does not depend on w or r. In the rewriting process, we replace each Laurent monomial t^β by the formal power series of $((t-1)+1)^\beta$ around 1. This shows that each g_γ is a $\mathbb{Z}$-linear combination of the f_β. In particular, for all γ we have $w(g_\gamma) \geq \min_\beta w(f_\beta)$, and for $r > 0$ this suffices to conclude that the minimum is attained.

Conversely, we claim that each f_β is a $\mathbb{Z}$-linear combination of the g_γ. This is immediate if all β with $f_\beta \neq 0$ are already in $(\mathbb{Z}_{\geq 0})^m$ (since then we are just rewriting polynomials, and the rewriting can be reversed). The general case can be reduced to this, since multiplication of power series with a fixed power series of the form $((t-1)+1)^\beta$ is a $\mathbb{Z}$-linear isomorphism with inverse equal to multiplication with $((t-1)+1)^{-\beta}$. Consequently, we find that the minimum is attained for $r = 0$, as well, and that $\min_\gamma w(g_\gamma) = \min_\beta w(f_\beta) = \mu(0, w)(f)$.

Combining the two $\mathbb{Z}$-linear transitions, all countably many g_γ are $\mathbb{Z}$-linear combinations of finitely many among them. If d is the maximum value of $|\gamma|$ among these finitely many, then we can replace the minimum defining $w_r(f)$ by the minimum over all γ with $|\gamma| \leq d$. Then it is evident that $w_r(f)$ depends continuously on the pair $(w, r) \in X^{\mathrm{an}} \times [0, \infty]$.

Now w_r is a point in X^{an} that depends continuously on (w, r). For $r = \infty$ we have

$$w_\infty(f) = w(g_0) = w\left(\sum_\beta f_\beta\right) = w(f),$$

so $w_\infty = w$. As mentioned above, we have $w_0 = \mu(0, w)$. Finally, we must argue that if w already lies in Z, that is, if $w = \mu(0, w)$, then $w_r = w$ for all $r \in (0, \infty]$. But in this case the $\gamma = 0$ term in the definition of w_r equals $\min_\beta w(f_\beta)$ and all other terms are (strictly) larger than this, so that $w_r = w$ as desired. □

We conclude this section with two remarks on quotients. The first concerns the categorical quotient $X//\mathbb{G}_{\mathrm{m}}^m$ of X by the action of $\mathbb{G}_{\mathrm{m}}^m$, i.e., the affine variety with coordinate ring equal to the ring of $\mathbb{G}_{\mathrm{m}}^m$-invariants in $K[X]$. The morphism $X \to X//\mathbb{G}_{\mathrm{m}}^m$ gives rise to a morphism of analytic spaces, which sends a valuation $w \in X^{\mathrm{an}}$ to its restriction to the $\mathbb{G}_{\mathrm{m}}^m$-invariants.

Lemma 3.4. *The map* $X^{\mathrm{an}} \to (X//\mathbb{G}_{\mathrm{m}}^m)^{\mathrm{an}}$ *factorises as*

$$X^{\mathrm{an}} \to Z \to Z/\mathbb{R}^m \to (X//\mathbb{G}_{\mathrm{m}}^m)^{\mathrm{an}}.$$

Proof. We need to show that, for $\tau \in \mathbb{R}^m$ and $w \in X^{\mathrm{an}}$, the restriction of $w' := \mu(\tau, w)$ to the $\mathbb{G}_{\mathrm{m}}^m$-invariants $f \in K[X]$ does not depend on τ and equals the restriction of w to $\mathbb{G}_{\mathrm{m}}^m$-invariants. But this is immediate: f has weight zero, and hence

$$w'(f) = \mu(\tau, w)(f) = w(f) + 0 \cdot \tau = w(f),$$

as desired. □

The second remark concerns the passage from affine cones to projective varieties. Suppose that $X \subseteq \mathbb{A}^n$ is an affine cone, and denote by $\mathbb{P}X \subseteq \mathbb{P}^{n-1}$ the corresponding projective variety. The points of $(\mathbb{P}X)^{\mathrm{an}}$ are equivalence classes of points of $X^{\mathrm{an}} \setminus \{\infty\}$.

Lemma 3.5. *The map* $Z \to (\mathbb{P}X)^{\mathrm{an}}$ *factorises as*

$$Z \to Z/U \to (\mathbb{P}X)^{\mathrm{an}},$$

where $U := A^{-1}\mathbb{R}(1, \ldots, 1)$.

Proof. We need to show that if $A\tau = (C, \ldots, C)$ for some $C \in \mathbb{R}$ and if $w \in Z$, then $w' := \mu(\tau, w)$ is equivalent to w. Let f be a homogeneous polynomial of degree d in the graded ring $K[X]$, and decompose $f = \sum_{\beta \in \mathbb{Z}^m} f_\beta$. Then $\beta \cdot \tau = dC$ for all β with f_β non-zero, and hence

$$w'(f) = \min_\beta (w(f_\beta) + \beta \cdot \tau) = dC + \min_\beta w(f_\beta) = \mu(0, w) = w.$$

□

4. Smearing a subspace around by a torus

Let $Y \subseteq \mathbb{A}^n$ be a linear subspace not contained in any coordinate hyperplane and let $\varphi : \mathbb{G}_{\mathrm{m}}^m \to \mathbb{G}_{\mathrm{m}}^n$ be a torus homomorphism given by an $n \times m$ integer matrix A. Define

$$X := \overline{\{\varphi(t)y \mid y \in Y, t \in \mathbb{G}_{\mathrm{m}}^m\}},$$

so that X is stable under the action of $\mathbb{G}_{\mathrm{m}}^m$. Let X^0, Y^0 be the open subsets of X, Y, respectively, where none of the coordinates vanish. Then we have $\mathrm{Trop}(X^0) = \mathrm{Trop}(X) \cap \mathbb{R}^n$ and $\mathrm{Trop}(Y^0) = \mathrm{Trop}(Y) \cap \mathbb{R}^n$ and

$$\mathrm{Trop}(X^0) = A\mathbb{R}^m + \mathrm{Trop}(Y^0);$$

this follows, for instance, from [20, Proposition 2.5]. Let $\mu : \mathbb{R}^m \times X^{\mathrm{an}} \to Z$ be the map constructed in Sect. 3. We then obtain a continuous map

$$\mathbb{R}^m \times \mathrm{Trop}(Y) \to Z,\ (\tau, \eta) \mapsto \mu(\tau, \sigma_Y(\eta)),$$

where σ_Y is the section of $Y^{\mathrm{an}} \to \mathrm{Trop}(Y)$ constructed in Sect. 2. We would like to use this map to construct a section $\mathrm{Trop}(X) \to Z$ of the surjection $Z \to \mathrm{Trop}(X)$, or at least a section $\mathrm{Trop}(X^0) \to Z^0$, where Z^0 is the preimage of X^0 in Z^0. There are two basic strategies for doing so. The first strategy is given in the following proposition.

Proposition 4.1. *If the map $\mathbb{R}^m \times \mathrm{Trop}(Y^0) \to \mathrm{Trop}(X^0),\ (\tau, \eta) \mapsto A\tau + \eta$ has a continuous section, then so does the map* trop : $Z^0 \to \mathrm{Trop}(X^0)$*. Moreover, if the former section can be chosen $\mathbb{R}^m$-equivariant, then so can the latter.*

Here the action of $\mathbb{R}^m$ on $\mathbb{R}^m \times \mathrm{Trop}(Y^0)$ is given by addition in the first coordinate and the trivial action on $\mathrm{Trop}(Y^0)$.

Proof. The composition

$$\sigma : (\mathrm{Trop}(X^0) \to \mathbb{R}^m \times \mathrm{Trop}(Y^0) \to Z^0)$$

of a continous section $\mathrm{Trop}(X^0) \to \mathbb{R}^m \times \mathrm{Trop}(Y^0)$ and the map $(\tau, \eta) \mapsto \mu(\tau, \sigma_Y(\eta))$ is a section $\mathrm{Trop}(X^0) \to Z^0$. The second statement is immediate. □

Here is an application of this construction. Recall from [10] that the *tropical rank* of a real matrix is the largest size of a square submatrix whose *tropical determinant* is attained by a single term.

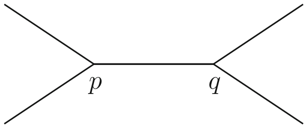
Proposition 4.2. *Let $m \le p$ natural numbers. Let $X \subseteq \mathbb{A}^{m\times p}$ be the matrix variety defined by the vanishing of all $m \times m$-minors. On X acts $\mathbb{G}_{\mathrm{m}}^m$ by scaling rows. Then the map* trop : $(X^0)^{\mathrm{an}} \to \mathrm{Trop}(X^0)$ *has a continuous section on the open subset U of* $\mathrm{Trop}(X^0)$ *consisting of matrices whose first $m - 1$ columns form a tropically non-singular matrix.*

Proof. Let Y be the linear subspace contained in X consisting of all matrices y such that $\mathbf{1}^T y = 0$, and let $\mathbb{G}_{\mathrm{m}}^m$ act on matrices by scaling rows. Let A be the corresponding $(mp)\times m$-matrix of integers. Then $X = \overline{\mathbb{G}_{\mathrm{m}}^m Y}$ and hence $\mathrm{Trop}(X^0) = A\mathbb{R}^m + \mathrm{Trop}(Y^0)$. Now $\mathrm{Trop}(Y^0)$ is the set of matrices η whose columns all lie in the tropical hyperplane where the minimum of the coordinates is attained at least twice. We will now argue that the map

$$\mathbb{R}^m \times \mathrm{Trop}(Y^0) \to \mathrm{Trop}(X^0),\ (\tau, \eta) \to A\tau + \eta$$

has an $\mathbb{R}^m$-equivariant section over the open set U, defined as follows. Let $\xi \in U$. Then for $\tau \in \mathbb{R}^m$ the condition that $\xi - A\tau$ lies in $\mathrm{Trop}(Y^0)$ is equivalent to the condition that for each $j = 1, \dots, p$ the minimum $\min_i(\xi_{ij} - \tau_i)$ is attained at least twice. This means that $-\tau$ lies on the intersection of the p tropical hyperplanes in $\mathbb{R}^m$ with coefficient vectors given by the columns of ξ. By the tropical non-singularity of the first $m-1$ columns of ξ, the intersection of the corresponding $m-1$ hyperplanes is already spanned by a single vector, $-\tau$, unique up to tropical scaling. For definiteness, choose τ_1 equal to ξ_{11}. We have that τ depends continuously on ξ. Indeed the *stable* intersection of $m - 1$ hyperplanes, that in this case coincides with the intersection, depends continuously on the hyperplanes (see [21, Section 5] and [17, Section 4]). Now $\xi - A\tau$ lies in $\mathrm{Trop}(Y^0)$ and we can apply Proposition 4.1 to obtain a $\mathbb{R}^m$-equivariant section $U \to (X^0)^{\mathrm{an}}$. □

Remark 4.3. The map $U \to \mathbb{R}^m$, $\xi \mapsto \tau$ constructed in the latter proof can in general *not* be extended to a continuous map $\mathrm{Trop}(X^0) \to \mathbb{R}^m$ with the property that $\xi - A\tau \in \mathrm{Trop}(Y^0)$ for all ξ. Indeed, consider the case where $m = p = 4$, so that X is the hypersurface defined by a single determinant. Take two column vectors $a, b \in \mathbb{R}^4$ in general position, so that the corresponding planes H_a, H_b in tropical projective 3-space intersect in a tropical projective line of the form:



Then the stable intersection of H_a, H_a, H_b is one of the two trivalent (projective) points, say p, and the stable intersection of H_a, H_b, H_b is the other point, q. Now consider the matrix $\xi = (a|a|b|b)$. Wiggling the first column slightly while keeping the remaining columns fixed, the matrix stays within $\mathrm{Trop}(X^0)$ but now with the first three columns defining hyperplanes that intersect in a single projective point near p. Hence we see that for ξ we need to take $-\tau$ in the stable intersection of H_a, H_a, H_b, i.e., in p, if we want it to depend continuously on ξ. But wiggling the last column instead, we find that we need to take $-\tau$ in q. Thus $-\tau$ cannot depend continuously on ξ.

We remark that the tropical multiplicity of such ξ is typically equal to two. After tropical scaling of rows and columns of ξ, and after permuting rows if necessary, we have

$$\xi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & b_2 \\ 0 & 0 & b_3 & b_3 \\ 0 & 0 & b_4 & b_4 \end{bmatrix}$$

where $0 \leq b_2 \leq b_3 \leq b_4$. The tropical determinant equals b_2. Moreover, if $b_2 < b_3$, then $\mathrm{in}_\xi \det(x) = (x_{13}x_{24} - x_{14}x_{23})(x_{31}x_{42} - x_{41}x_{32})$, which defines a scheme with two irreducible components. In view of [13, Theorem 10.6] it is conceivable that no continuous section of trop near ξ exists.

The second strategy for constructing a section $\mathrm{Trop}(X^0) \to (X^0)^{\mathrm{an}}$ is to show that the map $\mathbb{R}^m \times \mathrm{Trop}(Y^0) \to (X^0)^{\mathrm{an}}$ factors through the map $\mathbb{R}^m \times \mathrm{Trop}(Y^0) \to \mathrm{Trop}(X^0)$. We will now formulate sufficient conditions for this to happen.

The first of these conditions is purely polyhedral, namely, we require that for each $\eta \in \mathrm{Trop}(Y^0)$ the set

$$T_\eta := \{\tau \in \mathbb{R}^m \mid A\tau + \eta \in \mathrm{Trop}(Y^0)\},$$

which is the support of a polyhedral complex, is connected. Observe that these sets encode the ambiguity in the decomposition of ξ: if ξ equals both $\eta_1 + A\tau_1$ and $\eta_2 + A\tau_2$, then $\tau_1 - \tau_2 \in T_{\eta_1}$. Connectedness of T_η means that there exists a polyhedral path of decompositions of ξ from the first decomposition to the second. The second condition is more algebraic. Let $\eta \in \mathrm{Trop}(Y^0)$. Extend the valuation

$w := \sigma_Y(\eta)$ from $K[Y]$ to the field $K(Y)$; this can be done since it sends no non-zero polynomials to infinity. Let $y \in Y(K(Y))$ be the generic point of Y. The coordinates of y are thus $(x_1|_Y, \dots, x_n|_Y)$, and the vector of w-valuations of these coordinates is η. By slight abuse of notation, we write $w(y) = \eta$. Let $\tau \in \mathbb{R}^m$ be such that the line segment $[0, \tau]$ is contained in T_η. Then we require that for all sufficiently small $\epsilon > 0$ there exists a valued extension (L, w_L) of $K(Y)$ and a $t \in \mathbb{G}_\mathrm{m}^m(L)$ such that $w_L(t) = \epsilon\tau$ (with the same abuse of notation: the vector of valuations of the coordinates of t equals $\epsilon\tau$) and $\varphi(t)y \in Y^0(L)$.

Proposition 4.4. *Suppose that the torus homomorphism $\varphi : \mathbb{G}_\mathrm{m}^m \to \mathbb{G}_\mathrm{m}^n$ and the linear space Y satisfy the two aforementioned requirements. Then, for $\xi = A\tau_1 + \eta_1 \in \mathrm{Trop}(X^0)$ with $\tau_1 \in \mathbb{R}^m$ and $\eta_1 \in \mathrm{Trop}(Y^0)$ the expression*

$$\sigma(\xi) := \mu(\tau_1, \sigma_Y(\eta_1)) \in Z \subseteq X^{\mathrm{an}}$$

does not depend on the chosen decomposition of $\xi \in \mathrm{Trop}(X^0)$. The map $\sigma : \mathrm{Trop}(X^0) \to Z^0$ thus defined is a continuous, $\mathbb{R}^m$-equivariant section of the surjection $Z^0 \to \mathrm{Trop}(X^0)$.

Before we give the proof, we discuss a simple example in the plane.

Example 4.5. Let $Y \subseteq \mathbb{A}^2$ be given by the linear equation $x_1 - x_2 = 0$, and let $\varphi : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}^2$ be given by $\varphi(t) = (t, t^{-1})$, so that $A = (1, -1)^T$. Then we have

$$\mathrm{Trop}(Y) = \{(\eta_1, \eta_2) \in \mathbb{R}_\infty^2 \mid \eta_1 = \eta_2\} \quad \text{and} \quad A\mathbb{R}^1 = \{(\tau, -\tau) \mid \tau \in \mathbb{R}\}.$$

We have $X = \overline{\varphi(\mathbb{G}_\mathrm{m})Y} = \mathbb{A}^2$ and $\mathrm{Trop}(X^0) = A\mathbb{R}^1 + \mathrm{Trop}(Y^0)$ and $T_\eta = \{0\}$ for all $\eta \in \mathrm{Trop}(Y^0)$. In the second requirement we can just take $t = 1$ for all η. Thus both requirements are met. For $\xi = (\xi_1, \xi_2) = A\tau + \eta = (\tau + \eta_1, -\tau + \eta_1)$ and $f = \sum_{i,j} c_{ij} x_1^i x_2^j$ we find

$$\sigma(\xi)(f) = \min_{k \in \mathbb{Z}} \min_{i-j=k} (k\tau + v(c_{ij}) + (i+j)\eta_1) = \min_{i,j}(v(c_{ij}) + i\xi_1 + j\xi_2),$$

which extends to all of $\mathrm{Trop}(X)$, and in fact equals the section obtained in Sect. 2 when regarding X as a linear space. ◇

Proof of Proposition 4.4. For the first statement we need to prove that if ξ can also be decomposed as $A\tau_2 + \eta_2$ then

$$\mu(\tau_2, \sigma_Y(\eta_2)) = \mu(\tau_1, \sigma_Y(\eta_1)).$$

This is equivalent to

$$\mu(\tau_2 - \tau_1, \sigma_Y(\eta_2)) = \mu(0, \sigma_Y(\eta_1)).$$

Now $\tau_1 - \tau_2 \in T_{\eta_1}$, and since T_{η_1} is connected, by walking from 0 to $\tau_1 - \tau_2$ through T_{η_1} along a polyhedral path, it suffices to prove the following local version of this equality. Let $\eta \in \mathrm{Trop}(Y^0)$ and $\tau \in \mathbb{R}^m$ be such that the segment $[0, \tau]$ lies entirely in T_η. Then we want to show that

$$\mu(-\tau, \sigma_Y(A\tau + \eta)) = \mu(0, \sigma_Y(\eta)).$$

By definition of μ, it suffices to prove this when applied to a non-zero $f \in K[X]$ that is homogeneous with respect to the $\mathbb{G}_m^m$-action, say of weight β. We will prove, in fact, that the function

$$\ell : [0, 1] \to \mathbb{R},\ \epsilon \mapsto \mu(-\epsilon\tau, \sigma_Y(A\epsilon\tau + \eta))(f)$$

is constant on the interval $[0, 1]$. Since ℓ is a continuous function and $[0, 1]$ is connected, it suffices to prove that ℓ has a local minimum at every point in $[0, 1]$. We give the argument at the point 0; it follows at other points in a similar manner.

Set $w := \sigma_Y(\eta)$, and let $y \in Y(K(Y))$ be the generic point. Then for $\epsilon > 0$ sufficiently small a valued field extension $(L, w_L) \supseteq (K(Y), w)$ and $t \in \mathbb{G}_m^m(L)$ exist as in the second requirement, that is, with $w_L(t) = \epsilon\tau$ and $\varphi(t)y \in Y^0(L)$. After shrinking ϵ if necessary we may assume that η and $\epsilon A\tau + \eta$ are both compatible with the same basis $J \subseteq [n]$ of the matroid defined by Y. Expand the restriction $f|_Y$ as $\sum_{\alpha \in \mathbb{N}^J} c_\alpha x^\alpha$. Then on the one hand we have

$$w_L(f|_Y(\varphi(t)y)) = w_L(t^\beta f|_Y(y)) = \beta \cdot \epsilon\tau + \sigma_Y(\eta)(f),$$

where we have used that $\varphi(t)y \in Y(L)$ and that f is homogeneous of $\mathbb{G}_m^m$-weight β. On the other hand, we have

$$\begin{aligned} w_L(f|_Y(ty)) = w_L\left(\sum_{\alpha \in \mathbb{N}^J} c_\alpha t^{\alpha A} y^\alpha\right) &\leq \min_\alpha (v(c_\alpha) + \alpha \cdot A\epsilon\tau + \alpha \cdot \eta) \\ &= \sigma_Y(A\epsilon\tau + \eta)(f). \end{aligned}$$

Thus we find that

$$\ell(\epsilon) = \sigma_Y(A\epsilon\tau + \eta)(f) - \beta \cdot \epsilon\tau \geq \sigma_Y(\eta)(f) = \ell(0),$$

as desired. This shows that the section $\sigma : \mathrm{Trop}(X^0) \to Z$ is well-defined. To see that σ is continuous, decompose $\mathrm{Trop}(Y^0)$ into finitely many closed polyhedra P_i and let P_i' denote the image of

$$\mathbb{R}^m \times P_i \to \mathrm{Trop}(X^0),\ (\tau, \eta) \mapsto A\tau + \eta.$$

By basic linear algebra over $\mathbb{R}$, on each P_i' this map has a continuous (in fact, affine-linear) section $P_i' \to \mathbb{R}^m \times P_i$. This shows that the restriction of σ to each P_i' is continuous. Since the P_i' form a finite closed cover of $\mathrm{Trop}(X^0)$, the map σ is continuous on $\mathrm{Trop}(X^0)$.

Finally, we need to verify that σ is $\mathbb{R}^m$-equivariant. Let $\xi = A\tau + \eta \in \mathrm{Trop}(X^0)$ with $\tau \in \mathbb{R}^m$ and $\eta \in \mathrm{Trop}(Y^0)$. Let $\tau' \in \mathbb{R}^m$. Then we have

$$\begin{aligned} \sigma(A\tau' + \xi) = \sigma(A(\tau' + \tau) + \eta) &= \mu(\tau' + \tau, \sigma_Y(\eta)) \\ &= \mu(\tau', \mu(\tau, \sigma_Y(\eta))) = \mu(\tau', \sigma(\xi)), \end{aligned}$$

as desired. □

Remark 4.6. While Propositions 4.1 and 4.4 give sections only over $\mathrm{Trop}(X^0)$, we will see that, at least in the cases of Grassmannians of planes and of the variety of rank-two matrices, sections exists over all of $\mathrm{Trop}(X)$.

5. Grassmannians of planes

In this section we set $n := \binom{m}{2}$ and consider $\mathbb{A}^n$ with coordinates x_{ij} for $1 \leq i < j \leq n$. We also write $x_{ji} = -x_{ij}$ for $i > j$, and $\xi_{ji} = \xi_{ij}$ for tropical coordinates. Let $X := \widehat{\mathrm{Gr}}(2, m) \subseteq \mathbb{A}^n$ denote the affine cone over the Grassmannian of planes, given as the image of the polynomial map

$$\psi : (\mathbb{A}^m)^2 \to \mathbb{A}^n, (y, z) \mapsto (y_i z_j - y_j z_i)_{i<j}.$$

This map is $\mathbb{G}_{\mathrm{m}}^m$-equivariant with respect to the standard (diagonal) action of $\mathbb{G}_{\mathrm{m}}^m$ on $(\mathbb{A}^m)^2$ and the action of $\mathbb{G}_{\mathrm{m}}^m$ on $\mathbb{A}^n$ via $\varphi : \mathbb{G}_{\mathrm{m}}^m \to \mathbb{G}_{\mathrm{m}}^n$ given by $\varphi(t) := (t_i t_j)_{i<j}$. The dense subset of $(\mathbb{A}^m)^2$ where z has no non-zero entries equals $\mathbb{G}_{\mathrm{m}}^m \cdot (\mathbb{A}^m \times \{\mathbf{1}\})$. Consequently, if we set

$$Y := \psi(\mathbb{A}^m \times \{\mathbf{1}\}) \subseteq X,$$

where $\mathbf{1}$ is the all-one vector, then we have $X = \overline{\varphi(\mathbb{G}_{\mathrm{m}}^m) \cdot Y}$. Note that Y is a linear space, with generic point $(y_i - y_j)_{i<j}$; hence we are in the setting of Sect. 4. Let $\mu : \mathbb{R}^m \times X^{\mathrm{an}} \to Z \subseteq X^{\mathrm{an}}$ be the map from Sect. 3, which restricts to an action of $\mathbb{R}^m$ on Z, and let $A \in \mathbb{Z}^{n \times m}$ be the matrix corresponding to φ. We will prove the following theorem.

Theorem 5.1. *The surjective projection from $Z \subseteq \widehat{\mathrm{Gr}}(2, m)^{\mathrm{an}}$ to $\mathrm{Trop}(\widehat{\mathrm{Gr}}(2, m))$ has a continuous, $\mathbb{R}^m$-equivariant section.*

A version of this theorem first appeared in [7]; see Remark 5.3 below. Lifts from $\mathrm{Trop}(\widehat{\mathrm{Gr}}(2, m))$ into tropicalisations of other flag varieties were constructed in [15,16].

Our proof consists of two parts. We first construct a continuous section in the spirit of Proposition 4.1, which relies on the choice of a hyperplane in $\mathrm{Trop}(\mathbb{P}^{m-1})$. Then we use the technique of Proposition 4.4 to verify that the constructed section is, in fact, natural and independent of the choice of hyperplane. This then also implies $\mathbb{R}^m$-equivariance.

We will use that the matroid on the variables x_{ij} defined by Y is the graphical matroid of the complete graph K_m. This is immediate from the definition of Y, and was also exploited in [1, Section 4]. Thus a basis J as in Sect. 2 is a tree with vertex set $[m]$. We will write Γ instead of J. Given such a tree Γ, one finds all $\eta \in \mathrm{Trop}(Y)$ compatible with Γ as follows (see also Remark 2.2). First, give arbitrary values in $\mathbb{R}_\infty$ to all η_{ij} with ij an edge in the tree Γ. Then, for each ege ij in $K_m \setminus \Gamma$ set η_{ij} equal to the minimum of the η_{kl} over all edges kl in the simple path from i to j in Γ. See Fig. 1.

Up to tropical scaling, the points of $\mathrm{Trop}(X) \setminus \{\infty\}$ are in one-to-one correspondence with tropical projective lines in the simplex $\Delta := \mathrm{Trop}(\mathbb{P}^{m-1})$ (see [23, Theorem 3.8] for $\mathrm{Trop}(X^0)$). Under this correspondence the point $(\xi_{ij})_{i<j}$ gives rise to the tropical projective line consisting of points ζ for which $\min\{\xi_{ij}+\zeta_k, \xi_{ik}+\zeta_j, \xi_{jk}+\zeta_i\}$ is attained at least twice for each $1 \leq i < j < k \leq m$. We will use the following characterisation of $\mathrm{Trop}(Y) \subseteq \mathrm{Trop}(X)$.

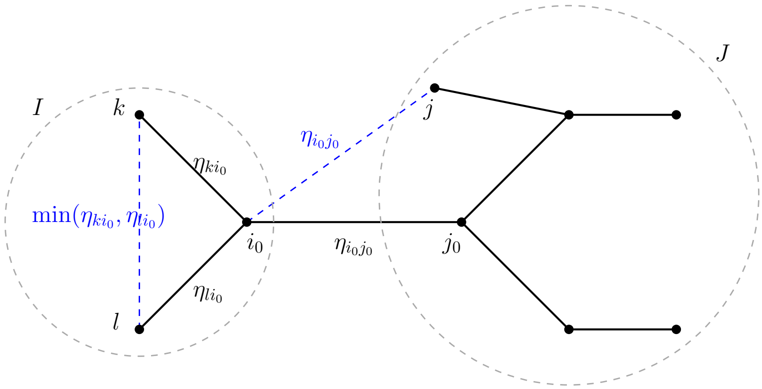


Fig. 1. A spanning tree in K_9 with minimal-weight edge $i_0 j_0$

Lemma 5.2. *Under the correspondence above, the points of* $\mathrm{Trop}(Y)$ *correspond bijectively to the tropical lines that pass through the all-zero point* 0.

Proof. First, for $\eta \in \mathrm{Trop}(Y)$, choose a lift $(x_{ij} := y_i - y_j)_{i<j}$ in Y with $v(x_{ij}) = \eta_{ij}$. Then the 3×3-subdeterminant of the matrix $(y|\mathbf{1}|\mathbf{1})^T$ in columns $i < j < k$ equals $0 = x_{ij} - x_{ik} + x_{jk}$. Hence $\min\{\xi_{ij}, \xi_{ik}, \xi_{jk}\}$ is attained at least twice, i.e., 0 lies on the tropical line corresponding to η.

Conversely, suppose that 0 lies on the tropical line corresponding to η, i.e., that for all $i < j < k$ the minimum $\min\{\eta_{ij}, \eta_{ik}, \eta_{jk}\}$ is attained at least twice. Equip K_m with edge weights given by η. Then in each triangle $\{i, j, k\}$ the minimum edge weight is attained at least twice. An easy induction then shows that in each *cycle* the minimum edge weight is also attained at least twice (see also Fig. 3 for a similar argument for the graphical matroid of the complete bipartite graph, where triangles are replace by four-cycles). Since these cycles are precisely the circuits of the matroid of Y, which form a tropical basis by [1,4], η lies in $\mathrm{Trop}(Y)$. □

Note that the action of $\mathbb{R}^m$ on $\mathrm{Trop}(X)$ is by translation of the tropical lines.

Proof of Theorem 5.1, construction of a continuous section. To go from $\xi \in \mathrm{Trop}(X)$ to a pair $(\tau, \eta) \in \mathbb{R}^m \times \mathrm{Trop}(Y)$ one is tempted to proceed as follows. Let ℓ be the line represented by ξ, let $\tau \in \mathbb{R}^m$ be such that $-\tau + \mathbb{R}(1, \ldots, 1)$ is a point on ℓ, and set $\eta := -A\tau + \xi$. Then η represents the translate of ℓ by $-\tau$, which therefore passes through 0. By construction, the pair (τ, η) satisfies $A\tau + \eta = \xi$, so that the valuation $\sigma(\xi) := \mu(\tau, \sigma_Y(\eta))$ maps to ξ.

There are various problems with this definition of σ, but we can sharpen it as follows. A first, minor problem is that if $\xi = \infty (= (\infty, \ldots, \infty))$, then ξ does not represent a line. In that case, we just set $\sigma(\xi)$ equal to $\infty \in X^{\mathrm{an}}$. The second, and more serious, problem is that ℓ may not contain points $\tau \in \mathbb{R}^m/\mathbb{R}(1, \ldots, 1)$. To remedy this, we will use a stratification of $X = \widehat{\mathrm{Gr}}(2, m)$ and $\mathrm{Trop}(X)$ defined as follows (and also used, in slightly different terminology, in [7]). For $x \in X$ let $J_x \subseteq [m]$ be the set of i for which there exists a $j \neq i$ with $x_{ij} \neq 0$. Note that J_x is either empty, or else has cardinality at least two.

For any subset $J \subseteq [m]$ of cardinality zero or at least two we define

$$X_J := \{x \in X \mid J_x = J\}.$$

This stratum is a locally closed subset of X, and X is the disjoint union of these strata. The stratum $X_\emptyset$ consists of 0 only, while for $|J| \geq 2$ the stratum X_J is the ψ-image of the subset of $(\mathbb{A}^m)^2$ where y, z are linearly independent and $(y_i, z_i) = (0, 0)$ if and only if $i \notin J$. So $X_{[m]}$ is the largest one among these strata, and it parameterises lines that intersect $\mathbb{G}_\mathrm{m}^m$. Each X_J is the dense set in the (cone over the) smaller Grassmannian of 2-spaces contained in $\mathbb{A}^J \times \{0\}^I$ consisting of all spaces that intersect $\mathbb{G}_\mathrm{m}^J \times \{0\}^I$. Similarly, points of $\mathrm{Trop}(X_J)$ parameterise tropical lines in the face Δ_J of Δ (where all I-coordinates are ∞) that intersect the relative interior of Δ_J. Let Y_J denote the J-analogue of Y, that is, the subspace of $K^{\binom{J}{2}}$ parameterised by $(y_j - y_{j'})_{j<j'}$, identified with a subspace of $K^{\binom{m}{2}}$ by extending with zero coordinates. Then $Y_J \setminus \{0\}$ is a subset of X_J, and in fact we have $\varphi(\mathbb{G}_\mathrm{m}^J) \cdot (Y_J \setminus \{0\}) = X_J$. Note also that Y_J is not a *subspace* of $Y = Y_{[m]}$ but rather its *image* under projecting some coordinates to 0. Note that both $Y_\emptyset = X_\emptyset = \{0\}$.

We choose $\tau \in \mathbb{R}_\infty^m$ as a function of ξ as follows. If $\xi = \infty$, then set $\tau := \infty$. Otherwise, let H be the tropical hyperplane in Δ with the tropical equation $\zeta_1 \oplus \cdots \oplus \zeta_m$, and let τ represent the stable intersection of H and the tropical line ℓ represented by ξ. By continuity of stable intersection, the projective point $\tau + \mathbb{R}(1, \ldots, 1)$ depends continuously on non-infinite ξ.

Next, we choose η as a function of ξ. If $\xi = \infty$, then set $\eta := \infty$. Otherwise, let J of cardinality at least two be such that $\xi \in \mathrm{Trop}(X_J)$. Then ℓ lies in Δ_J and intersects the relative interior of Δ_J. As a consequence, $\tau_j \neq \infty$ if and only if $j \in J$. Set $\eta_{ij} := \xi_{ij} - \tau_i - \tau_j$ for $i, j \in J$ and $\eta_{ij} := \infty$ if one of i, j lies in I. Then η lies in $\mathrm{Trop}(Y_J)$ (and this also holds for $\xi = \infty$, in which case $J = \emptyset$).

The pair (τ, η) thus constructed does not depend continuously on ξ, but we claim that the valuation

$$\sigma(\xi) := \mu(\tau, \sigma_{Y_J}(\eta)) \in X^{\mathrm{an}}$$

does. Here we abuse notation slightly, since τ will in general have some coordinates equal to ∞—but one readily verifies that, since A contains only non-negative entries, μ extends to $\mathbb{R}_\infty^m \times X^{\mathrm{an}}$. By construction, we have $A\tau + \eta = \xi$, and this implies that $\sigma(\xi) \in X^{\mathrm{an}}$ does indeed map to ξ.

First observe that tropically scaling all coordinates of τ with $c \in \mathbb{R}$ and all coordinates of η with $-2c$ leads to the same valuation. Now let $\xi^{(p)}$, $p = 1, 2, 3, \ldots$ be a sequence of points in $\mathrm{Trop}(X)$ that converges to a non-infinity limit $\xi \in \mathrm{Trop}(X_J)$ with $|J| \geq 2$. After deleting an initial segment of the sequence, we may assume that each $\xi^{(p)}$ lies in some $\mathrm{Trop}(X_{J^{(p)}})$ with $J^{(p)} \supseteq J$. Let $\eta^{(p)} \in \mathrm{Trop}(Y_{J^{(p)}})$ and $\tau^{(p)} \in \mathbb{R}^m$ be the corresponding points, so that $\xi^{(p)} = A\tau^{(p)} + \eta^{(p)}$ for all p. The projective points $\tau^{(p)} + \mathbb{R}(1, \ldots, 1)$ converge to $\tau + \mathbb{R}(1, \ldots, 1)$ (by continuity of stable intersection). Hence, after suitable tropical scalings of the $\tau^{(p)}$ and the $\eta^{(p)}$, we achieve that $\tau^{(p)} \to \tau$ for $p \to \infty$. Then for $i, j \in J$ we find that

$$\eta_{ij}^{(p)} = \xi_{ij}^{(p)} - \tau_i^{(p)} - \tau_j^{(p)} \to \xi_{ij} - \tau_i - \tau_j = \eta_{ij} \quad \text{for } p \to \infty.$$

We now argue that for each $\mathbb{G}_{\mathrm{m}}^m$-homogeneous element $f \in K[X]$ the value $\sigma(\xi^{(p)})(f)$ converges to $\sigma(\xi)(f)$. Let $\beta \in \mathbb{N}^m$ be the weight of f. If $\beta_i > 0$ for some $i \notin J$, then f lies in the ideal generated by the coordinates x_{kj} for which one of k, j does not lie in J. In this case, $\sigma(\xi)(f) = \infty$. To see that $\sigma(\xi^{(p)})(f)$ tends to infinity, expand $f = \sum_{ij} x_{ij} f_{ij}$ where the sum is over pairs (i, j) that are not both in J. Then we have

$$\sigma(\xi^{(p)})(f) \geq \min_{ij}(\xi_{ij}^{(p)} + \sigma(\xi^{(p)})(f_{ij})).$$

Since each $\xi_{ij}^{(p)}$ tends to infinity and each $\sigma(\xi^{(p)})(f_{ij})$ is bounded from below, we find the desired convergence (a similar convergence argument applies when the limit ξ equals ∞). If $\beta_i = 0$ for all $i \notin J$, then f depends only on the coordinates x_{ij} with $i, j \in J$, and it suffices to show that

$$\sigma_{Y_{J^{(p)}}}(\eta^{(p)})(f) \to \sigma_{Y_J}(\eta)(f), \quad p \to \infty.$$

Using the definition of σ and the fact that $\eta_{ij}^{(p)} \to \eta_{ij}$ for $p \to \infty$ and $i, j \in J$, this convergence follows if there exists a tree on $J^{(p)}$ compatible with $\eta^{(p)}$ which contains a spanning tree on $J \subseteq J^{(p)}$. But this is a consequence of the basis exchange axiom: start with any tree Γ on $J^{(p)}$ compatible with $\eta^{(p)}$. If the induced forest $\Gamma|_J$ on J is not connected, pick arbitrary endpoints $j, j' \in J$ that belong to different connected components of $\Gamma|_J$. Then replace, in Γ, an edge in the simple path from j to j' of smallest $\eta^{(p)}$-weight by jj' (which has the same weight). This creates a new Γ compatible with $\eta^{(p)}$ such that $\Gamma|_J$ has fewer connected components than before. Proceed in this fashion until $\Gamma|_J$ is connected. See Fig. 2 for an illustration of this procedure. This concludes the proof that σ is a continuous section $\mathrm{Trop}(X) \to Z$ of the surjection $Z \to \mathrm{Trop}(X)$. □

Proof of Theorem 5.1, naturality and equivariance. In the previous proof, we decomposed ξ as $A\tau + \eta$ by choosing for $-\tau$ a point on the tropical line ℓ represented by ξ. This point was obtained by stably intersecting ℓ with a hypersurface.

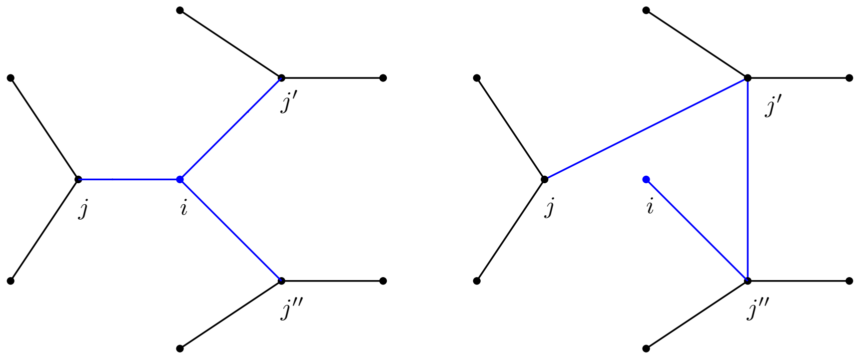


Fig. 2. Basis exchange in the case $J^{(p)} \setminus J = \{i\}$, with $\eta_{ij}^{(p)} \leq \eta_{ij'}^{(p)} \leq \eta_{ij''}^{(p)}$

By verifying the conditions of Proposition 4.4, we now show that the chosen decomposition is, in fact, irrelevant for the section σ.

First, if $\eta \in \mathrm{Trop}(Y^0)$ corresponds to a tropical projective line ℓ, then the $\tau \in \mathbb{R}^m$ for which $A\tau + \eta$ lies in $\mathrm{Trop}(Y)$ are those for which $-\tau$ lies in ℓ. Thus $T_\eta = -\ell$ is connected. This settles the first requirement for Proposition 4.4.

Now let $\eta \in \mathrm{Trop}(Y^0)$ and $\tau \in \mathbb{R}^m \setminus \{0\}$ such that $[0, \tau]$ lies in T_η. For sufficiently small $\epsilon > 0$ there exists a tree Γ compatible with both η and $\eta' := A\epsilon\tau + \eta$. Indeed, the points in $\mathrm{Trop}(Y^0)$ compatible with any given tree form (the support of) a closed, finite polyhedral complex. There are finitely many trees, and they give finitely many polyhedral complexes that together cover $\mathrm{Trop}(Y^0)$. But then $[\eta, A\tau+\eta]$ has an initial segment entirely contained in one of these complexes. By a similar argument, by shrinking ϵ, we may moreover assume that there exists an edge $i_0 j_0$ in Γ such that both

$$\eta_{i_0 j_0} \leq \eta_{ij} \text{ and } \eta'_{i_0 j_0} \leq \eta'_{ij} \quad \text{for all } ij \in \Gamma$$

(and hence also for all $ij \in K_m \setminus \Gamma$). The edge $i_0 j_0$ cuts the tree Γ into two connected components (see Fig. 1). Let $[m] = I \cup J$ be the vertex sets of these connected components, with $i_0 \in I$ and $j_0 \in J$. We claim that $\tau_i = \tau_{i_0}$ for all $i \in I$ and $\tau_j = \tau_{j_0}$ for all $j \in J$. Indeed, pick $j \in J$ and consider the cycle formed by i_0, j, and then back along Γ to i_0. We have $\eta'_{i_0 j} = \eta'_{i_0 j_0}$, since this is the edge of Γ in said cycle with smallest η'-weight. On the other hand, we have

$$\eta'_{i_0 j} = \eta_{i_0 j} + \epsilon(\tau_{i_0} + \tau_j) = \eta_{i_0 j_0} + \epsilon(\tau_{i_0} + \tau_j)$$

and

$$\eta'_{i_0 j_0} = \eta_{i_0 j_0} + \epsilon(\tau_{i_0} + \tau_{j_0}).$$

This shows that $\tau_j = \tau_{j_0}$. Similarly, we find that for all $i \in I$ we have $\tau_i = \tau_{i_0}$.

To construct t, we may assume that one of τ_{i_0}, τ_{j_0} is zero and the other positive—indeed, this can be achieved by adding a multiple of the all-one vector to τ, which can be mimicked by multiplying t with a scalar of the right valuation. Without loss of generality, suppose that $\epsilon\tau_{i_0} =: a$ is positive and τ_{j_0} is zero. Then adding $\epsilon A\tau$ to η has the effect of increasing all η_{ij} with $i, j \in I$ by $2a$, keeping all η_{ij} with $i, j \in J$ constant, and increasing all η_{ij} with $i \in I$ and $j \in J$ by a. As, by assumption, the minimal-weight edge in Γ remains the edge $i_0 j_0$, the minimal η-weight of an edge of Γ with both vertices in J must be at least $\eta_{i_0 j_0} + a$.

Now let $w = \sigma_Y(\eta)$ as in the proof of Theorem 2.1 and let $y \in Y^0$ be the generic point. Its coordinates are $x_{ij} = (y_i - y_j)_{ij}$ where the y_i are variables. It represents the subspace spanned by the rows of the matrix

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_m \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

A point t sends y into Y^0 if and only if it sends this subspace to another subspace containing the all-one vector, hence if and only if t^{-1} lies in the subspace. Hence we make the *Ansatz*

$$t_i = \frac{1}{c(y_i - y_{j_0}) + d},$$

where c, d will be chosen from K. Then by definition of w, we have

$$w(c(y_i - y_{j_0}) + d) = \min\{v(c) + w(y_i - y_{j_0}), v(d)\}.$$

Now for each i the expression $y_i - y_{j_0}$ expands as a sum of the x-variables corresponding to the edges on the path in Γ from i to j_0, and $w(y_i - y_{j_0})$ equals the minimal η-weight among these edges. For $i \in I$ this equals $\eta_{i_0 j_0}$, as this is the minimal-weight edge overall. Thus we choose $c \in K$ such that $v(c) + \eta_{i_0 j_0} = -a < 0$. For $i \in J$ the minimal weight along the path is at least $a + \eta_{i_0 j_0}$, so if we choose a $d \in K$ with valuation 0, then the denominator in the *Ansatz* gets the right valuation for both $i \in I$ and $i \in J$.

We have thus constructed a $t \in \mathbb{G}_\mathrm{m}(K(Y))$ with $w(t) = \epsilon\tau$ and $\varphi(t)y \in Y(K(Y))$. As all requirements of Proposition 4.4 are met, we have constructed an $\mathbb{R}^m$-equivariant section $\mathrm{Trop}(X^0) \to Z^0$. This section agrees with the restriction to $\mathrm{Trop}(X^0)$ of the section constructed in the previous proof. Hence the continuous section constructed there is $\mathbb{R}^m$-equivariant. □

Remark 5.3. In [7] the setting is projective rather than affine. Theorem 1.1 and Corollary 7.3 from that paper follow from our theorem by applying Lemmas 3.5 and 3.4, respectively.

6. Rank-two matrices

In this section we take $n = m \cdot p$ and consider $\mathbb{A}^n$ with coordinates x_{ij} with $1 \leq i \leq m$ and $1 \leq j \leq p$. Let $X \subset \mathbb{A}^n$ be the image of the polynomial map

$$\psi : (\mathbb{A}^m)^2 \times (\mathbb{A}^p)^2 \to \mathbb{A}^n, \quad (y, y'), (z, z') \mapsto (y_i z'_j - y'_i z_j)_{1 \leq i \leq m, 1 \leq j \leq p}.$$

It is the variety of matrices of rank at most two, and also the affine cone over the variety of secant lines of the Segre embedding of $\mathbb{P}^{m-1} \times \mathbb{P}^{p-1}$ in $\mathbb{P}^{n-1}$. It is an irreducible determinantal variety of dimension $2(m + p - 2)$.

Let Y be the subvariety of X defined as the image of

$$(\mathbb{A}^m \times \{\mathbf{1}\}) \times (\mathbb{A}^p \times \{\mathbf{1}\})$$

via ψ. Points of Y have coordinates $x_{ij} = (y_i - z_j)$ in $\mathbb{A}^n$, so that Y is the zero locus of the linear forms

$$x_{ij} + x_{lk} = x_{ik} + x_{lj}, \; 1 \leq i < l \leq m, 1 \leq j < k \leq p. \tag{2}$$

Consider also the homomorphism of tori given by

$$\varphi : \mathbb{G}_m^m \times \mathbb{G}_m^p \to \mathbb{G}_m^n, \, (t, s) \mapsto (t_i s_j)_{1 \leq i \leq m, 1 \leq j \leq p}.$$

The corresponding $n \times (m+p)$-matrix A has a one-dimensional kernel spanned by $(1, \ldots, 1, -1, \ldots, -1)$. We have $X = \overline{\varphi(\mathbb{G}_\mathrm{m}^m \times \mathbb{G}_\mathrm{m}^p) \cdot Y}$ (this can be proved using equivariance of ψ, as in Sect. 5), and

$$\mathrm{Trop}(X^0) = A(\mathbb{R}^m \times \mathbb{R}^p) + \mathrm{Trop}(Y^0)$$

where $X^0 \subseteq X$ and $Y^0 \subseteq Y$ are the loci where no coordinate is zero. Let $\mu : \mathbb{R}^{m+p} \times X^{\mathrm{an}} \to Z \subseteq X^{\mathrm{an}}$ be as constructed in Sect. 3. We will prove the following theorem.

Theorem 6.1. *The surjection* $X^{\mathrm{an}} \to \mathrm{Trop}(X)$*, where* X *is the variety of* $m \times p$*-matrices of rank at most two, has a continuous,* $\mathbb{R}^m$*-equivariant section* $\mathrm{Trop}(X) \to Z$.

Note that we do not claim that the section is also $\mathbb{R}^p$-equivariant. While this might be the case, our construction below does not yield this.

For the proof of this theorem, we need to understand the points in $\mathrm{Trop}(X)$ and its tropical subvariety $\mathrm{Trop}(Y)$. By [10, Corollary 3.8], a matrix $\xi \in \mathbb{R}^{m \times p}$ lies in $\mathrm{Trop}(X^0)$ if and only if it has tropical rank at most 2, i.e., if and only if all its 3×3-submatrices are tropically singular. This extends directly to all of $\mathrm{Trop}(X)$. To understand $\mathrm{Trop}(Y)$ note that the matroid defined by Y is the graphical matroid of the complete bipartite graph $K_{m,p}$; this is immediate from the parameterisation $x_{ij} = y_i - z_j$. In other words, $\eta \in \mathbb{R}_\infty^{m \times p}$ lies in $\mathrm{Trop}(Y)$ if and only if along each cycle in $K_{m,p}$ the minimal η-weight of an edge is attained at least twice. We claim that this is equivalent to the condition that in every 2×2-submatrix of η the minimal entry appears at least twice. Indeed, necessity of the latter condition is obvious, as any 2×2-submatrix records the weights of a 4-cycle in $K_{m,p}$. For sufficiency, assume that the minimal η-weight in every 4-cycle is attained at least twice, and let C be a general (simple, even) cycle in $K_{m,p}$. Label C as $i_1 - j_1 - i_2 - j_2 - \cdots - i_a - j_a - i_1$, where the is are in $[m]$ and the js are in $[n]$ and where $\alpha := \eta_{i_1,j_1}$ is the minimal weight of an edge in C. Assume, for a contradiction, that all other edges in C have η-weight strictly larger than α. Then in the 4-cycle $i_1 - j_1 - i_2 - j_2 - i_1$ the weight η_{i_1,j_2} must be α. Next, in the 4-cycle $i_1 - j_2 - i_3 - j_3 - i_1$ the weight η_{i_1,j_3} must also equal α, etc. In this manner we find that η_{i_1,j_a} must also equal α, a contradiction. See Fig. 3 for an illustration. Armed with this characterisation of $\mathrm{Trop}(Y)$ we will now prove the theorem.

Proof of Theorem 6.1. As in the proof of Theorem 5.1, we use a stratification of X. For $I \subseteq [m]$ and $J \subseteq [p]$ let X_{IJ} denote the locus in X consisting of x such that the rows of x labelled by $[m] \setminus I$ and the columns of x labelled by $[p] \setminus J$ are

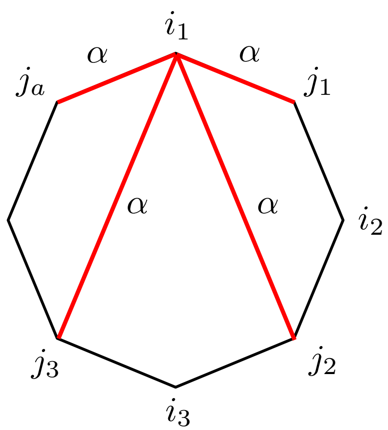


Fig. 3. Only four-cycles need to be tested for membership of $\mathrm{Trop}(Y^0)$

identically zero and the submatrix $x[I, J]$ does not have identically zero rows or columns. Let Y_{IJ} denote the (I, J)-analogue of Y. It is the image of Y under the map sending all coordinates outside the $[I, J]$-submatrix to zero.

For $\xi \in \mathrm{Trop}(X_{IJ})$ we let $\tau \in \mathbb{R}^m_\infty$ be the tropical product $\xi \odot (0, \ldots, 0)^T$, a point in the tropical convex hull of the columns of ξ. Then we have $\tau_i \neq \infty$ if and only if $i \in I$. Let $\xi' \in \mathrm{Trop}(X_{IJ})$ be the matrix obtained from ξ by subtracting τ_i from each ξ_{ij} with $i \in I, j \in J$. Then let $\rho \in \mathbb{R}^p_\infty$ be the tropical product $(0, \ldots, 0) \odot \xi'$, which records the minimal entry in each column of ξ'. Let η be the matrix obtained from ξ' by subtracting ρ_j from each ξ'_{ij} with $i \in I, j \in J$. By [10, Lemma 6.2], the matrix $\eta[I, J]$ has the property that in each of its 2×2-submatrices the minimal entry appears at least twice. By the discussion preceding the proof, η lies in $\mathrm{Trop}(Y_{IJ})$.

We set

$$\sigma(\xi) := \mu((\tau, \rho), \sigma_{Y_{IJ}}(\eta)),$$

and claim that this depends continuously on ξ. To see this, let $\xi^{(q)}$, $q = 1, 2, \ldots$ be a sequence in $\mathrm{Trop}(X)$ converging to $\xi \in \mathrm{Trop}(X_{IJ})$, and construct $\tau^{(q)}$ and $\rho^{(q)}$ and $\eta^{(q)}$ as above. After dropping finitely many initial terms, we have $\xi^{(q)} \in \mathrm{Trop}(X_{I^{(q)}J^{(q)}})$ with $I^{(q)} \supseteq I$ and $J^{(q)} \supseteq J$. For $i \in I$ and $j \in J$ we find that $\tau_i^{(q)} \to \tau_i$ for $q \to \infty$ and also $\lim_{q\to\infty} \rho_j^{(q)} = \rho_i$ and $\lim_{q\to\infty} \eta_{ij}^{(q)} = \eta_{ij}$. We will not need the limits of the remaining entries of $\tau^{(q)}, \rho^{(q)}, \eta^{(q)}$.

Let f be a $\mathbb{G}_\mathrm{m}^{m+p}$-weight (i.e., multi-homogeneous) element of $K[X]$. We have the same dichotomy as in the proof for the Grassmannian case: either f lies in the ideal generated by all variables x_{ij} with $i \notin I$ or $j \notin J$, and in this case

$$\sigma(\xi^{(q)})(f) \to \infty = \sigma(\xi)(f) \quad \text{for } q \to \infty;$$

or f lies in the ring generated by the x_{ij} with $i, j \in J$. In the latter case, it suffices to show that

$$\sigma_{Y_{I^{(q)}J^{(q)}}}(\eta^{(q)})(f) \to \sigma_{Y_{IJ}}(\eta)(f).$$

Proceeding as for the Grassmannian of 2-spaces, we find that there exists, for each q, a tree Γ_q compatible with $\eta^{(q)}$ that induces a tree (rather than a forest) on $I \cup J$. Using this tree, one finds that the left-hand side equals $\sigma_{Y_{IJ}}(\tilde{\eta}^{(q)})(f)$ where $\tilde{\eta}^{(q)}$ is derived from $\eta^{(q)}$ by setting the entries with $(i, j) \notin I \times J$ equal to infinity. Then the convergence follows by continuity of $\sigma_{Y_{IJ}}$ and the fact that $\tilde{\eta}^{(q)} \to \eta$ for $q \to \infty$.

The map $\mathrm{Trop}(X) \to \mathbb{R}^m_\infty, \xi \to \xi \odot (0, 0, \ldots, 0)^T = \tau$ is $\mathbb{R}^m$-equivariant, and this implies that σ is $\mathbb{R}^m$-equivariant. But the construction $\xi \mapsto \rho$ is not $\mathbb{R}^p$-equivariant. □

Remark 6.2. The proof above is not as satisfactory as the proof for Grassmannians of two-spaces in Sect. 5, which used the technique of Proposition 4.4 to prove that the defined section is independent of the decomposition $\xi = A\tau + \eta$ and hence equivariant. We have tried to mimick the proof for the Grassmannian, but failed because for suitable $\eta \in \mathrm{Trop}(Y^0)$ the set T_η can have dimension much larger than the expected dimension four. This implies that the second requirement in Proposition 4.4 cannot be satisfied. Of course, this does not rule out the existence of alternative techniques for proving $\mathbb{R}^{m+p}$-equivariance.

7. A-discriminants

Linear spaces smeared around by tori, as discussed in Sect. 4, arise in the study of A-discriminants from [12]. Let $\varphi : \mathbb{G}_m^m \to \mathbb{G}_m^n$ be a torus homomorphism with corresponding integer $n \times m$-matrix A, and let V be the closure in $\mathbb{A}^n$ of the image of φ, a toric variety. The linear action of $\mathbb{G}_m^m$ on $\mathbb{A}^n$ gives rise to an action on the dual space $(\mathbb{A}^n)^\vee$, given by a torus homomorphism $\varphi^\vee : \mathbb{G}_m^m \to \mathbb{G}_m^n$ corresponding to the matrix $-A$.

Let $Y \subseteq (\mathbb{A}^n)^\vee$ be the annihilator of the tangent space $T_{\varphi(1)}V$. Since A, when regarded as a matrix over K, is the derivative of φ at 1, Y is the orthogonal complement of the column space of A. For $t \in \mathbb{G}_m^m$, $\varphi(t)$ maps $T_{\varphi(1)}V$ into $T_{\varphi(t)}V$, hence we find that $\varphi^\vee(t)$ maps V into the annihilator of $T_{\varphi(t)}V$. Thus the variety X defined as the Zariski closure of the union of these annihilators equals $\overline{\varphi^\vee(\mathbb{G}_m^m) \cdot Y}$. This is known as the Horn uniformisation of the *dual variety* of V. It was used in [8] to characterise $\mathrm{Trop}(X^0)$ as

$$\mathrm{Trop}(X^0) = -A\mathbb{R}^m + \mathrm{Trop}(Y^0),$$

where, of course, the minus sign is only a reminder of the contragredience of the action of $\mathbb{G}_m^m$ on $(\mathbb{A}^n)^\vee$ and can also be left out. This leads to the following fundamental problem.

Problem 7.1. *For which torus homomorphisms $\varphi : \mathbb{G}_m^m \to \mathbb{G}_m^n$ does the map from the analytification of the dual variety X of $V = \overline{\mathrm{im}\varphi} \subseteq \mathbb{A}^n$ to $\mathrm{Trop}(X)$ admit a continuous, $\mathbb{R}^m$-equivariant section into the subset $Z \subseteq \mathrm{Trop}(X)$ defined in Sect. 3?*

We do not have any general results at this point. Instead, we now consider the very special case of *Cayley's hyperdeterminant*, and we stay away from zero coordinates.

Example 7.2. Let $n = 2^3$ and use coordinates $x_{ijk},\ i, j, k \in \{0, 1\}$ on $\mathbb{A}^8$. Let $m = 3 \cdot 2$ and use coordinates $t_i, u_j, v_k,\ i, j, k \in \{0, 1\}$ on $\mathbb{G}_m^6$. Let φ be the map $(t, u, v) \mapsto (t_i u_j v_k)_{i,j,k}$. Then V^0 is the variety of rank-one tensors of format $2 \times 2 \times 2$. The dual variety X is a hypersurface whose defining equation is Cayley's hyperdeterminant

$$\begin{aligned}\Delta = {} & x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{100}^2 x_{011}^2 \\ & - 2x_{000}x_{001}x_{110}x_{111} - 2x_{000}x_{010}x_{101}x_{111} - 2x_{000}x_{011}x_{100}x_{111} \\ & - 2x_{001}x_{010}x_{101}x_{110} - 2x_{001}x_{011}x_{110}x_{100} - 2x_{010}x_{011}x_{101}x_{100} \\ & + 4x_{000}x_{011}x_{101}x_{110} + 4x_{001}x_{010}x_{100}x_{111}.\end{aligned}$$

The tropical variety of X is known explicitly (though we will not use this knowledge): modulo its four-dimensional lineality space it is a 3-dimensional fan in 4-space. Intersecting with a 3-dimensional sphere yields a 2-dimensional spherical polyhedral complex, which consists of two nested tetrahedra glued by quadrangles along corresponding edges; see Fig. 4. This is the spherical complex of the normal fan of the bipyramid over a tetrahedron from [14, Section 2].

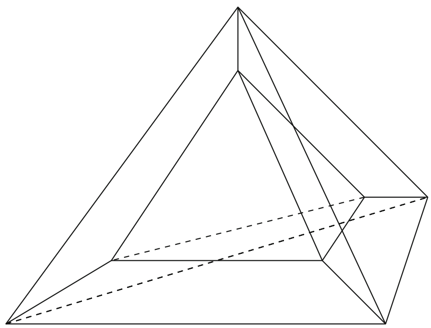


Fig. 4. The tropical variety of Cayley's hyperdeterminant has *eight triangles* and *six quadrangles*

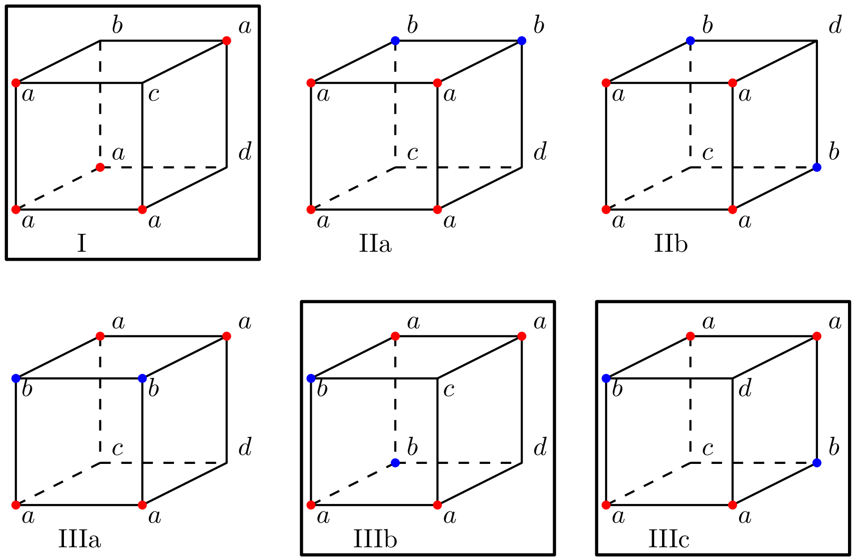


Fig. 5. The six orbits of maximal cones in $\mathrm{Trop}(Y^0)$, with $a \leq b \leq c \leq d$

The matrix A sends $\tau = (\rho, \delta, \nu) \in \mathbb{R}^6$ to the $2 \times 2 \times 2$-array with entries $(\rho_i + \delta_j + \nu_k)_{ijk}$. The kernel of this map consists of vectors of the form $(a\mathbf{1}, b\mathbf{1}, c\mathbf{1})$ with $a+b+c = 0$, so the column space $\operatorname{im} A$ has dimension 4. It defines the matroid on the vertices of the three-dimensional cube in which independence is affine independence. Since the complement of any four affinely independent vertices of the cube is again affinely independent, this matroid is self-dual. So the dual matroid, which is the matroid of the linear space Y, is the same matroid on 8 elements.

Up to symmetries of the cube, the seven-dimensional polyhedral fan $\mathrm{Trop}(Y^0)$ has six maximal cones, and they are depicted in Fig. 5. Among these, the cones of type IIa, IIb, and IIIa lie in $A\mathbb{R}^6$ plus the union of the cones of type I, IIIb, IIIc. For instance, take the array in type IIIa and add $(c-b)/2, 0, (b-c)/2$ to the positions with entries b,a,(c and d), respectively. The array thus added lies in the column space of A, and the result is an array in the boundary of type IIIb (with b and c replaced by $(b+c)/2$ and d replaced by $d + (b-c)/2$).

Now let C be a cone of type I, IIIb, or IIIc. Then the linear span of C intersects $A\mathbb{R}^6$ only in scalar multiples of the all-one array. This follows from the fact that the span in $\mathbb{R}^3$ of the differences of vertices of the cube with the same label (a or b) is all of $\mathbb{R}^3$ (this is not true for the other types!). Thus on $A\mathbb{R}^6 + C \subseteq \operatorname{Trop}(X^0)$ we can define a section σ_C into $\operatorname{Trop}(X^0)$ as follows: write ξ as $A\tau + \eta$ with $\eta \in C$ and set $\sigma(\xi) := \mu(\tau, \sigma_Y(\eta))$. Note that, for any $c \in \mathbb{R}$, subtracting $(c\mathbf{1}, c\mathbf{1}, c\mathbf{1})$ from τ and adding $3c$ times the all-one array to η yields the same value for $\sigma(\xi)$, so that σ is well-defined on $A\mathbb{R}^6 + C$.

Next we verify that if C' is a second cone of type I, IIIb, or IIIc, then σ_C and $\sigma_{C'}$ agree on the intersection $(A\mathbb{R}^6 + \sigma_C) \cap (A\mathbb{R}^6 + C')$. This is immediate if

$$(A\mathbb{R}^6 + C) \cap (A\mathbb{R}^6 + C') = A\mathbb{R}^6 + (C \cap C')$$

as the recipes defining σ_C and $\sigma_{C'}$ agree on the right-hand side. For each choice of C and C', a vector witnessing that the left-hand side is *strictly larger* than the right-hand side can be found by solving a number of linear programs. If none of these linear programs turns out to be feasible, then equality holds. We have performed this test for all choices of C in the cones I, IIIb, IIIc, and C' in one of the orbits of these cones. Together with Proposition 4.1 this proves the following theorem.

Theorem 7.3. *Let $X \subseteq K^{2\times2\times2}$ be the hypersurface defined by Cayley's hyperdeterminant, equipped with the natural action of $\mathbb{G}_m^2 \times \mathbb{G}_m^2 \times \mathbb{G}_m^2$. Let $X^{an} \to Z$ be the retraction defined relative to this torus action. Then the surjection $Z^0 \to \operatorname{Trop}(X^0)$ has a continuous, $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$-equivariant section.* $\diamondsuit$

Acknowledgments. We thank Joe Rabinoff for interesting discussions and, more specifically, for pointing out that the retract Z of X^{an} that we define in Sect. 3 is, in fact, a strong deformation retract. We also thank a referee for pointing out an error in an earlier version of this paper concerning the variety of matrices of less than full rank.

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