Cournot Oligopolies With Product Differentiation Under Uncertainty

C. CHIARELLA
School of Finance and Economics
University of Technology, Sydney
P.O. Box 123, Broadway, NSW 2007, Australia
carl.chiarella@uts.edu.au

F. SZIDAROVSZKY
Systems and Industrial Engineering Department
University of Arizona
Tucson, Arizona 85721-0020, U.S.A.
szidar@sie.arizona.edu

(Received March 2004; revised and accepted April 2005)

Abstract—This paper considers Cournot oligopolies with product differentiation when the firms have inexact knowledge of the price functions and there are random time lags in obtaining and implementing information on the firms' own outputs and prices as well as on the prices of the competitors. After the basic dynamic model without time lag is formulated, the asymptotic stability in the linear case is examined. The introduction of time lags makes the asymptotic behaviour of the steady state more complex. A detailed stability analysis is undertaken. In the case of symmetric firms and with a special weighting function, the possibility of the birth of limit cycle motion in output is also investigated in addition to stability analysis. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Oligopolies, Uncertainty, Information lag, Bifurcation, Limit cycles

1. INTRODUCTION

The research on static and dynamic oligopolies is one of the richest areas in the economics literature. Since the pioneering work of Cournot [1] many scientists have examined oligopoly models and their extensions. Okuguchi [2] has provided a comprehensive summary of results on single-product models, and their multi-product extensions with applications are discussed in [3]. In most of the models discussed in literature, it is assumed that the firms have full knowledge of all price functions as well as having instantaneous information available on the firms' own outputs and prices and on those of the competitors.

There have been some attempts at modeling the lack of precise knowledge in oligopolies. Cyert and De Groot [4,5] have discussed mainly the duopoly case, Kirman [6] considered differentiated products and linear demand functions, which firms misspecify and attempt to estimate. He also showed that the dynamic process may converge to the full information equilibrium or to some
other equilibrium. An economically guided learning process has been introduced by Gates et al. [7], and Kirman [8] has analysed how the equilibrium values are affected by mistaken beliefs. Léonard and Nishimura [9] have shown how the fundamental dynamic properties of the model can be drastically altered in discrete duopolies. Their model has been investigated in the case of continuous oligopolies by Chiarella and Szidarovszky [10].

It is well known (see, for example, [11]) that for linear systems, local and global asymptotic stability are equivalent, however, in the presence of nonlinearity, this is no longer true. For discrete time scales, global asymptotic stability of the equilibrium of the Cournot model as well as the emergence of periodic and chaotic behavior around it is examined by Kopel [12-14]. The structural stability of continuous oligopolies without information lags has been investigated by Krawiec and Szpydowski [15] by showing that the structure of the phase space is independent of the dimensions of the model and the dynamics of oligopoly models are structurally stable. Similar studies have been performed by Puu [16,17]. Russell et al. [18] have introduced fixed-time lags and examined the resulting dynamic process. Difference-differential equations with infinite spectra had to be studied. This approach, however, has two major drawbacks. First, the length of many time delays is uncertain, so it cannot be assumed to be a fixed number. Second, the presence of the infinite spectra makes the application of stability and bifurcation analysis analytically intractable. These drawbacks can be overcome by using continuously distributed time lags. This has the effect of distributing the time lags over a range of values. It also turns out that the spectrum of the resulting dynamic system is finite. Continuously distributed time lags were originally introduced by Cushing [19] in mathematical biology. Their first economic application was given by Invernizzi and Medio [20] and later they were successfully applied to dynamic oligopolies by Chiarella [21], Chiarella and Khomin [22], and Chiarella and Szidarovszky [23].

In this paper, the method introduced in [23] will be extended to oligopolies with product differentiation, and this methodology will be applied to cases when the firms have only imperfect knowledge of the price functions. The paper develops as follows. After the basic dynamic model is presented, the information lag will be formulated and included into the model. In the case of symmetric firms and special weighting functions, stability analysis will be presented then, as well as the possibility of the birth of limit cycles will be discussed.

2. THE OLIGOPOLY MODEL WITH PRODUCT DIFFERENTIATION

Consider an n-firm oligopoly with single differentiated products. Let \( f_i \) and \( C_i \) denote the price and cost functions of Firm \( i \), respectively, then its profit is given as

\[
\pi_i = x_i f_i (x_1, \ldots, x_n) - C_i (x_i),
\]

where \( x_1, \ldots, x_n \) denote the outputs of the firms. In this paper, we assume that at each time period each firm observes its own output and all prices, however, they do not know the price functions exactly. Let \( f_{ij} \) denote the price function of Product \( j \) as believed by Firm \( i \). In formulating the reply functions of the firms, we have to develop a mechanism to assess the outputs of the rivals. Let \( p_{1j}^*, \ldots, p_{nj}^* \) be the observed prices and \( x_i^* \) the output of Firm \( i \). Then, this firm thinks that the outputs of the rivals satisfy equations,

\[
f_{ij} (x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n) = p_{ij}^*,
\]

for \( j = 1, 2, \ldots, n \). So, there are \( n \) equations for \( n - 1 \) unknowns. Because the firm has an imexact knowledge of the price functions, these equations are usually contradictory and the firm may obtain no solution. Therefore, Firm \( i \) selects the \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \) values as its assessments of the outputs of the rivals which minimize the squared average error,

\[
\frac{1}{n} \sum_{j=1}^{n} \left[ f_{ij} (x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n) - p_{ij}^* \right]^2.
\]
Let \( h_i(x^*_i, p^*_1, \ldots, p^*_n) \) denote the minimal solution which is assumed now to be unique. With fixed \( x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \) let
\[
g_i(x_{-i}) = \arg \max_{x_i} \{ x_i f_i (x_{-i}, x_i) - C_i(x_i) \} \tag{4}
\]
denote the best response, which is unique if \( x_i f_i - C_i \) is strictly concave in \( x_i \). Therefore, the best response of Firm \( i \) turns out to be
\[
g_i(h_i(x_{-i}, p_1, \ldots, p_n)) = (g_i \circ h_i)(x_i, f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)). \tag{5}
\]
If each firm adjusts its output into the direction of its best reply, then the dynamism is modeled by the system of nonlinear ordinary differential equations,
\[
\dot{x}_i = k_i ((g_i \circ h_i)(x_i, f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) - x_i), \tag{6}
\]
for \( i = 1, 2, \ldots, n \), where \( k_i \) is a sign-preserving function.

**EXAMPLE 1.** Consider the linear case, when the true price function of Product \( j \) is
\[
f_j(x_1, \ldots, x_n) = \sum_{k=1}^{n} b^*_j x_k + a^*_j, \tag{7}
\]
but Firm \( i \) believes that this function is
\[
f_{ij}(x_1, \ldots, x_n) = \sum_{k=1}^{n} b_{ij} x_k + a_{ij}, \tag{8}
\]
for all \( i \) and \( j \). Assume that the cost function of Firm \( i \) is given by
\[
C_i(x_i) = \beta_i + \alpha_i x_i, \tag{9}
\]
for all \( i \). The believed profit of Firm \( i \) is given as
\[
x_i \left( \sum_{k \neq i} b_{ik} x_k + h_{iit} x_i + a_{it} \right) - (\beta_i + \alpha_i x_i). \tag{10}
\]

We assume \( b_{iit} < 0 \), that is, the firms believe that their own price is a decreasing function of their own output, is a reasonable assumption in most economies. Under this assumption, there is a unique profit maximizing \( x_i \), which is assumed to be positive. Then, it can be obtained from the first-order condition that
\[
2b_{iit} x_i + \sum_{k \neq i} b_{ik} x_k + a_{it} - \alpha_i = 0, \tag{11}
\]
implies that the best response is determined as
\[
g_i(x_{-i}) = -\frac{1}{2b_{iit}} \sum_{k \neq i} b_{ik} x_k + \frac{\alpha_i - a_{it}}{2b_{iit}} = -\frac{1}{2b_{iit}} b^*_i x_{-i} + \frac{\alpha_i - a_{it}}{2b_{iit}}, \tag{12}
\]
where \( b^*_i = (b_{i1}, \ldots, b_{i,i-1}, b_{i,i+1}, \ldots, b_{in}, n) \).

Equation (2) has now the special form,
\[
\sum_{k \neq i} b_{ijk} x_k = p^*_j - b_{ij} x^*_i - a_{ij}, \tag{13}
\]
for \( j = 1, 2, \ldots, n \).
For the sake of mathematical simplicity introduce the notation,

\[ \mathbf{p}^* = \begin{pmatrix} p_1^* \\ \vdots \\ p_n^* \end{pmatrix}, \quad b_i = \begin{pmatrix} b_{1i} \\ \vdots \\ b_{ni} \end{pmatrix}, \quad a_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} \]

and

\[ \mathbf{B}_i = \begin{pmatrix} b_{1i1} & b_{1i1} & \cdots & b_{1in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{ni1} & b_{ni1} & \cdots & b_{nin} \end{pmatrix}, \]

then the best least-squares fit of equation (13) is determined as

\[ \mathbf{b}_i^* (x_i^*, p_1^*, \ldots, p_n^*) = \left( \mathbf{B}_i^T \mathbf{B}_i \right)^{-1} \mathbf{B}_i^T \left( \mathbf{p}^* - \mathbf{b}_i^* x_i^* - a_i \right). \] (14)

Therefore, in equation (6),

\[ \begin{align*}
\frac{\partial x_i}{\partial t} &= -\frac{1}{2b_{ii}} b_i^* \left( \mathbf{B}_i^T \mathbf{B}_i \right)^{-1} \mathbf{B}_i^T \left( \mathbf{p}^* - \mathbf{b}_i^* x_i^* - a_i \right) \\
+ \frac{\alpha_i - a_{ii}}{2b_{ii}} &= -\frac{1}{2b_{ii}} b_i^* \left( \mathbf{B}_i^T \mathbf{B}_i \right)^{-1} \mathbf{B}_i^T \left( \mathbf{p}^* - \mathbf{b}_i^* x_i^* - a_i \right) + \frac{\alpha_i - a_{ii}}{2b_{ii}}, 
\end{align*} \] (15)

where we have used the simplifying notation,

\[ \mathbf{B}_i = \begin{pmatrix} b_{1i1} \cdots b_{1i1} \cdots b_{1in} \\ \vdots \vdots \vdots \\ b_{ni1} \cdots b_{ni1} \cdots b_{nin} \end{pmatrix}, \quad b_i^* = \begin{pmatrix} b^*_{1i} \\ \vdots \\ b^*_{ni} \end{pmatrix}, \quad \text{and} \quad a^* = \begin{pmatrix} a^*_{1i} \\ \vdots \\ a^*_{ni} \end{pmatrix}. \]

Consequently, the dynamics (6) of the system becomes

\[ \dot{x}_i = K_i \cdot \left( -\frac{1}{2b_{ii}} b_i^* \left( \mathbf{B}_i^T \mathbf{B}_i \right)^{-1} \mathbf{B}_i^T \left( \mathbf{p}^* - \mathbf{b}_i^* x_i^* - a^* - a_i \right) + \frac{\alpha_i - a_{ii}}{2b_{ii}} - x_i \right), \] (16)

where we assume that the adjustment function is linear: \( k_i(\Delta) = K_i \cdot \Delta \) with \( K_i > 0 \). The asymptotic behavior of the resulting linear system can be examined by using standard methodology.

Consider further the special case when all firms know the exact price functions. Then, \( b_i = b_i^* \), \( \mathbf{B}_i = \mathbf{B}_i^* \), \( b_i^* = (b_{i1}^*, \ldots, b_{i1}^*, b_{i1}^*, \ldots, b_{inn}^*) \), and \( a_i = a_i^* \) for all \( i \). Then, equation (16) simplifies as

\[ \dot{x}_i = K_i \cdot \left( -\frac{1}{2b_{ii}} b_i^* \frac{\partial}{\partial x - i} x_i + c_i \right), \] (17)

where

\[ c_i = \frac{1}{2b_{ii}} (\alpha_i - a_{ii}) \] (18)

is a constant for \( i = 1, 2, \ldots, n \).

Notice that the coefficient matrix of system (17) has the special form,

\[ K \cdot \mathcal{D} \cdot H, \] (19)

where

\[ K = \text{diag} (K_1, \ldots, K_n), \quad \mathcal{D} = \text{diag} \left( -\frac{1}{2b_{ii}^*}, \ldots, -\frac{1}{2b_{mn}^*} \right), \]
Since $b_{ii}^* < 0$ and $K_i > 0$ for all $i$, both matrices $K$ and $D$ are positive definite. It is well known (see, for example, [11]) that system (17) is asymptotically stable if $H + H^T$ is negative definite. This is the case, for example, if the matrix is strictly diagonally dominant. Hence, we have the following result.

**THEOREM 1.** Assume that for all $i$,

$$
\sum_{j \neq i} |b_{ij}^* + b_{ji}^*| < 4 |b_{ii}^*|.
$$

(20)

Then, system (17) is asymptotically stable.

Condition (20) can be interpreted as stating that the quantity of Product $i$ has a much higher effect on its own price than do the outputs of the other products.

### 3. THE EFFECT OF INFORMATION LAGS

In this section, we reexamine model (6) under the assumption that in the right-hand sides only delayed values of $x_1, \ldots, x_n$ are used. Such lags arise when the market reacts to delayed quantities in price formation, or the firms react to an average of old data in order to avoid an over-reaction to sudden changes in market conditions. By assuming continuously distributed time lags, in equation (6), $x_j(t)$ is replaced by a weighted average of earlier data, that is,

$$
x_{ij}^* (t) = \int_0^t w(t - s, T, m) x_j(s) \, ds,
$$

(21)

where $w$ is the appropriate weighting function, taken as a special gamma or exponential density function,

$$
w(t - s, T, m) = \begin{cases} 
\frac{1}{m!} \left( \frac{m}{T} \right)^{m+1} (t-s)^m e^{-m(t-s)/T}, & \text{if } m \geq 1, \\
\frac{1}{T} e^{-(t-s)/T}, & \text{if } m = 0.
\end{cases}
$$

(22)

Here, $T > 0$ and $m$ is a nonnegative integer. Notice that this weighting function has the following properties,

(a) for $m = 0$, weights are exponentially declining with the most weight given to the most current data;

(b) for $m \geq 1$, zero weight is assigned to the most current data, rising to maximum at $t - s = T$, and declining exponentially thereafter;

(c) as $m$ increases, the function becomes more peaked at $t - s = T$. As $m$ becomes large, the weighting function may for all practical purposes be regarded as very close to the Dirac delta function centered at $t - s = T$;

(d) as $T \to 0$, the function converges to the Dirac delta function;

(e) the area under the weighting function in interval $[0, \infty)$ is unity for all $T$ and $m$.

The same weighting function was used in earlier studies on the classical Cournot model with information lags in [22,23].

The asymptotic behavior of the resulting nonlinear Volterra type integro-differential equation is examined via linearization, since the system can be rewritten as a larger system of nonlinear
ordinary differential equations (see [23]). The linearized system has the form,

\[
\dot{x}_\delta(t) = K_t \left( \Delta_t + \sum_{j=1}^{n} \delta_{ij} \gamma_{ij} - 1 \right) \int_0^t w(t-s, T_{st}, m_{st}) x_{\delta_i}(s) \, ds
\]

\[+ \sum_{k \neq i} \sum_{j=1}^{n} \delta_{kj} \gamma_{jk} \int_0^t w(t-s, T_{sk}, m_{sk}) x_{\delta_k}(s) \, ds,\]

where \( x_{\delta_i} \) denotes the deviation of \( x_i \) from its equilibrium level, \( K_t = k_t'(0) \), and

\[
\Delta_t = \frac{\partial (g_t \circ h_t)}{\partial x_t}, \quad \delta_{ij} = \frac{\partial (g_t \circ h_t)}{\partial f_j}, \quad \gamma_{ij} = \frac{\partial f_j}{\partial x_t},
\]

at the steady state. In order to derive the characteristic equation, we seek the solution in the form,

\[
x_{\delta_i}(t) = v_i e^{\lambda t} \quad (i = 1, 2, \ldots, n).
\]

Substituting this form into equation (23) and allowing \( t \to \infty \), we have

\[
\left[ \lambda - K_t \left( \Delta_t + \sum_{j=1}^{n} \delta_{ij} \gamma_{ij} - 1 \right) \int_0^\infty w(s, T_{st}, m_{st}) e^{-\lambda s} \, ds \right] v_i
\]

\[= - \sum_{k \neq i} \left[ K_i \sum_{j=1}^{n} \delta_{kj} \gamma_{jk} \int_0^\infty w(s, T_{sk}, m_{sk}) e^{-\lambda s} \, ds \right] v_k = 0.
\]

By introducing the notation,

\[
A_i(\lambda) = \lambda - K_t \left( \Delta_t + \sum_{j=1}^{n} \delta_{ij} \gamma_{ij} - 1 \right) \left( \frac{\Lambda T_{st}}{r_{st}} + 1 \right)^{-(m_{st}+1)},
\]

with

\[
r_{st} = \begin{cases} 
  m_{st}, & \text{if } m_{st} \geq 1, \\
  1, & \text{if } m_{st} = 0,
\end{cases}
\]

and

\[
B_{ik} = -K_i \sum_{j=1}^{n} \delta_{ij} \gamma_{jk} \left( \frac{\Lambda T_{sk}}{r_{sk}} + 1 \right)^{-(m_{sk}+1)}
\]

with

\[
r_{sk} = \begin{cases} 
  m_{sk}, & \text{if } m_{sk} \geq 1, \\
  1, & \text{if } m_{sk} = 0,
\end{cases}
\]

equation (25) can be rewritten as a nonlinear eigenvalue problem,

\[
\det \begin{pmatrix}
A_1(\lambda) & B_{12}(\lambda) & \cdots & B_{1n}(\lambda) \\
B_{21}(\lambda) & A_2(\lambda) & \cdots & B_{2n}(\lambda) \\
\vdots & \vdots & \ddots & \vdots \\
B_{n1}(\lambda) & B_{n2}(\lambda) & \cdots & A_n(\lambda)
\end{pmatrix} = 0.
\]

In the general case, it is impossible to give a closed form simple representation of this equation, so in order to investigate the location of the eigenvalues, it will be necessary to resort to numerical methods.
EXAMPLE 2. Consider again the linear case examined in the previous example. From equation (15), we see that

$$\Delta_i = -\frac{1}{2b_{ij}} b_i^* \left( B_i^T B_i \right)^{-1} B_i^T b_i$$

and

$$\delta_{ij} = -\frac{1}{2b_{ij}} b_i^* \left( B_i^T B_i \right)^{-1} B_i^T e_j,$$

where $e_j$ is the $j$th basis vector, and from equation (7), we have

$$\gamma_{j1} = b_{j1}^*.$$

4. SPECIAL CASES

We assume first that for each firm, functions $B_i(\lambda)$ are identical for all values of $\lambda$. So, let $B_i(\lambda) \equiv B_i(\lambda)$. This is the case when from the viewpoint of all firms, the competitors are equal and considered identical. Then, the eigenvalue equation (28) becomes

$$\det \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & \cdots & B_1(\lambda) \\ B_2(\lambda) & A_2(\lambda) & \cdots & B_2(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ B_n(\lambda) & B_n(\lambda) & \cdots & A_n(\lambda) \end{pmatrix} = 0.$$ (32)

It has been shown in [23] that this equation can be rewritten as

$$\prod_{i=1}^{n} \left( A_i(\lambda) - B_i(\lambda) \right) \left[ 1 + \sum_{i=1}^{n} \frac{B_i(\lambda)}{A_i(\lambda) - B_i(\lambda)} \right] = 0,$$ (33)

therefore, the eigenvalue equation breaks up to $(n + 1)$ equations, namely,

$$A_i(\lambda) - B_i(\lambda) = 0 \quad (i = 1, 2, \ldots, n),$$ (34)

$$1 + \sum_{i=1}^{n} \frac{B_i(\lambda)}{A_i(\lambda) - B_i(\lambda)} = 0.$$ (35)

The general treatment of these equations can be performed in the same way as shown in [23], so the methodology is not repeated here. Instead, further special cases will be analyzed.

Consider next the symmetric case, when $K_1 = \cdots = K_n = K$ with $K > 0$, $\Delta_i + \sum_{j=1}^{n} \delta_{ij} \gamma_{j1} = \Delta_i \equiv \Delta$, and $\sum_{j=1}^{n} \delta_{ij} \gamma_{jk} \equiv \delta$, $T_{ii} \equiv T^*$, $m_{ii} \equiv m^*$, $T_{ik} \equiv T$, $m_{ik} \equiv m$, for all $i$ and $k \neq i$. Then, $r_i \equiv r^*$, $r_{ik} \equiv r$, and for all $i$,

$$A_i(\lambda) = \lambda - K\Delta \left( \frac{\lambda T^*}{r^*} + 1 \right)^{-(m^*+1)},$$ (36)

and

$$B_i(\lambda) = -K\delta \left( \frac{\lambda T}{r} + 1 \right)^{-(m+1)},$$ (37)

therefore, equations (34) and (35) are reduced to the two simple polynomial equations,

$$\lambda \left( \frac{\lambda T^*}{r^*} + 1 \right)^{m^*+1} \left( \frac{\lambda T}{r} + 1 \right)^{m+1} - K\Delta \left( \frac{\lambda T}{r} + 1 \right)^{m+1} + K\delta \left( \frac{\lambda T^*}{r^*} + 1 \right)^{m^*+1} = 0$$ (38)

and

$$\lambda \left( \frac{\lambda T^*}{r^*} + 1 \right)^{m^*+1} \left( \frac{\lambda T}{r} + 1 \right)^{m+1} - K\Delta \left( \frac{\lambda T}{r} + 1 \right)^{m+1} - (n-1)K\delta \left( \frac{\lambda T^*}{r^*} + 1 \right)^{m^*+1} = 0.$$ (39)
For the sake of simplicity, assume that the time lag in obtaining and implementing information on its own data is much shorter than that on the data of the other firms. So, we might assume that \( T^* = 0 \). Then, equations (38) and (39) can be summarized as

\[
(\lambda - K\Delta) \left( \frac{\lambda T}{r} + 1 \right)^{m+1} + K\delta N = 0,
\]

where \( N = 1 \) or \( 1 - n \).

First, assume that \( \delta = 0 \). Then, equation (40) has only real roots: \( \lambda = K\Delta \) and for \( T \neq 0 \), \( \lambda = -r/T \). So, if \( \Delta < 0 \), then the steady state is locally asymptotically stable and if \( \Delta > 0 \), then it is unstable. If \( \Delta = 0 \), then no definite conclusion can be drawn. So, in the ensuing discussions, we will always assume that \( \delta \neq 0 \).

If \( m = 0 \), this equation is quadratic,

\[
\lambda^2 T + \lambda (1 - KT\Delta) + K (\delta N - \Delta) = 0.
\]

It is well known (see, for example, [11]) that all eigenvalues have negative real parts if and only if

\[
1 - KT\Delta > 0, \quad \delta - \Delta > 0, \quad \delta (1 - n) - \Delta > 0,
\]

which occurs if and only if

\[
\Delta < \min \left\{ \frac{1}{KT}; \delta; \frac{\delta (1 - n)}{\Delta} \right\}.
\]

Under condition (43) the steady state is locally asymptotically stable. When the steady state loses asymptotic stability, then certain kinds of instability may occur. We are particularly interested in exploring the possibility of the birth of limit cycles, which result in fluctuating output patterns.

**Theorem 2.** Assume \( m=0 \) and the symmetric case with \( \Delta > 0 \), furthermore that one of the values \( \delta - \Delta \) and \( (1 - n)\delta - \Delta \) is positive and the other is negative. Then, there is a limit cycle in the neighborhood of the critical value \( T_{cr} = 1/(K\Delta) \).

**Proof.** In order to use the Hopf bifurcation theorem (see [24]), we are looking for pure complex roots. If \( \lambda = i\alpha \) is a nonzero root of equation (41), then

\[
-i\alpha^2 T + \alpha (1 - KT\Delta) + K (\delta N - \Delta) = 0.
\]

Equating the real and imaginary parts to zero, we have

\[
\alpha^2 = \frac{K (\delta N - \Delta)}{T} \quad \text{and} \quad 1 - KT\Delta = 0.
\]

Real \( \alpha \) exists if \( \delta N - \Delta > 0 \) and \( T = 1/(K\Delta) \). \( T > 0 \) requires that \( \Delta > 0 \). We select \( T \) as the bifurcation parameter, so the critical value of \( T \) for both values of \( N \) is

\[
T_{cr} = \frac{1}{K\Delta}.
\]

Differentiating equation (41) implicitly with respect to \( T \), we have

\[
\frac{\partial \lambda}{\partial T} = \frac{-\lambda^2 + \lambda K\Delta}{2\lambda T + (1 - KT\Delta)}.
\]

At the critical value with \( \lambda = i\alpha \),

\[
\left. \frac{\partial \lambda}{\partial T} \right|_{\lambda = i\alpha, T = T_{cr}} = \frac{\alpha^2 + i\alpha K\Delta}{i\alpha T_{cr}} = \frac{K\Delta}{2T_{cr}} - i\frac{\alpha}{2T_{cr}},
\]

with nonzero real part. Then, the Hopf bifurcation theorem implies the existence of limit cycle.
Note that the condition that only one of the values $\delta - \Delta$ and $(1 - n)\delta - \Delta$ is positive implies that there is only one pair of pure complex eigenvalues.

Assume next that $m = 1$, then equation (40) is cubic,

$$\lambda^3 T^2 + \lambda^2 (2T - K\Delta T^2) + \lambda (1 - 2TK\Delta) + K (\delta N - \Delta) = 0. \tag{49}$$

The local asymptotical stability of the steady state can be analysed by applying the Routh-Hurwitz criteria (see for example, [11]), the details of which are left to the reader as a simple exercise.

**Theorem 3.** Let $m = 1$ with the symmetric case.

(i) Assume first that $\delta > 0$. If $0 < \Delta < \delta$, then there is a positive critical value for $T$ with $N = 1$, which is the smaller root of equation (53) and there is limit cycle in its neighborhood. If $\Delta = 0$, then with $N = 1$ there is a critical value for $T$ given by equation (63), and there is a limit cycle around it. If $0 > \Delta > -(1/8)\delta$, then both roots of equation (53) are critical values for $N = 1$, and there is a limit cycle in the neighborhood of both of them.

(ii) Assume next that $\delta < 0$. If $0 < \Delta < \delta(1 - n)$, then there is a positive critical value for $T$ with $N = 1 - n$, which is the smaller root of equation (53), and there is a limit cycle in its neighborhood. If $\Delta = 0$, then with $N = 1 - n$ there is a critical value for $T$ given by equation (63), and there is a limit cycle around it. If $0 > \Delta > (1/8)\delta(n - 1)$, then both roots of equation (53) are critical values for $N = 1 - n$, and there is a limit cycle in the neighborhood of both of them.

**Proof.** To analyze the possibility of the birth of limit cycles assume again that $\lambda = \alpha i$ is a pure complex root. Then,

$$-i\alpha^2 T^2 - \alpha^2 (2T - K\Delta T^2) + \alpha (1 - 2TK\Delta) + K (\delta N - \Delta) = 0. \tag{50}$$

By equating the real and imaginary parts to zero, we obtain the condition,

$$\alpha^2 = \frac{1 - 2TK\Delta}{T^2} = \frac{K (\delta N - \Delta)}{2T - K\Delta T^2}. \tag{51}$$

In order to have real nonzero $\alpha$, we need

$$1 - 2TK\Delta > 0. \tag{52}$$

The critical value of $T$ is obtained from the second equation in (51) as the solution of the quadratic equation,

$$2T^2 K^2 \Delta^2 - TK (4\Delta + \delta N) + 2 = 0. \tag{53}$$

Positive solution for $T$ exists if and only if

$$4\Delta + \delta N > 0 \tag{54}$$

and

$$K^2 (4\Delta + \delta N)^2 - 16K^2 \Delta^2 = K^2 \delta N (\delta N + 8\Delta) \geq 0 \tag{55}$$

and in this case both roots are positive. First, notice that equation (53) can be written as

$$2 (TK\Delta - 1)^2 = TK\delta N, \tag{56}$$

implying that $\delta N \geq 0$. If $\delta N = 0$, then $TK\Delta = 1$, which contradicts condition (52). Hence, we must have $\delta N > 0$. Inequalities (54) and (55) can be now summarized as

$$\Delta \geq -\frac{1}{8} \delta N \tag{57}$$
Notice that if $\delta \neq 0$, then with $N = 1$ and $1 - n$, we get different critical values, and if $\delta = 0$, then equation (49) is identical for $N = 1$ and $1 - n$, so the critical values coincide in the two cases.

Differentiating equation (49) implicitly with respect to $T$, which is selected again as the bifurcation parameter, we have

$$
\frac{\partial \lambda}{\partial T} = \frac{-2\lambda^3 T - \lambda^2 (2 - 2K\Delta T) + 2\lambda K\Delta}{3\lambda^2 T^2 + 2\lambda (2T - K\Delta T^2) + (1 - 2TK\Delta)}
$$

$$
= \frac{2\lambda^2 (1 - K\Delta T) + 2\lambda (\alpha^2 T + \alpha K\Delta)}{(-3\lambda^2 T^2 + 1 - 2TK\Delta) + 2\alpha T (2 - K\Delta T)}
$$

(58)

with real part at the critical value,

$$
\text{Re} \left( \frac{\partial \lambda}{\partial T} \right) \bigg|_{T=T_{cr}(N)} = \frac{1 - (K\Delta)^2}{T^2 (K\Delta - 1) (K\Delta - 5)}. 
$$

(59)

Under condition (52), the denominator is always positive, and the numerator is nonzero if $K\Delta \neq -1$. From equation (53), it is easy to see that $K\Delta = -1$ if and only if $\Delta = -(1/8)\delta N$.

We will now elaborate in more details the conditions for positive critical $T$ values. We have to consider three cases depending on the sign of $\Delta$.

First, assume that $\Delta > 0$, then from equation (53) and condition (52), we should have

$$
TK = \frac{(4\Delta + \delta N) \pm \sqrt{\delta N (\delta N + 8\Delta)}}{4\Delta^2} < \frac{1}{2\Delta},
$$

(60)

which is equivalent to the condition,

$$
2\Delta + \delta N < \mp \sqrt{\delta N (\delta N + 8\Delta)}.
$$

(61)

Since both $\delta N$ and $\Delta$ are positive, the left-hand side is also positive, so (61) may hold with only the positive sign. In this case, (61) is equivalent to relation,

$$
\Delta < \delta N,
$$

(62)

in which case (57) is obviously satisfied, and (61) holds with the smaller root only.

If $\Delta < 0$, then from equation (53) and condition (52), we should have

$$
TK = \frac{(4\Delta + \delta N) \pm \sqrt{\delta N (\delta N + 8\Delta)}}{4\Delta^2} > \frac{1}{2\Delta},
$$

which holds for both positive roots.

If $\Delta = 0$, then equation (49) becomes

$$
\lambda^3 T^2 + \lambda^2 2T + \lambda + K\delta N = 0
$$

(63)

and (51) is specialized as

$$
\alpha^2 = \frac{1}{T^2} = \frac{K\delta N}{2T}.
$$

(64)

So, the critical value of $T$ equals

$$
T_{cr} (N) = \frac{2}{\delta NK},
$$

(65)

which is always positive, if $\delta N > 0$.

If $\delta > 0$, then $\delta N > 0$ for only $N = 1$, and if $\delta < 0$, then $\delta N > 0$ for only $N = 1 - n$. If $\delta = 0$, then $\delta N$ cannot be positive.
5. CONCLUSIONS

In this paper, \(n\)-firm Cournot oligopolies with product differentiation were examined under uncertainty, when each firm is able to measure its own output and all prices. Since firms use only believed price functions, the outputs of the rival firms are obtained by using a special least squares fit of the price equations. These estimates of the rivals' outputs are then used by each firm to determine the best response of the rival firms. It is also assumed that each firm adjusts its output into the direction of its best response, so a system of nonlinear ordinary differential equations is obtained to describe the output trajectories. In the linear case, sufficient conditions are given for the asymptotic stability of the steady state.

Then, it was assumed that there is an information lag in obtaining and implementing output information, or that the firms are reluctant to react to sudden changes so they react to averaged market fluctuations. This situation is modeled with continuously distributed time lags, and the resultant nonlinear Volterra-type integro-differential equations were examined by linearization and eigenvalues analysis. In the case of symmetric firms, analytical results were obtained. Conditions were derived for the local asymptotic stability of the equilibrium, and in the case of bifurcation, the possibility of cyclical behavior was explored.

REFERENCES
