1. Introduction

This paper is the first of a series in which Favard classes of singular integrals in several variables will be discussed systematically. The purpose is not only to extend the results of the one-dimensional theory established by one of the authors (e.g. [7], [8]) to \( n \) dimensions, but to try to bring within a single structure as much as possible of a wide flung development which will also give new insights into the one-dimensional situation.

Let \( E^n \) be the \( n \)-dimensional Euclidean space, whose elements will consistently be denoted by \( u, v, x \). We write e.g. \( u=(u_1, u_2, \ldots, u_n) \). If \( f \) is measurable on \( E^n \) we set

\[
\|f(\cdot)\|_p = \begin{cases} 
\left[ \int_{E^n} |f(x)|^p \, dx \right]^{1/p} & (1 < p < \infty) \\
\text{ess sup}_{x \in E^n} |f(x)| & (p = \infty),
\end{cases}
\]

and \( L_p(E^n) \) is the space of functions for which the norm \( \|f(\cdot)\|_p \) is finite. Consider an approximation process for a function \( f \in L_p(E^n) \), \( 1 < p < \infty \), defined by means of a singular integral of Fourier convolution type

\[
K(f;x;q) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} f(x-u) k(u;q) \, du,
\]

where \( q \) is a positive parameter and \( k(u;q) \) is said to be the kernel of the integral (1.1) subject to the following conditions [5, p. 1]:

\[
\begin{align*}
(i) \quad & \|k(\cdot; q)\|_1 \leq M, \quad \int_{E^n} k(u; q) \, du = (2\pi)^{n/2} \quad (\text{all } q > 0), \\
(ii) \quad & \lim_{q \to \infty} \int_{|u| > \delta} |k(u; q)| \, du = 0 \quad (\text{all } \delta > 0),
\end{align*}
\]

with constant \( M > 0 \).

1) The main results of this series of papers were announced by R. J. Nessel in talks held on September 15, 1964 at the Austrian Mathematical Congress, Graz, and on March 6, 1964 and August 5, 1965 at the Mathematical Research Institute, Oberwolfach, Black Forest. A brief summary will appear in a research announcement by the authors entitled: Favard classes for \( n \)-dimensional singular integrals, Bull. Amer. Math. Soc. 72 (1966), in print.
If \( f \in L_p(\mathbb{E}^n), 1 \leq p < \infty \), it is a well-known fact [5, p. 10] that under these conditions the singular integral (1.1) exists almost everywhere (\( = \text{a.e.} \)) and satisfies the relations

\[
\begin{align*}
\text{(i)} & \quad \|K(f; \cdot; \varrho)\|_p < \|k(\cdot; \varrho)\|_p \|f(\cdot)\|_p \\
\text{(ii)} & \quad \lim_{\varrho \to \infty} \|K(f; \cdot; \varrho) - f(\cdot)\|_p = 0.
\end{align*}
\]

It is exactly the latter relation which expresses the fact that (1.1) is an approximation process for the function \( f \). If, furthermore, the kernel \( k(x; \varrho) \) is bounded for every \( \varrho > 0 \), then (1.1) exists everywhere.

If a function \( k \in L_1(\mathbb{E}^n) \) is normalized by \( \int_{\mathbb{E}^n} k(u)du = (2\pi)^{n/2} \), then it defines a kernel of type (1.2) if we put \( k(u; \varrho) = \varrho^n k(\varrho u) \). For this important class of kernels (1.1) takes on the form [5, p. 2]

\[
K(f; x; \varrho) = \frac{\varrho^n}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} f(x-u) k(\varrho u) du.
\]

One well-known problem in approximation theory is the connection between the rapidity of the approximation in the \( L_p \)-norm of \( f \) by the general singular integral \( K(f; x; \varrho) \) as \( \varrho \to \infty \) and the structural properties of the function \( f \). Our aim in this series of papers is to discuss a particular but nevertheless important case of this general problem which concerns the optimal rate of approximation of non-trivial functions \( f \) by the general singular integral (1.1) and to determine the exact class \( F \) of functions \( f \) for which this optimal rate is precisely attained. This is the so-called saturation problem for the process (1.1) and \( F \) is the corresponding Favard class. This notion, first introduced by J. FAVARD [14], may be precisely defined in our situation as follows:

**Definition 1.1:** Let \( f \in L_p(\mathbb{E}^n), 1 \leq p < \infty \), and (1.1) be a given singular integral with kernel \( k(x; \varrho) \). Suppose there exists a monotone decreasing function \( \varphi(\varrho) \) with \( \lim_{\varrho \to \infty} \varphi(\varrho) = 0 \) and a class \( F \subset L_p(\mathbb{E}^n) \) of functions \( f \) such that

a) \( \|K(f; \cdot; \varrho) - f(\cdot)\|_p = o(\varphi(\varrho)) \) as \( \varrho \to \infty \) implies \( f(x) = 0 \) a.e.;

b) \( \|K(f; \cdot; \varrho) - f(\cdot)\|_p = O(\varphi(\varrho)) \) as \( \varrho \to \infty \) implies \( f \in F \);

c) \( f \in F \) implies \( \|K(f; \cdot; \varrho) - f(\cdot)\|_p = O(\varphi(\varrho)) \) as \( \varrho \to \infty \),

whereby \( F \) contains at least one element different from the null-function. Then the singular integral \( K(f; x; \varrho) \) is said to be saturated with order \( O(\varphi(\varrho)) \) in the space \( L_p(\mathbb{E}^n) \) and \( F \) is called its Favard (or saturation) class.

Although we are mainly interested in norm convergence we may at this stage emphasize that point-wise convergence of the general singular integral (1.1) also holds provided further conditions are satisfied by the kernel \( k \) or the function \( f \). See for instance [5, p. 2], [6, p. 64], [15].
A particular example of a singular integral whose Favard class will be determined in the second paper of the series, is that of Gauss-Weierstrass defined by

\[(1.5) \quad W(f; x; t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(x-u) e^{-((1/4t)u^2)} du\]

with kernel \(w(u) = 2^{n/2} \exp \{-u^2\}\) and \(e = (2 \cdot \sqrt{t})^{-1}, t > 0\). If \(f \in L_p(\mathbb{R}^n)\), according to (1.3) (ii), the \(W(f; x; t)\) converge in the \(L_p\)-norm to \(f\) as \(t \to 0^+\). Regarding point-wise convergence we may cite the following result (see [6, p. 103] or [20, p. 432] for a survey of the properties of \(W(f; x; t)\) which plays a definite role in some of the proofs to follow.

**Lemma 1.1:** If \(f \in L_p(\mathbb{R}^n), 1<p<\infty\), then \(W(f; x; t)\) converges a.e. to \(f(x)\) as \(t \to 0^+\). If \(f\) is defined, continuous and bounded on \(\mathbb{R}^n\), then

\[(1.6) \quad \lim_{t \to 0^+} W(f; x; t) = f(x)\]

for each \(x \in \mathbb{R}^n\).

At this point we may mention a paper by M. H. Taibleson [20] in which, among other results, classes of functions \(f\) are determined for which the approximation by the specific singular integrals of Gauss-Weierstrass and Cauchy-Poisson is of a definite non-optimal order. Both papers complement each other in the sense that one paper determines the class of functions in the case of saturated approximation, the other in the case of "non-saturated" approximation. As may be expected, the methods of proof of the two papers[1] are very different. Let it be said that partial results for the Cauchy-Poisson singular integral in two dimensions were given by one of the authors in [9].

Just as in [9], the essential tool for the solution of the saturation problem for the singular integral (1.1) will be the \(n\)-dimensional Fourier transform, which, for \(f \in L_1(\mathbb{R}^n)\), is defined by

\[(1.7) \quad \hat{f}(\nu) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \nu, x \rangle} f(x) \, dx,\]

\(\langle v, x \rangle \equiv v_1x_1 + \ldots + v_nx_n\) being the inner product of the vectors \(v, x \in \mathbb{R}^n\).

---

[1] On December 1, 1965 the authors received a preprint by Jörgen Löfström: "Some theorems on interpolation spaces with applications to approximation in \(L_p\)", in which an attempt is made to combine the Fourier transform method, presented in [8], with the theory of interpolation spaces and the theory of strongly continuous semi-groups of operators (see [1a] as well as [26], Note II) in order to obtain simultaneously results on saturated and non-saturated approximation in Euclidian \(n\)-space including also operators which are not of semi-group type. In the applications the saturation classes are for \(1<p\leq 2\) characterized by the domains of the infinitesimal generators of certain "associated" semi-groups.
The Fourier transform \( f^\wedge (v) \) of a function \( f \in L_p(\mathbb{R}^n) \), \( 1 < p < 2 \), will be given by the relation

\[
(1.8) \quad \lim_{N \to \infty} \| f^\wedge (\cdot) - \frac{1}{(2\pi)^{n/2}} \int_{|x| < N} e^{-i<v,x>} f(x) \, dx \|_q = 0,
\]

where \( p \) and \( q \) are conjugate numbers: \( p^{-1} + q^{-1} = 1 \), and the Fourier-Stieltjes transform of a bounded measure \( \mu \) by

\[
(1.9) \quad \mu^\wedge (v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i<v,x>} d\mu(x).
\]

Fourier transforms in \( L_1(\mathbb{R}^n) \) are discussed in detail in [3] and [6]. For references on Fourier-Stieltjes transforms we may mention the paper of S. Bochner included in [3] as an author's supplement. See also [5]. Fourier transforms in \( L_2(\mathbb{R}^n) \) are considered in e.g. [6] and [23]. The theory of Fourier transforms in \( L_p(\mathbb{R}^n) \), \( 1 < p < 2 \), seems to be well-known, though, to the best of our knowledge, there is no place in the literature where the properties of these transforms are explicitly stated and proved. Most of these properties are immediate extensions of the one-dimensional theory presented in [22], others may be obtained by the same methods which are successful in \( L_2 \)-theory.

Let us mention, to avoid misunderstandings, that we will refer to a relation such as

\[
(1.10) \quad \frac{1}{\mathbb{R}^n} \int f(u)g^\wedge(u) \, du = \frac{1}{\mathbb{R}^n} \int f^\wedge(u)g(u) \, du
\]

as a Parseval formula. (1.10) is true for \( f, g \in L_p(\mathbb{R}^n) \), \( 1 < p < 2 \) ([6], [22]). There are analogous formulae for Fourier-Stieltjes transforms.

This paper treating the general theory consists of two further sections. Section 2 is concerned with necessary and sufficient conditions for the representation of functions by Fourier-Stieltjes or Fourier integrals. These representation theorems will then be used to prove general saturation theorems for approximation processes given by (1.1). In the second paper we shall apply the general theory to some particular singular integrals such as those of Gauss-Weierstrass and Cauchy-Poisson. Then in a third paper we shall continue the general theory and treat radial kernels and will also give further applications, e.g. to the singular integral of Bochner-Riesz.

The authors would like to express their sincere gratitude to Dr. H. Berens for his many valuable suggestions.

2. Representation Theorems

As has been mentioned, in this section we will discuss the representation of a given function as a Fourier-(Stieltjes) integral. Let us begin with a
function \( g \in L_p(E^n) \), \( 1 < p < 2 \), and its Fourier transform \( \hat{g} = \hat{f} \) and let us form the integral

\[
F_R(x) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|\nu_j|}{R} \right] \right\} e^{i<x, \nu>} f(v) \, dv,
\]

which we may rewrite by using the Parseval formula (1.10) as

\[
\left( \frac{2}{\pi R} \right)^n \int_{E^n} g(x-u) \left\{ \prod_{j=1}^{n} \frac{\sin^2 (R/2) \nu_j}{\nu_j^2} \right\} \, du.
\]

Here we used the notation (see [13])

\[
\left\{ 1 - \frac{|t|}{R} \right\} = \begin{cases} 
(1 - (|t|/R)) & \text{for } |t| < R \\
0 & \text{for } |t| \geq R
\end{cases}
\]

and the relations

\[
\int_{-R}^{R} \left( 1 - \frac{|t|}{R} \right) e^{-ist} \, dt = \frac{4}{R} \frac{\sin^2 (R/2)s}{s^2}; \quad \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \, dt = \pi.
\]

By (1.3) we know that the integral (2.1) converges to the original function \( g \) in the mean of order \( p \) as \( R \to \infty \). Thus starting with an arbitrary measurable function \( f \) which is integrable over every finite interval, the question arises as to whether there exists a necessary and sufficient condition upon the integral (2.1) which guarantees the existence of a bounded measure \( \mu \) or of a function \( g \in L_p(E^n) \), \( 1 < p < 2 \), for which the Fourier-Stieltjes or Fourier transform is equal to the given function \( f \) a.e. The following theorems give an affirmative answer.

**Theorem 2.1:** Let \( f \) be measurable in \( E^n \) and summable over every finite interval. A necessary and sufficient condition that \( f(v) \) can be represented almost everywhere as a Fourier-Stieltjes transform

\[
\hat{f}(v) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{i<x, \nu>} \, d\mu(x)
\]

with \( \mu \) a bounded measure, is that

\[
\left\| \int_{E^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|\nu_j|}{R} \right] \right\} e^{i<x, \nu>} f(v) \, dv \right\|_1 = O(1) \quad (R \to \infty).
\]

If \( f \) is continuous for all \( v \in E^n \) then the representation \( f(v) = \hat{\mu}^{-1}(v) \) holds everywhere.

**Theorem 2.2:** Let \( f \) be measurable in \( E^n \) and summable over every finite interval. Then for \( 1 < p < 2 \) the condition

\[
\left\| \int_{E^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|\nu_j|}{R} \right] \right\} e^{i<x, \nu>} f(v) \, dv \right\|_{L^p} = O(1) \quad (R \to \infty)
\]
is necessary and sufficient such that a function \( g \) belonging to \( L_p(\mathbb{E}^n) \) exists with its Fourier transform almost everywhere equal to \( f \).

**Theorem 2.3:** Let \( f \) be measurable in \( \mathbb{E}^n \) and summable over every finite interval. Then the conditions

\[
(2.6) \quad \left\| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \left( 1 - \frac{|v_j|}{R} \right) \right\} e^{i<v,v>} f(v) \, dv \right\|_1 = O(1) \quad (R \to \infty),
\]

\[
(2.7) \quad \lim_{s \to \infty} \| F_R(\cdot) - F_S(\cdot) \|_1 = 0
\]

are necessary and sufficient that there exists a function \( g \in L_1(\mathbb{E}^n) \) such that \( f(v) = g^\wedge(v) \) a.e. If \( f \) is continuous then this representation holds everywhere.

Since the proofs of the necessity of all statements of the three theorems immediately follow by Parseval's formula and (1.3) we only need to prove the sufficiency of the stated conditions. Note that in case \( 1 < p < 2 \) the condition (2.5) always implies (2.7) for \( L_p \)-norms.

**Proof of Theorem 2.1:** Sufficiency: We will first consider the continuous case. Since

\[ f_R(v) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} e^{i<v,v>} F_R(-x) \, dx \]

and therefore \( |f_R(v)| \leq \| F_R(\cdot) \|_1 \) for all \( v \in \mathbb{E}^n \). Since this bound is by (2.4) uniform with respect to large \( R \) we first of all conclude the boundedness of \( f \).

On the other hand the functions \( F_R(x) \) define a set of absolutely continuous measures which are by (2.4) uniformly bounded in \( \mathbb{E}^n \) with respect to large \( R \). Therefore [5, p. 16] there exist a sequence of positive numbers \( \{R_j\} \) with \( \lim_{j \to \infty} R_j = \infty \) and a bounded measure \( \mu \) such that

\[
\lim_{j \to \infty} \int_{\mathbb{E}^n} h(u) \, F_{R_j}(u) \, du = \int_{\mathbb{E}^n} h(u) \, d\mu
\]

for every continuous function \( h \) which vanishes at infinity, thus for which \( \lim_{|u| \to \infty} h(u) = 0 \), i.e.: The set of functions \( \{F_R(x)\} \) contains a weakly* convergent subsequence (see also (3.15)). If we apply this limit relation to the particular functions \( \exp \{ -tu^2 - i<x,u> \} \) with arbitrary \( x \in \mathbb{E}^n \) and \( t > 0 \) and use the relation

\[
(2.8) \quad \int_{\mathbb{E}^n} e^{-t<v,v>} e^{-tx^2} \, dx = \left( \frac{\pi}{t} \right)^{n/2} e^{-(1/4)t} \quad (t > 0),
\]
then we obtain by the Parseval formula and Lebesgue’s dominated convergence theorem

\[
\frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} \mu^\sim(u) \, du = \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{-tu^2 - i<x, u>} \, d\mu(u) = \\
= \lim_{t \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{-tu^2 - i<x, u>} \, F_R(u) \, du = \\
= \lim_{t \to \infty} \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} f_R(u) \, du = \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} f(u) \, du
\]

for all $t > 0$, i.e.: the Gauss-Weierstrass integral of the continuous bounded function $(\mu^\sim - f)$ vanishes for all $t > 0$. But in view of Lemma 1.1 this implies $\mu^\sim = f$, completing the proof which in an obvious way covers the discontinuous case, too.

Next let us say a few words about the sufficiency of the conditions of Theorem 2.3. In view of Theorem 2.1 there first of all exists a bounded measure $\mu$ such that $f(v) = \mu^\sim(v)$ a.e. On the other hand, (2.7) and the completeness of $L_1(E^n)$ implies the existence of some $g \in L_1(E^n)$ such that $\lim_{R \to \infty} ||F_R(\cdot) - g(\cdot)||_1 = 0$. If we therefore repeat the last argument of the preceding proof we conclude that the Gauss-Weierstrass integral of $\{\mu^\sim(v) - g^\sim(v)\}$ vanishes for all $t > 0$, establishing Theorem 2.3.

**Proof of Theorem 2.2:** **Sufficiency:** Since $F_R \in L_p(E^n)$, $1 < p < 2$, we can form the Fourier transform $F_R^\sim$ and obtain

\[
\frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} F_R^\sim(u) \, du = \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{-tu^2 - i<x, u>} F_R(u) \, du = \\
= \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} f_R(u) \, du.
\]

Here we used the Parseval formula for Fourier transforms in $L_p(E^n)$ and $L_1(E^n)$. Now by Lemma 1.1 the left hand side tends to $F_R^\sim(x)$ a.e. as $t \to 0+$, since $F_R^\sim \in L_q(E^n)$, $p^{-1} + q^{-1} = 1$, whereas the right hand side tends to $f_R(x)$ a.e. as $t \to 0+$, since $f_R \in L_1(E^n)$. Therefore we conclude: $f_R(x) = F_R^\sim(x)$ a.e. and by Titchmarsh’s inequality: $||f_R(\cdot)||_q < ||F_R(\cdot)||_p$. Since this bound is by (2.5) uniform with respect to large $R$ we obtain by Fatou’s lemma: $f \in L_q(E^n)$.

Now the condition (2.5) states that the norms of the functions $F_R$ are uniformly bounded with respect to large $R$. Therefore, by the weak* compactness of the spaces $L_p(E^n)$, $1 < p < 2$, there exist ([21, p. 209], see also (3.16)) a sequence of positive numbers $\{R_j\}$ with $\lim R_j = \infty$ and a function $g \in L_p(E^n)$ such that

\[
\lim \int_{E^n} h(u) F_{R_j}(u) \, du = \int_{E^n} h(u) g(u) \, du
\]
for every function \( h \in L_q(E^n) \), \( p^{-1} + q^{-1} = 1 \). Again we take the particular functions \( h(x) = \exp \{-tu^2 - i < x, u > \} \) with arbitrary \( x \in E^n \) and \( t > 0 \). In the same way as in the previous proof we conclude

\[
\frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(x-u)^2} \{ g^*(u) - f(u) \} \, du = 0,
\]

which by Lemma 1.1, since \( (g^* - f) \in L_q(E^n) \), implies \( f = g^* \) a.e. and completes the proof.

Note that in case \( p = 2 \) the latter argument is superfluous. For we then obtain from \( f \in L_2(E^n) \) in virtue of Plancherel’s theorem [6, p. 117], that \( f \) is the Fourier transform of some function of \( L_2(E^n) \) (namely of \( f^*(-v) \)).

**Historical remarks:**

There is a considerable amount of literature concerning the representation of functions as Fourier-Stieltjes or Fourier integrals and the exact characterization of those functions. First of all, we must mention a paper of A. C. Berry [2] of 1931, then the important work of S. Bochner [3, p. 95] and [4] and the theory generated by it, e.g. [17], [18]. The conditions used here are due to H. Cramér [12] who also regards the \( n \)-dimensional question for \( p = 1 \), using a lemma of S. Bochner [3, p. 322] for the proofs. R. Doss [13] also treats the criterion of H. Cramér but he proves its sufficiency by reducing it to that of the theorem of A. C. Berry. All these authors are mainly interested in the different possible representations related to the case \( p = 1 \).

Our methods of proof were suggested by a paper of A. C. Offord [16] in which he treats the problem of defining a Fourier transform in \( L_p(-\infty, \infty) \), \( 1 < p < \infty \). They rest deeply upon certain selection principles such as the weak* compactness of the space \( L_p(E^n) \) for \( 1 < p < 2 \). Using these tools we may prove all cases \( 1 < p < 2 \) in a unified manner as we have already seen. Essentially the same methods were used by J. L. B. Cooper [11], who presented the one-dimensional theory. Let us point out that one may express the given representation theorems in a more general form by replacing the special Fejér kernel used here in e.g. (2.4) by general summation kernels. See e.g. [11], [12].

3. **General Saturation Theorems**

In discussing the actual saturation problem of the singular integral (1.1) further conditions must be satisfied by the kernel. In following the well developed one-dimensional theory ([8], [19]) it turns out as already mentioned, that the integral transform method first introduced by one of the authors in e.g. [7], [8], [10] and considered in the particular case of the Laplace transform by H. Berens and P. L. Butzer [1c], is also an appropriate tool for the \( n \)-dimensional situation. This method works in a
very similar fashion to that with which certain initial or boundary value problems of differential equations may be solved by using e.g. the Laplace transform. If for \(1 < p < 2\) one applies the \(n\)-dimensional Fourier transform to (1.1) one obtains, by the convolution theorem, a separation of the kernel and the particular function \(f\). Since saturation is a property of the approximation process, i.e. of the kernel, and not of the particular function \(f\) it seems to be reasonable to postulate further conditions upon the Fourier transform of the kernel. For this purpose, let \(\psi(v)\) be a function defined and continuous in \(E^n\) with isolated zeros such that

\[
\lim_{q \to \infty} \frac{k^\sim(v; q) - 1}{\varphi(q)} = \psi(v)
\]

for all \(v \in E^n\), where \(\varphi\) is defined as in Definition 1.1. Furthermore, suppose there exists a family \(\{v_e\}\) of uniformly bounded measures such that the representation

\[
k^\sim(v; q) - 1 = \psi(v) v_e^\sim(v)
\]

holds for every \(v \in E^n\) and \(q > 0\). Sometimes, we moreover need the absolute continuity of these measures, i.e. that there exists a family of integrable functions \(h(x; q)\) uniformly bounded in norm for \(q > 0\) such that

\[
k^\sim(v; q) - 1 = \psi(v) h^\sim(v; q)
\]

for all \(v \in E^n\) and \(q > 0\). These conditions suffice to solve the saturation problem for the process (1.1). Explicitly we have the following theorem concerning the saturation of (1.1) in \(L_1(E^n)\).

**Theorem 3.1:** Let \(f \in L_1(E^n)\) and the kernel \(k(x; q)\) of (1.1) satisfy (3.1).

a) If there exists a function \(g \in L_1(E^n)\) such that

\[
\lim_{q \to \infty} \left\| \frac{1}{\varphi(q)} \{K(f; \cdot; q) - f(\cdot)\} - g(\cdot) \right\|_1 = 0,
\]

then \(\psi(v)f^\sim(v) = g^\sim(v)\) for all \(v \in E^n\). In particular, \(\|K(f; \cdot; q) - f(\cdot)\|_1 = o(\varphi(q))\) as \(q \to \infty\) implies \(f(x) = 0\) a.e.

b) If

\[
\|K(f; \cdot; q) - f(\cdot)\|_1 = O(\varphi(q)) \quad (q \to \infty),
\]

then there exists a bounded measure \(\mu\) such that for all \(v \in E^n\)

\[
\psi(v)f^\sim(v) = \mu^\sim(v).
\]

c) If the kernel \(k(x; q)\) in addition satisfies (3.2) then the representation (3.5) with bounded measure \(\mu\) in turn implies the approximation (3.4).
Proof: a): By the convolution theorem the Fourier transform of the singular integral \( K(f; x; \varrho) \) is given by \( k^\wedge(v; \varrho)f^\wedge(v) \). Thus we have

\[
\left| \frac{1}{\varrho} \{ k^\wedge(v; \varrho) - 1 \} f^\wedge(v) - g^\wedge(v) \right| < \frac{1}{\varrho} \| K(f; \cdot; \varrho) - f(\cdot) \|_1 \cdot
\]

The hypothesis together with the property (3.1) therefore gives \( \varphi(v) f^\wedge(v) = g^\wedge(v) \) for all \( v \in \mathbb{E}^n \). In particular, if \( g(x) = 0 \) a.e., then the properties of the function \( \varphi \) and the continuity of \( f^\wedge \) imply \( f^\wedge(v) = 0 \) from which the result follows by the uniqueness theorem for Fourier transforms.

b): First of all we obtain from (3.4) as in part a)

\[
(3.6) \quad \left| \frac{1}{\varrho} \{ k^\wedge(v; \varrho) - 1 \} f^\wedge(v) \right| < \frac{1}{\varrho} \| K(f; \cdot; \varrho) - f(\cdot) \|_1 < M
\]

uniformly for large \( \varrho \). Now let us consider the integral

\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right\} e^{i<x, v>} \{ k^\wedge(v; \varrho) - 1 \} f^\wedge(v) dv =
\]

\[
= \left( \frac{2}{\pi R} \right)^n \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \frac{\sin^2(R/2)(x_j - u_j)}{(x_j - u_j)^2} \right\} \{ K(f; u; \varrho) - f(u) \} du =
\]

\[
= \sigma(\{ K(f; \cdot; \varrho) - f(\cdot) \}; x; R),
\]

where we have again used the Parseval formula (1.10). It follows that

\[
\left\| \frac{1}{\varrho} \sigma(\{ K(f; \cdot; \varrho) - f(\cdot) \}; \cdot; R) \right\|_1 < \frac{1}{\varrho} \| K(f; \cdot; \varrho) - f(\cdot) \|_1 < M
\]

and therefore

\[
(3.7) \quad \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right\} e^{i<x, v>} \cdot \frac{1}{\varrho} \{ k^\wedge(v; \varrho) - 1 \} f^\wedge(v) dv \right|_1 = O(1)
\]

uniformly for \( R > 0 \) and large \( \varrho \).

Our next aim is to interchange the limit \( \varrho \rightarrow \infty \) and the two integrations in (3.7). It follows by (3.1), (3.6) and Lebesgue’s dominated convergence theorem that

\[
\lim_{\varrho \rightarrow \infty} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right\} e^{i<x, v>} \frac{1}{\varrho} \{ k^\wedge(v; \varrho) - 1 \} f^\wedge(v) dv =
\]

\[
= \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right\} e^{i<x, v>} \varphi(v) f^\wedge(v) dv.
\]
If we now apply Fatou's lemma we obtain from (3.7)

\[
\left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} \left\{ \prod_{j=1}^n \left( 1 - \frac{|\nu_j|}{R} \right) \right\} e^{i\cdot \cdot \cdot} \psi(v) f^- (v) \, dv \right\|_1 < \\
< \liminf_{q \to -\infty} \left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} \left\{ \prod_{j=1}^n \left( 1 - \frac{|\nu_j|}{R} \right) \right\} e^{i\cdot \cdot \cdot} \psi(v) \, dv \right\|_1 
\]

\[
\cdot \frac{1}{q(q)} \{k^- (v; e) - 1\} f^- (v) \, dv \right\|_1 = O(1)
\]

uniformly for \( R > 0 \). Thus the continuous and by (3.6) and (3.1) bounded function \( \psi(v) f^- (v) \) satisfies Cramér's criterion (2.4) so that the representation (3.5) follows from Theorem 2.1.

c): Parseval's formulae and Lebesgue's dominated convergence theorem together with the properties (3.2) and (3.5) give

\[
\left\| \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(\cdot - u)^2} \frac{1}{q(q)} \{K(f; u; e) - f(u)\} \, du \right\|_1 = \\
= \left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{i\cdot \cdot \cdot} \psi(v) e^{i\cdot \cdot \cdot} \frac{1}{q(q)} \{k^- (v; e) - 1\} f^- (v) \, dv \right\|_1 = \\
= \left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{i\cdot \cdot \cdot} \psi(v) \, dv \right\|_1 = \\
= \left\| \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(\cdot - u)^2} d(\mu \circ \nu_e) (u) \right\|_1 < \int_{E^n} d\mu \int_{E^n} d\nu_e = O(1)
\]

uniformly for all positive \( t \) and \( q \). Here \( (\mu \circ \nu_v) \) denotes the convolution of the bounded measures \( \mu \) and \( \nu_v \) the Fourier-Stieltjes transform of which is given by the product \( \mu^- (v) \nu_v^- (v) \) (see [3, p. 325], [24, p. 109]). Since the Gauss-Weierstrass integral of the \( L_1 \)-function \( \{K(f; x; e) - f(x)\} \) converges a.e. to this function as \( t \to 0^+ \) we obtain by Fatou's lemma

\[
\left\| \frac{1}{q(q)} \{K(f; \cdot ; e) - f(\cdot )\} \right\|_1 < \liminf_{t \to 0^+} \left\| \frac{1}{(4\pi t)^{n/2}} \int_{E^n} e^{-(1/4t)(\cdot - u)^2} \cdot \frac{1}{q(q)} \{K(f; u; e) - f(u)\} \, du \right\|_1 = O(1)
\]

uniformly for all \( q > 0 \), which completes the proof of Theorem 3.1.

We observe that in part b), the inverse part of the theorem, only the condition (3.1) must be satisfied by the kernel whereas in part c), the direct part, the additional assumption (3.2) was needed. An analogous remark holds for the next theorem, which treats the case \( 1 < p < 2 \).

**Theorem 3.2:** Let \( f \in L_p(E^n), 1 < p < 2 \), and the kernel \( k(x; e) \) of (1.1) satisfy (3.1).
a) If there exists a function \( g \in L_p(E^n) \) such that
\[
\lim_{\varrho \to \infty} \left\| \frac{1}{\varphi(\varrho)} \left( K(f; \cdot; \varrho) - f(\cdot) \right) - g(\cdot) \right\|_p = 0
\]
then \( \psi(v)f(v) = g^\wedge(v) \) a.e. In particular, \( \|K(f; \cdot; \varrho) - f(\cdot)\|_p = o(\varphi(\varrho)) \) as \( \varrho \to \infty \) implies \( f(x) = 0 \) a.e.

b) If
\[
\|K(f; \cdot; \varrho) - f(\cdot)\|_p = O(\varphi(\varrho)) \quad (\varrho \to \infty),
\]
then there exists a function \( g \in L_p(E^n) \) such that almost everywhere
\[
\psi(v)f(v) = g^\wedge(v).
\]

c) If the kernel \( k(x; \varrho) \) in addition satisfies (3.3), then the representation (3.9) with \( g \in L_p(E^n) \) in turn implies the approximation (3.8).

Proof: a) As in the case \( p = 1 \), the convolution theorem states that the Fourier transform of \( K(f; x; \varrho) \) is given by \( k^\wedge(v; \varrho)f^\wedge(v) \) a.e. Thus we obtain by Titchmarsh's inequality and the hypothesis \( (p - 1 + q - 1 = 1) \)
\[
\left\| \frac{1}{\varphi(\varrho)} \{k^\wedge(\cdot; \varrho) - 1\}f^\wedge(\cdot) - g^\wedge(\cdot) \right\|_q < \left\| \frac{1}{\varphi(\varrho)} \{K(f; \cdot; \varrho) - f(\cdot)\} - g(\cdot) \right\|_p = o(1)
\]
as \( \varrho \to \infty \). But this, by a well-known theorem, implies the existence of a sequence \( \{\varrho_j\} \) of positive numbers with \( \lim_{\varrho_j \to \infty} \varrho_j = \infty \) such that
\[
\lim_{\varrho_j \to \infty} \frac{1}{\varphi(\varrho_j)} [k^\wedge(v; \varrho_j) - 1]f^\wedge(v) = g^\wedge(v)
\]
almost everywhere. Property (3.1) therefore gives \( \psi(v)f^\wedge(v) = g^\wedge(v) \) a.e. The same result may also be obtained by using Fatou's lemma. Furthermore, if in particular \( g(x) = 0 \) a.e., we conclude \( f^\wedge(v) = 0 \) a.e. and from the uniqueness theorem for the \( L_p \)-Fourier transforms \( f(x) = 0 \) a.e.

b) As in part a), Titchmarsh's inequality and the assumption (3.8) give
\[
(3.10) \quad \left\| \frac{1}{\varphi(\varrho)} \{k^\wedge(\cdot; \varrho) - 1\}f^\wedge(\cdot) \right\|_q < \left\| \frac{1}{\varphi(\varrho)} \{K(f; \cdot; \varrho) - f(\cdot)\} \right\|_p < M
\]
uniformly for large \( \varrho \). Moreover, it follows by (3.1), (3.10) and Fatou's lemma that \( \psi(v)f^\wedge(v) \in L_q(E^n) \). Proceeding as in the proof of part b) of Theorem 3.1 it follows in exactly the same way
\[
\left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} \prod_{j=1}^n \left[ 1 - \frac{|y_j|}{R} \right] e^{i\varrho \cdot v} \frac{1}{\varphi(\varrho)} \{k^\wedge(v; \varrho) - 1\}f^\wedge(v) \, dv \right\|_p = O(1)
\]
uniformly for \( R > 0 \) and large \( \varrho \). Now \( (\varphi(\varrho))^{-1}[k^\wedge(v; \varrho) - 1]f^\wedge(v) \) define a family of functions in \( L_q(E^n) \), \( 2 < q < \infty \), the norms of which are by (3.10) uniformly bounded with respect to large \( \varrho \) and which by (3.1) converge
point-wise to \( \psi(v) \hat{f}(v) \in L_q(E^n) \) a.e. as \( q \to \infty \). By a well-known theorem
they therefore converge weakly*, too, so that we obtain in particular
\[
\lim_{e \to \infty} \int_{E^n} \left( \prod_{i=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right) e^{i<x,v>} \frac{1}{q(v)} \{ \hat{k}(v; e) - 1 \} \hat{f}(v) \, dv = \int_{E^n} \left( \prod_{i=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right) e^{i<x,v>} \psi(v) \hat{f}(v) \, dv.
\]

An application of Fatou's lemma finally yields
\[
\left\| \frac{1}{(2\pi)^{n/2}} \int_{E^n} \left( \prod_{i=1}^{n} \left[ 1 - \frac{|v_j|}{R} \right] \right) e^{i<x,v>} \psi(v) \hat{f}(v) \, dv \right\|_p = O(1)
\]
uniformly for \( R > 0 \). Thus \( \{ \psi(v) \hat{f}(v) \} \) belongs to \( L_q(E^n) \) and satisfies Cramér's criterion (2.5) so that the representation (3.9) follows from Theorem 2.2.

c): Instead of carrying over the proof of part c) of Theorem 3.1 to the present case \( 1 < p < 2 \), which is obviously possible, we will indicate another way of proof, also applicable to \( p = 1 \). Using (3.3), (3.9) and the convolution theorem we have for almost all \( v \)
\[
\left[ \frac{1}{q(v)} \{ K(f; \cdot; e) - f(\cdot) \} \right]^\wedge(v) = \frac{1}{q(v)} \{ \hat{k}(v; e) - 1 \} \hat{f}(v) = \hat{h}(v; e) g(v) = \\begin{multline*}
\left[ \frac{1}{(2\pi)^{n/2}} \int_{E^n} g(x-u) h(u; e) \, du \right]^\wedge(v).
\end{multline*}
\]

By the uniqueness theorem we therefore obtain for fixed \( \varrho > 0 \)
\[
(3.11) \quad \frac{1}{q(v)} \{ K(f; x; \varrho) - f(x) \} = \frac{1}{(2\pi)^{n/2}} \int_{E^n} g(x-u) h(u; \varrho) \, du
\]
almost everywhere so that by the assumptions on \( h(x; \varrho) \) the approximation (3.8) immediately follows by (1.3).

Remark 3.1: As in the last paragraph the case \( p = 2 \) gives rise to a considerable simplification of the proofs and assumptions. In particular, instead of (3.3) the kernel only needs to satisfy
\[
(3.12) \quad \left| \frac{\hat{k}(v; \varrho) - 1}{q(v)} \right| < M |\psi(v)|
\]
for all \( v \in E^n \) and \( \varrho > 0 \) in order that the statement of part c) of Theorem 3.2 follows. Contrary to (3.3) the condition (3.12) is very easy to verify in the applications (see the second paper of the series).

Corollary 3.1: If the kernel \( k(x; \varrho) \) of the singular integral (1.1) satisfies the condition (3.1) and either (3.2) or (3.3) depending upon whether \( p = 1 \) or \( 1 < p < 2 \), then the saturation order of the singular integral (1.1) exists for \( 1 < p < 2 \) and is given by the function \( \psi(v) \) of (3.1). The corresponding Favard class \( F \), characterized by the function \( \psi(v) \), is precisely the class of functions \( f \) in \( L_1(E^n) \) or \( L_p(E^n) \) for which the representations (3.5) and (3.9), respectively, hold.
Here we will not enter into a detailed discussion as to whether non-trivial functions belong to the saturation class for arbitrary continuous \( \psi(v) \). There are no difficulties in the applications. In general, we may only mention that according to a result of S. Bochner [3, p. 114] the saturation class is non-void, if, for instance, \( \psi \) has continuous second partial derivatives in some arbitrary finite interval.

In the proof of part c) of Theorem 3.2 we did not only deduce the required approximation (3.8) but also the representation (3.11). If, therefore, the kernel \( k(x; \varrho) \) of (1.1) satisfies (3.3) with functions \( h(x; \varrho) \) which have all further properties (1.2) of a kernel we will obtain by (1.3) (ii):

Theorem 3.3: Let \( f \in L_p(\mathbb{E}^n) \), \( 1 \leq p < 2 \), and the kernel \( k(x; \varrho) \) of (1.1) satisfy (3.3) where \( h(x; \varrho) \) again is a kernel.

\( \psi(v)f^\varphi(v) = g^\varphi(v) \)
(a.e. in case \( 1 < p \leq 2 \)), then

(3.13) \[ \lim_{\varrho \to \infty} \left\| \frac{1}{\psi(\varrho)} \{ K(f; \cdot \varrho; \varrho) - f(\cdot \varrho) \} - g(\cdot \varrho) \right\| = 0. \]

b) For \( 1 < p < 2 \) the Favard class of the approximation process (1.1) is precisely described by the strong \( L_p \)-convergence of the functions \( (\psi(\varrho))^{-1} \{ K(f; x; \varrho) - f(x) \} \) as \( \varrho \to \infty \). For \( p = 1 \) the strong convergence of the latter functions characterizes those functions of the Favard class of (1.1) for which the measure in the representation (3.5) is absolutely continuous.

For the proof we need only mention that part b) is on the one hand a consequence of Corollary 3.1 and part a) and on the other hand of part a) of Theorem 3.1 and 3.2. Note that the assumptions of the theorem always imply the condition (3.1).

Part b) of the last theorem may be considered as a first contribution to the problem of finding characterizations of the Favard class of the singular integral (1.1) other than those given by Corollary 3.1. Thus we now suppose that the order and class of saturation of (1.1) are already known, and we are interested in further information about these classes. In this respect the result of part b) of Theorem 3.3 is unsatisfactory not only on account of the assumptions but also because of the restriction \( 1 < p < 2 \). But if we replace the strong convergence by the weak* convergence we may obtain characterizations true for all \( 1 < p < 2 \). They as well as Theorem 3.3 will at the same time give some new contributions to the one-dimensional theory, too. In this special situation the assumptions of these theorems are very easy to verify (see also the corresponding remark on Picard's singular integral in section 9).

Theorem 3.4: Let \( f \in L_p(\mathbb{E}^n) \), \( 1 \leq p < 2 \), and the kernel \( k(x; \varrho) \) of (1.1) satisfy the condition (3.1). Suppose further that the saturation order of (1.1) is given by \( \psi(\varrho) \) and that the corresponding Favard classes are characterized
by \( \psi(v) \) and the respective representations (3.5) or (3.9). Then these Favard classes consist precisely of those functions \( f \) for which the set

\[
(\psi(q))^{-1}\{K(f; x; q) - f(x)\}
\]

converges weakly* as \( q \to \infty \).

**Proof**: To begin with the case \( p = 1 \) let \( f \) belong to the Favard class of (1.1). Then on the one hand there exists a bounded measure \( \mu \) such that \( \psi(v)f'(v) = \mu'(v) \) for all \( v \in \mathbb{E}^n \) and on the other hand the functions \( (\psi(q))^{-1}\{K(f; x; q) - f(x)\} \) belonging to \( L_1(\mathbb{E}^n) \) define a set of absolutely continuous and bounded measures for which the total variations

\[
|||\psi(q)|||^{-1}\{K(f; q) - f(\cdot)\}|_1
\]

are by the definition of saturation uniformly bounded with respect to large \( q \). Therefore there exists a weakly* convergent subsequence [5, p. 16], i.e. there exist a sequence of positive numbers \( \{q_j\} \) with \( \lim q_j = \infty \) and a bounded measure \( \nu \) such that

\[
(3.14) \quad \lim_{j \to \infty} \int_{\mathbb{E}^n} h(u) \frac{1}{\psi(q_j)} \{K(f; u; q_j) - f(u)\} \, du = \int_{\mathbb{E}^n} h(u) \, d\nu
\]

for every continuous function \( h \) vanishing at infinity. Our next aim is to prove that the measures \( \mu \) and \( \nu \) must coincide. For this purpose, if we apply (3.14) to the special functions \( \exp \{-tu^2 - i <x,u>\} \) with arbitrary \( x \in \mathbb{E}^n \) and \( t > 0 \), then it follows by Parseval’s formula, (3.1), (3.6) and Lebesgue’s dominated convergence theorem that

\[
\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{E}^n} e^{-(1/4t)(x-u)^2} \psi(u) \, du = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} e^{-tuv} e^{-i<x,u>} \, d\nu(u) =
\]

\[
= \lim_{j \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} e^{-tuv} e^{-i<x,u>} \frac{1}{\psi(q_j)} \{K(f; u; q_j) - f(u)\} \, du =
\]

\[
= \lim_{j \to \infty} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{E}^n} e^{-(1/4t)(x-u)^2} \frac{1}{\psi(q_j)} \{k'(u; q_j) - 1\} f'(u) \, du =
\]

\[
= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{E}^n} e^{-(1/4t)(x-u)^2} \psi(u) f'(u) \, du = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{E}^n} e^{-(1/4t)(x-u)^2} \mu'(u) \, du.
\]

Thus the Gauss-Weierstrass integral of the bounded and continuous function \( \mu' - \nu' \) vanishes for all \( t > 0 \) which by Lemma 1.1 gives \( \mu'(x) = \nu'(x) \) for all \( x \in \mathbb{E}^n \) and \( \mu = \nu \) by the uniqueness theorem for Fourier-Stieltjes transforms [5, p. 24]. But \( \mu \) being the only weak* limit of the functions \( (\psi(q))^{-1}\{K(f; x; q) - f(x)\} \) implies the weak* convergence of the functions themselves, i.e. (3.14) holds in general for \( q \to \infty \).

Conversely, let us suppose that the functions \( (\psi(q))^{-1}\{K(f; x; q) - f(x)\} \) converge weakly* to some bounded measure \( \mu \) as \( q \to \infty \), i.e. there exists a bounded measure \( \mu \) such that

\[
(3.15) \quad \lim_{q \to \infty} \int_{\mathbb{E}^n} h(u) \frac{K(f; u; q) - f(u)}{\psi(q)} \, du = \int_{\mathbb{E}^n} h(u) \, d\mu
\]
for every continuous $h$ vanishing at infinity. But in view of the uniform boundedness principle [21, p. 205] and the Riesz representation theorem [21, p. 397] this implies that the norms $\| (\varphi(q))^{-1} K(f; \cdot; q) - f(\cdot) \|_1$ are uniformly bounded as $q \to \infty$, which proves the theorem for $p = 1$.

The case $1 < p < 2$ is treated in a very similar fashion. Here weak* convergence of the functions $(\varphi(q))^{-1} K(f; x; q) - f(x)$ means that there exists a function $g \in L_p(\mathbb{E}^n)$ such that

$$
\lim_{q \to \infty} \int_{\mathbb{E}^n} h(u) \frac{K(f; u; q) - f(u)}{\varphi(q)} \, du = \int_{\mathbb{E}^n} h(u)g(u) \, du
$$

for every $h \in L_q(\mathbb{E}^n)$, $p^{-1} + q^{-1} = 1$.

**Remark 3.2:** If the convergence in (3.1) is dominated in every finite interval which is the case in all of the applications to be considered, we may replace the second half of the preceding proof by the following more simple argument which uses only the methods of proof developed in this paper: Regarding e.g. $p = 1,$ let (3.15) be satisfied. Then we obtain in particular

$$
\left( \frac{2}{\pi R} \right)^n \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \frac{\sin^2 (R/2)(x_j - u_j)}{(x_j - u_j)^2} \right\} \, d\mu(u) =
$$

$$
= \lim_{q \to \infty} \left( \frac{2}{\pi R} \right)^n \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \frac{\sin^2 (R/2)(x_j - u_j)}{(x_j - u_j)^2} \right\} \frac{K(f; u; q) - f(u)}{\varphi(q)} \, du =
$$

$$
= \lim_{q \to \infty} \frac{1}{(2\pi)^n/2} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \right\} \frac{1 - |\varphi_j|}{R} \left\{ e^{i<x, v>} \frac{k^*(v; q) - 1}{\varphi(q)} f^*(v) \right\} \, dv =
$$

$$
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \right\} \frac{1 - |\varphi_j|}{R} \left\{ e^{i<x, v>} \varphi(v) f^*(v) \right\} \, dv.
$$

Here we used Parseval’s formula and the modified condition (3.1) which permits the application of Lebesgue’s dominated convergence theorem. Now

$$
\left\| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{E}^n} \left\{ \prod_{j=1}^{n} \right\} \frac{1 - |\varphi_j|}{R} \left\{ e^{i<x, v>} \varphi(v) f^*(v) \right\} \, dv \right\|_{L_1} < \int_{\mathbb{E}^n} |d\mu|,
$$

uniformly for all $R > 0$, from which the representation $\varphi(v) f^*(v) = \mu^*(v)$ follows by Theorem 2.1 and the other direction of the present theorem. The case $1 < p < 2$ is treated in a very similar fashion.

**Corollary 3.2:** Let $f \in L_p(\mathbb{E}^n), 1 < p < 2$, and the kernel $k(x; q)$ of (1.1) satisfy (3.3) where $h(x; q)$ again is a kernel. Then the functions $(\varphi(q))^{-1} K(f; x; q) - f(x)$ converge strongly as $q \to \infty$ if and only if they are weakly* convergent.

For the connections with semi-group theory we refer to section 4 in Note II of this series.
REFERENCES


