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Functional equations from generating functions: a novel approach to deriving identities for the Bernstein basis functions

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Abstract

The main aim of this paper is to provide a novel approach to deriving identities for the Bernstein polynomials using functional equations. We derive various functional equations and differential equations using generating functions. Applying these equations, we give new proofs for some standard identities for the Bernstein basis functions, including formulas for sums, alternating sums, recursion, subdivision, degree raising, differentiation and a formula for the monomials in terms of the Bernstein basis functions. We also derive many new identities for the Bernstein basis functions based on this approach. Moreover, by applying the Laplace transform to the generating functions for the Bernstein basis functions, we obtain some interesting series representations for the Bernstein basis functions.

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1 Introduction

The Bernstein polynomials have many applications: in approximations of functions, in statistics, in numerical analysis, in p-adic analysis and in the solution of differential equations. It is also well known that in Computer Aided Geometric Design (CAGD) polynomials are often expressed in terms of the Bernstein basis functions. These polynomials are called *Bezier curves and surfaces*. Convexity and its generalization play an important role in the theory of Bernstein polynomials. Therefore, a fixed point theorem and its applications are also very important in the theory of Bezier curves and surfaces.

Many of the known identities for the Bernstein basis functions are currently derived in an *ad hoc* fashion, using either the binomial theorem, the binomial distribution, tricky algebraic manipulations or blossoming. The main purpose of this work is to construct novel functional equations for the Bernstein polynomials. Using these functional equations and the Laplace transform, we develop a novel approach both to standard and to new identities for the Bernstein polynomials. Thus these polynomial identities are just the residue of a much more powerful system of functional equations.

The remainder of this study is organized as follows. We find several *functional equations* and *differential equations* for the Bernstein basis functions using generating functions. From these equations, many properties of the Bernstein basis functions are then derived. For instance, we give a new proof of the recursive definition of the Bernstein basis functions as well as a novel derivation for the two-term formula for the derivatives of the nth degree Bernstein basis functions. Using functional equations, we give new derivations for the sum and alternating sum of the Bernstein basis functions and a formula for the monomials in terms of the Bernstein basis functions. We also derive identities corresponding to the degree elevation and subdivision formulas for Bezier curves. We prove many new identities for the Bernstein basis functions. Finally, we give some applications of the Laplace transform to the generating functions for the Bernstein basis functions. We obtain interesting series representations for the Bernstein basis functions. We also give some remarks and observations related to the Fourier transform and complex generating functions for the Bernstein basis functions.

2 Generating functions

The Bernstein polynomials and related polynomials have been studied and defined in many different ways, for example, by q-series, complex functions, p-adic Volkenborn integrals and many algorithms. In this section, we provide novel generating functions for the Bernstein basis functions.

The Bernstein basis functions $B_k^n(x)$ are defined as follows.

Definition 2.1

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$k = 0, 1, \dots, n$, *cf.* [1–16].

Generating functions for the Bernstein basis functions can be defined as follows.

Definition 2.2

$$f_{\mathbb{B},k}(x,t) = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}. \tag{2}$$

Note that there is one generating function for each value of k.

Theorem 2.3

$$f_{\mathbb{B},k}(x,t) = \frac{t^k x^k e^{(1-x)t}}{k!}. \tag{3}$$

Proof By substituting (1) into the right-hand side of (2), we get

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\binom{n}{k} x^k (1-x)^{n-k} \right) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!} = \frac{(xt)^k}{k!}\sum_{n=k}^{\infty}(1-x)^{n-k}\frac{t^{n-k}}{(n-k)!}.$$

The right-hand side of the above equation is a Taylor series for $e^{(1-x)t}$, thus we arrive at the desired result. □

We give some alternative forms of the generating functions in (2) as follows:

$$\sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!}e^{xt} = \frac{t^k x^k e^t}{k!}, \tag{4}$$

$$\sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!}e^{-t} = \frac{t^k x^k e^{-xt}}{k!}, \tag{5}$$

and

$$\sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!}e^{(x-1)t} = \frac{t^k x^k}{k!}. \tag{6}$$

By using the above alternative forms, we derive some new identities for the Bernstein basis functions.

Remark 2.4 If we replace x by $\frac{x-a}{b-a}$ in (3), where $a < b$, then

$$\frac{t^k(\frac{x-a}{b-a})^k e^{(\frac{b-x}{b-a})t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x,a,b)\frac{t^n}{n!},$$

where $B_k^n(x,a,b)$ denotes the generalized Bernstein basis function defined by

$$B_k^n(x,a,b) = \binom{n}{k}\frac{(x-a)^k(b-x)^{n-k}}{(b-a)^m},$$

cf. [4].

A *Bernstein polynomial* $\mathcal{P}(x)$ is a polynomial represented in the Bernstein basis functions

$$\mathcal{P}(x) = \sum_{k=0}^{n} c_k^n B_k^n(x), \tag{7}$$

cf. [4].

Recently, Simsek [15, 16], Simsek *et al.* [14] and Acikgoz *et al.* [1] have also studied on the generating function for Bernstein basis type functions.

There are many applications of the Bernstein-type polynomials which are related to the theory of the Bezier curves, approximation theory, solving high even-order differential equations by using the Bernstein Galerkin method.

In approximation theory, the Weierstrass approximation theorem is very useful. The Bernstein basis functions are also related to this theorem. We give the following remarks for this theorem.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The sequence of the Bernstein polynomials of f is given by

$$\mathfrak{B}_n(x,f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_k^n(x),$$
$$\left|\mathfrak{B}_n(x,f) - f(x)\right| \to 0$$

as $n \to \infty$, which was proved by many methods, for example, by the Weierstrass approximation theorem, by the probabilistic interpretation to $\mathfrak{B}_n(x,f)$ (*cf.* [4, 11, 13]).

3 Identities for the Bernstein basis functions

In this section, we use the generating functions for the Bernstein basis functions to derive a family of functional equations. Using these equations, we derive a collection of identities for the Bernstein basis functions.

3.1 Sums and alternating sums

From (3), we get the following functional equations:

$$\sum_{k=0}^{\infty} f_{\mathbb{B},k}(x,t) = e^t \tag{8}$$

and

$$\sum_{k=0}^{\infty} (-1)^k f_{\mathbb{B},k}(x,t) = e^{(1-2x)t}. \tag{9}$$

Theorem 3.1 (Sum of the Bernstein basis functions)

$$\sum_{k=0}^{n} B_k^n(x) = 1.$$

Proof From (8), one finds that

$$\sum_{k=0}^{\infty} f_{\mathbb{B},k}(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}. \tag{10}$$

By combining (2) and (10), we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k^n(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 3.2 (Alternating sum of the Bernstein basis functions)

$$\sum_{k=0}^{n}(-1)^k B_k^n(x) = (1-2x)^n.$$

Proof By combining (9) and (10), we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k B_k^n(x)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{(1-2x)^n t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 3.3 Goldman [6], [5, Chapter 5, pp.299-306] derived the formula for the alternating sum of the Bernstein basis functions algebraically.

3.2 Subdivision

From (3), we have the following functional equation:

$$f_{\mathbb{B},j}(xy,t) = f_{\mathbb{B},j}(x,ty)e^{t(1-y)}. \tag{11}$$

From this functional equation, we get the following identity which is the basis for subdivision of Bezier curves, *cf.* [4–6, 15].

Theorem 3.4

$$B_j^n(xy) = \sum_{k=j}^{n} B_j^k(x)B_k^n(y).$$

Proof By equations (3) and (11),

$$\sum_{n=j}^{\infty} B_j^n(xy)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} B_j^n(x)y^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(1-y)^n t^n}{n!}\right).$$

Therefore

$$\sum_{n=j}^{\infty} B_j^n(xy)\frac{t^n}{n!} = \sum_{n=j}^{\infty}\left(\sum_{k=j}^{n} B_j^k(x)\frac{y^k(1-y)^{n-k}}{k!(n-k)!}\right)t^n.$$

Substituting (1) into the above equation, we arrive at the desired result. □

Remark 3.5 Theorem 3.4 is a bit tricky to prove with algebraic manipulations. Goldman [6], [5, Chapter 5, pp.299-306] proved this identity algebraically. He also proved the following related subdivision identities:

$$B_j^n\big((1-y)x+y\big) = \sum_{k=0}^{j} B_{j-k}^{n-k}(x)B_k^n(y)$$

and

$$B_j^n\big((1-y)x+yz\big)=\sum_{k=0}^{n}\left(\sum_{p+q=j}B_p^{n-k}(x)B_q^k(z)\right)B_k^n(y).$$

For additional identities, see [6], [5, Chapter 5, pp.299-306].

3.3 Formula for the monomials in terms of the Bernstein basis functions

Multiplying both sides of (3) by $\binom{k}{l}$, we get

$$\binom{k}{l}\frac{(xt)^k}{k!}e^{t(1-x)}=\binom{k}{l}\sum_{n=0}^{\infty}B_k^n(x)\frac{t^n}{n!}.$$

Summing both sides of the above equation over k, we obtain the following functional equation, which is used to derive a formula for the monomials in terms of the Bernstein basis functions:

$$\frac{x^l t^l}{l!}e^t=\sum_{k=0}^{\infty}\binom{k}{l}f_{\mathbb{B},k}(x,t). \tag{12}$$

Theorem 3.6

$$\binom{n}{l}x^l=\sum_{l=0}^{k}\binom{k}{l}B_k^n(x).$$

Proof Combining (2) and (12), we get

$$\frac{x^l}{l!}\sum_{n=0}^{\infty}\frac{t^{n+l}}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{k}{l}B_k^n(x)\right)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\left(\binom{n}{l}x^l\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{k}{l}B_k^n(x)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

3.4 Differentiating the Bernstein basis functions

In this section we give higher-order derivatives of the Bernstein basis functions. We begin by observing that

$$f_{\mathbb{B},k}(x,t)=g_k(t,x)h(t,x), \tag{13}$$

where

$$g_k(t,x)=\frac{t^k x^k}{k!}$$

and

$$h(t,x) = e^{(1-x)t}.$$

Using Leibnitz's formula for the lth derivative, with respect to x, we obtain the following *higher-order partial differential equation*:

$$\frac{\partial^l f_{\mathbb{B},k}(x,t)}{\partial x^l} = \sum_{j=0}^{l} \binom{l}{j} \left(\frac{\partial^j g_k(t,x)}{\partial x^j} \right) \left(\frac{\partial^{l-j} h(t,x)}{\partial x^{l-j}} \right). \tag{14}$$

From this equation, we arrive at the following theorem.

Theorem 3.7

$$\frac{\partial^l f_{\mathbb{B},k}(x,t)}{\partial x^l} = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} t^l f_{\mathbb{B},k-j}(x,t). \tag{15}$$

Proof Formula (15) follows immediately from (14). □

Applying Theorem 3.7, we obtain a new derivation for the higher-order derivatives of the Bernstein basis functions.

Theorem 3.8

$$\frac{d^l B_k^n(x)}{dx^l} = \frac{n!}{(n-l)!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} B_{k-j}^{n-l}(x). \tag{16}$$

Proof By substituting the right-hand side of (2) into (15), we get

$$\sum_{n=0}^{\infty} \left(\frac{d^l B_k^n(x)}{dx^l} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} B_{k-j}^n(x) \right) \frac{t^{n+l}}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \left(\frac{d^l B_k^n(x)}{dx^l} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} \binom{n}{l} l! B_{k-j}^{n-l}(x) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Substituting $l = 1$ into (16), we arrive at the following standard corollary.

Corollary 3.9

$$\frac{d}{dx} B_k^n(x) = n\left(B_{k-1}^{n-1}(x) - B_k^{n-1}(x) \right),$$

cf. [1–16].

3.5 Recurrence relation

In the previous section we computed the derivative of (13) with respect to x to derive a derivative formula for the Bernstein basis functions. In this section we are going to differentiate (13) with respect to t to derive a recurrence relation for the Bernstein basis functions.

Using Leibnitz's formula for the νth derivative, with respect to t, we obtain the following *higher-order partial differential equation*:

$$\frac{\partial^{\nu} f_{\mathbb{B},k}(x,t)}{\partial t^{\nu}} = \sum_{j=0}^{\nu} \binom{\nu}{j} \left(\frac{\partial^{j} g_k(t,x)}{\partial t^{j}}\right)\left(\frac{\partial^{\nu-j} h(t,x)}{\partial t^{\nu-j}}\right). \tag{17}$$

From the above equation, we have the following theorem.

Theorem 3.10

$$\frac{\partial^{\nu} f_{\mathbb{B},k}(x,t)}{\partial t^{\nu}} = \sum_{j=0}^{\nu} B_j^{\nu}(x) f_{\mathbb{B},k-j}(x,t). \tag{18}$$

Proof Formula (18) follows immediately from (17). □

Using definitions (3) and (1) in Theorem 3.10, we obtain a recurrence relation for the Bernstein basis functions.

Theorem 3.11

$$B_k^n(x) = \sum_{j=0}^{\nu} B_j^{\nu}(x) B_{k-j}^{n-\nu}(x). \tag{19}$$

Proof By substituting the right-hand side of (2) into (18), we get

$$\frac{\partial^{\nu}}{\partial t^{\nu}}\left(\sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\nu} B_j^{\nu}(x) B_{k-j}^{n}(x)\right)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=\nu}^{\infty} B_k^n(x)\frac{t^{n-\nu}}{(n-\nu)!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\nu} B_j^{\nu}(x) B_{k-j}^{n}(x)\right)\frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=\nu}^{\infty} B_k^n(x)\frac{t^{n-\nu}}{(n-\nu)!} = \sum_{n=\nu}^{\infty}\left(\sum_{j=0}^{\nu} B_j^{\nu}(x) B_{k-j}^{n-\nu}(x)\right)\frac{t^{n-\nu}}{(n-\nu)!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

We also computed the derivative of (3) with respect to t to derive the following higher-order partial differential equation:

$$\frac{\partial^{\nu} f_{\mathbb{B},k}(x,t)}{\partial t^{\nu}} = \sum_{j=0}^{\nu} \binom{\nu}{j} B_{\nu-j}^{\nu}(x) f_{\mathbb{B},k-\nu+j}(x,t).$$

By using the above equation, we derive another recurrence relation for the Bernstein basis functions as follows:

$$B_k^n(x) = \sum_{j=0}^{\nu} \binom{\nu}{j} B_{\nu-j}^{\nu}(x) B_{k-\nu+j}^{n-\nu}(x).$$

Remark 3.12 Setting $\nu = 1$ in (19), one obtains the standard recurrence

$$B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x).$$

3.6 Degree raising

In this section we present a functional equation which we apply to provide a new proof of the degree raising formula for the Bernstein polynomials.

From (3), we obtain the following *functional equation*:

$$(xt)^d f_{\mathbb{B},k}(x,t) = \frac{(k+d)!}{k!} f_{\mathbb{B},k+d}(x,t).$$

Therefore

$$x^d B_k^n(x) = \frac{n!(k+d)!}{k!(n+d)!} B_{k+d}^{n+d}(x). \tag{20}$$

Substituting $d = 1$ into the above equation, we have

$$xB_k^n(x) = \frac{k+1}{n+1} B_{k+1}^{n+1}(x). \tag{21}$$

The above relation can also be proved by (1), *cf.* [4–6].

From (3), we also get the following functional equation:

$$(xt)^{-d} f_{\mathbb{B},k}(x,t) = \frac{(k-d)!}{k!} f_{\mathbb{B},k-d}(x,t).$$

Therefore

$$(1-x)^d B_k^n(x) = \frac{n!(n+d-k)!}{(n+d)!(n-k)!} B_k^{n+d}(x).$$

Substituting $d = 1$, we have

$$(1-x)B_k^n(x) = \frac{(n+1-k)}{(n+1)} B_k^{n+1}(x). \tag{22}$$

Adding (21) and (22), we get the standard degree elevation formula

$$B_k^n(x) = \frac{1}{n+1}\bigl((k+1)B_{k+1}^{n+1}(x) + (n+1-k)B_k^{n+1}(x)\bigr).$$

4 New identities

In this section, using alternative forms of the generating functions, functional equations and the Laplace transform, we give many new identities for the Bernstein basis functions.

Using (3), we obtain the following *functional equations*:

$$f_{\mathbb{B},k_1}(x,t)f_{\mathbb{B},k_2}(x,t) = \binom{k_1+k_2}{k_1}\frac{1}{2^{k_1+k_2}}f_{\mathbb{B},k_1+k_2}(x,2t) \tag{23}$$

and

$$f_{\mathbb{B},k}(x,t)f_{\mathbb{B},k}(y,-t) = \frac{(-xyt^2)^k}{(k!)^2}e^{t(y-x)}. \tag{24}$$

Theorem 4.1

$$B^n_{k_1+k_2}(x) = \frac{2^{k_1+k_2-n}k_1!k_2!}{(k_1+k_2)!}\sum_{j=0}^{n}\binom{n}{j}B^j_{k_1}(x)B^{n-j}_{k_2}(x).$$

Proof By substituting the right-hand side of (2) into (23), we get

$$\sum_{n=0}^{\infty}B^n_{k_1}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}B^n_{k_2}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}B^n_{k_1+k_2}(x)\frac{2^{n-k_1-k_2}(k_1+k_2)!t^n}{n!k_1!k_2!}.$$

Therefore

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}B^j_{k_1}(x)B^{n-j}_{k_2}(x)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}B^n_{k_1+k_2}(x)\frac{2^{n-k_1-k_2}(k_1+k_2)!t^n}{n!k_1!k_2!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.2

$$(-xy)^k(y-x)^{n-2k} = \frac{(k!)^2}{(n)_{2k}}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}B^j_k(x)B^{n-j}_k(y),$$

where

$$(n)_{2k} = n(n-1)\cdots(n-2k+1),$$

and $(n)_0 = 1$.

Proof Combining (2) and (24), we get

$$\left(\sum_{n=0}^{\infty}B^n_k(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^nB^n_k(y)\frac{t^n}{n!}\right) = \frac{(-xy)^k}{(k!)^2}\sum_{n=0}^{\infty}\frac{(y-x)^nt^{n+2k}}{n!}.$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}B_k^j(x)B_k^{n-j}(y)\right)\frac{t^n}{n!} = \frac{(-xy)^k}{(k!)^2}\sum_{n=0}^{\infty}(n)_{2k}(y-x)^{n-2k}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.3 *Let $x \neq 0$. For all positive integers k and n, we have*

$$\sum_{j=0}^{n-k}\binom{n}{j}x^{j-k}B_k^{n-j}(x) = \binom{n}{k}.$$

Proof By using (4), we obtain

$$\sum_{n=0}^{\infty}B_k^n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!} = \frac{t^k x^k}{k!}\sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k}\binom{k}{j}x^j B_k^{n-j}(x)\right)\frac{t^n}{n!} = x^k\sum_{n=k}^{\infty}\binom{n}{k}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.4 *For all positive integers k and n, we have*

$$\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}B_k^{n-j}(x) = (-1)^{n-k}\binom{n}{k}x^n.$$

Proof By using (5), we get

$$\sum_{n=0}^{\infty}B_k^n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} = \frac{t^k x^k}{k!}\sum_{n=0}^{\infty}(-1)^n x^n\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k}(-1)^j\binom{k}{j}B_k^{n-j}(x)\right)\frac{t^n}{n!} = \sum_{n=k}^{\infty}\binom{n}{k}(-1)^{n-k}x^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.5 *For all positive integers k and n, we have*

$$\sum_{j=0}^{n-k}(-1)^j\binom{n}{j}(1-x)^j B_k^{n-j}(x) = \begin{cases} x^k, & \textit{for } n = k, \\ 0, & \textit{for } n \neq k. \end{cases}$$

Proof Proof of Theorem 4.5 is the same as that of Theorem 4.3. So we omit it. □

5 Applications of the Laplace transform to the generating functions for the Bernstein basis functions

In this section, we give some applications of the Laplace transform to the generating functions for the Bernstein basis functions. We obtain interesting series representations for the Bernstein basis functions.

Theorem 5.1 *Let $x \neq 0$. For all the positive integer k, we have*

$$\sum_{n=0}^{\infty} x B_k^n(x) = 1.$$

Proof Integrate equation (5) (by parts) with respect to t from 0 to ∞, we have

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{1}{n!} \int_0^{\infty} t^n e^{-t}\, dt = \frac{x^k}{k!} \int_0^{\infty} t^k e^{-xt}\, dt. \tag{25}$$

If we appropriately use the case

$$x > 0$$

of the following Laplace transform of the function $f(t) = t^k$:

$$\mathcal{L}\left(t^k\right) = \frac{k!}{x^{k+1}},$$

on the both sides of (25), we find that

$$\sum_{n=0}^{\infty} B_k^n(x) = \frac{1}{x}.$$

From the above equation, we arrive at the desired result. □

Theorem 5.2 *Let $x \neq 0$. For all the positive integer k, we have*

$$\sum_{n=0}^{\infty} (-1)^n \frac{B_k^n(x)}{x^{n+1}} = (-1)^k x^k.$$

Proof Proof of Theorem 5.2 is the same as that of Theorem 5.1. That is, if we replace t by $-t$ in equation (4) and integrate by parts with respect to t from 0 to ∞ using the Laplace transform of the function $f(t) = t^n$, then we arrive at the desired result. □

6 Further remarks and observations

The Fourier series of the Bernstein polynomials has been studied, without generating functions, by Izumi *et al.* [8]. They investigated many properties of the Fejer mean of the Fourier series of these polynomials. The Fourier transform of the Bernstein polynomials has also been given, without generating functions, by Chui *et al.* [3]. By replacing t by it

in (4)-(6), one may give applications of the Fourier transform to the *complex* generating functions for the Bernstein basis functions.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper himself. The author read and approved the final manuscript.

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