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On the spectral analysis of quantum field Hamiltonians

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Abstract

We define C^*-algebras on a Fock space such that the Hamiltonians of quantum field models with positive mass are affiliated to them. We describe the quotient of such algebras with respect to the ideal of compact operators and deduce consequences in the spectral theory of these Hamiltonians: we compute their essential spectrum and give a systematic procedure for proving the Mourre estimate.

Keywords: C^*-algebras; Fock space; Quantum fields; Spectral analysis; Essential spectrum; Mourre estimate

Contents

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1. Introduction

This paper is motivated and related to the work on the spectral and scattering theory of quantum field models initiated in [33,36] and further developed in [23–25]. Our purpose is to show that abstract C^*-algebra techniques allow one to obtain in this context quite general results in a rather simple and systematic way which avoids ad-hoc and intricate constructions. We use ideas introduced in [11,13] in the context of the N-body problem and in a more general setting in [28]. The main point of this approach is that understanding the quotient of a C^*-algebra with respect to the ideal of compact operators[1] gives a lot of information relevant to the spectral analysis of the operators affiliated to the algebra. In [28,29] the relevant C^*-algebras are generated by a set of "elementary" Hamiltonians specific to a certain physical situation. The "real" Hamiltonians are then the self-adjoint operators affiliated to the algebra. We adopt here the same strategy.

In order to avoid any misunderstanding we emphasize that the topics considered in this paper are quite far from the theory of relativistic quantum fields. As in the references quoted above (and in the reference section) our results are relevant only for quantum field models with a spatial cutoff and living in a Fock space (hopefully this last restriction will be removed in the near future). On the other hand, our approach clearly covers many physically interesting models of the many-body theory, our focus being on the study of systems with an infinite number of degrees of freedom and without particle number conservation.

Our results on the spectral analysis of quantum field Hamiltonians (QFH) are consequences of the theorem stated below.[2] Let $\mathcal{H}$ be a complex Hilbert space and let $\Gamma(\mathcal{H})$ be the symmetric or antisymmetric Fock space over $\mathcal{H}$. The field operators $\phi(u)$ and the Segal operators $\Gamma(A)$ are defined as usual. If $U = (u_1, \dots, u_n)$ belongs to the Cartesian power $\mathcal{H}^n$ we set $\phi(U) = \phi(u_1)\dots\phi(u_n)$; in the case $n = 0$ this is interpreted as $\phi(\emptyset) = 1_{\Gamma(\mathcal{H})}$. If $\|A\| < 1$ then $\phi(U)\Gamma(A)$ is a well-defined bounded operator. Let $\mathscr{K}(\mathcal{H})$ be the space of all compact operators on $\Gamma(\mathcal{H})$.

Theorem 1.1. *Let $\mathcal{O}$ be an abelian C^*-algebra on $\mathcal{H}$ such that its strong closure does not contain finite rank projections. Let $\mathscr{F}(\mathcal{O}) \subset B(\Gamma(\mathcal{H}))$ be the C^*-algebra generated by the operators $\phi(U)\Gamma(A)$ with U as above and $A \in \mathcal{O}$ with $\|A\| < 1$. Then there is a unique morphism $\mathcal{P} : \mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ such that $\mathcal{P}[\phi(U)\Gamma(A)] = A \otimes [\phi(U)\Gamma(A)]$ for all U, A. We have $\ker \mathcal{P} = \mathscr{K}(\mathcal{H})$, which defines a canonical embedding*

$$\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O}). \tag{1.1}$$

This statement has the advantage that it is simple and covers both the bosonic and fermionic cases. Alternative, technically more convenient, versions of Theorem 1.1 are Theorem 5.4 (see also Lemma 5.11) and Theorem 6.2. Instead of working separately with the Bose and Fermi case one may consider a supersymmetric (or $\mathbb{Z}_2$-graded) Hilbert space $\mathcal{H}$ as in [22] which gives a unified approach to the subject. Since this requires more preliminary developments, and since one gets the same result by taking a tensor product of the bosonic and fermionic Fock space, we did not present this version.

[1] More general ideals also play a rôle, cf. [3,11,13].

[2] In this introduction we shall freely use notions, notations and terminology which are defined in precise terms in the body of the paper, see especially Sections 2, 6 and 7.

In spite of the simplicity of its statement, Theorem 1.1 has important consequence in the spectral theory of QFH: it immediately gives a description of the essential spectrum of these Hamiltonians and also gives a systematic and simple way of proving the Mourre estimate for them with conjugate operators of the form $A = \mathrm{d}\Gamma(\mathfrak{a})$. Such an estimate allows one to prove absence of singular continuous spectrum and is an important step in the proof of asymptotic completeness, cf. [1,2,23,24].

The first difficulty one meets in the algebraic approach we use is the isolation of the correct "algebra of energy observables," in the terminology of [28,29]. In fact, if the algebra we start with is too large, then its quotient with respect to the compacts will probably be too complicated to be useful. On the other hand, we cannot choose it too small because then physically relevant Hamiltonians will not be affiliated to it. Since we have chosen the algebras $\mathscr{F}(\mathcal{O})$ in such a way that general classes of QFH are self-adjoint operators affiliated to them, it seems to us quite remarkable that the description of the quotient given in (1.1) is so simple.

One can also give a priori justifications of the choice of $\mathscr{F}(\mathcal{O})$, we describe two of them below. First, the algebra $\mathscr{F}(\mathcal{O})$ can be obtained by a procedure completely analogous to that used in [28] in the setting of quantum systems with a finite number of degrees of freedom. We interpret $\mathcal{H}$ as the one particle Hilbert space and $\mathcal{O}$ as the C^*-algebra generated by the one particle kinetic energy Hamiltonians.[3] We take as algebra of kinetic energy observables of the field $\Gamma(\mathcal{O}) = C^*(\Gamma(A) \mid A \in \mathcal{O}, \|A\| < 1)$, because this is the C^*-algebra generated by the operators of the form $\mathrm{d}\Gamma(h)$ with h a self-adjoint operator affiliated to $\mathcal{O}$ with $\inf h = m > 0$ (in this paper we restrict ourselves to the case of particles with strictly positive mass). Now we have to decide what kind of interactions we take into account. It is characteristic to quantum fields that the interaction term is some kind of generalized polynomial in the field operators. In the fermionic case we define the "algebra of elementary interactions" $\mathscr{F}(\mathcal{H})$ as the C^*-algebra generated by polynomials in the field operators. Since in the bosonic case the field operators are not bounded, we define $\mathscr{F}(\mathcal{H})$ in this case as the C^*-algebra generated by operators of the form $\int_E \mathrm{e}^{\mathrm{i}\phi(u)} f(u)\, d_E u$, where E is a finite-dimensional vector subspace of $\mathcal{H}$, $d_E u$ is the measure associated to the Euclidean structure we have on E, and f is an integrable function on E. Finally, the algebra of energy observables of the field should be the norm closed linear space of operators on $\Gamma(\mathcal{H})$ generated by the products FS with $F \in \mathscr{F}(\mathcal{H})$ and $S \in \Gamma(\mathcal{O})$. It is easy to see that this is exactly $\mathscr{F}(\mathcal{O})$.

A second characterization of the algebra $\mathscr{F}(\mathcal{O})$ is physically more satisfactory. Let us call *elementary quantum field Hamiltonian of type* $\mathcal{O}$ a self-adjoint operator of the form $H = \mathrm{d}\Gamma(h) + V$, where h is a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathcal{O}$ such that $h \geqslant m$ for some real $m > 0$ and $V \in \mathscr{F}(\mathcal{H})$ is a symmetric operator. This seems to be the smallest class of self-adjoint operators which may naturally be thought as QFH. But $\mathscr{F}(\mathcal{O})$ is just the C^*-algebra generated by these QFH (Proposition 3.10).

On the other hand, the condition which characterizes $\mathcal{P}$ in Theorem 1.1 can be stated in the following equivalent form: $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$ for each elementary QFH (Proposition 5.10 and Lemma 5.11). But this relation has a simple physical interpretation: it says that by taking the quotient with respect to the compacts one gets the Hamiltonian of the system consisting of a free particle with kinetic energy h and of the initial field (the interaction between them being cutoff). So one particle has been pull out from the field without modifying the Hamil-

[3] We assume here that $\mathcal{O}$ acts non-degenerately on $\mathcal{H}$, the only situation of physical interest. The model one should always have in mind is $\mathcal{H} = L^2(X)$ with X a locally compact abelian group, the configuration space of the particle, and $\mathcal{O} = C_0(X^*)$, the algebra of continuous, convergent to zero at infinity, functions of the momentum operator.

tonian of the field (which is possible because the field consists of a potentially infinite number of particles).

As we said above, the embedding (1.1) has interesting consequences in the spectral analysis of the self-adjoint operators affiliated to $\mathscr{F}(\mathcal{O})$. Thus it is important to show that physically realistic QFH belong to this class and this is not at all obvious because the elementary QFH which generate the algebra are just toy models, they only look like real QFH. In Section 7 we give several general criteria for an operator to be affiliated to $\mathscr{F}(\mathcal{O})$ which show that the class of affiliated Hamiltonians is large. As an application, we point out in Section 9 an abstract class of operators affiliated to $\mathscr{F}(\mathcal{O})$ which covers the Hamiltonian of the $P(\varphi)_2$ model with a spatial cutoff. In Section 10, where we show how to treat coupled systems in our framework, we prove that massive Pauli–Fierz Hamiltonians are affiliated to $\mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ ($\mathscr{L}$ is the Hilbert space of the confined system) and deduce the location of their essential spectrum and the Mourre estimate under conditions on the form factor weaker than usual (see assumption (PF) on p. 136).

We shall describe now the kind of results we get concerning the spectral properties of the operators affiliated to $\mathscr{F}(\mathcal{O})$ (precise statements and details are in Sections 7–10). For simplicity, in the rest of this Introduction we assume, besides the conditions of Theorem 1.1, that the algebra $\mathcal{O}$ is non-degenerate on $\mathcal{H}$ (this is trivially satisfied in all the examples we have in mind). Concerning the essential spectrum, the following is an immediate consequence of (1.1): if H is a self-adjoint operator affiliated to $\mathscr{F}(\mathcal{O})$ then

$$\sigma_{\mathrm{ess}}(H) = \sigma\big(\mathcal{P}(H)\big). \tag{1.2}$$

Here $\widetilde{H} \equiv \mathcal{P}(H)$ is a self-adjoint operator[4] on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ affiliated to $\mathcal{O} \otimes \mathscr{F}(\mathcal{O})$. If $\mathscr{X}$ is the spectrum of the abelian algebra $\mathcal{O}$ then $\mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \cong C_0(\mathscr{X}; \mathscr{F}(\mathcal{O}))$ and $\widetilde{H}$ is identified with a continuous family $\{\widetilde{H}(x)\}_{x \in \mathscr{X}}$ of self-adjoint operators on $\Gamma(\mathcal{H})$ affiliated to $\mathscr{F}(\mathcal{O})$. Then (1.2) can be written as

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{x \in \mathscr{X}} \sigma\big(\widetilde{H}(x)\big). \tag{1.3}$$

The Hamiltonians of the quantum field models usually considered in the literature are, however, much more specific than just affiliated to $\mathscr{F}(\mathcal{O})$: they are bounded from below and have the property that there is a self-adjoint operator h affiliated to $\mathcal{O}$ with $h \geqslant m > 0$ such that $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$. We call such QFH *standard* and we interpret h as the *one particle kinetic energy* associated to H and $m = \inf h > 0$ as the *one particle mass* (Definition 7.7). The simplest standard QFH are the elementary ones, but the class is much larger, for example the $P(\varphi)_2$ and Pauli–Fierz models as well as the fermionic models considered in [1,2] belong to this class (see Theorems 9.5 and 10.9). Now for a standard H we clearly have

$$\sigma_{\mathrm{ess}}(H) = \sigma(h) + \sigma(H). \tag{1.4}$$

This formula covers the models treated in [1,2,23–25]. The version (11.9) for systems with a particle number cutoff covers the spin–boson model [33,36].

[4] Or, in very singular situations that do not concern us here, a slightly more general object, since its domain could be not dense.

We then study the Mourre estimate for standard QFH (in this case the result is quite explicit, but more general situations may be treated, see Remark 8.10). As in [4,5,23,24,47] we consider only conjugate operators of the form $A = \mathrm{d}\Gamma(\mathfrak{a})$ where $\mathfrak{a}$ is a self-adjoint operator on $\mathcal{H}$. We assume:

$$\mathrm{e}^{-\mathrm{i}t\mathfrak{a}}\mathcal{O}\mathrm{e}^{\mathrm{i}t\mathfrak{a}} = \mathcal{O} \quad \text{for all real } t \text{ and } t \mapsto \mathrm{e}^{-\mathrm{i}t\mathfrak{a}}S\mathrm{e}^{\mathrm{i}t\mathfrak{a}} \text{ is norm continuous if } S \in \mathcal{O}. \tag{1.5}$$

This condition is quite easy to check in concrete situations, for example in the important case when $\mathcal{O} = C_0(\mathbb{R}^n)$ and $\mathfrak{a}$ is associated to a completely integrable vector field of class C^1.

The operators H and h must satisfy usual regularity conditions with respect to A and $\mathfrak{a}$, respectively, namely they have to be of class $C^1_{\mathrm{u}}(A)$ and $C^1_{\mathrm{u}}(\mathfrak{a})$, respectively. In the case of H for example, this means that the map $\Phi : t \mapsto \mathrm{e}^{-\mathrm{i}tA}(H + \mathrm{i})^{-1}\mathrm{e}^{\mathrm{i}tA}$ is of class C^1 in norm. This implies that the commutator $[H, \mathrm{i}A]$ is well defined as continuous sesquilinear form on $D(H)$, see Section 8 for a more precise assertion. We mention that in order to get the more subtle consequences of the Mourre estimate one has to impose stronger regularity assumptions (e.g. it suffices that Φ be of class C^2 in the strong operator topology).

Under these conditions one says that *the Mourre estimate holds* at energy $\lambda \in \mathbb{R}$ if

$$\text{there are } \varepsilon, \delta > 0 \text{ and a compact operator } K \text{ such that}$$
$$E(\lambda, \varepsilon)[H, \mathrm{i}A]E(\lambda, \varepsilon) \geqslant \delta E(\lambda, \varepsilon) + K. \tag{1.6}$$

Here E is the spectral measure of H and $E(\lambda, \varepsilon) = E([\lambda - \varepsilon, \lambda + \varepsilon])$. Then we define the *threshold set* $\tau_A(H)$ *of* H *with respect to* A as the set of real points where the Mourre estimate does not hold. Of course, $\tau_{\mathfrak{a}}(h)$ is similarly defined. We now state the most important particular case of Theorem 8.6.

Theorem 1.2. *Let H be a standard QFH with one particle kinetic energy h such that* $\inf h > 0$. *Let* $\mathfrak{a}$ *be a self-adjoint operator on* $\mathcal{H}$ *satisfying condition* (1.5) *and let* $A = \mathrm{d}\Gamma(\mathfrak{a})$. *Assume that H is of class $C^1_{\mathrm{u}}(A)$ and h is of class $C^1_{\mathrm{u}}(\mathfrak{a})$ with* $[h, \mathrm{i}\mathfrak{a}] \geqslant 0$. *Then*

$$\tau_A(H) = \left[\bigcup_{n=1}^{\infty} \tau^n_{\mathfrak{a}}(h)\right] + \sigma_{\mathrm{p}}(H), \quad \textit{where } \tau^n_{\mathfrak{a}}(h) = \tau_{\mathfrak{a}}(h) + \cdots + \tau_{\mathfrak{a}}(h) \quad (n \textit{ terms}). \tag{1.7}$$

So at each point outside the set $\tau_A(H)$ described in (1.7) the operator A is conjugate to H in the sense of Mourre, i.e. the estimate (1.6) holds. The relation (1.7) is quite intuitive physically speaking. It says that an energy λ is an A-threshold for H if and only if one can write it as a sum $\lambda = \lambda_1 + \cdots + \lambda_n + \mu$ where the λ_k are $\mathfrak{a}$-threshold energies of the free particle of kinetic energy h and μ is the energy of a bound state of the field. So at energy λ one can extract n free particles from the field such that each one has an $\mathfrak{a}$-threshold energy and such that the field remains in a bound state.

We have considered until now only one self-interacting field but many physically interesting models involve several fields interacting between themselves or with some external quantum system like an atom or a spin. In Section 10 we propose an abstract scheme for studying such situations and we treat in this setting the case of a positive mass Bose field interacting with a "confined" system (the massive Pauli–Fierz model). More precisely, let $\mathscr{L}$ be the Hilbert space of the states of the confined system and L a positive self-adjoint operator on $\mathscr{L}$ with purely discrete spectrum (the internal Hamiltonian). Then the total Hamiltonian of the coupled system is a self-adjoint operator on $\mathscr{H} = \Gamma(\mathcal{H}) \otimes \mathscr{L}$ of the form

$$H = \mathrm{d}\Gamma(h) \otimes 1 + 1 \otimes L + \phi(v) \equiv H_0 + \phi(v), \tag{1.8}$$

where h is as before and $v : D(L^{1/2}) \to D(h^{1/2})^* \otimes \mathscr{L}$ is a bounded operator such that

$$\lim_{r\to\infty} \left\| \left(h^{-1/2} \otimes 1\right) v (L + r)^{-1/2} \right\| < 1. \tag{1.9}$$

Under this condition one can define in a natural way $\phi(v)$ as a quadratic form which is form bounded with respect to H_0 with relative bound less than 1, hence the form sum $H_0 + \phi(v)$ defines a bounded from below self-adjoint operator denoted H in (1.8).

According to our strategy, in order to do the spectral analysis of such Hamiltonians we have first to isolate the C^*-algebra of energy observables of the coupled system. It is clear that the algebra of the confined system should be $K(\mathscr{L})$, the algebra of all compact operators on $\mathscr{L}$ (then L is affiliated to it). Then we take $\mathscr{F}(\mathcal{O}, \mathscr{L}) \equiv \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ as C^*-algebra of energy observables of the total system. The analogue of Theorem 1.1 in this situation is Theorem 10.4 and the notion of standard QFH is an obvious extension of that considered before. Thus, if we are able to show that the operator H defined by (1.8) is affiliated to $\mathscr{F}(\mathcal{O}, \mathscr{L})$, then everything follows from the general theory presented above. In fact, if we use the abbreviation $h + L = h \otimes 1 + 1 \otimes L$, we have

Theorem 1.3. *Let H be as in* (1.8) *with v satisfying* (1.9) *and such that for some $\alpha > 1/2$*

$$(h + L)^{-\alpha} v (L + 1)^{-1/2} \quad \textit{and} \quad (h + L)^{-1/2} v (L + 1)^{-\alpha} \textit{ are compact operators.} \tag{1.10}$$

Then H is affiliated to $\mathscr{F}(\mathcal{O}, \mathscr{L})$ and is a standard QFH with h as one particle kinetic energy. In particular $\sigma_{\mathrm{ess}}(H) = \sigma(h) + \sigma(H)$. Let $\mathfrak{a}$ be as in (1.5) *and let us set $A = \mathrm{d}\Gamma(\mathfrak{a}) \otimes 1$. If H is of class $C^1_{\mathrm{u}}(A)$ and h is of class $C^1_{\mathrm{u}}(\mathfrak{a})$ with $[h, \mathrm{i}\mathfrak{a}] \geqslant 0$, then* (1.7) *is valid.*

We wish to make some historical comments concerning the methods we use. First, the fact that quotients of C^*-algebras with respect to the ideal of compact operators play an important rôle is an old and quite natural idea in the context of the theory of pseudo-differential operators; the references [19,48] seem particularly relevant for us. Second, the first use of C^*-algebra methods in the spectral analysis of physically interesting models appears, as far as we know, in the work of J. Bellissard [7,8] on solid state physics (see [9] for more recent results and references). But the C^*-algebras and the C^*-algebra techniques used by Bellissard and his collaborators are very different from ours, e.g. K-theory plays an important rôle in their works but is probably irrelevant here (it would be nice if somebody would show the contrary). The usefulness of techniques like computation of quotients of C^*-algebras in the spectral theory of many body systems and quantum field models seems to have been first noticed in [11,13]. Note that some of the results described here were announced in [27–29].

In recent works on quantum field models [23,24,33] the techniques involved in the description of the essential spectrum and the proof of the Mourre estimate are of a quite different nature. It is natural to try to understand the connection between these two approaches and I shall make below a comment (or rather a conjecture) on this matter, but I have to say that the situation is not really clear to me.

It is helpful to understand first what happens in the N-body case. The corresponding C^*-algebra of energy observables may be thought as the C^*-algebra generated by the N-body Hamiltonians corresponding to all allowed interactions (see [11,21] for a precise definition). Then the analogue of Theorem 1.1 gives an embedding of the quotient of $\mathscr{C}$ with respect to the compacts in $\bigoplus_a \mathscr{C}_a$ where a runs over the set of partitions consisting of two fragments and the algebra $\mathscr{C}_a$ is similarly defined but with the supplementary condition that the interactions should not couple particles belonging to different fragments (the sum may be extended to partitions involving *at least* two fragments, and this could be relevant in the field case). Thus we have morphisms $\mathcal{P}_a : \mathscr{C} \to \mathscr{C}_a$ which implement this embedding. On the other hand, the geometric methods, as presented in [26] for example, give us a partition of unity $\{J_a\}$ on the N-body Hilbert space $\mathscr{H}$ with the following property: for each $T \in \mathscr{C}$ one has $T \sim \sum_a J_a \mathcal{P}_a(T) J_a$, where $\sim$ means equality modulo a compact (see [13, p. 59] for the proof). In other terms, the *linear* map

$$\bigoplus_a \mathscr{C}_a \ni (T_a) \mapsto \sum_a J_a T_a J_a \in B(\mathscr{H})$$

induces a cross-section of the canonical morphism $\mathscr{C} \to \bigoplus_a \mathscr{C}_a$. Thus, heuristically speaking, if we take the quotient with respect to the compacts we see directly what happens at infinity, while the use of a partition of unity just places us in a neighbourhood of infinity. It is easy to see that this argument also covers the non-symmetrized spin–boson models considered in [33,36] and the arguments from Sections 4 and 5 of [33] make this interpretation quite transparent.

The analogous construction in the field case should give a linear map $\mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \to B(\Gamma(\mathcal{H}))$ whose restriction to the range of $\mathcal{P}$ should be a cross-section of $\mathcal{P}$. Unfortunately such a construction does not seem straightforward to me, although the J map introduced in [37] and the partitions of unity constructed in [23,24] could be relevant.

The paper is organized as follows. In Section 2 we summarize the most important notations and results from the theory of symmetric Fock spaces following [6,15,35] and also the more recent [23,24]. We prefer to define the scalar product (2.13) on a Fock space as in [42] and the definition (2.6) of the annihilation and creation operators in terms of the field operators is slightly unusual, which explains some differences in the numerical factors. Similar conventions are adopted in the antisymmetric case presented in Section 6 where we use [43] as main reference.

In Section 3 we define the algebras $\mathscr{F}(\mathcal{O})$ and present some of their properties and alternative characterizations. In Section 4 we prove the main theorem for the algebra $\mathscr{A}(\mathcal{H}) \equiv \mathscr{F}(\mathbb{C}1_{\mathcal{H}})$, which is an important technical step but also has an intrinsic mathematical interest because we show that the quotient $\mathscr{A}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ is canonically isomorphic to $\mathscr{A}(\mathcal{H})$. We also give there some consequences of this fact in the spectral analysis of the elements of $\mathscr{A}(\mathcal{H})$. In Section 5 we prove our main technical result, Theorem 5.4. We consider only the bosonic case until Section 6 where we describe briefly the corresponding results in the fermionic case (which is nicer but easier).

Sections 7–11 are devoted to applications in the spectral analysis of quantum field models of Theorem 1.1. In Section 7 we give criteria for affiliation to $\mathscr{F}(\mathcal{O})$ and a general formula for the essential spectrum of the operators affiliated to this algebra (Theorem 7.6 and relation (7.6)). We also introduce there the important class of standard QFH and describe their essential spectrum. The main result of Section 8 is Theorem 8.6 which gives the Mourre estimate for such Hamiltonians. In Section 9 we show that a general class of QFH, including the $P(\varphi)_2$ model, are standard in the sense defined before, hence all these results apply to them. In Section 10 we sketch a method of analyzing several fields with couplings between them and external systems and consider in detail the massive Pauli–Fierz model. Note that the Pauli–Fierz Hamiltonian is also standard. In the last section we treat models with a particle number cutoff, which have some interesting features. We do not treat explicitly the fermionic case because it is easy to see that models like those considered in [1,2] are standard in our sense so their spectral properties (essential spectrum and Mourre estimate) follow from the general theorems of Sections 7 and 8.

2. Bosonic Fock space

2.1. Our notations are rather standard but we recall here some of them to avoid any ambiguity. If $\mathcal{E},\mathcal{F}$ are vector spaces then $L(\mathcal{E},\mathcal{F})$ is the space of linear maps $\mathcal{E}\to\mathcal{F}$ and we abbreviate $L(\mathcal{E})=L(\mathcal{E},\mathcal{E})$. If $\mathcal{E},\mathcal{F}$ are Banach spaces then $B(\mathcal{E},\mathcal{F})$ and $K(\mathcal{E},\mathcal{F})$ are the subspaces of $L(\mathcal{E},\mathcal{F})$ consisting of continuous or compact maps, respectively, and we set $B(\mathcal{E})=B(\mathcal{E},\mathcal{E})$, $K(\mathcal{E})=K(\mathcal{E},\mathcal{E})$. When needed for the clarity of the argument we denote by $1_{\mathcal{E}}$ the identity operator on a Banach space $\mathcal{E}$ or the identity element of an algebra $\mathcal{E}$. The domain of an operator T is denoted $D(T)$. The Hilbert spaces are complex Hilbert spaces unless the contrary is explicitly mentioned and the scalar product is linear in the second variable. If a symbol like $T^{(*)}$ appears in a relation, this means that the relation holds both for T and T^*. We denote by $C^*(T\mid T\in\mathscr{T},P_1,P_2,\ldots)$ the C^*-algebra generated by a family $\mathscr{T}$ of operators T which have the properties P_1,P_2, etc. The C^*-algebra generated by a self-adjoint operator H is $C_0(H)=\{f(H)\mid f\in C_0(\mathbb{R})\}$. More generally, the C^*-algebra generated by a family of self-adjoint operators is the smallest C^*-algebra which contains the resolvents of these operators. A *morphism* between two C^*-algebras is a $*$-morphism. $C_0(X)$ is the space of continuous complex-valued functions on the locally compact space X that converge to zero at infinity and $C_c(X)$ that of continuous functions with compact support.

We need a version of the polarization formula. Let X,Y be vector spaces, $Q:X\times\cdots\times X\to Y$ an n-linear symmetric map, and let us set $q(x)=Q(x,\ldots,x)$. Denote $|a|$ the cardinal of a set a. Then

$$(-1)^n n!\,Q(x_1,\ldots,x_n)=\sum_{a\subset\{1,\ldots,n\}}(-1)^{|a|}q\biggl(\sum_{i\in a}x_i\biggr). \tag{2.1}$$

2.2. Let $\mathcal{H}$ be a complex Hilbert space with scalar product $\langle\cdot|\cdot\rangle$ and let $U(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$. A (*regular*) *representation of the CCR over* $\mathcal{H}$, or a *Weyl system over* $\mathcal{H}$, is a couple $(\mathscr{H},W)$ consisting of a Hilbert space $\mathscr{H}$ and a map $W:\mathcal{H}\to U(\mathscr{H})$ which satisfies

$$W(u+v)=\mathrm{e}^{\mathrm{i}\Im\langle u|v\rangle}W(u)W(v)\quad\text{for all } u,v\in\mathcal{H} \tag{2.2}$$

and such that the restriction of W to each finite-dimensional subspace is strongly continuous. Then

$$W(0)=1,\qquad W(u)^*=W(-u),\qquad W(u)W(v)=\mathrm{e}^{-2\mathrm{i}\Im\langle u|v\rangle}W(v)W(u). \tag{2.3}$$

We denote $\mathscr{W}(\mathcal{H})$ the C^*-algebra generated by the operators $W(u)$ and we call it *Weyl algebra over* $\mathcal{H}$:

$$\mathscr{W}(\mathcal{H})=C^*\bigl(W(u)\bigm| u\in\mathcal{H}\bigr). \tag{2.4}$$

The C^*-algebras $\mathscr{W}(\mathcal{H})$ associated to two Weyl systems are canonically isomorphic, see [15] for a proof. This also gives canonical embeddings $\mathscr{W}(\mathcal{K}) \subset \mathscr{W}(\mathcal{H})$ for closed subspaces $\mathcal{K}$ of $\mathcal{H}$.

The *field operator* associated to the one particle state $u \in \mathcal{H}$ is defined as the unique self-adjoint operator $\phi(u)$ on $\mathscr{H}$ such that $W(tu) = \mathrm{e}^{\mathrm{i}t\phi(u)}$ for all real t. We have for all $u, v \in \mathcal{H}$:

$$W(u)\phi(v)W(u)^* = \phi(v) - 2\Im\langle u|v\rangle \quad \text{and} \quad \big[\phi(u), \phi(v)\big] = 2\mathrm{i}\Im\langle u|v\rangle. \tag{2.5}$$

The space $\mathscr{H}^\infty$ of vectors $f \in \mathscr{H}$ such that $u \mapsto W(u)f$ is a C^∞ map on each finite-dimensional subspace of $\mathcal{H}$ is a dense subspace of $\mathscr{H}$ stable under all the operators $W(u)$ and $\phi(u)$. Moreover, $\mathscr{H}^\infty$ is a core for each $\phi(u)$ (by Nelson lemma) and the second relation in (2.5) holds in operator sense on $\mathscr{H}^\infty$. The map $u \mapsto \phi(u) \in L(\mathscr{H}^\infty)$ is clearly *not* linear but only $\mathbb{R}$-linear, as it follows from (2.2) after replacing u, v by tu, tv with t real and then taking derivatives at $t = 0$.

The *annihilation* and *creation* operators associated to the one particle state u are defined by

$$a(u) = \big(\phi(u) + \mathrm{i}\phi(\mathrm{i}u)\big)/2, \qquad a^*(u) = \big(\phi(u) - \mathrm{i}\phi(\mathrm{i}u)\big)/2 \tag{2.6}$$

on $\mathscr{H}^\infty$ and then extended by taking closures. On $\mathscr{H}^\infty$ we have $\phi(u) = a(u) + a^*(u)$. The map $u \mapsto a^*(u) \in L(\mathscr{H}^\infty)$ is linear, $u \mapsto a(u) \in L(\mathscr{H}^\infty)$ is antilinear, and

$$\big[a(u), a^*(v)\big] = \langle u|v\rangle, \qquad \big[a(u), a(v)\big] = 0, \qquad \big[a^*(u), a^*(v)\big] = 0 \quad \text{on } \mathscr{H}^\infty. \tag{2.7}$$

On the other hand, from (2.5) we also get:

$$W(u)a^{(*)}(v)W(u)^* = a^{(*)}(v) - \langle v|\mathrm{i}u\rangle^{(*)}, \qquad \big[a^{(*)}(v), W(u)\big] = \langle v|\mathrm{i}u\rangle^{(*)}W(u). \tag{2.8}$$

Some of our later constructions will depend only on the existence of a *particle number operator for the Weyl system* W, which is a self-adjoint operator N on $\mathscr{H}$ such that

$$\mathrm{e}^{\mathrm{i}tN}W(u)\mathrm{e}^{-\mathrm{i}tN} = W\big(\mathrm{e}^{\mathrm{i}t}u\big) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathcal{H}. \tag{2.9}$$

Such an operator is clearly not uniquely defined and it is easy to prove that if it exists then N can be chosen such that its spectrum be either $\mathbb{N} = \{0, 1, 2, \ldots\}$ or $\mathbb{Z}$, see [18]. In [17] it is shown that we are in the first situation if and only if W is a direct sum of Fock representations (cf. below). Since

$$W\big(\mathrm{e}^{\mathrm{i}t}u\big) = W(u\cos t + \mathrm{i}\sin t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\|u\|^2\sin 2t}W(u\cos t)W(\mathrm{i}u\sin t)$$

by taking derivatives in (2.9) at $t = 0$ we get (this is easy to justify in the Fock representation):

$$W(u)NW(u)^* = N - \phi(\mathrm{i}u) + \|u\|^2, \qquad \big[N, W(u)\big] = W(u)\big(\phi(\mathrm{i}u) + \|u\|^2\big). \tag{2.10}$$

Replacing u by tu in the last equation and then taking the derivatives at $t = 0$ we get

$$\big[N, \mathrm{i}\phi(u)\big] = \phi(\mathrm{i}u), \qquad (N+1)a(u) = a(u)N, \qquad (N-1)a^*(u) = a^*(u)N. \tag{2.11}$$

A *vacuum state* for the Weyl system W is a vector $\Omega \in \mathscr{H}$ with $\|\Omega\| = 1$, $\Omega \in D(\phi(u))$ for all $u \in \mathcal{H}$, and such that the map $u \mapsto \phi(u)\Omega$ is linear. It is easy to prove that a vacuum state belongs to $\mathscr{H}^\infty$ and that a vector Ω of norm one is a vacuum state if and only if $\Omega \in \bigcap_u D(a(u))$ and $a(u)\Omega = 0$ for all u, see for example [24, Proposition 4.1].

A *Fock representation* of the CCR over $\mathcal{H}$ is a triple $(\mathscr{H}, W, \Omega)$ consisting of a Weyl system $(\mathscr{H}, W)$ over $\mathcal{H}$ and a vacuum state Ω which is cyclic for W. It is easy to show that two Fock representations are canonically isomorphic, more precisely if $(\mathscr{H}', W', \Omega')$ is a second Fock representation then there is a unique bijective isometry $J : \mathscr{H} \to \mathscr{H}'$ such that $J\Omega = \Omega'$ and $JW(u) = W'(u)J$ for all $u \in \mathcal{H}$. For this reason one may say *the* Fock representation and speak about "realizations" of this representation. The realizations are constructed such as to diagonalize various sets of operators. If $\mathcal{H}$ is infinite-dimensional then there are irreducible representations of the CCR which are not Fock.

The Fock space realization that we describe below is motivated by the following observations. Let $\mathscr{H}^0 = \mathbb{C}\Omega$ and for each integer $n \geqslant 1$ let $\mathscr{H}^n$ be the closed linear subspace of $\mathscr{H}$ generated by the vectors of the form $a^*(u_1)\dots a^*(u_n)\Omega$ with $u_k \in \mathcal{H}$. From (2.7) and since Ω is cyclic we get $\mathscr{H} = \bigoplus_{n=0}^\infty \mathscr{H}^n$ (Hilbert direct sum) and $\|a^*(u)^n\Omega\| = \sqrt{n!}\|u\|^n$. Let us denote $\mathfrak{S}(n)$ the set of permutations of $\{1, \dots, n\}$. Then, since the operators $a^*(u)$ are pairwise commuting, we have

$$\langle a^*(u_1)\dots a^*(u_n)\Omega \,|\, a^*(v_1)\dots a^*(v_n)\Omega \rangle = \sum_{\sigma\in\mathfrak{S}(n)} \langle u_1|v_{\sigma(1)}\rangle \dots \langle u_n|v_{\sigma(n)}\rangle. \tag{2.12}$$

2.3. Let $\mathcal{H}_{\mathrm{alg}}^\vee$ be the symmetric algebra[5] over the vector space $\mathcal{H}$. We denote by uv the product of two elements u, v of $\mathcal{H}_{\mathrm{alg}}^\vee$ and by u^n the nth power of an element $u \in \mathcal{H}_{\mathrm{alg}}^\vee$. The unit element is denoted either 1 or Ω. Let $\mathcal{H}_{\mathrm{alg}}^{\vee n}$ be the linear subspace spanned by the powers u^n with $u \in \mathcal{H}$. Note that $\mathcal{H}_{\mathrm{alg}}^{\vee 0} = \mathbb{C}\Omega$. Then $\mathcal{H}_{\mathrm{alg}}^\vee = \sum_{n\in\mathbb{N}} \mathcal{H}_{\mathrm{alg}}^{\vee n}$ (direct sum of linear spaces) and for $f \in \mathcal{H}_{\mathrm{alg}}^{\vee n}$ and $g \in \mathcal{H}_{\mathrm{alg}}^{\vee m}$ we have $fg \in \mathcal{H}_{\mathrm{alg}}^{\vee(n+m)}$. We set $\mathcal{H}_{\mathrm{alg}}^{\vee n} = \{0\}$ for $n < 0$, so $\mathcal{H}_{\mathrm{alg}}^\vee$ becomes a $\mathbb{Z}$-graded algebra.

We shall equip $\mathcal{H}_{\mathrm{alg}}^\vee$ with the unique scalar product such that $\mathcal{H}_{\mathrm{alg}}^{\vee n} \perp \mathcal{H}_{\mathrm{alg}}^{\vee m}$ if $n \neq m$ and

$$\langle u_1 \dots u_n | v_1 \dots v_n\rangle = \sum_{\sigma\in\mathfrak{S}(n)} \langle u_1|v_{\sigma(1)}\rangle \dots \langle u_n|v_{\sigma(n)}\rangle. \tag{2.13}$$

From the polarization formula (2.1) we see that this scalar product is uniquely determined by the condition $\langle u^n|v^m\rangle = n!\langle u|v\rangle^n \delta_{nm}$ for all $u, v \in \mathcal{H}$ and $n, m \geqslant 0$ (see also the characterization given on p. 100). Then it is easy to prove that

$$\|uv\| \leqslant \binom{n+m}{n}^{1/2} \|u\|\|v\| \quad \text{if } u \in \mathcal{H}_{\mathrm{alg}}^{\vee n} \text{ and } v \in \mathcal{H}_{\mathrm{alg}}^{\vee m}. \tag{2.14}$$

[5] This is a complex abelian unital algebra in which $\mathcal{H}$ is linearly embedded and which is uniquely determined (modulo canonical isomorphisms) by the following universal property: if $\xi : \mathcal{H} \to \mathcal{A}$ is a linear map with values in a unital algebra $\mathcal{A}$ such that $\xi(u)\xi(v) = \xi(v)\xi(u)$ for all u, v then there is a unique extension of ξ to a morphism of unital algebras $\mathcal{H}_{\mathrm{alg}}^\vee \to \mathcal{A}$ (see [10] for example). Concerning the notation uv we use for the product we note that in concrete situations, when some other product uv is already defined, this notation could be ambiguous. Then we replace it by $u \vee v$ and denote by $u^{\vee n}$ the powers of u.

We define the *Fock space* $\Gamma(\mathcal{H}) \equiv \mathcal{H}^{\vee}$ *over* $\mathcal{H}$ as the completion of $\mathcal{H}^{\vee}_{\mathrm{alg}}$ for the scalar product defined by (2.13). Let $\mathcal{H}^{\vee n}$ be the closure of $\mathcal{H}^{\vee n}_{\mathrm{alg}}$ in $\Gamma(\mathcal{H})$. Then we can write $\Gamma(\mathcal{H}) = \bigoplus_n \mathcal{H}^{\vee n}$, a Hilbert space direct sum. We shall also use the notations $\Gamma_n(\mathcal{H}) = \sum_{k=0}^n \mathcal{H}^{\vee n}$ and $\Gamma_{\mathrm{fin}}(\mathcal{H}) = \bigcup_n \Gamma_n(\mathcal{H})$. Note that $\mathcal{H}^{\vee 0} \equiv \mathbb{C}\Omega$. The vector Ω is the *vacuum state* and the orthogonal projection on it is $\omega = |\Omega\rangle\langle\Omega|$.

Using (2.14) we can extend by continuity the multiplication and get a structure of unital abelian algebra on $\Gamma_{\mathrm{fin}}(\mathcal{H})$ such that $\mathcal{H}^{\vee n}\mathcal{H}^{\vee m} \subset \mathcal{H}^{\vee(n+m)}$. Then (2.14) remains valid for all $u \in \mathcal{H}^{\vee n}$ and $v \in \mathcal{H}^{\vee m}$. We keep the notation uv for the product of two elements u and v of $\Gamma_{\mathrm{fin}}(\mathcal{H})$.

We denote by 1^n and 1_n the orthogonal projections of $\Gamma(\mathcal{H})$ onto the subspaces $\mathcal{H}^{\vee n}$ and $\Gamma_n(\mathcal{H})$, respectively. Thus $1_n = 1^0 + \cdots + 1^n$ and $1^0 = \omega$. The *number operator* is defined by $N = \sum_n n1^n$.

For each $u \in \mathcal{H}$ the *creation operator* $a^*(u)$ is the closure of the operator of multiplication by u on $\Gamma(\mathcal{H})$ and the *annihilation operator* $a(u)$ is its the adjoint of. Then $\Gamma_{\mathrm{fin}}(\mathcal{H})$ is included in the domains of $a^*(u)$ and $a(u)$, is left invariant by both operators, and the operator $a(u)$ is a derivation of the algebra $\Gamma_{\mathrm{fin}}(\mathcal{H})$. The *field operator* $\phi(u) = a(u) + a^*(u)$ is essentially self-adjoint on $\Gamma_{\mathrm{fin}}(\mathcal{H})$ and the following elementary estimate

$$\left\|\phi(u)^p v\right\| \leqslant \|2u\|^p \left\|\sqrt{(N+1)\dots(N+p)}\, v\right\| \tag{2.15}$$

is valid for all $u \in \mathcal{H}$, $v \in \Gamma(\mathcal{H})$, and $p \geqslant 1$ integer. Then $W(u) = \mathrm{e}^{\mathrm{i}\phi(u)}$ defines a Weyl system over $\mathcal{H}$.

2.4. If $A_i \in B(\mathcal{H})$ for $i = 1, \dots, n$ are given then there is a unique operator $A_1 \vee \cdots \vee A_n \in B(\mathcal{H}^{\vee n})$ such that $(A_1 \vee \cdots \vee A_n)u^n = (A_1 u)\dots(A_n u)$ for all $u \in \mathcal{H}$. We extend it to $\Gamma(\mathcal{H})$ by identifying $A_1 \vee \cdots \vee A_n \equiv A_1 \vee \cdots \vee A_n 1^n$. By convention $A^{\vee 0} = \omega$.

If $A \in B(\mathcal{H})$ then there is a unique unital endomorphism $\Gamma(A)$ of the algebra $\Gamma_{\mathrm{fin}}(\mathcal{H})$ such that $\Gamma(A)u = Au$ for all $u \in \mathcal{H}$ and such that the restriction of $\Gamma(A)$ to each $\Gamma_n(\mathcal{H})$ be continuous. One has $\Gamma(A)u^n = (Au)^n$ if $u \in \mathcal{H}$ and $\Gamma(A) = \bigoplus_{n \geqslant 0} A^{\vee n}$ in an obvious sense. The operator $\Gamma(A)$ is bounded on $\Gamma(\mathcal{H})$ if and only if $\|A\| \leqslant 1$ (we keep the notation $\Gamma(A)$ for its closure). Then $\|\Gamma(A)\| = 1$, $\Gamma(AB) = \Gamma(A)\Gamma(B)$, $\Gamma(1) = 1$ and $\Gamma(0) = \omega$. Note that $z^N = \Gamma(z)$ for $z \in \mathbb{C}$.

Moreover, there is a unique derivation $\mathrm{d}\Gamma(A)$ of the algebra $\Gamma_{\mathrm{fin}}(\mathcal{H})$ such that $\mathrm{d}\Gamma(A)u = Au$ for all $u \in \mathcal{H}$. Thus we have $\mathrm{d}\Gamma(A)u^n = n(Au)u^{n-1}$ if $n \geqslant 1$ and $\mathrm{d}\Gamma(A)\Omega = 0$. This operator is closable and we denote its closure by the same symbol. If A is self-adjoint then $\Gamma(\mathrm{e}^{\mathrm{i}A}) = \mathrm{e}^{\mathrm{i}\,\mathrm{d}\Gamma(A)}$.

The definition of $\mathrm{d}\Gamma(A)$ is extended as usual to operators A which are infinitesimal generators of contractive C_0-semigroups $\{\mathrm{e}^{tA}\}$ on $\mathcal{H}$: the operator $\mathrm{d}\Gamma(A)$ is defined by the rule $\Gamma(\mathrm{e}^{tA}) = \mathrm{e}^{t\,\mathrm{d}\Gamma(A)}$.

The following identities hold on $\Gamma_{\mathrm{fin}}(\mathcal{H})$ for all $A \in B(\mathcal{H})$ and $u \in \mathcal{H}$:

$$\Gamma(A)a^*(u) = a^*(Au)\Gamma(A), \qquad \Gamma(A)a\big(A^*u\big) = a(u)\Gamma(A). \tag{2.16}$$

If $A^*A = 1$ we also get $\Gamma(A)a(u) = a(Au)\Gamma(A)$ by replacing u by Au in the second identity, hence

$$\Gamma(A)\phi(u) = \phi(Au)\Gamma(A) \quad \text{and} \quad \Gamma(A)W(u) = W(Au)\Gamma(A) \quad \text{if } A^*A = 1. \tag{2.17}$$

More generally, if $A^*:\mathcal{H}\to\mathcal{H}$ is a surjective map then there is an operator $A^\dagger\in B(\mathcal{H})$ such that $A^*A^\dagger=1$ and then, if we denote $\phi_A(u)=a(A^\dagger)+a^*(Au)$ we get

$$\Gamma(A)a(u)=a\big(A^\dagger u\big)\Gamma(A)\quad\text{and}\quad \Gamma(A)\phi(u)=\phi_A(u)\Gamma(A). \tag{2.18}$$

Observe that if $A\in B(\mathcal{H})$ is invertible then $A^\dagger=(A^*)^{-1}$.

2.5. Let $\mathcal{K}\subset\mathcal{H}$ be a linear subspace. Then we have a canonical embedding $\mathcal{K}^\vee_{\mathrm{alg}}\subset\mathcal{H}^\vee_{\mathrm{alg}}$ obtained by identifying $\mathcal{K}^\vee_{\mathrm{alg}}$ with the unital subalgebra of $\mathcal{H}^\vee_{\mathrm{alg}}$ generated by $\mathcal{K}$. If $\mathcal{L}\subset\mathcal{H}$ is another linear subspace then $\mathcal{K}^\vee_{\mathrm{alg}}$ and $\mathcal{L}^\vee_{\mathrm{alg}}$ are subalgebras of the abelian algebra $\mathcal{H}^\vee_{\mathrm{alg}}$ so we have a natural unital morphism $\mathcal{K}^\vee_{\mathrm{alg}}\otimes\mathcal{L}^\vee_{\mathrm{alg}}\to\mathcal{H}^\vee_{\mathrm{alg}}$ (algebraic tensor product) which is injective if and only if $\mathcal{K}\cap\mathcal{L}=0$ and surjective if and only if $\mathcal{K}+\mathcal{L}=\mathcal{H}$. Thus $(\mathcal{K}\oplus\mathcal{L})^\vee_{\mathrm{alg}}=\mathcal{K}^\vee_{\mathrm{alg}}\otimes\mathcal{L}^\vee_{\mathrm{alg}}$.

Let $\mathcal{K}\subset\mathcal{H}$ be a closed subspace. Then the embedding $\mathcal{K}^\vee_{\mathrm{alg}}\subset\mathcal{H}^\vee_{\mathrm{alg}}$ obviously extends to an isometric embedding $\Gamma(\mathcal{K})\subset\Gamma(\mathcal{H})$. Moreover, the canonical algebraic identification $\mathcal{H}^\vee_{\mathrm{alg}}=\mathcal{K}^\vee_{\mathrm{alg}}\otimes\mathcal{K}^{\perp\vee}_{\mathrm{alg}}$ extends to a Hilbert space identification $\Gamma(\mathcal{H})=\Gamma(\mathcal{K})\otimes\Gamma(\mathcal{K}^\perp)$. Indeed, the scalar product (2.13) has been chosen such that the identification map be isometric (the norm of a tensor product of Hilbert spaces being defined in the standard way). In fact (2.13) is the unique scalar product on $\mathcal{H}^\vee_{\mathrm{alg}}$ such that $\|\Omega\|=1$, a vector $u\in\mathcal{H}$ has the same norm in $\mathcal{H}$ and in $\mathcal{H}^\vee_{\mathrm{alg}}$, and for each closed subspace $\mathcal{K}\subset\mathcal{H}$

$$\langle uv|u'v'\rangle=\langle u|u'\rangle\langle v|v'\rangle=\langle u\otimes v|u'\otimes v'\rangle\quad\text{for all } u\in\mathcal{K}^\vee_{\mathrm{alg}},\ v\in\big(\mathcal{K}^\perp\big)^\vee_{\mathrm{alg}}.$$

In order to avoid ambiguities we indicate, when necessary, by a subindex the Hilbert space on which the various objects depend, for example $W_\mathcal{H}$, $N_\mathcal{H}$ and so on. We also use abbreviations like $N'_\mathcal{K}=N_{\mathcal{K}^\perp}$, $\Omega'_\mathcal{K}=\Omega_{\mathcal{K}^\perp}$, etc. Then, relatively to the factorization $\Gamma(\mathcal{H})=\Gamma(\mathcal{K})\otimes\Gamma(\mathcal{K}^\perp)$, we have for $u\in\mathcal{K}$

$$W_\mathcal{H}(u)=W_\mathcal{K}(u)\otimes 1,\qquad \phi_\mathcal{H}(u)=\phi_\mathcal{K}(u)\otimes 1,\qquad a^{(*)}_\mathcal{H}(u)=a^{(*)}_\mathcal{K}(u)\otimes 1. \tag{2.19}$$

Note also the relations $\Omega_\mathcal{H}=\Omega_\mathcal{K}\otimes\Omega'_\mathcal{K}$ and $\omega_\mathcal{H}=\omega_\mathcal{K}\otimes\omega'_\mathcal{K}$. If $A=B\oplus C$ in $\mathcal{H}=\mathcal{K}\oplus\mathcal{K}^\perp$ then

$$\Gamma(A)=\Gamma(B)\otimes\Gamma(C),\qquad \mathrm{d}\Gamma(A)=\mathrm{d}\Gamma(B)\otimes 1+1\otimes\mathrm{d}\Gamma(C). \tag{2.20}$$

In particular $z^{N_\mathcal{H}}=z^{N_\mathcal{K}}\otimes z^{N'_\mathcal{K}}$ for $|z|\leqslant 1$ and $N_\mathcal{H}=N_\mathcal{K}\otimes 1+1\otimes N'_\mathcal{K}$.

After the identification $\Gamma(\mathcal{H})=\Gamma(\mathcal{K})\otimes\Gamma(\mathcal{K}^\perp)$ the embedding $\Gamma(\mathcal{K})\subset\Gamma(\mathcal{H})$ is nothing else but $\Gamma(\mathcal{K})\equiv\Gamma(\mathcal{H})\otimes\Omega'_\mathcal{K}$. Then extending an operator T defined on the subspace $\Gamma(\mathcal{K})$ by zero on the orthogonal subspace of $\Gamma(\mathcal{H})$ amounts to identifying $T\equiv T\otimes\omega'_\mathcal{K}$. This is coherent with the first relation in (2.20): $\Gamma(B\oplus 0)=\Gamma(B)\otimes\Gamma(0)=\Gamma(B)\otimes\omega'_\mathcal{K}$.

Let $\mathscr{K}(\mathcal{H})=K(\Gamma(\mathcal{H}))$ be the C^*-algebra of compact operators on $\Gamma(\mathcal{H})$. Clearly

$$\mathscr{K}(\mathcal{H})=\mathscr{K}(\mathcal{K})\otimes\mathscr{K}\big(\mathcal{K}^\perp\big). \tag{2.21}$$

As explained above, we have a natural identification of $\mathscr{K}(\mathcal{K})$ with a C^*-subalgebra $\mathscr{K}_\mathcal{K}(\mathcal{H})$ of $\mathscr{K}(\mathcal{H})$, a compact operator on $\Gamma(\mathcal{H})$ being identified with its extension by zero on $\Gamma(\mathcal{K})^\perp$:

$$\mathscr{K}_\mathcal{K}(\mathcal{H})\equiv\mathscr{K}(\mathcal{K})\otimes\omega'_\mathcal{K}\subset\mathscr{K}(\mathcal{H}). \tag{2.22}$$

Lemma 2.1. *$\{\mathscr{K}_E(\mathcal{H})\}$, where E runs over the set of finite-dimensional subspaces of $\mathcal{H}$, is an increasing family of C^*-algebras and the closure of its union is $\mathscr{K}(\mathcal{H})$.*

Proof. It suffices to note that the spaces $\Gamma(E)$, with $E \subset \mathcal{H}$ finite-dimensional, form an increasing family of closed subspaces of $\Gamma(\mathcal{H})$ whose union is dense in $\Gamma(\mathcal{H})$. □

3. The algebras $\mathscr{F}(\mathcal{O})$

We fix a complex Hilbert space $\mathcal{H}$ and to each C^*-algebra $\mathcal{O}$ of operators on it we associate a C^*-algebra of operators on the bosonic Fock space $\Gamma(\mathcal{H})$ according to the following rule:

$$\Gamma(\mathcal{O}) = C^*\big(\Gamma(A) \bigm| A \in \mathcal{O},\ \|A\| < 1\big). \tag{3.1}$$

Since $\Gamma(A)\Gamma(B) = \Gamma(AB)$ and $\Gamma(A)^* = \Gamma(A^*)$ this is in fact the norm closed linear space generated by the operators $\Gamma(A)$ with $A \in \mathcal{O}$ and $\|A\| < 1$. We shall prove in a moment that

$$\begin{aligned} \Gamma(\mathcal{O}) = {} & \text{closure of the linear space generated by the } \Gamma(A) \\ & \text{with } A \in \mathcal{O} \text{ and } 0 \leqslant A \leqslant \|A\| < 1. \end{aligned} \tag{3.2}$$

Proposition 3.1. *The map $\mathcal{O} \mapsto \Gamma(\mathcal{O})$ is increasing and we have*

$$\Gamma\big(\{0\}\big) = \mathbb{C}\omega, \qquad \Gamma(\mathbb{C}1_{\mathcal{H}}) = C_0(N) = \big\{\theta(N) \bigm| \theta \in C_0(\mathbb{N})\big\}. \tag{3.3}$$

If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{O} = \mathcal{O}_1 \oplus \mathcal{O}_2$ for some C^*-subalgebras $\mathcal{O}_i \subset B(\mathcal{H}_i)$, then

$$\Gamma(\mathcal{O}) = \Gamma(\mathcal{O}_1) \otimes \Gamma(\mathcal{O}_2), \tag{3.4}$$

where the tensor product is defined by the identification $\Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$.

Proof. The first assertion is obvious and the first relation in (3.3) follows from $\Gamma(0) = \omega$. Since the closed subspace generated by the functions $\lambda \mapsto \lambda^n$ with $0 < \lambda < 1$ is dense in $C_0(\mathbb{N})$ we see that the second relation in (3.3) is true. To prove (3.4) we use (2.20) and the fact that for $A = A_1 \oplus A_2$ we have $\|A\| = \sup(\|A_1\|, \|A_2\|)$ so that $\|A\| < 1$ if and only if $\|A_1\| < 1$ and $\|A_2\| < 1$. □

We shall give a more explicit description of $\Gamma(\mathcal{O})$ for an arbitrary $\mathcal{O}$ below. Observe first that the linear subspace of $B(\mathcal{H}^{\vee n})$ generated by the operators of the form $A_1 \vee \cdots \vee A_n$ with $A_i \in \mathcal{O}$ is a $*$-algebra. Indeed, this follows from $(A_1 \vee \cdots \vee A_n)^* = A_1^* \vee \cdots \vee A_n^*$ and

$$n!(A_1 \vee \cdots \vee A_n)(B_1 \vee \cdots \vee B_n) = \sum_{\sigma \in \mathfrak{S}(n)} (A_1 B_{\sigma(1)}) \vee \cdots \vee (A_n B_{\sigma(n)}) \tag{3.5}$$

which is obvious if $A_1 = \cdots = A_n$ and $B_1 = \cdots = B_n$ and the general case follows by applying twice the polarization formula (2.1). Thus the norm closed linear space generated by the operators $A_1 \vee \cdots \vee A_n$ with $A_i \in \mathcal{O}$ is a C^*-algebra that we shall denote $\mathcal{O}^{\vee n}$. We make the convention $\mathcal{O}^{\vee 0} = \mathbb{C}1^0 = \mathbb{C}\omega$.

Proposition 3.2. *$\mathcal{O}^{\vee n}$ is the norm closed linear space of operators on $\mathcal{H}^{\vee n}$ generated by the operators $A^{\vee n}$ with $A \in \mathcal{O}$ and $A \geqslant 0$. Moreover, we have* (3.2) *and*

$$\Gamma(\mathcal{O}) = \bigoplus_n \mathcal{O}^{\vee n} \equiv \left\{ \sum_n A_n 1^n \;\middle|\; A_n \in \mathcal{O}^{\vee n},\ \|A_n\| \to 0 \right\}. \tag{3.6}$$

Proof. Let $\mathcal{L}$ be the linear space of operators on $\mathcal{H}^{\vee n}$ generated by the operators $A^{\vee n}$ with $A \in \mathcal{O}$ and $A \geqslant 0$. From the polarization formula (2.1) we first deduce that the operators $A_1 \vee \cdots \vee A_n$ with $A_i \in \mathcal{O}$ and $A_i \geqslant 0$ belong to $\mathcal{L}$ and then, by n-linearity, that the same assertion holds without the condition $A_i \geqslant 0$. This proves the first assertion of the proposition.

Let $\mathscr{L}$ be the norm closed linear space generated by the operators $\Gamma(A)$ such that $A \in \mathcal{O}$ and $0 \leqslant A \leqslant a$ for some $a < 1$. Let $A \geqslant 0$ with $\|A\| < 1$. For $0 \leqslant t \leqslant 1$ we then have $\Gamma(tA) = \sum t^n A^{\vee n}$, so the map $t \mapsto \Gamma(tA) \in \mathscr{L}$ is of class C^∞ and its derivative of order n at $t = 0$ is equal to $n!A^{\vee n}$. Clearly then we get $A^{\vee n} \in \mathscr{L}$ for all $A \in \mathcal{O}$, $A \geqslant 0$. From what we proved before we get $\mathcal{O}^{\vee n} \subset \mathscr{L}$. Then if $A \in \mathcal{O}$, $\|A\| < 1$ we have $\Gamma(A)1_n \in \mathscr{L}$ and $\|\Gamma(A) - \Gamma(A)1_n\| \leqslant \|A\|^{n+1} \to 0$, so $\Gamma(A) \in \mathscr{L}$. This clearly proves $\mathscr{L} = \Gamma(\mathcal{O})$, i.e. (3.2). The inclusion $\subset$ in (3.6) is obvious and the inverse inclusion follows from the preceding arguments. $\square$

We are mainly interested in C^*-algebras of operators on $\Gamma(\mathcal{H})$ of the following form:

$$\mathscr{F}(\mathcal{O}) = C^*\big(W(u)\Gamma(A) \;\big|\; u \in \mathcal{H},\ A \in \mathcal{O},\ \|A\| < 1\big). \tag{3.7}$$

Observe that $\Gamma(\mathcal{O}) \subset \mathscr{F}(\mathcal{O})$.

Proposition 3.3. (1) *If $\mathcal{O}_1 \subset \mathcal{O}_2$ are C^*-subalgebras of $B(\mathcal{H})$ then $\mathscr{F}(\mathcal{O}_1) \subset \mathscr{F}(\mathcal{O}_2)$.*

(2) *We have $\mathscr{F}(\{0\}) = \mathscr{K}(\mathcal{H})$, in particular $\mathscr{K}(\mathcal{H}) \subset \mathscr{F}(\mathcal{O})$ for all $\mathcal{O}$.*

(3) *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{O} = \mathcal{O}_1 \oplus \mathcal{O}_2$ for some C^*-subalgebras $\mathcal{O}_i \subset B(\mathcal{H}_i)$, then*

$$\mathscr{F}(\mathcal{O}) = \mathscr{F}(\mathcal{O}_1) \otimes \mathscr{F}(\mathcal{O}_2), \tag{3.8}$$

where the tensor product is defined by the identification $\Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$.

Proof. The first assertion is obvious and an easy proof of (2) involves coherent vectors [35]. Indeed,

$$W(u)\Omega = \mathrm{e}^{-\|u\|^2/2}\mathrm{e}^{\mathrm{i}u} \equiv \mathrm{e}^{-\|u\|^2/2} \sum_n \frac{\mathrm{i}^n}{n!} u^n$$

and the linear span of these vectors is dense in $\Gamma(\mathcal{H})$. Thus the norm closed linear subspace of $B(\Gamma(\mathcal{H}))$ generated by the operators $W(u)\omega = |W(u)\Omega\rangle\langle\Omega|$ is equal to the space of rank one operators of the form $|u\rangle\langle\Omega|$ with $u \in \Gamma(\mathcal{H})$. But the C^*-algebra generated by these operators is exactly $\mathscr{K}(\mathcal{H})$. Finally, to prove (3) we argue as in the proof of Proposition 3.1 by using (2.19) and (2.20) in order to get $W(u)\Gamma(A) = [W(u_1)\Gamma(A_1)] \otimes [W(u_2)\Gamma(A_2)]$ if $u = u_1 \oplus u_2$ and $A = A_1 \oplus A_2$. $\square$

If $\mathcal{O} \subset B(\mathcal{H})$ is a C^*-subalgebra then let $\mathcal{H}_{\mathcal{O}}$ be the closed linear space generated by the vectors Au with $A \in \mathcal{O}$, $u \in \mathcal{H}$. One says that $\mathcal{O}$ is *non-degenerate* (or acts non-degenerately on $\mathcal{H}$) if

$\mathcal{H}_{\mathcal{O}} = \mathcal{H}$. Denote $\mathcal{O}_0$ the algebra $\mathcal{O}$ when viewed as a C^*-algebra of operators on $\mathcal{H}_{\mathcal{O}}$. Thus $\mathcal{O}_0$ acts non-degenerately on $\mathcal{H}_{\mathcal{O}}$ and we have $\mathcal{O}|\mathcal{H}_{\mathcal{O}}^{\perp} = \{0\}$, hence by (2) and (3) of Proposition 3.3:

$$\mathscr{F}(\mathcal{O}) = \mathscr{F}(\mathcal{O}_0) \otimes \mathscr{K}\big(\mathcal{H}_{\mathcal{O}}^{\perp}\big) \text{ relatively to } \Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_{\mathcal{O}}) \otimes \Gamma\big(\mathcal{H}_{\mathcal{O}}^{\perp}\big). \tag{3.9}$$

In some of our results we shall assume that $\mathcal{O}$ is non-degenerate but one may use (3.9) to extend them to possibly degenerate algebras. We shall not do it explicitly in order to simplify the arguments and also because this is of no interest in the applications we have in mind. In fact, we interpret $\mathcal{O}$ as the C^*-algebra generated by the allowed one particle Hamiltonians of the field, in particular there should be self-adjoint operators h on $\mathcal{H}$ affiliated to $\mathcal{O}$. But this implies that $\mathcal{O}$ is non-degenerate (see Section 7).

Proposition 3.4. *If $\mathcal{O}$ is non-degenerate then $\mathscr{F}(\mathcal{O})$ is the norm closed linear subspace generated by the operators of the form $W(u)\Gamma(A)$ with $u \in \mathcal{H}$ and $A \in \mathcal{O}$ such that $A \geqslant 0$ and $\|A\| < 1$.*

Proof. Let $\mathscr{M}$ be the norm closed linear subspace generated by the operators of the form $W(u)\Gamma(A)$ with A as in the statement of the lemma. Clearly $\mathscr{M} \subset \mathscr{F}(\mathcal{O})$ and (3.2) implies that $\mathscr{M}$ contains a set which generates $\mathscr{F}(\mathcal{O})$ as a C^*-algebra, so it suffices to show that $\mathscr{M}$ is a $*$-algebra. Proposition 3.2 shows that $W(u)\Gamma(A)1^n \equiv W(u)A^{\vee n} \in \mathscr{M}$ if $u \in \mathcal{H}$ and $A \in \mathcal{O}$. By computing derivatives with respect to $t_1, \ldots, t_p$ of $W(t_1u_1 + \cdots + t_pu_p)$ and by using the estimate

$$\big\|\phi(u)^p 1_n\big\| \leqslant \sqrt{p!}\|2\sqrt{n+1}\,u\|^p \tag{3.10}$$

which is a consequence of (2.15) we get $\phi(u_1)\ldots\phi(u_p)\Gamma(A)1^n \in \mathscr{M}$ for all $u_1, \ldots, u_p \in \mathcal{H}$. And this is equivalent to $a^*(u)^p a(v)^q 1^n \Gamma(A) \in \mathscr{M}$ for all u, v, p, q, n.

Now let $A, B \in \mathcal{O}$ be positive and $\varepsilon > 0$ real. Then (2.16) and $1^n a^*(u)^p a(v)^q = a^*(u)^p a(v)^q 1^{n-p+q}$ imply

$$1^n \Gamma(A + \varepsilon B) a^*(u)^p a\big((A + \varepsilon B)v\big)^q = a^*\big((A + \varepsilon B)u\big)^p a(v)^q 1^{n-p+q} \Gamma(A) \in \mathscr{M}.$$

Thus $1^n\Gamma(A + \varepsilon B)a^*(u)^p a(w)^q \in \mathscr{M}$ for each w in the closure of the range of an operator of the form $A + \varepsilon B$ (because the preceding expression is norm continuous as function of w). Now let J_ν be an approximate unit for $\mathcal{O}$ [40, pp. 77, 78], let $\mathcal{R}_\nu$ be the closure of the range of $A + \varepsilon J_\nu$, and $\mathcal{N}_\nu = \ker(A + \varepsilon J_\nu)$, so that $\mathcal{R}_\nu = \mathcal{N}_\nu^{\perp}$. We have $v \in \mathcal{N}_\nu$ if and only if $\langle v|Av\rangle = \langle v|J_\nu v\rangle = 0$ hence $\mathcal{N}_\mu \subset \mathcal{N}_\nu$ if $\mu \geqslant \nu$. Moreover, $\mathcal{N}_\nu$, and hence $\mathcal{R}_\nu$, is independent of ε. And we have $1^n\Gamma(A + \varepsilon J_\nu)a^*(u)^p a(w)^q \in \mathscr{M}$ for each $w \in \mathcal{R}_\nu$ by what we proved before. If we make here $\varepsilon \to 0$ then we get norm convergence and so $1^n\Gamma(A)a^*(u)^p a(w)^q \in \mathscr{M}$ for $w \in \mathcal{R}_\nu$. On the other hand, $\bigcap_\nu \mathcal{N}_\nu = \{0\}$ because $\mathcal{O}$ is non-degenerate and so $\lim_\nu J_\nu v = v$ for all $v \in \mathcal{H}$. It follows that $\{\mathcal{R}_\nu\}$ is an increasing family of closed subspaces of $\mathcal{H}$ whose union is dense in $\mathcal{H}$. Thus we have $1^n\Gamma(A)a^*(u)^p a(w)^q \in \mathscr{M}$ for w in the union and then by norm continuity for all $w \in \mathcal{H}$. Clearly then we get $1^n\Gamma(A)\phi(u)^p \in \mathscr{M}$ for all $A \in \mathcal{O}$ with $A \geqslant 0$ and $u \in \mathcal{H}$. From (3.10) we see that $1^n W(u) = \sum_p 1^n(\mathrm{i}\phi(u))^p/p!$ the series being convergent in norm. Hence $1^n\Gamma(A)W(u) \in \mathscr{M}$ for all $u \in \mathcal{H}$ and positive $A \in \mathcal{O}$. By arguments already used in the proof of Proposition 3.2 we obtain $1^n A^{\vee n} W(u) \in \mathscr{M}$ for arbitrary $A \in \mathcal{O}$. This clearly implies $\Gamma(A)W(u) \in \mathscr{M}$ if $A \in \mathcal{O}$ and $\|A\| < 1$.

To summarize, $\mathscr{M}$ is equal to the norm closed linear subspace generated by the operators $W(u)\Gamma(A)$ with $A \in \mathcal{O}$, $\|A\| < 1$, and we have proved that $\Gamma(A)W(u) \in \mathscr{M}$ under the same conditions. Thus $\mathscr{M}$ is stable under taking adjoints. For a product $W(u)\Gamma(A)W(v)\Gamma(B)$ we write $\Gamma(A)W(v)$ as limit of linear combinations of operators $W(w)\Gamma(C)$ with $C \in \mathcal{O}$, $\|C\| < 1$, and use (2.2) and $\Gamma(C)\Gamma(B) = \Gamma(CB)$. This gives $W(u)\Gamma(A)W(v)\Gamma(B) \in \mathscr{M}$, hence $\mathscr{M}$ is a C^*-algebra. □

Remark 3.5. The arguments of the preceding proof show that if $\mathcal{O}$ is non-degenerate then $\mathscr{F}(\mathcal{O})$ is the norm closed linear span of the operators $\phi(u)^n\Gamma(A)$ with $u \in \mathcal{H}$, $n \in \mathbb{N}$ and $A \in \mathcal{O}$ with $\|A\| < 1$.

Remark 3.6. Proposition 3.4 is not valid if $\mathcal{O}$ is degenerate. Indeed, with the notations of (3.9) and if $u = u_0 + u_1$ with $u_0 \in \mathcal{H}_{\mathcal{O}}$, $u_1 \in \mathcal{H}_{\mathcal{O}}^{\perp}$, then for $A \in \mathcal{O}$ with $\|A\| < 1$ we have

$$W(u)\Gamma(A) = \big[W(u_0)\Gamma(A_0)\big] \otimes \big[W(u_1)\Gamma(0)\big] = \big[W(u_0)\Gamma(A_0)\big] \otimes |W(u_1)\Omega\rangle\langle\Omega|$$

and the operators $|W(u_1)\Omega\rangle\langle\Omega|$ do not generate linearly $\mathscr{K}(\mathcal{H}_{\mathcal{O}}^{\perp})$.

Lemma 3.7. *Assume $A, B \in B(\mathcal{H})$ and $\|A\| \leqslant c$, $\|B\| \leqslant c$ with $c < 1$. If we set $\tilde{c} = \sup_{k\geqslant 1} kc^{k-1}$ then*

$$\big\|\Gamma(A) - \Gamma(B)\big\| \leqslant \tilde{c}\|A - B\|.$$

For $u, v \in \mathcal{H}$ and $n \in \mathbb{N}$ we have

$$\big\|\big(W(u) - W(v)\big)1_n\big\| \leqslant \big|\Im\langle u|v\rangle\big| + 2\sqrt{n+1}\|u - v\|.$$

If $\|A\| < 1$ the map $u \mapsto W(u)\Gamma(A)$ is norm continuous on $\mathcal{H}$ and $\|\phi(u)^p\Gamma(A)\| < \infty$ for all p.

Proof. To prove the first part it suffices to show that $\|A^{\vee k} - B^{\vee k}\| \leqslant kc^{k-1}\|A - B\|$ if $k \geqslant 1$. But this follows from $A^{\vee k} - B^{\vee k} = \sum_{j=0}^{k-1} B^{\vee j} \vee (A - B) \vee A^{\vee(k-1-j)}$. For the proof of the second estimate we note that (2.2) implies $\|(W(u) - W(v))1_n\| \leqslant |\mathrm{e}^{\mathrm{i}\Im\langle v|u\rangle} - 1| + \|(W(u - v))1_n - 1_n\|$ and then we use

$$\big\|W(u)1_n - 1_n\big\| = \left\| \int_0^1 W(tu)\mathrm{i}\phi(u)1_n\,dt \right\| \leqslant \big\|\phi(u)1_n\big\| \leqslant 2\sqrt{n+1}\|u\|.$$

Next observe that $W(u)\Gamma(A) = W(u)1_n\Gamma(A) + W(u)\Gamma(A)1_n^{\perp}$ and $\|W(u)\Gamma(A)1_n^{\perp}\| \leqslant \|A\|^{n+1}$. Finally, the estimate

$$\big\|\phi(u)^p\lambda^N\big\| \leqslant \|2u\|^p\big\|\sqrt{(N+1)\dots(N+p)}\lambda^N\big\| \leqslant \sqrt{p!}\|2u\|^p\big\|(N+1)^{p/2}\lambda^N\big\| \tag{3.11}$$

is a straightforward consequence of (2.15), and this proves the last assertion of the lemma. □

We define now an analogue in the present setting of the graded Weyl algebra which has been introduced and studied for finite-dimensional symplectic spaces $\mathcal{H}$ in [13,30]. The following construction makes sense for an arbitrary Weyl system $(\mathscr{H}, W)$. A finite-dimensional real vector subspace E of $\mathcal{H}$ inherits an Euclidean structure so it is equipped with a canonical translation invariant measure $d_E u$ and the corresponding $L^1(E)$ space is well defined. Since the map $u \mapsto W(u)$ is strongly continuous on E, we can define $W(f) = \int_E W(u) f(u)\, d_E u \in B(\mathscr{H})$ if $f \in L^1(E)$. Let

$$\mathscr{F}(E, \mathcal{H}) = \text{norm closure of } \{W(f) \mid f \in L^1(E)\}. \tag{3.12}$$

From (2.2) one may deduce that $\mathscr{F}_E(\mathcal{H})$ is a C^*-algebra and that we have (the proof given in [13] for finite-dimensional $\mathcal{H}$ extends without any modification to our context):

(i) $\mathscr{F}(E, \mathcal{H}) \cdot \mathscr{F}(F, \mathcal{H}) \subset \mathscr{F}(E + F, \mathcal{H})$,
(ii) if $\mathcal{L}$ is a finite family of finite-dimensional real subspaces of $\mathcal{H}$ then $\sum_{E \in \mathcal{L}} \mathscr{F}(E, \mathcal{H})$ is a norm closed subspace and the sum is a direct sum of linear spaces.

We define the *graded Weyl algebra* $\mathscr{F}(\mathcal{H}) \equiv \mathscr{W}_{\mathrm{gr}}(\mathcal{H})$ as the norm closure of $\sum_E \mathscr{F}(E, \mathcal{H})$, where E runs over the set of all finite-dimensional *complex* subspaces of $\mathcal{H}$. Then $\mathscr{F}(\mathcal{H})$ is equipped with a graded C^*-algebra structure in the sense of [21, Definition 3.1]. $\mathscr{F}(\mathcal{H})$ is unital because $\mathscr{F}(\{0\}, \mathcal{H}) = \mathbb{C}$.

In the Fock representation we have a quite explicit description of the algebras $\mathscr{F}(E, \mathcal{H})$. This follows, as explained in [13], from the fact that a complex finite-dimensional subspace of $\mathcal{H}$ is symplectic:

$$\mathscr{F}(E, \mathcal{H}) = \mathscr{K}(E) \otimes 1 \text{ relatively to the factorization } \Gamma(\mathcal{H}) = \Gamma(E) \otimes \Gamma(E^{\perp}). \tag{3.13}$$

Finally, we define $\mathscr{W}_{\max}(\mathcal{H})$, the largest C^*-algebra of operators which can be naturally associated to the Weyl system in the Fock representation. In particular, $\mathscr{W}_{\max}(\mathcal{H})$ contains $\mathscr{W}(\mathcal{H})$ and $\mathscr{F}(\mathcal{H})$. If f is a bounded Borel regular measure on $\mathcal{H}$ (for the norm topology) and $v \in \Gamma_{\mathrm{fin}}(\mathcal{H})$ then the integral $W(f)v = \int_{\mathcal{H}} W(u)v\, df(u)$ is well defined because, by Lemma 3.7, the map $u \mapsto W(u)v$ is bounded and continuous on $\mathcal{H}$. Clearly $\|W(f)v\| \leqslant \|v\| \|f\|$ where $\|f\|$ is the variation of f, so $v \mapsto W(f)v$ extends to a bounded operator $W(f)$ on $\Gamma(\mathcal{H})$. It is easy to show that the set of operators $W(f)$ is a $*$-algebra and we define $\mathscr{W}_{\max}(\mathcal{H})$ as its norm closure.

If $\mathscr{M}, \mathscr{N}$ are C^*-subalgebras of a given C^*-algebra we denote by $\mathscr{M} \cdot \mathscr{N}$ the linear subspace consisting of the operators of the form $S_1 T_1 + \cdots + S_n T_n$ with $S_i \in \mathscr{M}$, $T_i \in \mathscr{N}$ and $n \geqslant 1$, and by $[\![\mathscr{M} \cdot \mathscr{N}]\!]$ the norm closure of this linear subspace.

Proposition 3.8. *If $\mathcal{O}$ is non-degenerate then*

$$\mathscr{F}(\mathcal{O}) = [\![\mathscr{F}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!] = [\![\mathscr{W}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!] = [\![\mathscr{W}_{\max}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!]. \tag{3.14}$$

Proof. We first observe that $W(u)\Gamma(A) \in [\![\mathscr{F}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!]$ if $u \in \mathcal{H}$ and $\|A\| < 1$. Indeed, since $W(tu)\Gamma(A)$ is a norm continuous function of t (see Lemma 3.7), the sequence $\int_{\mathbb{R}} W(tu) f_k(t)\, dt\, \Gamma(A)$ converges in norm to $W(u)\Gamma(A)$ if f_k is a sequence in $L^1(\mathbb{R})$ which converges to the Dirac measure at $t = 1$. Thus $\mathscr{F}(\mathcal{O}) \subset [\![\mathscr{F}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!]$ by Proposition 3.4. The converse inclusion follows from the norm continuity of the map $u \mapsto W(u)\Gamma(A)$ (use again Lemma 3.7). For the same reason we have $W(f)\Gamma(A) \in [\![\mathscr{W}(\mathcal{H}) \cdot \Gamma(\mathcal{O})]\!]$ for an arbitrary bounded Borel regular measure on $\mathcal{H}$. □

Proposition 3.10 will justify the physical interpretation of the algebra $\mathscr{F}(\mathcal{O})$ as C^*-algebra of energy observables of the field with one particle kinetic energy affiliated to $\mathcal{O}$. Recall that QFH is an abbreviation for "quantum field Hamiltonian."

Definition 3.9. We shall call *elementary quantum field Hamiltonian of type* $\mathcal{O}$ a self-adjoint operator of the form $H = \mathrm{d}\Gamma(h) + V$ where: (i) h is a self-adjoint operator on $\mathcal{H}$ with $h \geqslant m$ for some real $m > 0$ and $h^{-1} \in \mathcal{O}$; (ii) V a symmetric operator such that $V = W(f)$ with $f \in L^1(E)$ for some finite-dimensional linear space $E \subset \mathcal{H}$.

For a self-adjoint operator h such that $h \geqslant m > 0$ the relations $h^{-1} \in \mathcal{O}$ and $\mathrm{e}^{-h} \in \mathcal{O}$ are equivalent and imply $\theta(h) \in \mathcal{O}$ for all $\theta \in C_0(\mathbb{R})$. If an elementary QFH of type $\mathcal{O}$ exists then $\mathcal{O}$ contains a positive injective operator, e.g. $A = h^{-1}$, and this clearly implies that $\mathcal{O}$ is non-degenerate. Reciprocally:

Proposition 3.10. *If $\mathcal{O}$ contains a positive injective operator then $\mathscr{F}(\mathcal{O})$ is the C^*-algebra generated by the elementary QFH of type $\mathcal{O}$. Hence $\mathscr{F}(\mathcal{O}) = C^*(\mathrm{e}^{-H} \mid H$ is an elementary QFH).*

Proof. Let $H_s = \mathrm{d}\Gamma(h) + sV \equiv H_0 + sV$ where h, V are as in Definition 3.9 and s is a real number. If z is far enough from the spectrum of H_0 then we have a norm convergent expansion for $R_s = (z - H)^{-1}$:

$$R_s = R_0(1 - VR_0)^{-1} = \sum_{n \geqslant 0} s^n R_0(VR_0)^n. \tag{3.15}$$

We have $\mathrm{e}^{-tH_0} = \Gamma(\mathrm{e}^{-th}) \in \Gamma(\mathcal{O})$ if $t > 0$ because $\mathrm{e}^{-th} \in \mathcal{O}$ and has norm < 1, so $R_0 \in \Gamma(\mathcal{O})$. From Proposition 3.8 we then get $R_s \in \mathscr{F}(\mathcal{O})$, hence the C^*-algebra $\mathscr{C}$ generated by the elementary QFH is contained in $\mathscr{F}(\mathcal{O})$.

We now prove the converse inclusion. Let h and H_s be as above, so that $R_s \in \mathscr{C}$ for all s. By taking the first order derivative at $s = 0$ in (3.15) we get $R_0 V R_0 \in \mathscr{C}$. By definition we have $\theta(H_0) \in \mathscr{C}$ for any $\theta \in C_0(\mathbb{R})$, hence we also have $\theta(H_0)R_0VR_0 \in \mathscr{C}$. By choosing θ conveniently in $C_\mathrm{c}(\mathbb{R})$ and then by an approximation argument we get $\eta(H_0)VR_0 \in \mathscr{C}$ for all $\eta \in C_0(\mathbb{R})$.

Let η_n be a sequence of continuous functions with $0 \leqslant \eta_n \leqslant 1$, $\eta_n(x) = 1$ if $|x| \leqslant n$, and $\eta_n(x) = 0$ if $|x| \geqslant n + 1$. Our next purpose is to prove that $\eta_n(H_0)VR_0 \to VR_0$ in norm. The operator $(N + 1)R_0$ is bounded, hence it is easy to see that it suffices to show that $\|1_n^\perp V(N + 1)^{-1}\| \to 0$ as $n \to \infty$. We have $V = W(f) = \int_E W(u) f(u)\, d\lambda_E(u)$ for some subspace E of finite dimension and $f \in L^1(E)$ and it is clear that for the proof of this assertion it suffices to assume that f has compact support. We have

$$\left\|1_n^\perp W(u)(N + 1)^{-1}\right\| \leqslant (n + 1)^{-1} + \left\|1_n^\perp \left[W(u), (N + 1)^{-1}\right]\right\|.$$

On the other hand $[N, W(u)] = W(u)(\phi(\mathrm{i}u) + \|u\|^2)$ hence by using (3.10) we get:

$$\begin{aligned}\left\|1_n^\perp \left[W(u), (N + 1)^{-1}\right]\right\| &= \left\|1_n^\perp (N + 1)^{-1} W(u)\big(\phi(\mathrm{i}u) + \|u\|^2\big)(N + 1)^{-1}\right\| \\ &\leqslant (n + 1)^{-1}\left\|\big(\phi(\mathrm{i}u) + \|u\|^2\big)(N + 1)^{-1}\right\| \leqslant (n + 1)^{-1}\big(2\|u\| + \|u\|^2\big).\end{aligned}$$

Thus we have

$$\left\| 1_n^\perp W(u)(N+1)^{-1} \right\| \leqslant \left(1 + \|u\|\right)^2 (n+1)^{-1}$$

from which we get $\|1_n^\perp V(N+1)^{-1}\| \to 0$. This finishes the proof of $\lim \eta_n(H_0) V R_0 \to V R_0$ in norm which in turn implies $V R_0 \in \mathscr{C}$.

Thus we have $V R_0 \in \mathscr{C}$ and then $V \mathrm{e}^{-H_0} = V R_0 \cdot (z - H_0)\mathrm{e}^{-H_0} \in \mathscr{C}$. Since $\mathrm{e}^{-H_0} = \Gamma(\mathrm{e}^{-h})$ we obtain $V\Gamma(A) \in \mathscr{C}$ for any operator A of the form $A = \mathrm{e}^{-h}$ with h a self-adjoint operator on $\mathcal{H}$ such that $h \geqslant m > 0$ and $\mathrm{e}^{-h} \in \mathcal{O}$. In other terms, we have $V\Gamma(A) \in \mathscr{C}$ for any operator $A \in \mathcal{O}$ such that A is positive and injective and such that $\|A\| < 1$. Indeed, it suffices then to choose $h = -\log A$. Now let $A \in \mathcal{O}$ be positive and $\|A\| < 1$. By assumption, $\mathcal{O}$ contains a positive injective operator S. If $\varepsilon > 0$ is small enough then $A_\varepsilon = A + \varepsilon S$ belongs to $\mathcal{O}$, is positive and injective, and $\|A_\varepsilon\| \leqslant c < 1$ uniformly in ε. Then $V\Gamma(A_\varepsilon) \in \mathscr{C}$ and from Lemma 3.7 we get $V\Gamma(A) \in \mathscr{C}$. Finally, (3.2) shows that $VT \in \mathscr{C}$ for all $T \in \Gamma(\mathcal{O})$. From Proposition 3.8 we obtain $\mathscr{F}(\mathcal{O}) \subset \mathscr{C}$. □

4. $\mathscr{A}(\mathcal{H})$ and its canonical endomorphism

We set $\mathscr{A}(\mathcal{H}) = \mathscr{F}(\mathbb{C}1_{\mathcal{H}})$. From Proposition 3.8 we get:

$$\mathscr{A}(\mathcal{H}) = \left[\!\left[\mathscr{F}(\mathcal{H}) \cdot C_0(N) \right]\!\right] = \left[\!\left[\mathscr{W}(\mathcal{H}) \cdot C_0(N) \right]\!\right] = \left[\!\left[\mathscr{W}_{\max}(\mathcal{H}) \cdot C_0(N) \right]\!\right]. \tag{4.1}$$

Alternative descriptions of $\mathscr{A}(\mathcal{H})$ are consequences of the results form Section 3. For example, $\mathscr{A}(\mathcal{H})$ is the norm closed subspace generated by each of the following classes of operators:

(i) $\phi(u)^n \theta(N)$ with $u \in \mathcal{H}$, $n \in \mathbb{N}$ and $\theta \in C_{\mathrm{c}}(\mathbb{R})$;
(ii) $a^*(u)^p a(v)^q 1^n$ with $u, v \in \mathcal{H}$ and $p, q, n \geqslant 0$.

Proposition 4.1. *$\mathscr{K}(\mathcal{H}) \subset \mathscr{A}(\mathcal{H})$ and $\mathscr{K}(\mathcal{H}) = \mathscr{A}(\mathcal{H})$ if and only if $\mathcal{H}$ is finite-dimensional.*

Proof. The first assertion is clear by Proposition 3.3. $\mathcal{H}$ is finite-dimensional if and only if $1^1 \in \mathscr{K}(\mathcal{H})$ and then $C_0(N) \subset \mathscr{K}(\mathcal{H})$. Since $1^1 \in \mathscr{A}(\mathcal{H})$, the second assertion of the proposition follows. □

If E is a finite-dimensional (complex) subspace of $\mathcal{H}$ let us define

$$\mathscr{A}_E(\mathcal{H}) = \left[\!\left[\mathscr{W}(E) \cdot C_0(N) \right]\!\right] = \left[\!\left[\mathscr{F}(E, \mathcal{H}) \cdot C_0(N) \right]\!\right]. \tag{4.2}$$

The equality follows from the arguments of the proof of Proposition 3.8. Note that $\mathscr{A}_{\{0\}}(\mathcal{H}) = C_0(N)$. With the notation $N'_E = N_{E^\perp}$ introduced in Section 2, we have:

Proposition 4.2. *$\mathscr{A}_E(\mathcal{H}) = \mathscr{K}(E) \otimes C_0(N'_E)$ relatively to $\Gamma(\mathcal{H}) = \Gamma(E) \otimes \Gamma(E^\perp)$. In other terms*:

$$\mathscr{A}_E(\mathcal{H}) = \bigoplus_n \mathscr{K}(E) \otimes 1^n_{E^\perp} = \left\{ \sum_n K_n \otimes 1^n_{E^\perp} \mid K_n \in \mathscr{K}(E),\ \|K_n\| \to 0 \right\}, \tag{4.3}$$

where $1^n_{E^\perp}$ is the projection onto the n particle subspace of $\Gamma(E^\perp)$, in particular $1^0_{E^\perp} = \omega'_E$. If $\mathcal{H}$ is infinite-dimensional then

$$\mathscr{A}_E(\mathcal{H}) \cap \mathscr{K}(\mathcal{H}) = \mathscr{K}(E) \otimes \omega'_E \equiv \mathscr{K}_E(\mathcal{H}). \tag{4.4}$$

Proof. By an argument used before $\mathscr{A}_E$ is the closed linear space generated by the operators $T\lambda^N$ with $T \in \mathscr{F}(E, \mathcal{H})$ and $0 < \lambda < 1$. By (3.13) this is the same as the closed linear space generated by $(K\lambda^{N_E}) \otimes \lambda^{N'_E}$ with K compact on $\Gamma(E)$. Replacing K by $K\theta(N_E)\lambda^{-N_E}$ with θ with compact support and then making $\theta \to 1$ we see that $\mathscr{A}_E$ is generated by the operators $K \otimes \lambda^{N'_E}$, which proves the assertion of the proposition. □

We now prove that $\mathscr{A}(\mathcal{H})$ is the inductive limit of the family of C^*-algebras $\{\mathscr{A}_E(\mathcal{H})\}$.

Proposition 4.3. *If $E \subset F$ are finite-dimensional subspaces of $\mathcal{H}$ then $\mathscr{A}_E(\mathcal{H}) \subset \mathscr{A}_F(\mathcal{H})$. And we have*

$$\mathscr{A}(\mathcal{H}) = \overline{\bigcup_E \mathscr{A}_E(\mathcal{H})}. \tag{4.5}$$

Proof. We begin with a general remark. Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. If z is a complex number such that $|z| < 1$ then $z^N = z^{N_\mathcal{K}} \otimes z^{N'_\mathcal{K}} \in C_0(N_\mathcal{K}) \otimes C_0(N'_\mathcal{K})$. This clearly implies

$$C_0(N) \subset C_0(N_\mathcal{K}) \otimes C_0\big(N'_\mathcal{K}\big). \tag{4.6}$$

Now let us set $G = F \ominus E$. From $\mathcal{H} = E \oplus G \oplus F^\perp$ we get $\Gamma(\mathcal{H}) = \Gamma(E) \otimes \Gamma(G) \otimes \Gamma(F^\perp)$, hence:

$$\begin{aligned}\mathscr{A}_E(\mathcal{H}) &= \mathscr{K}(E) \otimes C_0\big(N'_E\big) \subset \mathscr{K}(E) \otimes C_0(N_G) \otimes C_0\big(N'_F\big)\\ &\subset \mathscr{K}(E) \otimes \mathscr{K}(G) \otimes C_0\big(N'_F\big) = \mathscr{K}(F) \otimes C_0\big(N'_F\big) = \mathscr{A}_F(\mathcal{H}).\end{aligned}$$

We have used (4.6), the fact that $C_0(N_G) \subset \mathscr{K}(G)$ since G is finite-dimensional, and (2.21). □

If $\mathcal{P}$ is an endomorphism of $\mathscr{A}(\mathcal{H})$, then the following conditions are equivalent:

(i) $\mathcal{P}(W(u)\lambda^N) = \lambda W(u)\lambda^N$ for each $u \in \mathcal{H}$ and $0 < \lambda < 1$;
(ii) $\mathcal{P}(W(u)\theta(N)) = W(u)\theta(N+1)$ for each $u \in \mathcal{H}$ and $\theta \in C_0(\mathbb{N})$.

Indeed, since $\theta_\lambda(n) = \lambda^n$ defines a function in $C_0(\mathbb{N})$, we see that (ii) ⇒ (i). To prove the converse, it suffices to note that the closed subspace generated by the functions θ_λ, $0 < \lambda < 1$, is dense in $C_0(\mathbb{N})$.

If a morphism $\mathcal{P}:\mathscr{A}(\mathcal{H})\to\mathscr{A}(\mathcal{H})$ satisfying the conditions (i) or (ii) above exists then *it is unique and surjective* by (4.1). We shall call it *the canonical endomorphism of* $\mathscr{A}(\mathcal{H})$. If $\mathcal{H}$ is finite-dimensional then $\mathscr{A}(\mathcal{H})=\mathscr{K}(\mathcal{H})$ has no nontrivial ideals, so the canonical endomorphism cannot exist.

Theorem 4.4. *If $\mathcal{H}$ is infinite-dimensional then the canonical endomorphism of $\mathscr{A}(\mathcal{H})$ exists and its kernel is $\mathscr{K}(\mathcal{H})$. Hence we have a canonical identification*

$$\mathscr{A}(\mathcal{H})/\mathscr{K}(\mathcal{H})\cong\mathscr{A}(\mathcal{H}). \tag{4.7}$$

Proof. Let τ be the endomorphism of $C_0(\mathbb{N})$ defined by $(\tau\theta)(m)=\theta(m+1)$. If $\mathcal{K}\neq\{0\}$ then $C_0(N_{\mathcal{K}})$ is isomorphic with $C_0(\mathbb{N})$ hence we get a realization of τ as endomorphism of $C_0(N_{\mathcal{K}})$. For each finite-dimensional subspace E let $\mathcal{P}_E=1\otimes\tau$, which is an endomorphism of $\mathscr{A}_E=\mathscr{K}(E)\otimes C_0(N'_E)$. We have $\ker\mathcal{P}_E=\mathscr{K}(E)\otimes\ker\tau$ because tensor product with $\mathscr{K}(E)$ preserves exact sequences [40, Theorem 6.5.2]. Since $\tau\theta(N'_E)=\theta(N'_E+1)$ we have $\ker\tau=\mathbb{C}\omega'_E$, so $\ker\mathcal{P}_E=\mathscr{K}(E)\otimes\omega'_E=\mathscr{A}_E\cap\mathscr{K}(\mathcal{H})$ because of (4.4).

Let F be a second finite-dimensional subspace such that $E\subset F$. Then we have $\mathscr{A}_E\subset\mathscr{A}_F$ and we shall prove that $\mathcal{P}_E$ is the restriction of $\mathcal{P}_F$ to $\mathscr{A}_E$. From (4.2) and arguments used before we see that $\mathscr{A}_E$ is the norm closed linear space generated by the operators $T=W(u)\lambda^N$ with $u\in E$ and $0<\lambda<1$, hence it suffices to show that $\mathcal{P}_E$ and $\mathcal{P}_F$ are equal on such elements. We have $T=(W(u)\lambda^{N_E})\otimes\lambda^{N'_E}$ relatively to the tensor factorization $\Gamma(\mathcal{H})=\Gamma(E)\otimes\Gamma(E^\perp)$ hence

$$\mathcal{P}_E(T)=\bigl(W(u)\lambda^{N_E}\bigr)\otimes\lambda^{N'_E+1}=W(u)\lambda^{N+1}.$$

An identical computation gives $\mathcal{P}_F(T)=W(u)\lambda^{N+1}$, which proves our assertion.

Now from Proposition 4.3 it follows that there is a unique endomorphism $\mathcal{P}$ of $\mathscr{A}$ such that $\mathcal{P}|\mathscr{A}_E=\mathcal{P}_E$. It is clear that $\mathcal{P}$ is the canonical endomorphism of $\mathscr{A}$. From Lemma 2.1 it follows that $\mathcal{P}(K)=0$ if K is a compact operator. Reciprocally, assume that $\mathcal{P}(K)=0$ and let $\varepsilon>0$. From (4.5) it follows that there is E and $K_E\in\mathscr{A}_E$ such that $\|K-K_E\|<\varepsilon$. Thus $\|\mathcal{P}_E(K_E)\|<\varepsilon$. The kernel of $\mathcal{P}_E$ is $\mathscr{K}_E=\mathscr{A}_E\cap\mathscr{K}(\mathcal{H})$ and $\mathcal{P}_E$ induces an isometric map from the quotient $\mathscr{A}_E/\mathscr{K}_E$ onto $\mathscr{A}_E$. From the definition of the quotient norm it follows that there is $L\in\mathscr{K}_E$ such that $\|K_E-L\|<2\varepsilon$. This implies $\|K-L\|<3\varepsilon$ and since L is a compact operator and ε is arbitrary, we see that K is compact. □

Remark 4.5. The following explicit expression of $\mathcal{P}$ has been noticed by George Skandalis:

$$\mathcal{P}(T)u=\operatorname*{s-lim}_{e\rightharpoonup 0}a(e)Ta^*(e)u\qquad\text{for all } T\in\mathscr{A}(\mathcal{H})\text{ and } u\in\Gamma_{\mathrm{fin}}(\mathcal{H}). \tag{4.8}$$

This is similar to relation (2.2) in [13]. The notation $e\rightharpoonup 0$ means that $\|e\|=1$ and that e converges to zero in the weak[6] topology. (4.8) follows easily from (2.8), (2.11) and $\text{s-lim}_{e\rightharpoonup 0}a(e)a^*(e)1_n=1_n$.

[6] More precisely, the limit is taken along the filter consisting of the intersections of the neighborhoods of zero in the weak topology with the unit sphere of $\mathcal{H}$. One may also replace it by the finer filter consisting of the subsets of the unit sphere which are orthogonal to finite-dimensional subspaces of $\mathcal{H}$, the proof is then simpler.

We give an application of Theorem 4.4 in spectral theory. Let $\mathcal{H}$ be infinite-dimensional.

Lemma 4.6. *If $T \in \mathscr{A}(\mathcal{H})$ then $\lim_{k\to\infty} \|\mathcal{P}^k(T)\| = 0$. Moreover, $1^n \in \mathscr{A}(\mathcal{H})$ and $\mathcal{P}^k(1^n) = 1^{n-k}$.*

Proof. From the characterizations of $\mathscr{A}(\mathcal{H})$ given in (4.1) we see that it suffices to consider T of the form $T = W(u)\theta(N)$ with $\theta \in C_c(\mathbb{N})$. Then $\mathcal{P}^k(T) = W(u)\theta(N+k) = 0$ for k large. □

Proposition 4.7. *The spectrum of an element of $\mathscr{A}(\mathcal{H})$ is countable. If $T \in \mathscr{A}(\mathcal{H})$ then its essential spectrum is equal to the spectrum of $\mathcal{P}(T)$.*

Proof. Let $\sigma_{\mathrm{ess}}(T)$ be the essential spectrum of an operator T and $\sigma_{\mathrm{d}}(T)$ its discrete spectrum, so that $\sigma(T)$ is equal to the *disjoint* union $\sigma_{\mathrm{d}}(T) \sqcup \sigma_{\mathrm{ess}}(T)$ and $\sigma_{\mathrm{d}}(T)$ does not have accumulation points outside $\sigma_{\mathrm{ess}}(T)$. If $T \in \mathscr{A}(\mathcal{H})$ then $\sigma_{\mathrm{ess}}(T) = \sigma(\mathcal{P}(T))$ hence we get by induction:

$$\sigma(T) = \sigma_{\mathrm{d}}(T) \sqcup \sigma\big(\mathcal{P}(T)\big) = \left[\bigsqcup_{k=0}^{n} \sigma_{\mathrm{d}}\big(\mathcal{P}^k(T)\big)\right] \sqcup \sigma\big(\mathcal{P}^{n+1}(T)\big)$$

which proves the assertion of the proposition. □

Remark 4.8. The following comments on the algebra $\mathscr{A}(\mathcal{H})$ play no role in this paper but are of some intrinsic interest. The advantage in using the graded Weyl algebra $\mathscr{F}(\mathcal{H})$ instead of other Weyl algebras which can be found in the literature is that N implements a norm continuous action of the unit circle on it. Indeed, (2.9) gives for $z \in \Sigma = \{z \in \mathbb{C} \mid |z| = 1\}$ and $u \in \mathcal{H}$

$$z^N W(u) \bar{z}^N = W(zu). \tag{4.9}$$

If E is a (complex) finite-dimensional subspace of $\mathcal{H}$ then E is stable under multiplication by z and for $f \in L^1(E)$ we have

$$z^N W(f) \bar{z}^N = \int_E W(zu) f(u)\, d_E u = \int_E W(u) f(\bar{z}u)\, d_E u \equiv W(f_z).$$

Since $\|W(f_z) - W(f)\| \leqslant \|f_z - f\|_{L^1} \to 0$ as $z \to 1$ we see that $z \mapsto z^N W(f) \bar{z}^N$ is norm continuous.

Thus $\alpha_z(T) = z^N T \bar{z}^N$ induces a norm continuous action of Σ on $\mathscr{F}(\mathcal{H})$ which is compatible with the grading (i.e. each $\mathscr{F}(E, \mathcal{H})$ is stable under the action). In particular, the crossed product C^*-algebra $\mathscr{F}(\mathcal{H}) \rtimes \Sigma$ is well defined. The algebra $\mathscr{A}(\mathcal{H})$ is a quotient of this crossed product: there is a unique morphism $\mathscr{F}(\mathcal{H}) \rtimes \Sigma \to \mathscr{A}(\mathcal{H})$ such that the image of $T \otimes \eta$ be $T\widetilde{\eta}(N) \equiv T \int_\Sigma z^N \eta(z)\, dz$ for all $T \in \mathscr{F}(\mathcal{H})$, $\eta \in L^1(\Sigma)$, see [29, Theorem 2.9]. This morphism is surjective but not injective.

The similarly defined morphism $\mathscr{F}(E, \mathcal{H}) \rtimes \Sigma \to \mathscr{A}_E(\mathcal{H})$ can be used in order to give a more conceptual proof of the existence of the morphism $\mathcal{P}_E$ constructed at the beginning of the proof of Theorem 4.4. I am indebted to G. Skandalis for a comment which clarified this point to me.

5. Canonical morphism of $\mathscr{F}(\mathcal{O})$

We now extend the results of Section 4 to a larger class of C^*-algebras $\mathcal{O}$ of operators on $\mathcal{H}$.

Definition 5.1. If a morphism $\mathcal{P}:\mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ with the property

$$\mathcal{P}\big(W(u)\Gamma(A)\big) = A \otimes \big[W(u)\Gamma(A)\big] \quad \text{if } u \in \mathcal{H} \text{ and } A \in \mathcal{O} \text{ with } \|A\| < 1 \tag{5.1}$$

exists, then it is uniquely determined and we call it the *canonical morphism* of $\mathscr{F}(\mathcal{O})$.

Example 5.2. Assume that $\mathcal{P}$ exists and recall that $\Gamma(\mathcal{O}) \subset \mathscr{F}(\mathcal{O})$. Then $\mathcal{P}(\Gamma(A)) = A \otimes \Gamma(A)$ if $A \in \mathcal{O}$ and $\|A\| < 1$. Replacing A by tA and taking derivatives at $t = 0$ we obtain $\mathcal{P}(A^{\vee 0}) \equiv \mathcal{P}(\omega) = 0$ and $\mathcal{P}(A^{\vee n}) = A \otimes A^{\vee(n-1)}$ if $n \geqslant 1$ (recall that $A^{\vee 0} = \omega$). From the polarization formula we then get

$$n\mathcal{P}(A_1 \vee \cdots \vee A_n) = \sum_k A_k \otimes [A_1 \vee \cdots \vee A_{k-1} \vee A_{k+1} \vee \cdots \vee A_n] \tag{5.2}$$

for all $A_1, \ldots, A_n \in \mathcal{O}$.

Remark 5.3. If needed we denote $\mathcal{P}_{\mathcal{O}}$ the morphism from Definition 5.1. Observe that if $\mathcal{O}_1 \subset \mathcal{O}_2$ and if the canonical morphism $\mathcal{P}_{\mathcal{O}_2}$ exists, then $\mathcal{P}_{\mathcal{O}_1}$ exists too and we have $\mathcal{P}_{\mathcal{O}_1} = \mathcal{P}_{\mathcal{O}_2}|\mathscr{F}(\mathcal{O}_1)$.

Theorem 5.4. *If $\mathcal{O}$ is an abelian C^*-algebra on $\mathcal{H}$ and its strong closure does not contain finite rank operators then the canonical morphism $\mathcal{P}$ exists and* $\ker \mathcal{P} = \mathscr{K}(\mathcal{H})$. *This gives a canonical embedding*

$$\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O}). \tag{5.3}$$

Remark 5.5. Observe that $\mathcal{H}$ cannot be finite-dimensional. In the rest of this remark we assume $\mathcal{O}$ non-degenerate and denote $\mathcal{O}'$ and $\mathcal{O}''$ its commutant and bicommutant. Note that

$$\mathscr{K}(\mathcal{H}) \subset \mathscr{F}(\mathcal{O}) \subset \mathscr{F}(\mathcal{O}''). \tag{5.4}$$

The strong closure of $\mathcal{O}$ is $\mathcal{O}''$, thus in Theorem 5.4 we have to assume that $\mathcal{O}''$ does not contain finite rank operators. Clearly this is equivalent to $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$. Observe that if there is a sequence of unitary operators $U_n \in \mathcal{O}'$ such that $\text{w-lim}_{n\to\infty} U_n = 0$ then this assumption is satisfied. On the other hand, if $\mathcal{H}$ is separable then $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$ if and only if there is a self-adjoint operator $S \in \mathcal{O}'$ with purely absolutely continuous spectrum; and if this is the case then $\mathrm{e}^{\mathrm{i}tS} \in \mathcal{O}'$ and $\text{w-lim}_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tS} = 0$.

Lemma 5.6. *Let $\mathcal{O}$ be an abelian finite-dimensional C^*-algebra on $\mathcal{H}$ with $1_{\mathcal{H}} \in \mathcal{O}$. Let $P_1, \ldots, P_n$ be the minimal projections of $\mathcal{O}$ and $\mathcal{H}_k = P_k\mathcal{H}$. Then $\mathcal{H} = \bigoplus_k \mathcal{H}_k$ and we have*

$$\mathscr{F}(\mathcal{O}) = \bigotimes_k \mathscr{A}(\mathcal{H}_k) \quad \textit{relatively to } \Gamma(\mathcal{H}) = \bigotimes_k \Gamma(\mathcal{H}_k). \tag{5.5}$$

Proof. Recall that we have $P_k \neq 0$, $P_i P_j = 0$ if $i \neq j$ and $P_1 + \cdots + P_n = 1_{\mathcal{H}}$. Moreover, each element of $\mathcal{O}$ is a linear combination of these projections. Thus we can write $\mathcal{O}$ as a direct sum of C^*-algebras $\mathcal{O} \equiv \bigoplus_k \mathbb{C} P_k$ and then we may use (2) of Proposition 3.3. More explicitly, if $A \in \mathcal{O}$ then $A = \sum_k z_k P_k$ and we have $\|A\| = \sup_k |z_k|$. Assume $\|A\| < 1$ and let $u \equiv \sum_k u_k$, then we get from (2.19) and (2.20)

$$W(u)\Gamma(A) = \bigotimes_k \big[W(u_k)\Gamma(z_k P_k)\big] \equiv \bigotimes_k \big[W(u_k)\Gamma(z_k)\big]$$

where we have identified $P_k = 1_{\mathcal{H}_k}$. Then (5.5) follows easily from this relation. □

Lemma 5.7. *Theorem* 5.4 *is true if $\mathcal{O}$ is finite-dimensional and* $1_{\mathcal{H}} \in \mathcal{O}$.

Proof. We keep the notations of Lemma 5.6 and observe that each $\mathcal{H}_k$ is infinite-dimensional because $\mathcal{O}$ does not contain finite-dimensional projections. By Theorem 4.4 the canonical endomorphism $\mathcal{P}_k$ of $\mathscr{A}(\mathcal{H}_k)$ exists. We shall now use Proposition 10.1: define $\mathcal{P}'_k$ as in that theorem and note that $\mathscr{J}_k = \mathscr{K}(\mathcal{H}_k)$ and $\widetilde{\mathscr{A}(\mathcal{H}_k)} = \mathscr{A}(\mathcal{H}_k)$. Proposition 4.2 implies that each $\mathscr{A}(\mathcal{H}_k)$ is nuclear. Taking into account Lemma 5.6 and Proposition 10.1 we get a morphism

$$\begin{aligned}\mathcal{P} \equiv \bigoplus_{k=1}^{n} \mathcal{P}'_k : \mathscr{F}(\mathcal{O}) &\to \bigoplus_{k=1}^{n} \mathscr{A}(\mathcal{H}_1) \otimes \cdots \otimes \mathscr{A}(\mathcal{H}_k) \otimes \cdots \otimes \mathscr{A}(\mathcal{H}_n) \\ &\equiv \mathbb{C}^n \otimes \mathscr{F}(\mathcal{O}) \cong \mathcal{O} \otimes \mathscr{F}(\mathcal{O})\end{aligned}$$

whose kernel is $\mathscr{K}(\mathcal{H}_1) \otimes \cdots \otimes \mathscr{K}(\mathcal{H}_n) = \mathscr{K}(\mathcal{H})$. Then, with the notations of the proof of Lemma 5.6:

$$\begin{aligned}\mathcal{P}\big(W(u)\Gamma(A)\big) &= \bigoplus_{k=1}^{n} \mathcal{P}'_k\big[\otimes_k\big[W(u_i)\Gamma(z_i)\big]\big] \\ &= \bigoplus_{k=1}^{n} \big[W(u_1)\Gamma(z_1)\big] \otimes \cdots \otimes \big[z_k W(u_k)\Gamma(z_k)\big] \otimes \cdots \otimes \big[W(u_n)\Gamma(z_n)\big] \\ &= (z_1 P_1 + \cdots + z_n P_n) \otimes \big(W(u)\Gamma(A)\big) = A \otimes \big(W(u)\Gamma(A)\big).\end{aligned}$$

Thus $\mathcal{P}$ is the canonical morphism of $\mathscr{F}(\mathcal{O})$. □

Proof of Theorem 5.4. If the theorem has been proved for non-degenerate $\mathcal{O}$ then the general case is a consequence of the factorization (3.9) and of Proposition 10.1 with $n = 2$, $\mathscr{C}_1 = \mathscr{F}(\mathcal{O}_0)$, $\mathscr{C}_2 = \mathscr{J}_2 = \mathscr{K}(\mathcal{H}_{\mathcal{O}}^{\perp})$. Thus we may assume that $\mathcal{O}$ is non-degenerate. Then, due to Remark 5.3, it suffices to assume that $\mathcal{O}$ is a von Neumann algebra, i.e. $\mathcal{O} = \mathcal{O}''$. Let $\mathscr{L}$ be the set of all finite-dimensional $*$-subalgebras of $\mathcal{O}$ which contain $1_{\mathcal{H}}$. Then $\mathscr{L}$ is a lattice for the order relation given by inclusion. Indeed, $\mathscr{L}$ is stable under (arbitrary) intersections and if $\mathcal{M}, \mathcal{N} \in \mathscr{L}$ then their upper bound $\mathcal{R}$ is constructed as follows: if $\mathscr{P}(\mathcal{M})$, $\mathscr{P}(\mathcal{N})$ are the sets of minimal projections of $\mathcal{M}, \mathcal{N}$ then we define $\mathscr{P}(\mathcal{R})$ as the set consisting of the non-zero projections of the form PQ with $P \in \mathscr{P}(\mathcal{M})$, $Q \in \mathscr{P}(\mathcal{N})$ and take $\mathcal{R}$ equal to the linear span of $\mathscr{P}(\mathcal{R})$. The total algebra $\mathcal{O}$ is the norm closure of the union of the algebras in $\mathscr{L}$, because each $A \in \mathcal{O}$ is normal, its spectral measure E_A has values in $\mathcal{O}$, and so is a norm limit of finite sums of the form $B = \sum_k z_k E_A(\Delta_k)$ with $z_k \in \mathbb{C}$ and $\Delta_k \subset \mathbb{C}$ Borel sets. Note also that the standard construction of such sums will produce operators with $\|B\| \leqslant \|A\|$.

From Proposition 3.3 we see that $\{\mathscr{F}(\mathcal{M}) \mid \mathcal{M} \in \mathscr{L}\}$ is a filtered increasing family of C^*-subalgebras of $\mathscr{F}(\mathcal{O})$. The definition (3.7), Lemma 3.7, and the remark made above concerning the norm of B imply that $\mathscr{F}(\mathcal{O})$ is the norm closure of the union of these subalgebras. In other terms, $\mathscr{F}(\mathcal{O})$ is the inductive limit of the net $\{\mathscr{F}(\mathcal{M})\}_{\mathcal{M}\in\mathscr{L}}$. Lemma 5.7 gives us for each $\mathcal{M} \in \mathscr{L}$ a canonical morphism $\mathcal{P}_{\mathcal{M}}$ and from Remark 5.3 it follows that $\mathcal{P}_{\mathcal{O}}(T) \equiv \mathcal{P}_{\mathcal{M}}(T)$ is independent of $\mathcal{M}$ if $T \in \bigcup_{\mathcal{M}} \mathscr{F}(\mathcal{M})$. It remains to extend $\mathcal{P}_{\mathcal{O}}$ to all $\mathscr{F}(\mathcal{O})$ by continuity and to check condition (i) of Proposition 5.10 by an obvious density and continuity argument. □

Remark 5.8. This is a natural extension of Remark 4.5. Let χ be a state on a C^*-algebra $\mathcal{O} \subset B(\mathcal{H})$ and let $\{e\}$ be a net of unit vectors in $\mathcal{H}$ such that $e \rightharpoonup 0$ and such that the state associated to e on $\mathcal{O}$ converges weakly to χ (G. Skandalis has shown me that each state χ on a C^*-algebra $\mathcal{O}$ with $\mathcal{O} \cap K(\mathcal{H}) = \{0\}$ can be expressed in this way). Then

$$\operatorname*{s-lim}_{e\rightharpoonup 0} a(e)\big[W(u)\Gamma(T)1_n\big]a^*(e) = \chi(T)W(u)\Gamma(T)1_{n-1} \quad \text{for all } u \in \mathcal{H} \text{ and } T \in \mathcal{O}.$$

Denote $\mathrm{I}_{\mathcal{O}}$ the identity morphism on $\mathcal{O}$ and for each integer $k \geqslant 1$ let us define

$$\mathcal{P}_k = \mathrm{I}_{\mathcal{O}}^{\otimes(k-1)} \otimes \mathcal{P} : \mathcal{O}^{\otimes(k-1)} \otimes \mathscr{F}(\mathcal{O}) \to \mathcal{O}^{\otimes k} \otimes \mathscr{F}(\mathcal{O}). \tag{5.6}$$

This is a morphism with $\mathcal{O}^{\otimes(k-1)} \otimes \mathscr{K}(\mathcal{H})$ as kernel (tensor product with an abelian algebra preserves exact sequences). Note that $\mathcal{O}^{\otimes(k-1)} \otimes \mathscr{K}(\mathcal{H}) \subset B(\mathcal{H}^{\otimes(k-1)} \otimes \Gamma(\mathcal{H}))$ does not contain compact operators if $k \geqslant 1$ and if we are in the conditions of Theorem 5.4. The following extends Lemma 4.6.

Proposition 5.9. *Under the conditions of Theorem* 5.4 *the map*

$$\mathcal{P}^k = \mathcal{P}_k \circ \cdots \circ \mathcal{P}_1 : \mathscr{F}(\mathcal{O}) \to \mathcal{O}^{\otimes k} \otimes \mathscr{F}(\mathcal{O}) \tag{5.7}$$

is a morphism uniquely determined by the property: $\mathcal{P}^k(W(u)\Gamma(A)) = A^{\otimes k} \otimes [W(u)\Gamma(A)]$ *if* $u \in \mathcal{H}$ *and* $A \in \mathcal{O}$, $\|A\| < 1$. *We have* $\lim_{k\to\infty} \|\mathcal{P}^k(T)\| = 0$ *for all* $T \in \mathscr{F}(\mathcal{O})$.

Proof. It remains only to prove the last relation. Clearly it suffices to consider only operators of the form $T = W(u)\Gamma(A)$. But then we have $\|\mathcal{P}^k(W(u)\Gamma(A))\| \leqslant \|A\|^k \|\Gamma(A)\|$. □

We mention a description of the canonical morphism $\mathcal{P}$ in the spirit of Proposition 3.10. Below $\mathcal{O}$ is any C^*-algebra on $\mathcal{H}$. At point (ii) we use the extension of the action of $\mathcal{P}$ to unbounded operators affiliated to $\mathscr{F}(\mathcal{O})$ (see Section 7): so (ii) is just (i) written at generator level (see the proof of Proposition 7.10).

Proposition 5.10. *Assume that $\mathcal{O}$ contains a positive injective operator. If $\mathcal{P} : \mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ is a morphism then $\mathcal{P}$ is the canonical morphism if and only if it satisfies the following equivalent conditions*:

(i) $\mathcal{P}(\mathrm{e}^{-H}) = \mathrm{e}^{-h} \otimes \mathrm{e}^{-H}$ *if* $H = \mathrm{d}\Gamma(h) + V$ *is an elementary QFH*;
(ii) $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$ *if* $H = \mathrm{d}\Gamma(h) + V$ *is an elementary QFH.*

Proof. Note that $\mathcal{P}$ is uniquely determined by the condition (i) because of Proposition 3.10. If $H = \mathrm{d}\Gamma(h) + V \equiv H_0 + V$ then (7.3) holds in norm because H_0 is bounded from below and V is bounded. If $\mathcal{P}$ is the canonical morphism, and since $\mathrm{e}^{-tH_0} = \Gamma(\mathrm{e}^{-th})$, we obtain (i) from

$$\mathcal{P}\big[\big(\mathrm{e}^{-V/n}\mathrm{e}^{-H_0/n}\big)^n\big] = \big[\mathcal{P}\big(\mathrm{e}^{-V/n}\mathrm{e}^{-H_0/n}\big)\big]^n = \big[\mathrm{e}^{-h/n} \otimes \big(\mathrm{e}^{-V/n}\mathrm{e}^{-H_0/n}\big)\big]^n. \tag{5.8}$$

Reciprocally, assume that $\mathcal{P}$ is a morphism and (i) holds. Let H be as in (i) and set $\widetilde{H} = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$. The operators $H, \widetilde{H}$ are bounded from below and $\mathcal{P}(\mathrm{e}^{-H}) = \mathrm{e}^{-\widetilde{H}}$. Since $\mathcal{P}$ is a morphism and the function $x \mapsto \mathrm{e}^{-x}$ algebraically generates $C_0([a,\infty[)$ if $a \in \mathbb{R}$, we get $\mathcal{P}(\theta(H)) = \theta(\widetilde{H})$ for all $\theta \in C_0(\mathbb{R})$. In particular, if z is a complex number with sufficiently large negative real part we can take $\theta(x) = (z - x)^{-1}$ and get $\mathcal{P}[(z - H)^{-1}] = (z - \widetilde{H})^{-1}$. Denote $R_z = (z - H_0)^{-1}$ and $\widetilde{R}_z = (z - \widetilde{H}_0)^{-1}$, where $\widetilde{H}_0 = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H_0$. Then we make a norm convergent series expansion to get

$$\mathcal{P}\sum_k R_z[VR_z]^k = \sum_k \widetilde{R}_z\big[(1_{\mathcal{H}} \otimes V)\widetilde{R}_z\big]^k.$$

We replace V by sV and take derivatives at $s = 0$ to obtain $\mathcal{P}[R_z V R_z] = \widetilde{R}_z(1_{\mathcal{H}} \otimes V)\widetilde{R}_z$. On the other hand, by taking $V = 0$ in this argument we get $\mathcal{P}(\theta(H_0)) = \theta(\widetilde{H}_0)$ for all $\theta \in C_0(\mathbb{R})$. Thus

$$\mathcal{P}\big[\theta(H_0)R_z V R_z\big] = \theta(\widetilde{H}_0)\widetilde{R}_z(1_{\mathcal{H}} \otimes V)\widetilde{R}_z.$$

By arguments already used in the proof of Proposition 3.10 we get first

$$\mathcal{P}\big[\eta(H_0)VR_z\big] = \eta(\widetilde{H}_0)(1_{\mathcal{H}} \otimes V)\widetilde{R}_z$$

for $\eta \in C_0(\mathbb{R})$ and then we see that this relation remains true for $\eta = 1$. Thus we have $\mathcal{P}[VR_z] = (1_{\mathcal{H}} \otimes V)\widetilde{R}_z$ for all complex numbers z with sufficiently large negative real part. By standard arguments we then get $\mathcal{P}[V\theta(H_0)] = (1_{\mathcal{H}} \otimes V)\theta(\widetilde{H}_0)$ for all $\theta \in C_0(\mathbb{R})$, in particular $\mathcal{P}[V\mathrm{e}^{-H_0}] = (1_{\mathcal{H}} \otimes V)\mathrm{e}^{-\widetilde{H}_0}$. But this is the same as

$$\mathcal{P}\big[V\Gamma\big(\mathrm{e}^{-h}\big)\big] = (1_{\mathcal{H}} \otimes V)\big(\mathrm{e}^{-h} \otimes \Gamma\big(\mathrm{e}^{-h}\big)\big) = \mathrm{e}^{-h} \otimes \big[V\Gamma\big(\mathrm{e}^{-h}\big)\big].$$

Thus $\mathcal{P}[V\Gamma(A)] = A \otimes [V\Gamma(A)]$ if $A = \mathrm{e}^{-h}$. By first choosing h conveniently and then by using the same argument as in the last part of the proof of Proposition 3.10 we see that the preceding relation holds for all $A \in \mathcal{O}$ with $\|A\| < 1$ and $A \geqslant 0$. As in Example 5.2 this implies

$$n\mathcal{P}\big[V(A_1 \vee \cdots \vee A_n)\big] = \sum_k A_k \otimes \big[V(A_1 \vee \cdots \vee A_{k-1} \vee A_{k+1} \vee \cdots \vee A_n)\big]$$

first for $A_k \geqslant 0$ and then for all $A_k \in \mathcal{O}$. Thus $\mathcal{P}[VA^{\vee n}] = A \otimes [VA^{\vee(n-1)}]$ for all $A \in \mathcal{O}$ from which we clearly get $\mathcal{P}[V\Gamma(A)] = A \otimes [V\Gamma(A)]$ if $A \in \mathcal{O}$ and $\|A\| < 1$. That this holds also for $V = W(u)$ follows easily as in the proof of Proposition 3.8. So $\mathcal{P}$ is the canonical morphism. □

We give one more characterization of $\mathcal{P}$ which is sometimes useful (e.g. it implies Theorem 1.1). The proof involves the same ideas as that of Proposition 3.4 so we do not give details.

Lemma 5.11. *$\mathscr{F}(\mathcal{O})$ coincides with the C^*-algebra generated by the operators of the form $\phi(u)^n\Gamma(A)$ with $u \in \mathcal{H}, n \in \mathbb{N}$ and $A \in \mathcal{O}$ with $\|A\| < 1$. A morphism $\mathcal{P}:\mathcal{O} \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ is the canonical morphism if and only if it satisfies $\mathcal{P}(\phi(u)^n\Gamma(A)) = A \otimes [\phi(u)^n\Gamma(A)]$ for all such u, n, A.*

6. The fermionic case

6.1. The fermionic version of the theory seems to me most pleasant esthetically speaking and certainly much easier. As before $\mathcal{H}$ is a complex Hilbert space with scalar product $\langle\cdot|\cdot\rangle$. A *representation of the CAR over $\mathcal{H}$*, or a *Clifford system over $\mathcal{H}$*, is a couple $(\mathscr{H}, \phi)$ consisting of a Hilbert space $\mathscr{H}$ and an $\mathbb{R}$-linear map $\phi:\mathcal{H} \to B(\mathscr{H})$ which satisfies

$$\phi(u)^* = \phi(u) \quad \text{and} \quad \phi(u)^2 = \|u\|^2 \quad \text{for all } u \in \mathcal{H}. \tag{6.1}$$

We set $[A, B]_+ = AB + BA$. Then the second condition above is equivalent to

$$\big[\phi(u), \phi(v)\big]_+ = 2\Re\langle u|v\rangle \quad \text{for all } u, v \in \mathcal{H}. \tag{6.2}$$

Note that the map $\phi:\mathcal{H} \to B(\mathscr{H})$ is an isometry, which makes the theory much simpler. We define the *annihilation* and *creation* operators associated to the one particle state u by the relations (2.6), so that $\phi(u) = a(u) + a^*(u)$. Then $a^*:\mathcal{H} \to B(\mathscr{H})$ is a linear continuous map, $a:\mathcal{H} \to B(\mathscr{H})$ is antilinear and continuous, and $a^*(u)$ is just the adjoint of the operator $a(u)$. We have

$$\big[a(u), a^*(v)\big]_+ = \langle u|v\rangle, \qquad \big[a(u), a(v)\big]_+ = 0, \qquad \big[a^*(u), a^*(v)\big]_+ = 0. \tag{6.3}$$

A *number operator for the Clifford system* $(\mathscr{H}, \phi)$ is a self-adjoint operator N on $\mathscr{H}$ satisfying

$$\mathrm{e}^{\mathrm{i}tN}\phi(u)\mathrm{e}^{-\mathrm{i}tN} = \phi\big(\mathrm{e}^{\mathrm{i}t}u\big) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathcal{H}. \tag{6.4}$$

As in the bosonic case we have

$$\big[N, \mathrm{i}\phi(u)\big] = \phi(\mathrm{i}u), \qquad (N+1)a(u) = a(u)N, \qquad (N-1)a^*(u) = a^*(u)N. \tag{6.5}$$

A *vacuum state* for the Clifford system $(\mathscr{H}, \phi)$ is a vector $\Omega \in \mathscr{H}$ with $\|\Omega\| = 1$ such that the map $u \mapsto \phi(u)\Omega$ is linear and this condition is equivalent to $a(u)\Omega = 0$ for all u.

6.2. We define the *Clifford algebra over $\mathcal{H}$* by

$$\mathscr{F}(\mathcal{H}) = C^*\big(\phi(u)\ \big|\ u \in \mathcal{H}\big). \tag{6.6}$$

We refer to [43] for a presentation of the theory of Clifford algebras suited to our context. In their terminology, $\mathscr{F}(\mathcal{H})$ is the Clifford algebra generated by the *real* vector space $\mathcal{H}$ equipped with the scalar product $\Re\langle u|v\rangle$. In particular, if the (complex) dimension of $\mathcal{H}$ is n then $\mathscr{F}(\mathcal{H})$ is of dimension 2^{2n}. The C^*-algebras $\mathscr{F}(\mathcal{H})$ associated to two Clifford systems over $\mathcal{H}$ are canonically isomorphic in a natural sense, which explains why $(\mathscr{H}, \phi)$ is not included in the notation. The algebra $\mathscr{F}(\mathcal{H})$ has a rich and interesting structure: it is central and simple, it has a unique tracial state, and it is $\mathbb{Z}_2$-graded ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), i.e. there is a unique automorphism γ of

$\mathscr{F}(\mathcal{H})$ such that $\gamma(\phi(u)) = -\phi(u)$ for all $u \in \mathcal{H}$. Clearly $\gamma^2 = 1$ and if we set $\mathscr{F}_\pm(\mathcal{H}) = \{T \in \mathscr{F}(\mathcal{H}) \mid \gamma(T) = \pm T\}$ then we get a linear direct sum decomposition $\mathscr{F}(\mathcal{H}) = \mathscr{F}_+(\mathcal{H}) + \mathscr{F}_-(\mathcal{H})$.

If $\mathcal{K}$ is a closed vector subspace of $\mathcal{H}$ we identify $\mathscr{F}(\mathcal{K})$ with the C^*-subalgebra of $\mathscr{F}(\mathcal{H})$ generated by the operators $\phi(u)$ with $u \in \mathcal{K}$. If $E \subset F$ are finite-dimensional subspaces of $\mathcal{H}$ then $\mathscr{F}(E) \subset \mathscr{F}(F)$ are finite-dimensional $*$-subalgebras of $\mathscr{F}(\mathcal{H})$ and

$$\mathscr{F}(\mathcal{H}) = \overline{\bigcup_E \mathscr{F}(E)} \tag{6.7}$$

where E runs over the set of finite-dimensional subspaces of $\mathcal{H}$. In particular, $\mathscr{F}(\mathcal{H})$ is nuclear.

6.3. One defines the Fock representation exactly as in the bosonic case; the uniqueness modulo canonical isomorphisms is obvious. The construction of the "particle Fock realization" is parallel to that in the Bose case, one just has to replace "symmetric" and the symbol $\vee$ by "antisymmetric" and $\wedge$ (the details can be found in [43]). So $\mathcal{H}^\wedge_{\mathrm{alg}}$ is the antisymmetric (or exterior) algebra[7] over the vector space $\mathcal{H}$, we use the notation uv for the product of two elements u, v of $\mathcal{H}^\wedge_{\mathrm{alg}}$ (or $u \wedge v$ if ambiguities occur in concrete situations), and the unit element is denoted either 1 or Ω. Then $\mathcal{H}^{\wedge n}_{\mathrm{alg}}$ is the linear subspace spanned by the products $u_1 \dots u_n$ with $u_i \in \mathcal{H}$ and $\mathcal{H}^\wedge_{\mathrm{alg}}$ is equal to the linear direct sum $\sum_{n\in\mathbb{N}} \mathcal{H}^{\wedge n}_{\mathrm{alg}}$. We shall equip $\mathcal{H}^\wedge_{\mathrm{alg}}$ with the unique scalar product such that $\mathcal{H}^{\wedge n}_{\mathrm{alg}} \perp \mathcal{H}^{\wedge m}_{\mathrm{alg}}$ for $n \neq m$ and

$$\langle u_1 \dots u_n | v_1 \dots v_n \rangle = \sum_{\sigma \in \mathfrak{S}(n)} \varepsilon_\sigma \langle u_1 | v_{\sigma(1)} \rangle \dots \langle u_n | v_{\sigma(n)} \rangle, \tag{6.8}$$

where ε_σ is the signature of the permutation σ. The estimate (2.14) remains valid in the present situation.

We define the *Fock space* $\Gamma(\mathcal{H}) \equiv \mathcal{H}^\wedge$ *over* $\mathcal{H}$ as the completion of $\mathcal{H}^\wedge_{\mathrm{alg}}$ for the scalar product defined by (6.8). Then $\mathcal{H}^{\wedge n}$ is the closure of $\mathcal{H}^{\wedge n}_{\mathrm{alg}}$ in $\Gamma(\mathcal{H})$, we have $\Gamma(\mathcal{H}) = \bigoplus_n \mathcal{H}^{\wedge n}$ (Hilbert space direct sum), and the spaces $\Gamma_n(\mathcal{H})$ and $\Gamma_{\mathrm{fin}}(\mathcal{H})$ are defined as in the symmetric case. Similarly for the number operator N and the projections 1_n, 1^n, ω. Note that $\Gamma_{\mathrm{fin}}(\mathcal{H})$ is a unital algebra but not abelian: it is a $\mathbb{Z}$-graded *anticommutative* algebra, i.e. we have $uv = (-1)^{nm} vu$ if $u \in \mathcal{H}^{\wedge n}$ and $v \in \mathcal{H}^{\wedge m}$.

The creation–annihilation operators $a^{(*)}(u)$ and the field operator $\phi(u)$ are defined exactly as in the bosonic case. Important differences are the boundedness of these operators: $\|a^{(*)}(u)\| = \|u\|$, and the fact that $a(u)$ is an *antiderivation*:

$$a(u)(vw) = \bigl(a(u)v\bigr)w + (-1)^n v\bigl(a(u)w\bigr) \quad \text{if } v \in \mathcal{H}^{\wedge n},\ w \in \Gamma_{\mathrm{fin}}(\mathcal{H}). \tag{6.9}$$

If $A_1, \dots, A_n \in B(\mathcal{H})$ then there is a unique operator $A_1 \wedge \dots \wedge A_n \in B(\mathcal{H}^{\wedge n})$ such that

$$(A_1 \wedge \dots \wedge A_n)(u_1 \dots u_n) = (n!)^{-1} \sum_{\sigma \in \mathfrak{S}(n)} \varepsilon_\sigma (A_1 u_{\sigma(1)}) \dots (A_n u_{\sigma(n)}) \tag{6.10}$$

[7] The definition is similar to that in the symmetric case, cf. the footnote on p. 98, just replace the commutativity condition $\xi(u)\xi(v) = \xi(v)\xi(u)$ by $\xi(u)\xi(v) = -\xi(v)\xi(u)$.

for all $u_1, \dots, u_n \in \mathcal{H}$. We extend it to $\Gamma(\mathcal{H})$ by identifying $A_1 \wedge \cdots \wedge A_n \equiv A_1 \wedge \cdots \wedge A_n 1^n$. If $A_1 = \cdots = A_n \equiv A$ we denote $A^{\wedge n}$ this operator. Note that $A^{\wedge n}$ is uniquely defined by the relation $A^{\wedge n}(u_1 \dots u_n) = (Au_1)\dots(Au_n)$ for all $u_1, \dots, u_n \in \mathcal{H}$. Observe that $A_1 \wedge \cdots \wedge A_n$ is a symmetric function of $A_1, \dots, A_n$ hence one may use the polarization formula in this case too.

As in the bosonic case, for each $A \in B(\mathcal{H})$ there is a unique unital endomorphism $\Gamma(A)$ of the algebra $\Gamma_{\mathrm{fin}}(\mathcal{H})$ such that $\Gamma(A)u = Au$ for all $u \in \mathcal{H}$ and such that the restriction of $\Gamma(A)$ to each $\Gamma_n(\mathcal{H})$ be continuous. In fact $\Gamma(A) = \bigoplus_{n \geqslant 0} A^{\wedge n}$. Clearly $\Gamma(AB) = \Gamma(A)\Gamma(B)$, $\Gamma(1) = 1$, $\Gamma(0) = \omega$, and $z^N = \Gamma(z)$ for $z \in \mathbb{C}$. The relations (2.16)–(2.18) remain valid. The operator $\Gamma(A)$ is bounded on $\Gamma(\mathcal{H})$ if $\|A\| \leqslant 1$. Finally, there is a unique derivation $\mathrm{d}\Gamma(A)$ of the algebra $\Gamma_{\mathrm{fin}}(\mathcal{H})$ such that $\mathrm{d}\Gamma(A)u = Au$ if $u \in \mathcal{H}$. Hence $\mathrm{d}\Gamma(A)(u_1 \dots u_n) = \sum_k u_1 \dots (Au_k) \dots u_n$ if $n \geqslant 1$ and $\mathrm{d}\Gamma(A)\Omega = 0$. We denote also by $\mathrm{d}\Gamma(A)$ the closure of this operator. If A is not bounded but generates a contractive C_0-semigroup on $\mathcal{H}$ then $\mathrm{d}\Gamma(A)$ is defined by $\Gamma(\mathrm{e}^{tA}) = \mathrm{e}^{t\,\mathrm{d}\Gamma(A)}$.

If $\mathcal{K} \subset \mathcal{H}$ is a closed subspace we identify $\mathcal{K}^{\wedge}_{\mathrm{alg}}$ with the subalgebra of $\mathcal{H}^{\wedge}_{\mathrm{alg}}$ generated by $\mathcal{K}$ and then by taking the closure in $\Gamma(\mathcal{H})$ we get an isometric embedding $\Gamma(\mathcal{K}) \subset \Gamma(\mathcal{H})$. The scalar product (6.8) has been chosen such that

$$\langle uv|u'v'\rangle = \langle u|u'\rangle\langle v|v'\rangle = \langle u \otimes v|u' \otimes v'\rangle \quad \text{for all } u \in \Gamma_{\mathrm{fin}}(\mathcal{K}),\, v \in \Gamma_{\mathrm{fin}}\big(\mathcal{K}^{\perp}\big)$$

hence the linear map $\Gamma_{\mathrm{fin}}(\mathcal{K}) \otimes_{\mathrm{alg}} \Gamma_{\mathrm{fin}}(\mathcal{K}^{\perp}) \to \Gamma_{\mathrm{fin}}(\mathcal{H})$ associated to the bilinear map $(u, v) \mapsto uv$ extends to a linear bijective isometry $\Gamma(\mathcal{K}) \otimes \Gamma(\mathcal{K}^{\perp}) \to \Gamma(\mathcal{H})$. This gives us a canonical Hilbert space identification $\Gamma(\mathcal{H}) = \Gamma(\mathcal{K}) \otimes \Gamma(\mathcal{K}^{\perp})$. Note that the product on $\Gamma_{\mathrm{fin}}(\mathcal{K}) \otimes_{\mathrm{alg}} \Gamma_{\mathrm{fin}}(\mathcal{K}^{\perp})$ induced by the embedding in $\Gamma_{\mathrm{fin}}(\mathcal{H})$ is the anticommutative tensor algebra product, see [10]. Note that $\Omega_{\mathcal{H}} = \Omega_{\mathcal{K}} \otimes \Omega'_{\mathcal{K}}$ and everything we said starting with (2.20) until the end of Section 2 remains valid.

It is also trivial to check that, as in bosonic case, for each $u \in \mathcal{K}$ we have $a^{(*)}_{\mathcal{H}}(u) = a^{(*)}_{\mathcal{K}}(u) \otimes 1$ and $\phi_{\mathcal{H}}(u) = \phi_{\mathcal{K}}(u) \otimes 1$ relatively to the factorization $\Gamma(\mathcal{H}) = \Gamma(\mathcal{K}) \otimes \Gamma(\mathcal{K}^{\perp})$. On the other hand, if $u \in \mathcal{K}^{\perp}$ it is easy to check that $a^{(*)}_{\mathcal{H}}(u) = (-1)^{N_{\mathcal{K}}} \otimes a^{(*)}_{\mathcal{K}^{\perp}}(u)$. Thus for $u \in \mathcal{K}$ and $v \in \mathcal{K}^{\perp}$ we have

$$\phi_{\mathcal{H}}(u + v) = \phi_{\mathcal{K}}(u) \otimes 1 + (-1)^{N_{\mathcal{K}}} \otimes \phi_{\mathcal{K}^{\perp}}(u). \tag{6.11}$$

6.4. The theory developed in Sections 3–5 has a complete analogue in the present setting. Many things become in fact simpler and look more natural due to the boundedness of the field operators. So in what follows we state the results and make some comments concerning the proofs.

If $\mathcal{O}$ is a C^*-algebra on $\mathcal{H}$ then $\Gamma(\mathcal{O})$ is defined as in (3.1) and Proposition 3.2 (with $\vee$ replaced by $\wedge$) remains true because $A_1 \wedge \cdots \wedge A_n$ is a symmetric function of $A_1, \dots, A_n$. Then we define:

$$\mathscr{F}(\mathcal{O}) = C^*\big(S\Gamma(A) \bigm| S \in \mathscr{F}(\mathcal{H}),\, A \in \mathcal{O},\ \|A\| < 1\big) \tag{6.12}$$

and we set $\mathscr{A}(\mathcal{H}) = \mathscr{F}(\mathbb{C}1_{\mathcal{H}})$. *If $\mathcal{O}$ is non-degenerate then we have*

$$\mathscr{F}(\mathcal{O}) = \big[\!\big[\mathscr{F}(\mathcal{H}) \cdot \Gamma(\mathcal{O})\big]\!\big]. \tag{6.13}$$

The proof is a much simplified version of that of Proposition 3.4. We now consider Proposition 3.3.

Proof of the fermionic version of Proposition 3.3. $\mathscr{F}(\{0\})$ is the C^*-algebra generated by the operators $\phi(u_1)\dots\phi(u_n)\omega$ (where the product may be empty) and the linear span of these operators coincides with the linear span of $a^*(u_1)\dots a^*(u_n)\omega = |u_1\dots u_n\rangle\langle\Omega|$, from which (2) of Proposition 3.3 in the Fermi case follows easily. Now we prove (3) of Proposition 3.3. Basically this follows from

$$\begin{aligned}\phi(u)\Gamma(A) &= \big(\phi(u_1)\otimes 1 + (-1)^{N_1}\otimes\phi(u_2)\big)\Gamma(A_1)\otimes\Gamma(A_2)\\ &= \big[\phi(u_1)\Gamma(A_1)\big]\otimes\Gamma(A_2) + \Gamma(-A_1)\otimes\big[\phi(u_2)\Gamma(A_2)\big]\end{aligned}$$

but the complete argument is complicated by the fact that we have to consider arbitrary polynomials in the fields. Consider a product $\phi(w_1)\dots\phi(w_n)\Gamma(A)$ and decompose $w_k = u_k + v_k$, $A = B\oplus C$ with $u_1,\dots,u_n\in\mathcal{H}_1$, $v_1,\dots,v_n\in\mathcal{H}_2$, and $B\in\mathcal{O}_1, C\in\mathcal{O}_2$ with norms <1. Due to (6.11) and since $(-1)^{N_{\mathcal{H}_1}} = \Gamma(-1_{\mathcal{H}_1})$ we have $\phi(w_k) = \phi(u_k)\otimes 1 + \Gamma(-1)\otimes\phi(v_k)$ with some simplifications in the notations. If we develop the product $\phi(w_1)\dots\phi(w_n)$ and if we take into account the relation $\Gamma(-1)\phi(u_k) = \phi(-u_k)\Gamma(-1)$ we clearly get a sum of terms of the form (ordered products)

$$\Big[\prod_{j\in\alpha}\phi(\tilde{u}_j)\Big]\otimes\Big[\prod_{k\in\beta}\phi(v_k)\Big]\cdot\Gamma(\pm 1)\otimes 1$$

where α is a subset of $\{1,\dots,n\}$, β is the complementary subset, and $\tilde{u}_j$ is either u_j or $-u_j$. Since $\Gamma(\pm 1)\otimes 1\cdot\Gamma(A) = \Gamma(\pm B)\otimes\Gamma(C)$ we see that $\phi(w_1)\dots\phi(w_n)\Gamma(A)\in\mathscr{F}(\mathcal{O}_1)\otimes\mathscr{F}(\mathcal{O}_2)$ and the proof is finished by an obvious argument. $\square$

We mention one more fact, which is also true in the bosonic case but with a more complicated proof.

Proposition 6.1. *If $\mathcal{O}$ is non-degenerate then $\mathscr{F}(\mathcal{O})$ is the C^*-algebra generated by the operators of the form $\Gamma(A)$ or $\phi(u)\Gamma(A)$ with $u\in\mathcal{H}$ and $A\in\mathcal{O}$, $A\geqslant 0$, $\|A\|<1$.*

Proof. We give the proof under the supplementary assumption that $\mathcal{O}$ contains a positive injective operator (this is the only situation relevant in field theory; in general one has to use an approximate unit as in the proof of Proposition 3.4). Let $\mathscr{C}$ be the C^*-algebra generated by the operators of the form $\Gamma(A)$ or $\phi(u)\Gamma(A)$ with $u\in\mathcal{H}$ and $A\in\mathcal{O}$, $A\geqslant 0$, $\|A\|<1$. Due to (3.2) it is sufficient to show that any product $\phi(u_1)\dots\phi(u_n)\Gamma(A)$ with A as above belongs to $\mathscr{C}$. We show this in the case of two field factors $\phi(u)\phi(v)\Gamma(A)$, the general case is similar. We have $A = (\sqrt{A})^2$ and $\sqrt{A}\in\mathcal{O}$, is positive, and has norm strictly less than 1. By writing $\phi(u)\phi(v)\Gamma(A) = \phi(u)[\phi(v)\Gamma(\sqrt{A})]\Gamma(\sqrt{A})$ we see that it suffices to show the following: for each $v\in\mathcal{H}$ and $B\in\mathcal{O}$ with $B\geqslant 0$, $\|B\|<1$, the operator $\phi(v)\Gamma(B)$ belongs to the norm closure $\mathscr{L}$ of the linear span of the operators of the form $\Gamma(A)\phi(u)$ with u, A as before. We have $\phi(v)\Gamma(B) = a(v)\Gamma(B) + a^*(v)\Gamma(B)$ and so it suffices to have $a^{(*)}(v)\Gamma(B)\in\mathscr{L}$. In the case of $a(u)\Gamma(B)$ this is obvious by (2.16). Now let $S\in\mathcal{O}$ be positive and injective and let $\varepsilon>0$ real.

Then (2.16) implies $a^*((B+\varepsilon S)w)\Gamma(B+\varepsilon S) = \Gamma(B+\varepsilon S)a^*(w) \in \mathscr{L}$ for all $w \in \mathcal{H}$. The operator $B+\varepsilon S$ is positive and injective hence it has dense range. The map $u \mapsto a^*(u) \in B(\Gamma(\mathcal{H}))$ is norm continuous, hence we get $a^*(v)\Gamma(B+\varepsilon S) \in \mathscr{L}$ for all $v \in \mathcal{H}$. From Lemma 3.7 we easily get $\Gamma(B+\varepsilon S) \to \Gamma(B)$ in norm as $\varepsilon \to 0$, hence $a^*(v)\Gamma(B) \in \mathscr{L}$. □

One may define elementary QFH as in Definition 3.9 by asking $V \in \mathscr{F}(\mathcal{H})$ or $V \in \mathscr{F}(E)$ for some finite-dimensional subspace E of $\mathcal{H}$. And then Proposition 3.10 remains true (only a minor modification of the end of the proof is required). We may now state the fermionic version of our main result.

Theorem 6.2. *If $\mathcal{O}$ is an abelian C^*-algebra on $\mathcal{H}$ and its strong closure does not contain finite rank operators, then there is a unique morphism $\mathcal{P}: \mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ such that*

$$\mathcal{P}\big[S\Gamma(A)\big] = A \otimes \big[S\Gamma(A)\big] \quad \text{if } S \in \mathscr{F}(\mathcal{H}) \text{ and } A \in \mathcal{O},\ \|A\| < 1. \tag{6.14}$$

We have $\ker \mathcal{P} = \mathscr{K}(\mathcal{H})$*, which gives us a canonical embedding*

$$\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O}). \tag{6.15}$$

If $\mathcal{O}$ is non-degenerate then one may require (6.14) *to hold only for $S = \phi(u)^k$* (the powers $\phi(u)^k$ with $k \in \mathbb{N}$ are multiples of $\phi(u)$ or of the identity). The second characterization of $\mathcal{P}$ presented in Proposition 5.10 remains valid. The canonical endomorphism $\mathcal{P}$ of $\mathscr{A}(\mathcal{H})$ satisfies $\mathcal{P}(S\theta(N)) = S\theta(N+1)$ for all $S \in \mathscr{F}(\mathcal{H})$ and $\theta \in C_0(\mathbb{N})$.

The strategy of the proof of Theorem 6.2 is identical to that from the symmetric case. We first treat the case of $\mathscr{A}(\mathcal{H})$ as in Section 4 with the help of the algebras

$$\mathscr{A}_E(\mathcal{H}) = [\![\mathscr{F}(E) \cdot C_0(N)]\!] = \mathscr{K}(E) \otimes C_0\big(N'_E\big) \text{ relatively to } \Gamma(\mathcal{H}) = \Gamma(E) \otimes \Gamma\big(E^\perp\big).$$

Here E is finite-dimensional and $\mathscr{F}(E) \equiv \mathscr{F}(E) \otimes 1_{E^\perp}$ the $\mathscr{F}(E)$ from the right-hand side being the algebra of all operators on the finite-dimensional space $\Gamma(E)$. In particular we now have $N_E \in \mathscr{F}(E)$, in fact $N_E = \sum_{k=0}^n a^*(e_k)a(e_k)$ if $e_1, \dots, e_n$ is an orthonormal basis of E. For a general algebra $\mathcal{O}$ we proceed as in Section 5.

We now prove that $\mathscr{A}(\mathcal{H})$ has a natural $\mathbb{Z}_2$-grading and we state the fermionic version of Remark 4.5.

Proposition 6.3. *There is a unique automorphism γ of $\mathscr{A}(\mathcal{H})$ such that $\gamma(S\theta(N)) = \gamma(S)\theta(N)$ for all $S \in \mathscr{F}(\mathcal{H})$ and $\theta \in C_0(\mathbb{N})$. We have $\gamma^2 = 1$ and for each $T \in \mathscr{A}(\mathcal{H})$:*

$$\mathcal{P}(T) = \operatorname*{s-lim}_{e \rightharpoonup 0} a(e)\gamma(T)a^*(e). \tag{6.16}$$

Proof. From the fermionic version of (4.5) it follows that it suffices to define γ on $\mathscr{A}_E(\mathcal{H})$ for each finite-dimensional E. Since, as explained above, we then have $\mathscr{A}_E(\mathcal{H}) = \mathscr{K}(E) \otimes C_0(N'_E)$, the existence is rather obvious. However, the following explicit construction, cf. [43, Theorem 1.1.10], gives more information. Observe first that if $e \in \mathcal{H}$ and $\|e\| = 1$ then $\phi(e)\phi(\mathrm{i}e) = \mathrm{i}[a(e), a^*(e)]$, hence $\phi(e)\phi(\mathrm{i}e) = \phi(ze)\phi(\mathrm{i}ze)$ for all complex z with $|z| = 1$. Let $e_1, \dots, e_n$ be an orthonormal basis of E and $w = \phi(e_1)\phi(\mathrm{i}e_1)\dots\phi(e_n)\phi(\mathrm{i}e_n)$. It is clear that w is a unitary element of $\mathscr{F}(E)$ with $w^* = w$ if n is even and $w^* = -w$ if n is odd. The relation $wSw^* = \gamma(S)$

for $S \in \mathscr{F}(E)$ is easy to check (or see [43, Theorem 1.1.10]). By using the expression given above for N_E we get $wN_Ew^* = N_E$ and it is clear that $wN'_Ew^* = N'_E$. Thus $wNw^* = N$ and we may define $\gamma(T) = wTw^*$ for all $T \in \mathscr{F}(E)$.

We have $a(e)u_0 \dots u_n = \sum_k (-1)^k u_0 \dots \langle e|u_k\rangle \dots u_n$ hence s-$\lim_{e \rightharpoonup 0} a(e) = 0$. From the anti-commutation relation $a(e)a^*(e) + a^*(e)a(e) = 1$ we get s-$\lim_{e \rightharpoonup 0} a(e)a^*(e) = 1$. Thus $\mathcal{P}$ *defined* by (6.16) is an endomorphism of $\mathscr{A}(\mathcal{H})$. Note that

$$\left\| a(e)\phi(u) + \phi(u)a(e) \right\| = \left| \langle e|u\rangle \right| \to 0 \quad \text{if } e \rightharpoonup 0.$$

Finally, by using (6.5) it follows easily that $\mathcal{P}$ is the canonical endomorphism of $\mathscr{A}(\mathcal{H})$. □

It is clear that everything we said in Section 5 starting with Proposition 5.9 remains true or has an analogue in the fermionic case.

7. Self-adjoint operators affiliated to $\mathscr{F}(\mathcal{O})$

7.1. It will be convenient to use the notion of observable affiliated to a C^*-algebra as introduced in [13] and further studied in [3,21]. In this paper a self-adjoint operator is supposed to be densely defined but not densely defined operators appear by taking (norm) resolvent limits or images through C^*-algebra morphisms. An observable is a Hilbert space independent formulation of the notion of "not necessarily densely defined self-adjoint operator."

An *observable affiliated to a C^*-algebra* $\mathscr{C}$ is a morphism $H : C_0(\mathbb{R}) \to \mathscr{C}$. We set $H(\theta) = \theta(H)$ although H cannot be realized as a self-adjoint operator in general. Observables have the advantage that one can consider their images through morphisms: if $\mathcal{P} : \mathscr{C} \to \mathscr{D}$ is a morphism, then $\mathcal{P}(H)$ is the observable affiliated to $\mathscr{D}$ defined by $\theta(\mathcal{P}(H)) = \mathcal{P}(\theta(H))$ (this operation makes no sense at the Hilbert space level). The *spectrum* of H is the set $\sigma(H)$ of real points λ such that $\theta(H) \neq 0$ if $\theta(\lambda) \neq 0$. A sequence $\{H_n\}$ of observables affiliated to $\mathscr{C}$ is *convergent* if $\lim_n \theta(H_n)$ exists (in norm) for each $\theta \in C_0(\mathbb{R})$. Then $\theta(H) = \lim_n \theta(H_n)$ is an observable affiliated to $\mathscr{C}$ and we write $H = \lim_n H_n$.

Let $\mathscr{C}$ be a C^*-algebra of operators on a Hilbert space $\mathscr{H}$. We say that a self-adjoint operator H on $\mathscr{H}$ is *affiliated*[8] to $\mathscr{C}$ if $(H - z)^{-1} \in \mathscr{C}$ for some $z \in \mathbb{C} \setminus \sigma(H)$. This is equivalent to $\theta(H) \in \mathscr{C}$ for all $\theta \in C_0(\mathbb{R})$ and this gives us a morphism $\theta \mapsto \theta(H)$, hence H defines an observable affiliated to $\mathscr{C}$ and this observable determines the self-adjoint operator H uniquely. So the set of self-adjoint operators affiliated to $\mathscr{C}$ is a subset of the set of observables affiliated to $\mathscr{C}$. But there are observables affiliated to $\mathscr{C}$ which do not correspond to self-adjoint operators on $\mathscr{H}$ (and these could be physically interesting). See [3, p. 364] and [14] for details on this question.

It is clear that the spectrum of H as self-adjoint operator on $\mathscr{H}$ and as observable affiliated to $\mathscr{C}$ are identical. If $\{H_n\}$ is a sequence of self-adjoint operators affiliated to $\mathscr{C}$ then the sequence of observables H_n converges if and only if the sequence of operators H_n converges in norm resolvent sense.

If one insists in working with self-adjoint operators the following notion is useful. We say that an observable or a self-adjoint operator H is *strictly affiliated to* $\mathscr{C}$ if the linear space generated by the products $\theta(H)T$ with $\theta \in C_0(\mathbb{R})$ and $T \in \mathscr{C}$ is dense in $\mathscr{C}$. If there is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ then $\mathscr{C}$ is non-degenerate on $\mathscr{H}$.

[8] This should not be confused with the terminology of Woronowicz, see [21].

We refer to [21, Appendix] for a proof of the following fact:

If H is a self-adjoint operator strictly affiliated to $\mathscr{C}$ and if $\mathcal{P}$ is a non-degenerate representation of $\mathscr{C}$ on a Hilbert space $\mathscr{K}$, then there is a unique self-adjoint operator $\mathcal{P}(H)$ on $\mathscr{K}$ such that $\mathcal{P}(\phi(H)) = \phi(\mathcal{P}(H))$ for all $\phi \in C_0(\mathbb{R})$. Moreover, $\mathcal{P}(H)$ is strictly affiliated to the C^*-algebra $\mathcal{P}(\mathscr{C})$.

From now on we assume that $\mathscr{C} \subset B(\mathscr{H})$ is non-degenerate on $\mathscr{H}$. Then the *multiplier algebra*[9] of $\mathscr{C}$ is defined by

$$\mathscr{M} = \{ B \in B(\mathscr{H}) \mid BC \in \mathscr{C} \text{ and } CB \in \mathscr{C} \text{ if } C \in \mathscr{C} \}. \tag{7.1}$$

Each non-degenerate representation $\mathcal{P}$ of $\mathscr{C}$ on a Hilbert space $\mathscr{K}$ extends in a unique way to a representation (also denoted $\mathcal{P}$) of $\mathscr{M}$ on $\mathscr{K}$ such that $\mathcal{P}(B)\mathcal{P}(C) = \mathcal{P}(BC)$ for all $B \in \mathscr{M}$ and $C \in \mathscr{C}$.

Lemma 7.1. *Assume that H_0 is a self-adjoint operator* (*strictly*) *affiliated to $\mathscr{C}$ and that $V = V^*$ belongs to the multiplier algebra of $\mathscr{C}$. Then $H = H_0 + V$ is* (*strictly*) *affiliated to $\mathscr{C}$. If $\mathcal{P}$ is a non-degenerate representation of $\mathscr{C}$ then $\mathcal{P}(H) = \mathcal{P}(H_0) + \mathcal{P}(V)$.*

This is an easy consequence of $R(z) = \sum R_0(z)(V R_0(z))^k$ for large z, where $R(z) = (z - H)^{-1}$ and $R_0(z) = (z - H_0)^{-1}$. See [21] for the proof of the strict affiliation.

We quote below several affiliation criteria which are convenient for quantum field models.

Theorem 7.2. *Let H_0 and V be bounded from below self-adjoint operators on $\mathscr{H}$ such that the operator $H = H_0 + V$ with domain $D(H_0) \cap D(V)$ is self-adjoint* (*in particular, the intersection has to be dense in $\mathscr{H}$*). *If $\mathrm{e}^{-tH_0}\mathrm{e}^{-2tV}\mathrm{e}^{-tH_0} \in \mathscr{C}$ for all $t > 0$ then H is affiliated to $\mathscr{C}$.*

This follows from a result of Rogava [45] (see [38] for more recent results) which says that

$$\begin{aligned}\mathrm{e}^{-2tH} &= \lim_{n\to\infty} \big[\mathrm{e}^{-tH_0/n}\mathrm{e}^{-2tV/n}\mathrm{e}^{-tH_0/n}\big]^n \\ &= \lim_{n\to\infty} \big[\big(\mathrm{e}^{-tV/n}\mathrm{e}^{-tH_0/n}\big)^*\big(\mathrm{e}^{-tV/n}\mathrm{e}^{-tH_0/n}\big)\big]^n\end{aligned} \tag{7.2}$$

holds in norm for all $t > 0$. Under the same conditions we also have norm convergence in

$$\mathrm{e}^{-tH} = \lim_{n\to\infty} \big[\mathrm{e}^{-tV/n}\mathrm{e}^{-tH_0/n}\big]^n. \tag{7.3}$$

Other affiliation criteria can be found in [21], for example:

Theorem 7.3. *Let $H_0 \geqslant 0$ be a self-adjoint operator affiliated to $\mathscr{C}$ and let V be a symmetric form such that $-aH_0 - b \leqslant V \leqslant bH_0 + b$ for some real numbers $0 < a < 1$ and $b > 0$. Assume that $U \equiv (H_0 + 1)^{-1/2} V (H_0 + 1)^{-1/2}$ belongs to the multiplier algebra $\mathscr{M}$. Then $H = H_0 + V$ defined in form sense is a self-adjoint operator affiliated to $\mathscr{C}$. If H_0 is strictly affiliated to $\mathscr{C}$ then $U \in \mathscr{M}$ if and only $\theta(H_0) V (H_0 + 1)^{-1/2} \in \mathscr{C}$ for all $\theta \in C_{\mathrm{c}}(\mathbb{R})$ and then H is strictly affiliated to $\mathscr{C}$.*

[9] This is isomorphic with the abstractly defined multiplier algebra, cf. [39], but we shall not use this fact.

Now let us fix a probability measure space Q and consider the associated scale of L^p spaces. Let H_0 be a positive self-adjoint operator on L^2 which generates a *hypercontractive* semigroup in the following sense: for each $t > 0$ the operator e^{-tH_0} is a contraction in each L^p and there are $p > 2$ and $t > 0$ such that $\mathrm{e}^{-tH_0} L^2 \subset L^p$. We shall say that a real function V on Q is *admissible* if V and e^{-V} belong to L^p for all $p < \infty$ (observe that if V is bounded from below the second condition is automatically satisfied). Under these conditions on H_0 and V it can be shown that $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$ and its closure H is bounded from below, see [44, Theorem X.58]. Then [44, Theorem X.60]:

Theorem 7.4. *Assume that H is as above, let $\{V_n\}$ be a sequence of admissible functions, and let H_n be the closure of the operator $H_0 + V_n$. Assume that there is $p > 2$ such that $\|V_n - V\|_{L^p} \to 0$ and $\sup_n \|\mathrm{e}^{-V_n}\|_{L^p} < \infty$. Then $\lim H_n = H$ in norm resolvent sense.*

7.2. We consider now the case of interest in this paper. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{O}$ an abelian *non-degenerate* C^*-algebra on $\mathcal{H}$ such that $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$. We take $\mathscr{H} = \Gamma(\mathcal{H})$, which is either the bosonic or the fermionic Fock space, and $\mathscr{C} = \mathscr{F}(\mathcal{O})$. Then according to Theorems 5.4 and 6.2 we have a canonical morphism $\mathcal{P} : \mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ whose kernel is $\mathscr{K}(\mathcal{H}) \equiv K(\Gamma(\mathcal{H}))$. The algebra $\mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ is naturally realized on the Hilbert space $\mathcal{H} \otimes \Gamma(\mathcal{H})$ and thus we get an embedding

$$\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \subset B\big(\mathcal{H} \otimes \Gamma(\mathcal{H})\big). \tag{7.4}$$

Thus we may think of $\mathcal{P}$ as a representation of $\mathscr{F}(\mathcal{O})$ on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ with range $\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H})$ included (strictly in general) in $\mathcal{O} \otimes \mathscr{F}(\mathcal{O})$.

Lemma 7.5. *$\mathscr{F}(\mathcal{O})$ is non-degenerate on $\Gamma(\mathcal{H})$ and the representation $\mathcal{P}$ of $\mathscr{F}(\mathcal{O})$ on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ is non-degenerate. If $h \geqslant m > 0$ is a self-adjoint operator on $\mathcal{H}$ strictly affiliated to $\mathcal{O}$ then $H_0 = \mathrm{d}\Gamma(h)$ is strictly affiliated to $\Gamma(\mathcal{O})$ and to $\mathscr{F}(\mathcal{O})$.*

Proof. The action of the algebra $\mathscr{F}(\mathcal{O})$ on $\Gamma(\mathcal{H})$ is non-degenerate because $\mathscr{K}(\mathcal{H}) \subset \mathscr{F}(\mathcal{O})$. The action of $\mathcal{P}(\mathscr{F}(\mathcal{O}))$ on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ is also non-degenerate because this algebra contains the operators of the form $S \otimes \Gamma(S)$ with $S \in \mathcal{O}$ and $\|S\| < 1$ and if we take a sequence $\{S_n\}$ of such operators with $S_n \to 1_{\mathcal{H}}$ strongly then $S_n \otimes \Gamma(S_n)$ converges strongly to the identity operator on $\mathcal{H} \otimes \Gamma(\mathcal{H})$.

If h is strictly affiliated to $\mathcal{O}$ then the linear span of the operators $\theta(h)T$ with $\theta \in C_0(\mathbb{R})$ and $T \in \mathcal{O}$ is dense in $\mathcal{O}$. If h is also bounded from below this clearly implies $\|\mathrm{e}^{-\varepsilon h}T - T\| \to 0$ as $\varepsilon \to 0$ (and reciprocally). If $h \geqslant m > 0$ then from Lemma 3.7 we clearly get $\|\mathrm{e}^{-\varepsilon H_0}\Gamma(A) - \Gamma(A)\| \to 0$ as $\varepsilon \to 0$ if $A \in \mathcal{O}$, $\|A\| < 1$, and from this we deduce that H_0 is strictly affiliated to $\Gamma(\mathcal{O})$. Finally, we make a general remark:

if H is an observable strictly affiliated to $\Gamma(\mathcal{O})$ then it is strictly affiliated to $\mathscr{F}(\mathcal{O})$.

Indeed, we have $\Gamma(\mathcal{O}) \subset \mathscr{F}(\mathcal{O})$ and the natural (left or right) action of $\Gamma(\mathcal{O})$ on $\mathscr{F}(\mathcal{O})$ is non-degenerate, cf. Proposition 3.8. $\square$

Thus, if H is a self-adjoint operator on $\Gamma(\mathcal{H})$ strictly affiliated to $\mathscr{F}(\mathcal{O})$ then $\mathcal{P}(H)$ is a self-adjoint operator on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ strictly affiliated to the quotient algebra $\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H})$. If H

is only affiliated to $\mathscr{F}(\mathcal{O})$ then $\mathcal{P}(H)$ is only an observable affiliated to $\mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H})$ and in general cannot be realized as a self-adjoint operator on $\mathcal{H} \otimes \Gamma(\mathcal{H})$. In any case, as the simplest application in spectral theory of Theorems 5.4 and 6.2, we have the following description of the essential spectrum of H.

Theorem 7.6. *We have $\sigma_{\mathrm{ess}}(H) = \sigma(\mathcal{P}(H))$ if $H \in \mathscr{F}(\mathcal{O})$ or H is affiliated to $\mathscr{F}(\mathcal{O})$.*

This result can be made more explicit in the following terms. Since $\mathcal{O}$ is an abelian C^*-algebra its spectrum $\mathscr{X}$ is a locally compact topological space and we have a canonical identification

$$\mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \cong C_0\big(\mathscr{X}; \mathscr{F}(\mathcal{O})\big), \tag{7.5}$$

where $C_0(\mathscr{X}; \mathscr{F}(\mathcal{O}))$ is the C^*-algebra of norm continuous functions $F : \mathscr{X} \to \mathscr{F}(\mathcal{O})$ which tend to zero at infinity. Assume for simplicity that $\widetilde{H} \equiv \mathcal{P}(H)$ is a self-adjoint operator on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ (which holds if H is strictly affiliated to $\mathscr{F}(\mathcal{O})$), then $\widetilde{H}$ is identified with a continuous family $\{\widetilde{H}(x)\}_{x \in \mathscr{X}}$ of self-adjoint operators affiliated to $\mathscr{F}(\mathcal{O})$ and we have

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{x \in \mathscr{X}} \sigma\big(\widetilde{H}(x)\big). \tag{7.6}$$

See [3, 8.2.4] for details and for the proof that the union is closed ($\widetilde{H}$ could be only an observable).

7.3. The simplest operators affiliated to $\mathscr{F}(\mathcal{O})$ are the elementary QFH, and their images through $\mathcal{P}$ are described in Proposition 5.10. We give other examples below and in later sections. Since we think of $\mathscr{F}(\mathcal{O})$ as the C^*-algebra of energy observables of a quantum field, any observable affiliated to it should be interpreted as the Hamiltonian of some quantum field model with one particle kinetic energy affiliated to $\mathcal{O}$. Thus Theorem 7.6 and the formula (7.6) should cover a large class of models. However, the Hamiltonians of the usual models are of the same nature as the elementary QFH (only much more singular). We isolate this class of operators in the next definition.

Definition 7.7. A self-adjoint operator H on $\Gamma(\mathcal{H})$ is a *standard quantum field Hamiltonian* (*SQFH*) if H is bounded from below and affiliated to $\mathscr{F}(\mathcal{O})$ and if there is a self-adjoint operator $h \geqslant 0$ on $\mathcal{H}$ affiliated to $\mathcal{O}$ such that $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$. Under these conditions we shall also say that H *is of type* $\mathcal{O}$ and that h is the *one particle kinetic energy* and $m = \inf h$ the *one particle mass* associated to H.

If we apply Theorem 7.6 to SQFH Hamiltonians we get:

Theorem 7.8. *If H is a SQFH with one particle kinetic energy h and one particle mass m then*

$$\sigma_{\mathrm{ess}}(H) = \sigma(h) + \sigma(H) = \big\{\lambda + \mu \;\big|\; \lambda \in \sigma(h),\ \mu \in \sigma(H)\big\}. \tag{7.7}$$

In particular, if $m > 0$ then $\inf H$ is an eigenvalue of finite multiplicity of H isolated from the rest of the spectrum. If $\sigma(h) = [m, \infty[$ then $\sigma_{\mathrm{ess}}(H) = [m + \inf H, \infty[$.

The class of SQFH is quite large and many singular physically interesting Hamiltonians are affiliated to it. We shall give such examples in the next sections and we devote the rest of this section to some preliminary results in this direction.

Lemma 7.9. *The multiplier algebra of $\mathscr{F}(\mathcal{O})$ contains $\mathscr{W}_{\max}(\mathcal{H})$ in the bosonic case and $\mathscr{F}(\mathcal{H})$ in the fermionic case. If V belongs to one of these classes we have $\mathcal{P}(V) = 1_{\mathcal{H}} \otimes V$.*

Proof. In the bosonic case it suffices to consider $V = W(f)$ with f a bounded Borel regular measure on $\mathcal{H}$ and to show that for $T = \Gamma(A)S$ with $S \in \mathscr{W}(\mathcal{H})$ and $A \in \mathcal{O}$, $\|A\| < 1$ we have $VT \in \mathscr{F}(\mathcal{O})$ and $\mathcal{P}(VT) = (1_{\mathcal{H}} \otimes V)\mathcal{P}(T)$. We have $VT = \int W(u)\Gamma(A)S\,\mathrm{d}f(u)$ the integral being convergent in norm by Lemma 3.7, and $W(u)\Gamma(A)S \in \mathscr{F}(\mathcal{O})$, hence $VT \in \mathscr{F}(\mathcal{O})$ and

$$\begin{aligned}\mathcal{P}(VT) &= \int \mathcal{P}\big(W(u)\Gamma(A)S\big)\,\mathrm{d}f(u) = \int A \otimes \big(W(u)\Gamma(A)S\big)\,\mathrm{d}f(u)\\ &= A \otimes \big(V\Gamma(A)S\big) = \big(1_{\mathcal{H}} \otimes V\big)\big(A \otimes \big(\Gamma(A)S\big)\big) = (1_{\mathcal{H}} \otimes V)\mathcal{P}(T).\end{aligned}$$

The proof in the fermionic case is similar and easier. □

Proposition 7.10. *Let h be a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathcal{O}$ and such that* $\inf h > 0$. *Let $V = V^*$ be an element of the multiplier algebra of $\mathscr{F}(\mathcal{O})$. Then $H = \mathrm{d}\Gamma(h) + V$ is affiliated to $\mathscr{F}(\mathcal{O})$ and we have $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes \mathrm{d}\Gamma(h) + \mathcal{P}(V)$. In particular, if $V \in \mathscr{W}_{\max}(\mathcal{H})$ in the bosonic case and $V \in \mathscr{F}(\mathcal{H})$ in the fermionic case, then we have $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$, so H is a SQFH.*

Proof. The operator $H_0 = \mathrm{d}\Gamma(h)$ has the property $\mathrm{e}^{-tH_0} = \Gamma(\mathrm{e}^{-th})$ for $t > 0$ and $\mathrm{e}^{-th} \in \mathcal{O}$ and has norm < 1, so that

$$\mathcal{P}\big(\mathrm{e}^{-tH_0}\big) = \mathrm{e}^{-th} \otimes \Gamma\big(\mathrm{e}^{-th}\big) = \mathrm{e}^{-th} \otimes \mathrm{e}^{-tH_0}.$$

Thus $\mathcal{P}(H_0) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H_0$ and then we use Lemmas 7.1 and 7.9. □

Proposition 7.11. *Let V be a bounded from below self-adjoint operator on $\Gamma(\mathcal{H})$ affiliated to $\mathscr{W}_{\max}(\mathcal{H})$ in the Bose case and to $\mathscr{F}(\mathcal{H})$ in the Fermi case. Let h be a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathcal{O}$ with $h \geqslant m > 0$ and let us set $H_0 = \mathrm{d}\Gamma(h)$. If $H = H_0 + V$ is self-adjoint on $D(H_0) \cap D(V)$ then H is a SQFH of type $\mathcal{O}$ with h as one particle kinetic energy.*

Proof. That H is affiliated to $\mathscr{F}(\mathcal{O})$ is a consequence of Theorem 7.2. Then $\widetilde{H} = \mathcal{P}(H)$ is an observable affiliated to $\mathscr{F}(\mathcal{O})$ but we do not yet know if it can be realized as a self-adjoint operator on $\mathcal{H} \otimes \Gamma(\mathcal{H})$. In any case, the semigroup $\{\mathrm{e}^{-t\widetilde{H}}\}_{t>0}$ is well defined (it could be zero on a nontrivial subspace) and (7.3) implies:

$$\begin{aligned}\mathrm{e}^{-t\widetilde{H}} &= \mathcal{P}\big(\mathrm{e}^{-tH}\big) = \lim_{n\to\infty}\big[\mathcal{P}\big(\mathrm{e}^{-tV/n}\mathrm{e}^{-tH_0/n}\big)\big]^n\\ &= \lim_{n\to\infty}\big[\mathcal{P}\big(\mathrm{e}^{-tV/n}\Gamma\big(\mathrm{e}^{-th/n}\big)\big)\big]^n\\ &= \lim_n\big[\mathrm{e}^{-th/n} \otimes \big(\mathrm{e}^{-tV/n}\Gamma\big(\mathrm{e}^{-th/n}\big)\big)\big]^n\\ &= \lim_n \mathrm{e}^{-th} \otimes \big[\mathrm{e}^{-tV/n}\mathrm{e}^{-tH_0/n}\big]^n = \mathrm{e}^{-th} \otimes \mathrm{e}^{-tH}.\end{aligned}$$

Since this holds for all $t > 0$ we get $\widetilde{H} = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$. □

The fact that the class of SQFH contains singular physically interesting Hamiltonians is mainly due to its stability under norm resolvent convergence.

Proposition 7.12. *Assume that $\{H_n\}$ is a sequence of SQFH of type $\mathcal{O}$ with the same one particle kinetic energy h and such that $H_n \to H$ in norm resolvent sense, where H is a self-adjoint operator on $\Gamma(\mathcal{H})$. Then H is SQFH of type $\mathcal{O}$ with one particle kinetic energy h.*

Proof. Due to norm resolvent convergence the operators H_n are uniformly bounded from below and $\mathrm{e}^{-tH_n} \to \mathrm{e}^{-tH}$ in norm for each $t > 0$. Thus $\mathrm{e}^{-tH} \in \mathscr{F}(\mathcal{O})$ hence H is affiliated to $\mathscr{F}(\mathcal{O})$ and we have

$$\mathcal{P}\big(\mathrm{e}^{-tH}\big) = \lim_n \mathcal{P}\big(\mathrm{e}^{-tH_n}\big) = \mathrm{e}^{-th} \otimes \mathrm{e}^{-tH_n} = \mathrm{e}^{-th} \otimes \mathrm{e}^{-tH}$$

for all $t > 0$. This is equivalent to $\mathcal{P}(H) = h \otimes 1_{\Gamma(\mathcal{H})} + 1_{\mathcal{H}} \otimes H$. $\square$

8. Mourre estimate for operators affiliated to $\mathscr{F}(\mathcal{O})$

8.1. We begin with some basic facts concerning the Mourre estimate as presented in [3, Chapter 7]. Improvements of the theory including an extension to conjugate operators A which are only maximal symmetric can be found in [31] (this is especially useful for the treatment of zero mass fields).

Fix a self-adjoint operator A (the conjugate operator) on a Hilbert space $\mathscr{H}$. An operator $S \in B(\mathscr{H})$ is of class $C^1(A)$ if the map $t \mapsto \mathrm{e}^{-\mathrm{i}tA} S \mathrm{e}^{\mathrm{i}tA}$ is strongly C^1. If this map is of class C^1 in norm, we say that S is of class $C^1_{\mathrm{u}}(A)$. It is easy to see that S is of class $C^1(A)$ if and only if the commutator $[A, S]$, which is well defined as sesquilinear form on $D(A)$, extends to a bounded operator $[A, S]^\circ$ on $\mathscr{H}$.

Now let H be a second self-adjoint operator on $\mathscr{H}$ (the Hamiltonian). We say that H is of class $C^1(A)$ or $C^1_{\mathrm{u}}(A)$ if $(H - z)^{-1}$ has the corresponding property (here z is any number not in the spectrum of H). It is possible to characterize the $C^1(A)$ property in terms of the commutator $[A, H]$, we recall here only what is strictly necessary (see [31]). If H is of class $C^1(A)$ then $D(H) \cap D(A)$ is a core for H and the commutator $[A, H]$, defined as sesquilinear form on $D(H) \cap D(A)$, extends to a continuous sesquilinear form $[A, H]^\circ$ on $D(H)$ equipped with the graph topology [31, Proposition 2.19]. Moreover, we have:

$$\big[A, (H - z)^{-1}\big]^\circ = -(H - z)^{-1}[A, H]^\circ (H - z)^{-1}. \tag{8.1}$$

From now on we keep the notation $[A, H]$ for the extension $[A, H]^\circ$.

We define $\tilde{\rho}^A_H : \mathbb{R} \to (-\infty, \infty]$ as follows: $\tilde{\rho}^A_H(\lambda)$ is the upper bound of the numbers a for which there are a real function $\theta \in C_{\mathrm{c}}(\mathbb{R})$ with $\theta(\lambda) \neq 0$ and a compact operator K such that

$$\theta(H)[H, \mathrm{i}A]\theta(H) \geqslant a\theta(H)^2 + K.$$

In other terms, $\tilde{\rho}^A_H(\lambda)$ is the best constant in the Mourre estimate. Then let $\rho^A_H(\lambda)$ be the upper bound of the numbers a such that the preceding inequality holds for some θ and $K = 0$. So we get a second function $\rho^A_H : \mathbb{R} \to (-\infty, \infty]$ such that $\rho^A_H \leqslant \tilde{\rho}^A_H$. We have $\rho^A_H(\lambda) < \infty$ if and only if $\lambda \in \sigma(H)$ and $\tilde{\rho}^A_H(\lambda) < \infty$ if and only if $\lambda \in \sigma_{\mathrm{ess}}(H)$, see Lemma 7.2.1 and Proposition 7.2.6 in [3].

The two functions defined above are lower semi-continuous. Thus the set $\tau_A(H)$ where $\tilde{\rho}^A_H(\lambda) \leqslant 0$ is closed and will be called the set of A-*thresholds* of H. If $\lambda \notin \tau_A(H)$ we say that A

is conjugate to H *at* λ. The closed set $\kappa_A(H)$ of A-*critical points* of H is given by the condition $\rho_H^A(\lambda) \leqslant 0$.

Clearly $\tau_A(H) \subset \kappa_A(H)$. In order to understand how much differ these sets we introduce the following notion. Say that $\lambda \in \mathbb{R}$ is an *M-eigenvalue* of H if it is an eigenvalue and $\widetilde{\rho}_H^A(\lambda) > 0$. By the virial theorem, these eigenvalues are of finite multiplicity and are not accumulation points of eigenvalues. Thus the set $\mu_A(H)$ of all M-eigenvalues of H is discrete. The next result [3, Theorem 7.2.13] says that the functions ρ_H^A and $\widetilde{\rho}_H^A$ differ only on the small set $\mu^A(H)$. Let $\sigma_{\mathrm{p}}(H)$ be the set of eigenvalues of H.

Proposition 8.1. *We have* $\rho_H^A(\lambda) = 0$ *if* λ *is a M-eigenvalue of* H *and otherwise* $\rho_H^A(\lambda) = \widetilde{\rho}_H^A(\lambda)$. *Moreover,* $\rho_H^A(\lambda) > 0$ *if and only if* $\widetilde{\rho}_H(\lambda) > 0$ *and* $\lambda \notin \sigma_{\mathrm{p}}(H)$. *In particular* ($\sqcup$ *means disjoint union*):

$$\kappa_A(H) = \tau_A(H) \cup \sigma_{\mathrm{p}}(H) = \tau_A(H) \sqcup \mu_A(H). \tag{8.2}$$

We shall also need the following result, which is a particular case of [3, Theorem 8.3.6] (see also [12, Theorem 3.4] for a simpler proof in an important particular case).

Proposition 8.2. *Let* $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ *and let* H_i, A_i *be self-adjoint operators on* $\mathscr{H}_i$ *such that* H_i *is bounded from below and of class* $C_{\mathrm{u}}^1(A_i)$. *Consider the self-adjoint operators* $H = H_1 \otimes 1 + 1 \otimes H_2$ *and* $A = A_1 \otimes 1 + 1 \otimes A_2$ *on* $\mathscr{H}$. *Then* H *is of class* $C_{\mathrm{u}}^1(A)$ *and*

$$\rho_H^A(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2} \left[\rho_{H_1}^{A_1}(\lambda_1) + \rho_{H_2}^{A_2}(\lambda_2)\right]. \tag{8.3}$$

8.2. We shall explain now how one may compute the function $\widetilde{\rho}_H^A$ using C^*-algebra methods. This technique has been introduced in [13] in the context of the N-body problem and further developed in [3, Chapter 8]. The main point of this approach is that it avoids the use of auxiliary objects like partitions of unity. The presentation below is adapted to our needs, that from [3,13] is more general since it does not require the quotient algebra to be represented on a Hilbert space.

Let $\mathscr{C}$ be a C^*-algebra such that $K(\mathscr{H}) \subset \mathscr{C} \subset B(\mathscr{H})$. Then the quotient C^*-algebra $\widetilde{\mathscr{C}} = \mathscr{C}/K(\mathscr{H})$ is well defined. If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ then one can consider its image $\widetilde{H} = \mathcal{P}(H)$ through the canonical morphism $\mathcal{P} : \mathscr{C} \to \widetilde{\mathscr{C}}$. Then $\widetilde{H}$ is an observable affiliated to $\widetilde{\mathscr{C}}$ and the essential spectrum of H is equal to the spectrum of $\widetilde{H}$. We shall assume that a faithful non-degenerate realization of $\widetilde{\mathscr{C}}$ on some Hilbert space $\widetilde{\mathscr{H}}$ is given and that the observable $\widetilde{H}$ is realized as a self-adjoint operator (which we denote also by $\widetilde{H}$) on $\widetilde{\mathscr{H}}$.

Let A be a self-adjoint operator on $\mathscr{H}$ with $\mathrm{e}^{-\mathrm{i}tA}\mathscr{C}\mathrm{e}^{\mathrm{i}tA} = \mathscr{C}$ for each real t and such that the map $t \mapsto \mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}$ be norm continuous for each $S \in \mathscr{C}$. Since $\mathrm{e}^{-\mathrm{i}tA}K(\mathscr{H})\mathrm{e}^{\mathrm{i}tA} = K(\mathscr{H})$, there is a norm continuous one-parameter group of automorphisms α_t of $\widetilde{\mathscr{C}}$ such that $\mathcal{P}(\mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}) = \alpha_t(\widetilde{S})$ for all t and $S \in \mathscr{C}$. Finally, assume that the group α_t is unitarily implemented in the representation on $\widetilde{\mathscr{H}}$ (this is not needed in the more abstract theory presented in [3,13]). More precisely, our hypotheses are:

$$(\mathrm{CA}) \quad \begin{cases} A \text{ is a self-adjoint operator on } \mathscr{H} \text{ with } \mathrm{e}^{-\mathrm{i}tA}\mathscr{C}\mathrm{e}^{\mathrm{i}tA} = \mathscr{C} \text{ for all } t; \\ \text{the map } t \mapsto \mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA} \text{ is norm continuous for each } S \in \mathscr{C}; \\ \widetilde{A} \text{ is self-adjoint on } \widetilde{\mathscr{H}} \text{ and } \mathcal{P}(\mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}) = \mathrm{e}^{-\mathrm{i}t\widetilde{A}}\,\mathcal{P}(S)\,\mathrm{e}^{\mathrm{i}t\widetilde{A}} \text{ for all } t \text{ and } S \in \mathscr{C}. \end{cases}$$

The next proposition follows immediately from the preceding definitions and comments.

Proposition 8.3. *Assume that H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ and of class $C^1_{\mathrm{u}}(A)$. If $\widetilde{H}$ is a self-adjoint operator on $\widetilde{\mathscr{H}}$ then $\widetilde{H}$ is of class $C^1_{\mathrm{u}}(\widetilde{A})$ and $\widetilde{\rho}^A_H = \rho^{\widetilde{A}}_{\widetilde{H}}$.*

8.3. We shall apply the preceding general theory in the situation of interest for us in this paper. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{O}$ an abelian non-degenerate C^*-algebra of operators on $\mathcal{H}$ such that $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$. Let $\mathscr{H} = \Gamma(\mathcal{H})$ be the symmetric or antisymmetric Fock space over $\mathcal{H}$ and $\mathscr{C} = \mathscr{F}(\mathcal{O})$. We shall consider only conjugate operators of the form:

$$\text{(OA)}\quad \begin{cases} A = \mathrm{d}\Gamma(\mathfrak{a}) \text{ where } \mathfrak{a} \text{ is a self-adjoint operator on } \mathcal{H} \text{ such that } \mathrm{e}^{-\mathrm{i}t\mathfrak{a}}\mathcal{O}\mathrm{e}^{\mathrm{i}t\mathfrak{a}} = \mathcal{O} \\ \text{and such that the map } t \mapsto \mathrm{e}^{-\mathrm{i}t\mathfrak{a}}S\mathrm{e}^{\mathrm{i}t\mathfrak{a}} \text{ is norm continuous for all } S \in \mathcal{O}. \end{cases}$$

Lemma 8.4. *We have $\mathrm{e}^{-\mathrm{i}tA}\mathscr{F}(\mathcal{O})\mathrm{e}^{\mathrm{i}tA} = \mathscr{F}(\mathcal{O})$ for all real t and the map $t \mapsto \mathrm{e}^{-\mathrm{i}tA}T\mathrm{e}^{\mathrm{i}tA}$ is norm continuous for all $T \in \mathscr{F}(\mathcal{O})$.*

Proof. Note that $\mathrm{e}^{\mathrm{i}tA} = \Gamma(\mathrm{e}^{\mathrm{i}t\mathfrak{a}})$. In the bosonic case it suffices to take $T = W(u)\Gamma(S)$ with $u \in \mathcal{H}$ and $S \in \mathcal{O}$ with $\|S\| < 1$. Then, due to (2.17), we have

$$\mathrm{e}^{-\mathrm{i}tA}T\mathrm{e}^{\mathrm{i}tA} = W\big(\mathrm{e}^{-\mathrm{i}t\mathfrak{a}}u\big)\Gamma\big(\mathrm{e}^{-\mathrm{i}t\mathfrak{a}}S\mathrm{e}^{\mathrm{i}t\mathfrak{a}}\big) \tag{8.4}$$

and we get norm continuity by Lemma 3.7. In the fermionic case we may assume $T = \phi^k(u)\Gamma(S)$ with $k = 0, 1$ and the argument is even simpler. □

Lemma 8.5. *Let H be a self-adjoint operator affiliated to $\mathscr{F}(\mathcal{O})$. Then H is of class $C^1_{\mathrm{u}}(A)$ if and only if H is of class $C^1(A)$ and the operator $[A, (H - z)^{-1}]$ given by* (8.1) *belongs to $\mathscr{F}(\mathcal{O})$.*

Proof. If $S = (H - z)^{-1}$ then $S(t) \equiv \mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}$ belongs to $\mathscr{F}(\mathcal{O})$ for all real t. If H is of class $C^1_{\mathrm{u}}(A)$ then $[S, \mathrm{i}A]$ is the norm derivative at $t = 0$ of the map $t \mapsto S(t)$ hence belongs to $\mathscr{F}(\mathcal{O})$. On the other hand, if H is of class $C^1(A)$ then $[S(t), \mathrm{i}A]$ is the strong derivative of the map $t \mapsto S(t)$ hence we have $S(t) - S = \int_0^t \mathrm{e}^{-\mathrm{i}\tau A}[S, \mathrm{i}A]\mathrm{e}^{\mathrm{i}\tau A}$ in the strong topology. If $[S, \mathrm{i}A] \in \mathscr{F}(\mathcal{O})$ then by Lemma 8.4 the integrand here is norm continuous, hence the integral exists in norm, so $t \mapsto S(t)$ is norm C^1. □

From Theorems 5.4 and 6.2 and from relations like (8.4) (bosonic case) we get canonical identifications:

$$\widetilde{\mathscr{C}} \equiv \mathcal{P}\big(\mathscr{F}(\mathcal{O})\big) \subset \mathcal{O} \otimes \mathscr{F}(\mathcal{O}), \qquad \widetilde{\mathscr{H}} = \mathcal{H} \otimes \mathscr{H} \equiv \mathcal{H} \otimes \Gamma(\mathcal{H}),$$
$$\widetilde{A} = \mathfrak{a} \otimes 1 + 1 \otimes A. \tag{8.5}$$

Our main result on the Mourre estimate for SQFH follows.

Theorem 8.6. *Let H be a SQFH of type $\mathcal{O}$ with one particle kinetic energy h and one particle mass $m = \inf h > 0$. Assume that condition* (OA) *from p.* 127 *is fulfilled, that H is of class $C^1_{\mathrm{u}}(A)$, and that h is of class $C^1_{\mathrm{u}}(\mathfrak{a})$ and such that $\rho^{\mathfrak{a}}_h \geqslant 0$. Then $\kappa_{\mathfrak{a}}(h) = \tau_{\mathfrak{a}}(h)$, we have $\rho^A_H \geqslant 0$ and*

$$\tau_A(H) = \left[\bigcup_{n=1}^{\infty} \tau^n_{\mathfrak{a}}(h)\right] + \sigma_{\mathrm{p}}(H), \tag{8.6}$$

where $\tau_{\mathfrak{a}}^n(h)=\tau_{\mathfrak{a}}(h)+\cdots+\tau_{\mathfrak{a}}(h)$ *(n terms). Alternatively, if we set* $H_0=\mathrm{d}\Gamma(h)$ *then*

$$\tau_A(H_0)=\bigcup_{n=1}^{\infty}\tau_{\mathfrak{a}}^n(h) \quad \textit{and} \quad \tau_A(H)=\tau_A(H_0)+\sigma_{\mathrm{p}}(H). \tag{8.7}$$

Proof. The operator h cannot have eigenvalues of finite multiplicity because the corresponding spectral projection would be in $\mathcal{O}''$ which does not contain finite-dimensional projections. Hence from Proposition 8.1 we get $\widetilde{\rho}_h^{\mathfrak{a}}=\rho_h^{\mathfrak{a}}$, in particular $\kappa_{\mathfrak{a}}(h)=\tau_{\mathfrak{a}}(h)$. Since H is a SQFH we have $\widetilde{H}=h\otimes 1_{\Gamma(\mathcal{H})}+1_{\mathcal{H}}\otimes H$. By taking into account (8.5) we deduce from Propositions 8.3 and 8.2 that

$$\widetilde{\rho}_H^A(\lambda)=\inf_{\lambda=\lambda_1+\lambda_2}\big[\rho_h^{\mathfrak{a}}(\lambda_1)+\rho_H^A(\lambda_2)\big]=\inf_{\mu}\big[\rho_h^{\mathfrak{a}}(\lambda-\mu)+\rho_H^A(\mu)\big]. \tag{8.8}$$

In this proof we simplify notations and set $\widetilde{\rho}=\widetilde{\rho}_H^A$, $\rho=\rho_H^A$, and $\rho_h=\rho_h^{\mathfrak{a}}$. Also, without loss of generality, we shall assume that $\inf H=0$. Then $\sigma_{\mathrm{ess}}(H)\subset[m,\infty[$ due to Theorem 7.8. Thus the functions ρ on the interval $\lambda<0$ and $\widetilde{\rho}$ and ρ_h on $\lambda<m$ are equal to infinity, in particular

$$\widetilde{\rho}(\lambda)=\inf_{0\leqslant\mu\leqslant\lambda-m}\big[\rho_h(\lambda-\mu)+\rho(\mu)\big] \tag{8.9}$$

with the convention that the infimum over an empty set is equal to infinity. Observe that if $\lambda<m$ then λ is either in the resolvent set of H, and then $\rho(\lambda)=\infty$, or λ is in the discrete spectrum of H, hence is an M-eigenvalue of H, so $\rho(\lambda)=0$ by Proposition 8.1. Thus $\rho(\lambda)\geqslant 0$ if $\lambda<m$. Assume now that we have shown that $\rho(\lambda)\geqslant 0$ if $\lambda<km$ for an integer $k\geqslant 1$. If $\lambda<km+m$ then in (8.9) only $\mu<km$ will appear and so $\rho(\mu)\geqslant 0$. Since $\rho_h\geqslant 0$ by hypothesis, we get $\rho(\lambda)\geqslant 0$ if $\lambda<(k+1)m$. By induction we finally obtain $\rho(\lambda)\geqslant 0$ for all λ.

We thus have $0\leqslant\rho\leqslant\widetilde{\rho}$ and $\rho_h\geqslant 0$. Hence $\tau(H)\equiv\tau_A(H)$ is the set of λ such that $\widetilde{\rho}(\lambda)=0$ and $\kappa(H)\equiv\kappa_A(H)$ is the set of λ such that $\rho(\lambda)=0$. Moreover, $\tau(h)\equiv\tau_{\mathfrak{a}}(h)=\kappa_{\mathfrak{a}}(h)$ is the set of λ such that $\rho_h(\lambda)=0$. Then the first equality in (8.8) clearly gives: $\rho(\lambda)=0$ if and only if one can write $\lambda=\lambda_1+\lambda_2$ with $\rho_h(\lambda_1)=0$ and $\rho(\lambda_2)=0$ (these functions are lower semi-continuous). Finally, from (8.2) we obtain:

$$\tau(H)=\tau(h)+\kappa(H)=\tau(h)+\big[\tau(H)\cup\sigma_{\mathrm{p}}(H)\big]=\big[\tau(h)+\sigma_{\mathrm{p}}(H)\big]\cup\big[\tau(h)+\tau(H)\big]. \tag{8.10}$$

This equation for the set $\tau(H)$ has as unique solution $\bigcup_{n=1}^{\infty}[\tau_{\mathfrak{a}}^n(h)+\sigma_{\mathrm{p}}(H)]$ obtained by iteration. This gives (8.6), for (8.7) note that 0 is the only eigenvalue of H_0. □

Remark 8.7. The relation (8.6) describing the set $\tau_A(H)$ of A-thresholds of H has a simple physical interpretation. It says that an energy λ is an A-threshold if and only if one can write it as a sum $\lambda=\lambda_1+\cdots+\lambda_n+\mu$ where the λ_k are $\mathfrak{a}$-threshold energies of the free particle and μ is the energy of a bound state of the field. This means that at energy λ one can pull out n free particles from the field, each one having an $\mathfrak{a}$-threshold energy, such that the field remains in a bound state.

Remark 8.8. Outside the threshold set $\tau_A(H)$ one expects H to have nice spectral properties. A rather weak condition which implies the absolute continuity of the spectrum of H outside $\tau_A(H)$ (and many other properties) is that H be of class $\mathcal{C}^{1,1}(A)$, which means that the map

$t \mapsto \mathrm{e}^{-\mathrm{i}tA}(H+\mathrm{i})^{-1}\mathrm{e}^{\mathrm{i}tA}$ is of Besov class $B_\infty^{1,1}$ in norm (this is slightly more restrictive than the $C_\mathrm{u}^1(A)$ class; the boundedness of the double commutator $[A,[A,(H-z)^{-1}]]$ implies it). In particular, in order to exclude the existence of the singularly continuous spectrum, it is important to be sure that $\tau_A(H)$ is a small set. Note that $\tau_A(H)$ is always closed and that it is countable if $\tau_\mathfrak{a}(h)$ is countable and $\mathcal{H}$ separable. In fact, in the most important physical cases we have $\tau_\mathfrak{a}(h)=\{m\}$ and then $\tau_A(H)=m\mathbb{N}^*+\sigma_\mathrm{p}(H)$.

As an example, we consider the important particular case when $\mathcal{H}$ is a Sobolev space over an Euclidean space $X=\mathbb{R}^s$, e.g. $\mathcal{H}=L^2(X)$. The $P(\varphi)_2$ model as treated in [24] is covered by this example. Then we take $\mathcal{O}=C_0(X^*)$ (space of continuous functions of the momentum operator P which tend to zero at infinity). A self-adjoint operator h on $\mathcal{H}$ with $\inf h=m>0$ is strictly affiliated to $C_0(X^*)$ if and only if $h=h(P)$ where $h:X\to\mathbb{R}$ is a continuous function such that $|h(p)|\to\infty$ when $|p|\to\infty$.

We shall assume that $h:X\to\mathbb{R}$ is a function of class C^1 in the usual sense. Let $\tau(h)$ be the set of critical values of the function h in the usual sense, i.e. the numbers of the form $h(p)$ with $\nabla h(p)=0$. In this context it is natural to consider one particle conjugate operators of the form $\mathfrak{a}=F(P)Q+QF(P)$ with F a vector field of class $C_\mathrm{c}^\infty(X)$. The corresponding operators $A=\mathrm{d}\Gamma(\mathfrak{a})$ will be called *of class VF* (vector fields). The following is a consequence of Theorem 8.6.

Corollary 8.9. *In the preceding framework, let H be a SQFH with one particle kinetic energy h. Then $\sigma_\mathrm{ess}(H)=[m+\inf H,\infty[$. Assume that H is of class $C_\mathrm{u}^1(A)$ if A is of class VF and let*

$$\tau(H)=\left[\bigcup_{n=1}^{\infty}\tau^n(h)\right]+\sigma_\mathrm{p}(H), \tag{8.11}$$

where $\tau^n(h)=\tau(h)+\cdots+\tau(h)$ (n terms). Then H admits a conjugate operator of class VF at each point not in $\tau(H)$. If H is of class $\mathcal{C}^{1,1}(A)$ (e.g. if $[A,[A,(H-z)^{-1}]]$ is bounded) for each operator A of class VF then H has no singular continuous spectrum outside $\tau(H)$.

Remark 8.10. It is possible to prove the Mourre estimate for more general Hamiltonians H affiliated to $\mathscr{F}(\mathcal{O})$ if the operator A satisfies the condition (OA). We use again Proposition 8.3 by taking into account the identifications made in (8.5). But now one step in the preceding arguments is missing because in general $\widetilde{H}$ is no more representable in the form $h\otimes 1_{\Gamma(\mathcal{H})}+1_\mathcal{H}\otimes H'$ with operators h and H' affiliated to $\mathcal{O}$ and $\mathscr{F}(\mathcal{O})$, respectively, so we cannot use the Proposition 8.2. However, by using the techniques from [20, Sections 5 and 6] one can sometimes overcome this difficulty. For example, if $\widetilde{H}=h\otimes M+1_\mathcal{H}\otimes H'$ with $M\geqslant c>0$ then one can proceed as in [20, Section 6] (in fact, the situation here is much simpler). The main point is that Proposition 8.3 shows that we only have to estimate from below the commutator $[\widetilde{H},\mathrm{i}\widetilde{A}]$ which has the following special structure:

$$[\widetilde{H},\mathrm{i}\widetilde{A}]=[\widetilde{H},\mathrm{i}\mathfrak{a}\otimes 1_{\Gamma(\mathcal{H})}]+[\widetilde{H},1_\mathcal{H}\otimes \mathrm{i}A]. \tag{8.12}$$

As already mentioned in the comments after Theorem 7.6, if H is strictly affiliated to $\mathscr{F}(\mathcal{O})$ the quotient $\widetilde{H}$ is identified to a continuous family $\{\widetilde{H}(x)\}_{x\in\mathscr{X}}$ of self-adjoint operators $\widetilde{H}(x)$ on $\Gamma(\mathcal{H})$ strictly affiliated to $\mathscr{F}(\mathcal{O})$. Since $\mathfrak{a}$ "acts" only on the variable x (by condition (OA)) and due to Lemma 8.4, each term on the right-hand side of (8.12) formally belongs to $\mathscr{F}(\mathcal{O})$ and one may impose conditions which ensure strict positivity of the sum. All this can be done rigorously by working with the resolvent of H instead of H, as in [20, Section 5], and in fact the situation here is simpler than in the case of an N-body dispersive Hamiltonian.

9. QFH associated to Lagrangian subspaces of $\mathcal{H}$

Our purpose in this section is to show that Hamiltonians like that of the $P(\varphi)_2$ model are covered by our formalism. We shall consider only the bosonic situation. We first recall another classical procedure for constructing realizations of the Fock representation of the CCR, the so-called *field realizations*. The idea is to use maximal abelian subalgebras of the Weyl algebra $\mathscr{W}(\mathcal{H})$ defined on p. 96. Note that $\mathscr{W}(\mathcal{H})$ depends (modulo canonical isomorphisms) only on the symplectic structure of $\mathcal{H}$ defined by the symplectic form $\sigma(u,v) = \Im\langle u|v\rangle$. Recall that a real linear subspace $\mathcal{E}$ of $\mathcal{H}$ is called *isotropic* if $\sigma(u,v) = 0$ for all $u, v \in \mathcal{E}$ and that a maximal isotropic subspace is called *Lagrangian*. A straightforward argument gives:

Lemma 9.1. *For any isotropic subspace $\mathcal{E}$ we have $\mathcal{E} \cap \mathrm{i}\mathcal{E} = \{0\}$ and $\|u + \mathrm{i}v\|^2 = \|u\|^2 + \|v\|^2$ for all $u, v \in \mathcal{E}$; and $\mathcal{E}$ is Lagrangian if and only if $\mathcal{H} = \mathcal{E} + \mathrm{i}\mathcal{E}$ and then $\mathcal{E}$ is closed. If c is a conjugation (antilinear isometry such that $c^2 = 1$) then $\mathcal{H}_c = \{u \in \mathcal{H} \mid cu = u\}$ is a Lagrangian subspace of $\mathcal{H}$ and reciprocally, each Lagrangian subspace of $\mathcal{H}$ is of this form for a uniquely determined c.*

For each real linear subspace $\mathcal{E} \subset \mathcal{H}$ let $\mathscr{W}(\mathcal{E})$ be the closed linear subspace of $\mathscr{W}(\mathcal{H})$ generated by the operators $W(u)$ with $u \in \mathcal{E}$. This is obviously a C^*-subalgebra of $\mathscr{W}(\mathcal{H})$.

Lemma 9.2. *Let $\mathcal{E}$ be a real linear subspace of $\mathcal{H}$. Then $\mathscr{W}(\mathcal{E})$ is abelian if and only if $\mathcal{E}$ is isotropic and $\mathscr{W}(\mathcal{E})$ is maximal abelian in $\mathscr{W}(\mathcal{H})$ if and only if $\mathcal{E}$ is Lagrangian.*

Proof. Assume that $\mathscr{W}(\mathcal{E})$ is abelian and let $u, v \in \mathcal{E}$. From (2.2) we get $\mathrm{e}^{\mathrm{i}\Im\langle u|tv\rangle} = 1$ for all $t \in \mathbb{R}$ hence $\Im\langle u|v\rangle = 0$, so $\mathcal{E}$ is isotropic. If $\mathcal{E}$ is Lagrangian then $\mathscr{W}(\mathcal{E})$ is maximal abelian in $\mathscr{W}(\mathcal{H})$ because $\mathscr{W}(\mathcal{E})''$ is maximal abelian on the Fock space $\Gamma(\mathcal{H})$. Finally, assume that $\mathcal{E}$ is not Lagrangian, so that $\mathcal{K} = \mathcal{E} + \mathrm{i}\mathcal{E} \neq \mathcal{H}$. If $u \in \mathcal{H} \setminus \mathcal{K}$ then, as shown in the proof of Proposition 5.2.9 from [15], one has $W(u) \notin \mathscr{W}(\mathcal{K})$ so $W(u) \notin \mathscr{W}(\mathcal{E})$. If $\mathcal{K}$ is not dense in $\mathcal{H}$ we may choose $u \perp \mathcal{K}$ and get $W(u)$ in the commutant of $\mathscr{W}(\mathcal{E})$ but not in $\mathscr{W}(\mathcal{E})$. If $\mathcal{K}$ is dense in $\mathcal{H}$ then $\mathcal{E}$ cannot be closed and we choose u in the closure of $\mathcal{E}$ but not in $\mathcal{E}$. Since the closure of $\mathcal{E}$ is isotropic we see that $[W(u), W(v)] = 0$ for all $v \in \mathcal{E}$. But since the sum $\mathcal{K} = \mathcal{E} + \mathrm{i}\mathcal{E}$ is direct $W(u) \notin \mathscr{W}(\mathcal{E})$. □

In the rest of this section we fix a Lagrangian subspace $\mathcal{E}$ of $\mathcal{H}$. It is not difficult to show that the von Neumann algebra $\mathscr{W}(\mathcal{E})''$ generated by $\mathscr{W}(\mathcal{E})$ on $\Gamma(\mathcal{H})$ is maximal abelian and that Ω is a cyclic and separating vector for it. Then $\langle T\rangle = \langle \Omega|T\Omega\rangle$ defines a faithful state on $\mathscr{W}(\mathcal{E})''$ and we denote $L^p(\mathcal{E})$ the L^p spaces associated to the couple $(\mathscr{W}(\mathcal{E})'', \langle\cdot\rangle)$. These spaces are intrinsically defined by abstract integration theory [41] and can be realized as usual L^p spaces over a probability measure space Q which we shall not specify[10] because this is of no interest here (we refer to [24,46] for details on these questions). However, we men-

[10] We emphasize that if $\mathcal{H}$ is infinite-dimensional one can never take $Q = \mathcal{E}$ in any natural sense, so the notation $L^p(\mathcal{E})$ could be misleading. Of course, one may take Q equal to the spectrum of the C^*-algebra $\mathscr{W}(\mathcal{E})$, but this is not a really convenient choice. On the other hand, the theory of Gaussian cylindrical measures on $\mathcal{E}$ offers many useful realizations.

tion that at the abstract level we have canonical identifications $L^\infty(\mathcal{E}) = \mathscr{W}(\mathcal{E})''$ and if $1 \leqslant p < \infty$ then $L^p(\mathcal{E})$ is the completion of $L^\infty(\mathcal{E})$ for the norm $\|T\|_p = \langle |T|^p\rangle^{1/p}$. Moreover, from $\langle W(v)^* W(u)\rangle = \langle W(v)\Omega|W(u)\Omega\rangle$ it follows that the map $W(u) \mapsto W(u)\Omega$ extends to a unitary map $L^2(\mathcal{E}) \to \Gamma(\mathcal{H})$ which will be used from now on to identify these two Hilbert spaces. Thus we have

$$\mathscr{W}(\mathcal{E})'' \equiv L^\infty(\mathcal{E}) \subset L^p(\mathcal{E}) \subset L^2(\mathcal{E}) \equiv \Gamma(\mathcal{H}) \subset L^q(\mathcal{E}) \subset L^1(\mathcal{E})$$
$$\text{if } 1 < q < 2 < p < \infty. \tag{9.1}$$

We get a realization on $L^2(\mathcal{E})$ of the Fock representation by transport from $\Gamma(\mathcal{H})$ with the help of the identification map defined above. This *$\mathcal{E}$-realization* is a "field realization" in the sense that the field operators $\phi(u)$ are realized as operators of multiplication by (equivalence classes of) real measurable functions defined on a probability space Q. Note that the "momentum operators" defined by

$$\pi(u) = \phi(\mathrm{i}u) = \mathrm{i}\big(a^*(u) - a(u)\big) \quad \text{for } u \in \mathcal{E}$$

can be realized as differential operators for certain choices of Q. One has the commutation relations

$$\big[\phi(u), \phi(v)\big] = \big[\pi(u), \pi(v)\big] = 0 \quad \text{and} \quad \big[\phi(u), \pi(v)\big] = 2\mathrm{i}\langle u|v\rangle \quad \text{if } u, v \in \mathcal{E}.$$

Example 9.3. This is the most elementary situation which is of physical interest. Let h be a self-adjoint operator on $\mathcal{H}$ which leaves $\mathcal{E}$ invariant (i.e. is real with respect to the conjugation associated to $\mathcal{E}$) and has pure point spectrum. Then there is an orthonormal basis $\{e_k\}_{k\in K}$ of the real Hilbert space $\mathcal{E}$ and a function $h : K \to \mathbb{R}$ such that $h = \sum_k h(k)|e_k\rangle\langle e_k|$ as operator on $\mathcal{H}$. Let us set $a_k = a(e_k)$, $\phi_k = \phi(e_k/\sqrt{2})$, and $\pi_k = \pi(e_k/\sqrt{2})$. Then $H_0 = \mathrm{d}\Gamma(h)$ has the following familiar expression:

$$H_0 = \sum_k h(k)\, \mathrm{d}\Gamma\big(|e_k\rangle\langle e_k|\big) = \sum_k h(k) a_k^* a_k = \frac{1}{2}\sum_k h(k)\big(\pi_k^2 + \phi_k^2 - 1\big),$$

where ϕ_k, π_k are self-adjoint operators satisfying the commutation relations $[\phi_j, \phi_k] = [\pi_j, \pi_k] = 0$ and $[\phi_j, \pi_k] = \mathrm{i}\delta_{jk}$. This is the kinetic energy operator of the (discretized) field and the total Hamiltonian is obtained by adding a "generalized polynomial" V in the field operators ϕ_k.

We want to show that much more general Hamiltonians constructed by procedures similar to that of Example 9.3 are SQFH in our sense. Let $\mathcal{O}$ be an abelian non-degenerate C^*-algebra on $\mathcal{H}$ such that $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$. In the statement of the next result we use the terminology of abstract integration theory; we refer to [41] for a short review of the main facts.

Theorem 9.4. *Let $H_0 = \mathrm{d}\Gamma(h)$ where h is a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathcal{O}$ and satisfying $m \equiv \inf h > 0$ and $h^{-1}\mathcal{E} \subset \mathcal{E}$. Let V be a self-adjoint operator on $\Gamma(\mathcal{H})$ which is bounded from below, affiliated to $\mathscr{W}(\mathcal{E})''$, and has the property $V \in L^p(\mathcal{E})$ for all $p < \infty$. Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$ and its closure H is a SQFH of type $\mathcal{O}$ with one particle kinetic energy h.*

Proof. We shall use Theorem 7.4 with $H_0 = \mathrm{d}\Gamma(h)$. The conditions imposed on h imply that H_0 generates a hypercontractive semigroup due to Nelson's theorem [46, Theorem 1.17]. Then V, viewed as function on Q, is admissible by hypothesis, so H is essentially self-adjoint on $D(H_0) \cap D(V)$. Now assume that $V \in L^\infty = \mathscr{W}(\mathcal{E})''$. Kaplansky's density theorem [40, Theorem 4.3.3] implies that the closed ball of radius $\|V\|$ in $\mathscr{W}(\mathcal{E})$ is strongly dense in the closed ball of radius $\|V\|$ in $\mathscr{W}(\mathcal{E})''$. Since the function $1 \equiv \Omega$ belongs to L^2 it follows that there is a *sequence* $\{V_n\}$ of self-adjoint operators V_n in $\mathscr{W}(\mathcal{E})$ with $\|V_n\| \leqslant \|V\|$ such that $\|V_n - V\|_{L^2} \to 0$. But we have $\|V_n - V\|_{L^\infty} \leqslant 2\|V\|$ hence we get by interpolation $\|V_n - V\|_{L^p} \to 0$ for all $p < \infty$. Let $H_n = H_0 + V_n$, then Theorem 7.4 implies that $H_n \to H$ in norm resolvent sense. From Proposition 7.10 it follows that each H_n is a SQFH hence H is a SQFH of type $\mathcal{O}$ with one particle kinetic energy h by Proposition 7.12. In the general case, we consider the operators $V_n = \inf(V, n) \in L^\infty$ which obviously have the properties required in Theorem 7.4. Thus $H_n \to H$ in norm resolvent sense and we use again Proposition 7.12. $\square$

The preceding theorem covers $P(\varphi)_2$ models with a spatial and an ultraviolet cutoff in any dimension. In space–time dimension 2 it is possible to remove the ultraviolet cutoff staying in the Fock space. The fact that the corresponding Hamiltonian is a SQFH in the sense of Definition 7.7 follows from:

Theorem 9.5. *Let H_0 be as in Theorem* 9.4 *and let V be a self-adjoint operator on $\Gamma(\mathcal{H})$ affiliated to $\mathscr{W}(\mathcal{E})''$ with the property $V \in L^p(\mathcal{E})$ for all $p < \infty$. Assume that there is a sequence of operators V_n with the same properties as V and that there is some $q > 2$ such that*: (i) *each V_n is bounded from below*; (ii) $\sup_n \|\mathrm{e}^{-V_n}\|_{L^q} < \infty$; (iii) $\|V_n - V\|_{L^q} \to 0$. *Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$ and its closure H is a SQFH of type $\mathcal{O}$ with one particle kinetic energy h.*

This follows immediately from Theorems 9.4 and 7.4 and Proposition 7.12. Christian Gérard sent me[11] a short proof of the fact that the conditions of this theorem are satisfied in the two-dimensional $P(\varphi)_2$ model with a spatial cutoff with V_n defined with the help of ultraviolet cutoffs.

10. Coupling of systems and Pauli–Fierz model

10.1. Our treatment of the coupling between several fields and other external systems is based on the following elementary fact (which follows by induction from [28, Theorem 2.3]). By *ideal* we mean a closed bilateral ideal.

Proposition 10.1. *Assume that $\mathscr{C}_1, \ldots, \mathscr{C}_n$ are nuclear C^*-algebras equipped with ideals $\mathscr{J}_1, \ldots, \mathscr{J}_n$. Let $\mathcal{P}_k : \mathscr{C}_k \to \widetilde{\mathscr{C}_k} \equiv \mathscr{C}_k / \mathscr{J}_k$ be the canonical surjection and let $\mathcal{P}'_k = 1_{\mathscr{C}_1} \otimes \cdots \otimes \mathcal{P}_k \otimes \cdots \otimes 1_{\mathscr{C}_n}$ be the tensor product of this morphism with the identity maps, so that*

$$\mathcal{P}'_k \colon \mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_n \to \mathscr{C}_1 \otimes \cdots \otimes \widetilde{\mathscr{C}_k} \otimes \cdots \otimes \mathscr{C}_n$$

[11] By fax, on March 15, 2001 (*sic*).

is a morphism. Then the kernel of the morphism

$$\mathcal{P} \equiv \bigoplus_{k=1}^{n} \mathcal{P}'_k : \mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_n \to \bigoplus_{k=1}^{n} \mathscr{C}_1 \otimes \cdots \otimes \widetilde{\mathscr{C}_k} \otimes \cdots \otimes \mathscr{C}_n$$

is equal to $\mathscr{J}_1 \otimes \cdots \otimes \mathscr{J}_n$.

Corollary 10.2. *Assume that each* $\mathscr{C}_k$ *is realized on a Hilbert space* $\mathscr{H}_k$ *and* $\mathscr{J}_k = K(\mathscr{H}_k)$. *Let* H *be a self-adjoint operator on* $\mathscr{H} = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$ *affiliated to* $\mathscr{C} = \mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_n$ *and let us denote* $\widetilde{H}_k = \mathcal{P}'_k(H)$, *which is an observable affiliated to* $\mathscr{C}_1 \otimes \cdots \otimes \widetilde{\mathscr{C}_k} \otimes \cdots \otimes \mathscr{C}_n$. *Then*

$$\sigma_{\text{ess}}(H) = \bigcup_k \sigma(\widetilde{H}_k). \tag{10.1}$$

For this it suffices to note that $K(\mathscr{H}) = K(\mathscr{H}_1) \otimes \cdots \otimes K(\mathscr{H}_n)$.

For simplicity we take $n = 2$, we assume that we are in the framework of Corollary 10.2, and that the quotient $\widetilde{\mathscr{C}_k}$ is realized on a Hilbert space $\widetilde{\mathscr{H}_k}$. Then $\mathcal{P} = \mathcal{P}'_1 \oplus \mathcal{P}'_2$ gives an embedding of the quotient algebra $\widetilde{\mathscr{C}} = \mathscr{C}/K(\mathscr{H})$ as follows:

$$\widetilde{\mathscr{C}} \subset (\widetilde{\mathscr{C}_1} \otimes \mathscr{C}_2) \oplus (\mathscr{C}_1 \otimes \widetilde{\mathscr{C}_2}). \tag{10.2}$$

The C^*-algebra from the right-hand side is realized on the Hilbert space

$$\widetilde{\mathscr{H}} = (\widetilde{\mathscr{H}_1} \otimes \mathscr{H}_2) \oplus (\mathscr{H}_1 \otimes \widetilde{\mathscr{H}_2}). \tag{10.3}$$

Thus if H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ then its image $\mathcal{P}(H) = \widetilde{H}_1 \oplus \widetilde{H}_2 \equiv \widetilde{H}$, an observable affiliated to $\widetilde{\mathscr{C}}$, is expected to be realized as a self-adjoint operator on $\widetilde{\mathscr{H}}$ (this is always the case if we accept not densely defined self-adjoint operators).

We shall explain now how to prove the Mourre estimate in such situations. We assume that the data $\mathscr{C}_k, \mathcal{P}_k, \mathscr{H}_k, A_k, \widetilde{\mathscr{H}_k}, \widetilde{A}_k$ satisfy condition (CA), p. 126. If $A = A_1 \otimes 1_{\mathscr{H}_2} + 1_{\mathscr{H}_1} \otimes A_2$ on $\mathscr{H}$ then $\mathrm{e}^{\mathrm{i}tA} = \mathrm{e}^{\mathrm{i}tA_1} \otimes \mathrm{e}^{\mathrm{i}tA_2}$, hence $\mathrm{e}^{-\mathrm{i}tA}\mathscr{C}\mathrm{e}^{\mathrm{i}tA} = \mathscr{C}$ and the map $t \mapsto \mathrm{e}^{-\mathrm{i}tA}T\mathrm{e}^{\mathrm{i}tA} = \mathscr{C}$ is norm continuous for all $T \in \mathscr{C}$. Let us set

$$A_1^\circ = \widetilde{A}_1 \otimes 1_{\mathscr{H}_2} + 1_{\widetilde{\mathscr{H}_1}} \otimes A_2, \qquad A_2^\circ = A_1 \otimes 1_{\widetilde{\mathscr{H}_2}} + 1_{\mathscr{H}_1} \otimes \widetilde{A}_2,$$
$$\widetilde{A} = A_1^\circ \oplus A_2^\circ. \tag{10.4}$$

Then $\widetilde{A}$ is a self-adjoint operator on $\widetilde{\mathscr{H}}$ such that $\mathcal{P}(\mathrm{e}^{-\mathrm{i}tA}T\mathrm{e}^{\mathrm{i}tA}) = \mathrm{e}^{-\mathrm{i}t\widetilde{A}}\mathcal{P}(T)\mathrm{e}^{\mathrm{i}t\widetilde{A}}$ for all $T \in \mathscr{C}$. So if H is of class $C^1_{\mathrm{u}}(A)$ then $\widetilde{H}$ is of class $C^1_{\mathrm{u}}(\widetilde{A})$, each $\widetilde{H}_k$ is of class $C^1_{\mathrm{u}}(A_k^\circ)$. Let us set $\rho_k = \rho_{\widetilde{H}_k}^{A_k^\circ}$. Then, by using Proposition 8.3 and [3, Proposition 8.3.5] we obtain:

$$\widetilde{\rho}_H^A = \rho_{\widetilde{H}}^{\widetilde{A}} = \min(\rho_1, \rho_2). \tag{10.5}$$

Thus we are reduced to finding estimates from below for the functions ρ_k which can be done by using its relation with the corresponding function $\widetilde{\rho}_k$ as explained in the first part of Section 8. For this we need to know more about the operators $\widetilde{H}_k$ and we shall consider this question below only in the much more elementary case of the Pauli–Fierz Hamiltonians. Couplings with N-body systems as in [4,5,47] should be covered by the preceding formalism (we did not check the details).

10.2. An often studied situation is that of a field coupled with a small confined system. Confinement means that the Hamiltonian of the small system has purely discrete spectrum, hence we take as C^*-algebra of energy observables of the small system the algebra of compact operators. Since taking tensor products with a nuclear algebra preserves short exact sequences, we have slightly more than in the general case.

Proposition 10.3. *Let $\mathscr{C}$ be a C^*-algebra of operators on a Hilbert space $\mathscr{H}$ such that $K(\mathscr{H}) \subset \mathscr{C}$ and let us denote $\widetilde{\mathscr{C}} = \mathscr{C}/K(\mathscr{H})$. Let $\mathscr{L}$ be a second Hilbert space and H a self-adjoint operator on $\mathscr{H} \otimes \mathscr{L}$ affiliated to $\mathscr{C} \otimes K(\mathscr{L})$. Let $\widetilde{H} = \mathcal{P}(H)$ where $\mathcal{P} \equiv \mathcal{P} \otimes \mathrm{Id} : \mathscr{C} \otimes K(\mathscr{L}) \to \widetilde{\mathscr{C}} \otimes K(\mathscr{L})$ is the canonical morphism. Then $\sigma_{\mathrm{ess}}(H) = \sigma(\widetilde{H})$.*

We apply this to a bosonic or fermionic field coupled with a confined system. The next result is an immediate consequence of Theorems 5.4 and 6.2 and of Proposition 10.3.

Theorem 10.4. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{O} \subset B(\mathcal{H})$ a non-degenerate abelian C^*-algebra such that $\mathcal{O}'' \cap K(\mathcal{H}) = \{0\}$. Let $\mathscr{L}$ be a second Hilbert space and $\mathscr{H} = \Gamma(\mathcal{H}) \otimes \mathscr{L}$. Then there is a unique morphism $\mathcal{P} : \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ such that $\mathcal{P}[(F\Gamma(A)) \otimes L] = A \otimes (F\Gamma(A)) \otimes L$ for all $F \in \mathscr{F}(\mathcal{H})$, $A \in \mathcal{O}$ with $\|A\| < 1$, and $L \in K(\mathscr{L})$. One has* $\ker \mathcal{P} = K(\mathscr{H})$. *If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ then $\sigma_{\mathrm{ess}}(H) = \sigma(\mathcal{P}(H))$.*

Remark 10.5. We shall adopt, in the framework of Theorem 10.4, exactly the same definition of *standard QFH* as in Definition 7.7, we just replace the algebra $\mathscr{F}(\mathcal{O})$ with $\mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$. Then clearly *Theorem* 7.8 *remains true without any change*. The conjugate operators which are well adapted to the present situation are of the form $A \otimes 1_{\mathscr{L}}$ where A is as in assumption (OA), p. 127. We keep the notation A for them and note that *Theorem* 8.6 *and Corollary* 8.9 *remain valid without any change*.

Our purpose now is to show that the Hamiltonians of the massive Pauli–Fierz models are covered by Theorem 10.4. We shall consider the abstract version of this model introduced in [23] and further studied in [16,25,32,34]. We treat only the case of a boson field, the fermionic case is easier (just replace $\vee$ by $\wedge$ and note that many assertions become obvious). The following is a standard fact.

Lemma 10.6. *For each $p, q \in \mathbb{N}$ there is a unique linear continuous map $\mathcal{S}_{p,q} : \mathcal{H}^{\vee p} \otimes \mathcal{H}^{\vee q} \to \mathcal{H}^{\vee(p+q)}$ such that $\mathcal{S}_{p,q}(u \otimes v) = uv$ for all $u \in \mathcal{H}^{\vee p}$ and $v \in \mathcal{H}^{\vee q}$. One has $\|\mathcal{S}_{p,q}\| = \binom{p+q}{p}^{1/2}$.*

We consider the framework of Theorem 10.4 (bosonic case) and take $\mathscr{F}(\mathcal{O}, \mathscr{L}) = \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ as algebra of energy observables of our system. We recall [23] that for each operator $u \in B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$ the creation operator $a^*(u)$ acting in $\mathscr{H}$ is defined as the closure of the algebraic direct sum of the operators

$$a_n^*(u) : \mathcal{H}^{\vee n} \otimes \mathscr{L} \to \mathcal{H}^{\vee(n+1)} \otimes \mathscr{L} \quad \text{defined by } a_n^*(u) = (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) \circ (1_{\mathcal{H}^{\vee n}} \otimes u). \tag{10.6}$$

The difference in coefficients with respect to [32, (3.1)] is due to our choice of scalar product in the Fock space. Since no ambiguity may occur we shall identify $N = N \otimes 1_{\mathscr{L}}$. Then clearly we have

$$\left\| a^*(u)(N+1)^{-1/2} \right\| = \|u\|_{B(\mathscr{L},\mathcal{H}\otimes\mathscr{L})}. \tag{10.7}$$

Let $a(u)$ be the adjoint of the operator $a^*(u)$ and let $\phi(u) = a(u) + a^*(u)$. The domains of these operators contain $\mathscr{H}_{\text{fin}}$, the algebraic direct sum of the spaces $\mathcal{H}^{\vee n} \otimes \mathscr{L}$, and it is easy to see that $\phi(u)$ is essentially self-adjoint on this domain; we use the same notation for its closure. It is clear that the commutation relations (2.11) remain valid. Below and later on we shall identify $\Gamma(A) = \Gamma(A) \otimes 1_{\mathscr{L}}$ except in the situations when the clarity of the text requires more precision.

Lemma 10.7. *If $u \in K(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$ and $A \in \mathcal{O}$, $\|A\| < 1$, then $a^{(*)}(u)\Gamma(A) \in \mathscr{F}(\mathcal{O}, \mathscr{L})$ and*

$$\mathcal{P}\bigl[a^{(*)}(u)\Gamma(A)\bigr] = A \otimes \bigl[a^{(*)}(u)\Gamma(A)\bigr] \quad \text{on } \mathcal{H} \otimes \mathscr{H}. \tag{10.8}$$

Proof. From (10.7) we get

$$\left\| a^{(*)}(u)\Gamma(A) \right\| \leqslant \left\| a^{(*)}(u)(N+1)^{-1/2} \right\| \left\| (N+1)^{1/2}\Gamma(A) \right\| \leqslant C\|u\|_{B(\mathscr{L},\mathcal{H}\otimes\mathscr{L})}$$

hence the map $u \mapsto a^{(*)}(u)\Gamma(A)$ is norm continuous on $B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$. Thus it suffices to prove the assertions of the lemma for u of the form $u = f \otimes K$ with $f \in \mathcal{H}$ and K a compact operator on $\mathscr{L}$. More precisely, $u \in B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$ is defined by: $u(e) = f \otimes K(e)$. Then it is easy to check that $a^{(*)}(u) = a^{(*)}(f) \otimes K$ hence $a^{(*)}(u)\Gamma(A) = [a^{(*)}(f)\Gamma(A)] \otimes K \in \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$. □

Lemma 10.8. *For each $u \in B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$ the following relations are satisfied.*

(i) *Let $S, T \in B(\mathscr{L})$ and $A \in B(\mathcal{H})$ with $\|A\| < 1$. Then*

$$\bigl(\Gamma(A) \otimes S\bigr)a^*(u)(1_{\Gamma(\mathcal{H})} \otimes T) = a^*\bigl((A \otimes S)uT\bigr)\bigl(\Gamma(A) \otimes 1_{\mathscr{L}}\bigr). \tag{10.9}$$

(ii) *Let h, L be self-adjoint operators on $\mathcal{H}$ and $\mathscr{L}$, respectively, such that $h \geqslant m > 0$ and $L \geqslant 0$ and let $H_0 = \mathrm{d}\Gamma(h) \otimes 1_{\mathscr{L}} + 1_{\Gamma(\mathcal{H})} \otimes L$. Then for all $f \in \mathscr{H}_{\text{fin}}$ and all numbers $r > 0$ we have*

$$\bigl|\langle f|\phi(u)f\rangle\bigr| \leqslant C(u,r)\bigl\langle f\big|(H_0 + r)f\bigr\rangle, \tag{10.10}$$

where $C(u,r) = \|(h^{-1/2} \otimes 1_{\mathscr{L}})u(L+r)^{-1/2}\|^2$ and the right-hand side is allowed to be $+\infty$.

The proof of (i) is a mechanical application of the definitions; note that both sides of (10.9) are bounded operators. The second assertion is a particular case of [32, Proposition 4.1], but see also [25, Proposition 4.1] and [16, Theorem 2.1].

The second part of Lemma 10.8 allows us to define $\phi(u)$ as a continuous sesquilinear form on $D(H_0^{1/2})$ for an arbitrary continuous linear map[12] $u : \mathscr{L}_1 \to \mathcal{H}_1^* \otimes \mathscr{L}$. Here $\mathscr{L}_1 = D(L^{1/2})$ and $\mathcal{H}_1 = D(h^{1/2})$ are equipped with the graph topologies, $\mathcal{H}_1^*$ is the space adjoint to $\mathcal{H}_1$, and

[12] The theory of Pauli–Fierz Hamiltonians for such "form factors" has first been developed in [16], but we shall not follow their method. However, the reader might prefer the direct arguments and the more detailed presentation from [16].

we embed as usual $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_1^*$. Then $B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L}) \subset B(\mathscr{L}_1, \mathcal{H}_1^* \otimes \mathscr{L})$ densely in the strong operator topology and if $B(R)$ is the closed ball of radius R in $B(\mathscr{L}_1, \mathcal{H}_1^* \otimes \mathscr{L})$ then $B_0(R) = B(R) \cap B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$ is strongly dense[13] in $B(R)$.

Let, for example, $\mathscr{D}$ be the symmetric algebra over $\mathcal{H}_1$ algebraically tensorized with $\mathscr{L}_1$. This is a core for $H_0^{1/2}$ consisting of linear combinations of decomposable vectors. Fix $f \in \mathscr{D}$ and consider the map $u \mapsto \langle f|\phi(u)f\rangle$ defined for the moment only on $B(\mathscr{L}, \mathcal{H} \otimes \mathscr{L})$. It is clear from the definition (10.6) that this map is continuous for the strong operator topology induced by $B(\mathscr{L}_1, \mathcal{H}_1^* \otimes \mathscr{L})$. Thus, by the preceding considerations, (10.10) remains valid for $u \in B(\mathscr{L}_1, \mathcal{H}_1^* \otimes \mathscr{L})$ with the same constant $C(u, r)$.

One can define $\phi(u)$ in a second way (which below gives the same H). The graph norm on $\mathcal{H}_1$ defined by $h^{1/2}$ is such that the embedding $\mathcal{H}_1 \subset \mathcal{H}$ is contractive. Then we get injective contractive linear maps $\mathcal{H}_1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_1^*$ hence contractive dense embeddings $\Gamma(\mathcal{H}_1) \subset \Gamma(\mathcal{H}) \subset \Gamma(\mathcal{H}_1^*)$. On the other hand, we have a natural identification $\Gamma(\mathcal{H}_1)^* = \Gamma(\mathcal{H}_1^*)$. If $u : \mathscr{L}_1 \to \mathcal{H}_1^* \otimes \mathscr{L}$ then (10.6) clearly gives a continuous map $a_n^*(u) : \mathcal{H}^{\vee n} \otimes \mathscr{L}_1 \to (\mathcal{H}_1^*)^{\vee(n+1)} \otimes \mathscr{L}$ hence we obtain as usual a linear map $a^*(u) : \Gamma_{\mathrm{fin}}(\mathcal{H}) \otimes \mathscr{L}_1 \to \Gamma_{\mathrm{fin}}(\mathcal{H}_1^*) \otimes \mathscr{L}$. Then we define $\phi(u)$ as a quadratic form on $\Gamma_{\mathrm{fin}}(\mathcal{H}_1) \otimes \mathscr{L}_1$ (which is a core for H_0) by taking $\langle f|\phi(u)f\rangle = 2\Re\langle f|a^*(u)f\rangle$.

We summarize below our assumptions concerning massive Pauli–Fierz models:

$$
\text{(PF)} \quad \begin{cases}
\mathcal{H} \text{ and } \mathscr{L} \text{ are Hilbert spaces, } \Gamma(\mathcal{H}) \text{ is the symmetric Fock space, } \mathscr{H} = \Gamma(\mathcal{H}) \otimes \mathscr{L}; \\
\mathcal{O} \subset B(\mathcal{H}) \text{ is a non-degenerate abelian } C^*\text{-algebra such that } \mathcal{O}'' \cap K(\mathcal{H}) = \{0\}; \\
h \geqslant m > 0 \text{ is a self-adjoint operator on } \mathcal{H} \text{ strictly affiliated to } \mathcal{O}; \\
L \geqslant 0 \text{ is a self-adjoint operator on } \mathscr{L} \text{ with purely discrete spectrum;} \\
v \in B(D(L^{1/2}), D(h^{1/2})^* \otimes \mathscr{L}) \text{ is such that } \lim_{r \to \infty} C(v, r) < 1; \\
(h+L)^{-\alpha} v (L+1)^{-1/2} \text{ and } (h+L)^{-1/2} v (L+1)^{-\alpha} \text{ are compact operators if } \alpha > 1/2.
\end{cases}
$$

Here and later we use the abbreviation $h + L = h \otimes 1_{\mathscr{L}} + 1_{\mathcal{H}} \otimes L$.

Theorem 10.9. *Assume that conditions* (PF) *are fulfilled. Then* $H_0 = \mathrm{d}\Gamma(h) \otimes 1_{\mathscr{L}} + 1_{\Gamma(\mathcal{H})} \otimes L$ *is a positive self-adjoint operator on* $\mathscr{H}$ *strictly affiliated to* $\mathscr{F}(\mathcal{O}, \mathscr{L})$ *and* $\phi(v)$ *is a symmetric quadratic form on* $D(H_0^{1/2})$ *such that* $\pm\phi(v) \leqslant aH_0 + b$ *for some* $0 < a < 1$, $b > 0$. *The form sum* $H = H_0 + \phi(v)$ *is a self-adjoint operator on* $\mathscr{H}$ *strictly affiliated to* $\mathscr{F}(\mathcal{O}, \mathscr{L})$ *and* H *is a standard QFH with* h *as one particle kinetic energy* (*see Remark* 10.5). *In particular* $\sigma_{\mathrm{ess}}(H) = \sigma(h) + \sigma(H)$. *Finally, assume that* A *is as in condition* (OA), *p.* 127, *and let us identify* $A \otimes 1_{\mathscr{L}} = A$. *If* H *is of class* $C_{\mathrm{u}}^1(A)$ *and* h *is of class* $C_{\mathrm{u}}^1(\mathfrak{a})$ *with* $\rho_h^{\mathfrak{a}} \geqslant 0$, *then the conclusions of Theorem* 8.6 *are valid.*

Proof. We assume, without loss of generality, that $L \geqslant 1$. We have $\mathrm{e}^{-tH_0} = \Gamma(\mathrm{e}^{-th}) \otimes \mathrm{e}^{-tL} \in \mathscr{F}(\mathcal{O}, \mathscr{L})$ for all $t > 0$ and strict affiliation follows by noting that $\|\mathrm{e}^{-tH_0} T \otimes K - T \otimes K\| \to 0$ if $t \to 0$ for all $T \in \mathscr{F}(\mathcal{O})$ and $K \in K(\mathscr{L})$, see the proof of Lemma 7.5. The assertion concerning the existence of H as self-adjoint operator is clear by the preceding discussion (see also [16]). We shall now prove the strict affiliation of H to $\mathscr{F}(\mathcal{O}, \mathscr{L})$ and we do this by checking the conditions of Theorem 7.3, more precisely we shall prove that $\theta(H_0)\phi(v)H_0^{-1/2} \in \mathscr{F}(\mathcal{O}, \mathscr{L})$

[13] Indeed, it suffices to approximate T with $[(1+\varepsilon h)^{-1} \otimes 1_{\mathscr{L}}]T(1+\varepsilon L)^{-1}$.

if $\theta \in C_0(\mathbb{R})$. We shall prove by two different methods that $\mathrm{e}^{-H_0} a^*(v) H_0^{-1/2} \in \mathscr{F}(\mathcal{O}, \mathscr{L})$ and $H_0^{-1/2} a^*(v) \mathrm{e}^{-H_0} \in \mathscr{F}(\mathcal{O}, \mathscr{L})$, which clearly suffices.

We first show that $L H_0^{-1}$ belongs to the multiplier algebra of $\mathscr{F}(\mathcal{O}, \mathscr{L})$, where $L \equiv 1_{\Gamma(\mathcal{H})} \otimes L$. It suffices to prove that $(L H_0^{-1})(S \otimes T) \in \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ for dense sets of operators S and T in $\mathscr{F}(\mathcal{O})$ and $K(\mathscr{L})$, respectively. Note that the linear span of the operators $T = L^{-1} K$ with K compact on $\mathscr{L}$ is dense in $K(\mathscr{L})$ because it contains the rank one operators of the form $|f\rangle\langle g|$ with f in the range of L^{-1}, which is dense in $\mathscr{L}$. Since $(L H_0^{-1})(S \otimes T) = H_0^{-1}(S \otimes K)$ for such T, it suffices to prove that $\mathrm{e}^{-H_0}(S \otimes K) \in \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$, because then this will remain valid if e^{-H_0} is replaced by any $\theta(H_0)$ with $\theta \in C_0(\mathbb{R})$. But $\mathrm{e}^{-H_0}(S \otimes K) = (\Gamma(\mathrm{e}^{-h}) S) \otimes (\mathrm{e}^{-L} K)$ clearly belongs to $\mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$.

Now by using (10.9) we get

$$\begin{aligned} \mathrm{e}^{-H_0} a^*(v) H_0^{-1/2} &= \bigl(\Gamma\bigl(\mathrm{e}^{-h}\bigr) \otimes \mathrm{e}^{-L}\bigr) a^*(v) \bigl(1_{\Gamma(\mathcal{H})} \otimes L^{-1/2}\bigr) \cdot \bigl(L H_0^{-1}\bigr)^{1/2} \\ &= a^*\bigl(\mathrm{e}^{-h-L} v L^{-1/2}\bigr) \Gamma\bigl(\mathrm{e}^{-h}\bigr) \cdot \bigl(L H_0^{-1}\bigr)^{1/2}, \end{aligned}$$

where $L H_0^{-1}$ is interpreted as above. Since $\mathrm{e}^{-h-L} v L^{-1/2}$ is compact we can use Lemma 10.7 and then it suffices to note that $(L H_0^{-1})^{1/2}$ is also a multiplier for the algebra $\mathscr{F}(\mathcal{O}, \mathscr{L})$.

Next we consider the case of $H_0^{-1/2} a^*(v) \mathrm{e}^{-H_0}$. In order to simplify the writing we shall sometimes identify $1_n \equiv 1_n \otimes 1_{\mathscr{L}}$ and similarly for 1^n. Since $H_0 1_n^{\perp} \geqslant (n+1) m 1_n^{\perp}$ we easily see that $H_0^{-1/2} a^*(v) \mathrm{e}^{-H_0}$ is the norm limit as $n \to \infty$ of $H_0^{-1/2} a^*(v) \mathrm{e}^{-H_0} 1_n$. But 1_n is a finite sum of projections 1^k, so it suffices to show that $T \equiv H_0^{-1/2} a^*(v) \mathrm{e}^{-H_0} 1^n$ belongs to $\mathscr{F}(\mathcal{O}, \mathscr{L})$ for each n. From (10.6) we get

$$\begin{aligned} T &= H_0^{-1/2} (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) \bigl(1^n \otimes v\bigr) \bigl[\Gamma\bigl(\mathrm{e}^{-h}\bigr) 1^n\bigr] \otimes \mathrm{e}^{-L} \\ &= H_0^{-1/2} (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) \bigl(1^n \otimes M\bigr) \bigl(1^n \otimes \bigl[M^{-1} v L^{-\alpha}\bigr]\bigr) \bigl(\Gamma\bigl(\mathrm{e}^{-h}\bigr) \otimes L^{\alpha} \mathrm{e}^{-L}\bigr), \end{aligned}$$

where $M = h^{1/2} + L^{1/2}$ is an operator acting in $\mathcal{H} \otimes \mathscr{L}$ such that $(h+L)^{1/2} \leqslant M \leqslant \sqrt{2}(h+L)^{1/2}$. Thus, by hypothesis, $v_0 = M^{-1} v L^{-\alpha}$ is a compact operator $\mathscr{L} \to \mathcal{H} \otimes \mathscr{L}$. In the rest of this proof we realize $\mathcal{H}^{\vee k}$ as the subspace of $\mathcal{H}^{\otimes k}$ consisting of symmetric tensors (the norm being modified by a factor $\sqrt{k!}$, but this does not matter here), and then we have $H_0^{-1/2}(\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) = (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) H_0^{-1/2}$ in a natural sense and we have

$$T = (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) H_0^{-1/2} \bigl(1^n \otimes M\bigr) \bigl(1^n \otimes v_0\bigr) \bigl(\Gamma\bigl(\mathrm{e}^{-h}\bigr) \otimes L^{\alpha} \mathrm{e}^{-L}\bigr).$$

The operator $H_0^{-1/2}(1^n \otimes M)$, acting in $\mathcal{H}^{\otimes(n+1)} \otimes \mathscr{L}$, is bounded and v_0 is norm limit of linear combinations of operators of the form $u_0 \otimes K_0$ where $u_0 \in \mathcal{H}$ and $K_0 \in K(\mathscr{L})$ (see the proof of Lemma 10.7). Thus it suffices to prove that $T \in \mathscr{F}(\mathcal{O}, \mathscr{L})$ under the assumption $v_0 = u_0 \otimes K_0$ and clearly we may also assume $u_0 \in D(h^{1/2})$ and $K = L^{1/2} K_0$ compact. If we set $u = h^{1/2} u_0$ then we obtain:

$$\begin{aligned} T &= H_0^{-1/2} (\mathcal{S}_{n,1} \otimes 1_{\mathscr{L}}) \bigl(1^n \otimes [u \otimes K_0 + u_0 \otimes K]\bigr) \bigl(\Gamma\bigl(\mathrm{e}^{-h}\bigr) \otimes L^{\alpha} \mathrm{e}^{-L}\bigr) \\ &= H_0^{-1/2} a^*(u \otimes K_0 + u_0 \otimes K) \cdot \Gamma\bigl(\mathrm{e}^{-h}\bigr) \otimes 1_{\mathscr{L}} \cdot 1_{\Gamma(\mathcal{H})} \otimes \bigl(L^{\alpha} \mathrm{e}^{-L}\bigr). \end{aligned}$$

From Lemma 10.7, and since $1_{\Gamma(\mathcal{H})} \otimes (L^\alpha e^{-L})$ is multiplier for $\mathscr{F}(\mathcal{O}, \mathscr{L})$, we get $T \in \mathscr{F}(\mathcal{O}, \mathscr{L})$.

To prove that H is a SQFH it remains to show that $\mathcal{P}(H) = h \otimes 1_{\mathscr{H}} + 1_{\mathcal{H}} \otimes H$ (then the formula for the essential spectrum is a consequence, cf. Remark 10.5). Let $\lambda \geqslant 0$ real and let us set $\Lambda = (H_0 + \lambda)^{-1/2}$ (recall that in this proof we assume $H_0 \geqslant 1$) and $U = \Lambda\phi(v)\Lambda$. By Theorem 7.3 and by what we proved above, U belongs to the multiplier algebra $\mathscr{M}$ of $\mathscr{F}(\mathcal{O}, \mathscr{L})$. Indeed, this argument gives directly $\Lambda \in \mathscr{M}$ if $\lambda = 0$ and for the general case it suffices to write $U = (H_0^{1/2}\Lambda)(H_0^{-1/2}\phi(v)H_0^{-1/2})(H_0^{1/2}\Lambda)$ and to note that $H_0^{1/2}\Lambda \in \mathscr{M}$ because H_0 is strictly affiliated to $\mathscr{F}(\mathcal{O}, \mathscr{L})$. We have $e^{-H_0} = \Gamma(e^{-h}) \otimes e^{-L}$ hence from Theorem 10.4 we get $\mathcal{P}(e^{-H_0}) = e^{-h} \otimes e^{-H_0}$ hence

$$\widetilde{H}_0 \equiv \mathcal{P}(H_0) = h \otimes 1_{\mathscr{H}} + 1_{\mathcal{H}} \otimes H_0, \qquad \widetilde{\Lambda} \equiv \mathcal{P}(\Lambda) = (\widetilde{H}_0 + \lambda)^{-1/2}.$$

We shall prove below that

$$\widetilde{U} \equiv \mathcal{P}(U) = \widetilde{\Lambda}\big(1_{\mathcal{H}} \otimes \phi(v)\big)\widetilde{\Lambda} \equiv \widetilde{\Lambda}\widetilde{\phi}(v)\widetilde{\Lambda}, \tag{10.11}$$

where $\mathcal{P}$ is canonically extended to $\mathscr{M}$ as mentioned before Lemma 7.1. Assuming that this has been done, choose λ such that $\|U\| < 1$ (this is possible because $\pm\phi(v) \leqslant aH_0 + b$ with $a < 1$). Then clearly we have a norm convergent expansion

$$(H + \lambda)^{-1} = \Lambda(1 + U)^{-1}\Lambda = \sum (-1)^n \Lambda U^n \Lambda$$

which implies

$$\mathcal{P}\big((H + \lambda)^{-1}\big) = \sum (-1)^n \mathcal{P}(\Lambda)\mathcal{P}(U)^n\mathcal{P}(\Lambda) = \sum (-1)^n \widetilde{\Lambda}\widetilde{U}^n\widetilde{\Lambda} = (\widetilde{H} + \lambda)^{-1},$$

where $\widetilde{H} = \widetilde{H}_0 + \widetilde{\phi}(v)$ and this finishes the proof of the relation $\mathcal{P}(H) = h \otimes 1_{\mathscr{H}} + 1_{\mathcal{H}} \otimes H$. Note that $\pm\widetilde{\phi}(v) \leqslant a\widetilde{H}_0 + b$ with the same a, b as above.

It remains to prove (10.11). Since $a^*(v) = (\phi(v) + i\phi(iv))/2$ we have $\Lambda a^*(v)\Lambda \in \mathscr{M}$ and its adjoint is $\Lambda a(v)\Lambda$. Thus (10.11) is a consequence of

$$\mathcal{P}\big(\Lambda a^*(v)\Lambda\big) = \widetilde{\Lambda}\big(1_{\mathcal{H}} \otimes a^*(v)\big)\widetilde{\Lambda}, \tag{10.12}$$

which is what we show now. From (10.9) we have

$$e^{-H_0}a^*(v)\big(1_{\Gamma(\mathcal{H})} \otimes L^{-1/2}\big) = a^*\big(e^{-h-L}vL^{-1/2}\big)\big(\Gamma\big(e^{-h}\big) \otimes 1_{\mathscr{L}}\big).$$

The operator $1_{\Gamma(\mathcal{H})} \otimes L^{-1/2}$ belongs to $\mathscr{M}$ and it is easy to check that

$$\mathcal{P}\big(1_{\Gamma(\mathcal{H})} \otimes L^{-1/2}\big) = 1_{\mathcal{H}} \otimes 1_{\Gamma(\mathcal{H})} \otimes L^{-1/2}.$$

From now on we simplify notations and no more write the tensor product symbols when they are obvious from the context. Then

$$e^{-\widetilde{H}_0}\mathcal{P}\big(\Lambda a^*(v)\Lambda\big)L^{-1/2} = \mathcal{P}\big(e^{-H_0}\Lambda a^*(v)\Lambda L^{-1/2}\big) = \mathcal{P}\big(\Lambda a^*\big(e^{-h-L}vL^{-1/2}\big)\Gamma\big(e^{-h}\big)\Lambda\big).$$

Due to (10.8) this is equal to

$$\mathcal{P}(\Lambda)\mathcal{P}\big(a^*\big(\mathrm{e}^{-h-L}vL^{-1/2}\big)\Gamma\big(\mathrm{e}^{-h}\big)\big)\mathcal{P}(\Lambda) = \widetilde{\Lambda}\cdot\mathrm{e}^{-h}\otimes\big[a^*\big(\mathrm{e}^{-h-L}vL^{-1/2}\big)\Gamma\big(\mathrm{e}^{-h}\big)\big]\cdot\widetilde{\Lambda}$$

which in turn is equal to

$$\widetilde{\Lambda}\cdot\mathrm{e}^{-h}\otimes\big[\mathrm{e}^{-H_0}a^*(v)L^{-1/2}\big]\cdot\widetilde{\Lambda} = \widetilde{\Lambda}\mathrm{e}^{-\widetilde{H}_0}\big(1_{\mathcal{H}}\otimes a^*(v)\big)L^{-1/2}\widetilde{\Lambda}.$$

Thus we have proved:

$$\mathrm{e}^{-\widetilde{H}_0}\mathcal{P}\big(\Lambda a^*(v)\Lambda\big)L^{-1/2} = \widetilde{\Lambda}\mathrm{e}^{-\widetilde{H}_0}\big(1_{\mathcal{H}}\otimes a^*(v)\big)L^{-1/2}\widetilde{\Lambda} = \mathrm{e}^{-\widetilde{H}_0}\widetilde{\Lambda}\big(1_{\mathcal{H}}\otimes a^*(v)\big)\widetilde{\Lambda}L^{-1/2}.$$

Since the operators $\mathrm{e}^{-\widetilde{H}_0}$ and $L^{-1/2}$ are injective, we get (10.12).

The last assertion of the theorem concerns the Mourre estimate. It is clear by Remark 10.5. □

Remark 10.10. We note that the description of the essential spectrum given in Theorem 10.9 is an improvement of the *massive* case of [16, Theorem 2.3], where it is assumed that $h^{-1/2}v(L+1)^{-1/2}$ is compact, but not of [32, Proposition 4.9], which does not require $(L+1)^{-1}$ to be compact.

11. Systems with a particle number cutoff

In this section we fix an abelian non-degenerate C^*-algebra $\mathcal{O}$ of operators on the infinite-dimensional space $\mathcal{H}$ with $\mathcal{O}''\cap K(\mathcal{H})=\{0\}$ and let Γ be the symmetric or antisymmetric Fock space functor. We are interested in models where the number of particles is at most n, a given positive integer. Then the Hilbert space of the states of the system is $\Gamma_n(\mathcal{H})$ and the algebra of energy observables must be a C^*-algebra of operators on this space. Let $\mathscr{K}_n(\mathcal{H}) = K(\Gamma_n(\mathcal{H}))$ be the algebra of compact operators on $\Gamma_n(\mathcal{H})$.

We define for each integer $n\geqslant 0$ a C^*-subalgebra of $\mathscr{F}(\mathcal{O})$ by the following rule:

$$\mathscr{F}_n(\mathcal{O}) = 1_n\mathscr{F}(\mathcal{O})1_n. \tag{11.1}$$

Let $\mathscr{F}_n(\mathcal{O})=0$ for $n<0$. Thus $\mathscr{F}_n(\mathcal{O})$ lives in the subspace $\Gamma_n(\mathcal{H})$ (i.e. it is non-degenerate on $\Gamma_n(\mathcal{H})$ and its restriction to the orthogonal subspace is zero) and

$$\mathscr{F}_0(\mathcal{O}) = \mathbb{C}\omega, \qquad \mathscr{F}_n(\mathcal{O})\subset\mathscr{F}_{n+1}(\mathcal{O}) \quad\text{and}\quad \mathscr{F}(\mathcal{O}) = \overline{\bigcup_n \mathscr{F}_n(\mathcal{O})}. \tag{11.2}$$

Note that $\mathscr{K}_n(\mathcal{H}) = 1_n\mathscr{K}(\mathcal{H})1_n$ and this is an ideal of $\mathscr{F}_n(\mathcal{O})$.

In particular, the algebra $\mathscr{A}_n(\mathcal{H}) = \mathscr{F}_n(\mathbb{C}1_{\mathcal{H}}) = 1_n\mathscr{A}(\mathcal{H})1_n$ is a C^*-subalgebra of $\mathscr{A}(\mathcal{H})$ which lives in the subspace $\Gamma_n(\mathcal{H})$, has 1_n as unit element, and contains $\mathscr{K}_n(\mathcal{H})$ as an ideal. Moreover,

$$\mathscr{A}_0(\mathcal{H}) = \mathbb{C}\omega, \qquad \mathscr{A}_n(\mathcal{H})\subset\mathscr{A}_{n+1}(\mathcal{H}), \qquad \mathscr{A}(\mathcal{H}) = \overline{\bigcup_n \mathscr{A}_n(\mathcal{H})}. \tag{11.3}$$

These algebras can be defined independently of the material from the preceding sections. First, it is not difficult to prove that $\mathscr{A}_n(\mathcal{H})$ is the unital C^*-algebra generated by the operators $\phi_n(u) = 1_n\phi(u)1_n$. If $\Gamma_n(\mathcal{O})$ is the C^*-algebra generated by the operators $\Gamma_n(S) = \bigoplus_{k\leqslant n} S^{\vee k}$

with $S \in \mathcal{O}$, then we have $\mathscr{F}_n(\mathcal{O}) = [\![\mathscr{A}_n(\mathcal{H}) \cdot \Gamma_n(\mathcal{O})]\!]$. With $\mathcal{P}_n = \mathcal{P}|\mathscr{F}_n(\mathcal{O})$, we get from Theorems 5.4 and 6.2:

Proposition 11.1. *There is a unique morphism $\mathcal{P}_n : \mathscr{F}_n(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}_{n-1}(\mathcal{O})$ such that*

$$\mathcal{P}_n\big(\phi_n(u)^k \Gamma_n(S)\big) = S \otimes \big(\phi_{n-1}(u)^k \Gamma_{n-1}(S)\big) \tag{11.4}$$

for all $u \in \mathcal{H}$, $k \geqslant 0$, $S \in \mathcal{O}$. We have $\ker(\mathcal{P}_n) = \mathscr{K}_n(\mathcal{H})$*, hence we get canonical embedding*

$$\mathscr{F}_n(\mathcal{O})/\mathscr{K}_n(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}_{n-1}(\mathcal{O}). \tag{11.5}$$

The case of the algebras $\mathscr{A}_n(\mathcal{H})$ is particularly nice (we use Remark 4.5):

Corollary 11.2. *There is a unique morphism $\mathcal{P}_n : \mathscr{A}_n(\mathcal{H}) \to \mathscr{A}_{n-1}(\mathcal{H})$ such that $\mathcal{P}_n[\phi_n(u)] = \phi_{n-1}(u)$ for all $u \in \mathcal{H}$. This morphism is unital, surjective, it has $\mathscr{K}_n(\mathcal{H})$ as kernel, and is explicitly given by*

$$\mathcal{P}_n(T) = \operatorname*{s\text{-}lim}_{e \rightharpoonup 0} a(e) T a^*(e) \quad \text{for all } T \in \mathscr{A}_n(\mathcal{H}). \tag{11.6}$$

Thus we get a sequence of canonical surjective morphisms

$$0 \leftarrow \mathscr{A}_0(\mathcal{H}) \leftarrow \mathscr{A}_1(\mathcal{H}) \leftarrow \cdots \leftarrow \mathscr{A}_{n-1}(\mathcal{H}) \leftarrow \mathscr{A}_n(\mathcal{H}) \leftarrow \cdots \tag{11.7}$$

which induce canonical isomorphisms $\mathscr{A}_n(\mathcal{H})/\mathscr{K}_n(\mathcal{H}) \cong \mathscr{A}_{n-1}(\mathcal{H})$.

Remark 11.3. Theorem 1.2 from [27] looks more general then Proposition 11.1, but I found a gap in my proof of that theorem, cf. the comment on p. 162 in [29]. In fact, I know how to deduce Proposition 11.1 from [29, Proposition 3.32] (which is elementary and easy to prove), but the argument is much more involved than the methods used in the present paper (and the assumptions that $\mathcal{O}$ is abelian and that there are no finite rank operators in the von Neumann algebra generated by $\mathcal{O}$ cannot be avoided).

We finish with some applications in spectral theory. An advantage in having a particle number cutoff is that the strict positivity of the one particle mass is no more necessary, in fact the one particle kinetic energy h can be an arbitrary bounded from below self-adjoint operator affiliated to $\mathcal{O}$. On the other hand, the notion of *standard* QFH as introduced in Definition 7.7 does not make sense now. Instead, in the present context it is natural to consider the following class of *elementary* QFH with a particle number cutoff: these are the operators of the form $H_n = \mathrm{d}\Gamma_n(h) + V_n$ where h is a self-adjoint bounded from below operator affiliated to $\mathcal{O}$ and $V_n \in \mathscr{A}_n(\mathcal{H})$ is bounded and symmetric. It is clear that, as in the preceding sections, one may consider much more general interactions, but this is of no interest here.

Such a V_n being fixed, we define $V_k = \mathcal{P}^{n-k}(V_n) \in \mathscr{A}_k(\mathcal{H})$ for $k \leqslant n$. Note that if V_n is a polynomial in the operators $\{\phi_n(u)\}_{u \in \mathcal{H}}$ then V_k is the same polynomial in which each $\phi_n(u)$ has been replaced by $\phi_k(u)$. Or if $V_n = 1_n V 1_n$ for some $V \in \mathscr{F}(\mathcal{H})$, then $V_k = 1_k V 1_k$.

Let us set $H_k = \mathrm{d}\Gamma_k(h) + V_k$, this is a self-adjoint operator on $\Gamma_k(\mathcal{H})$. Of course, $H_0 = V_0 = c\omega$ for some complex number c. The techniques used before easily give that H_k is affiliated to $\mathscr{F}_k(\mathcal{O})$ and

$$\mathcal{P}(H_k) = h \otimes 1_{\Gamma_{k-1}(\mathcal{H})} + 1_{\mathcal{H}} \otimes H_{k-1} \quad \text{for } 1 \leqslant k \leqslant n. \tag{11.8}$$

In particular, we get an HVZ type description of the essential spectrum of the operator H_n:

$$\sigma_{\mathrm{ess}}(H_n) = \sigma(h) + \sigma(H_{n-1}). \tag{11.9}$$

Note how much simpler is this formula than in the n-body situation.

The treatment of the Mourre estimate is entirely similar to that from Section 8, so we give only the result. We consider only conjugate operators of the form $A_n = \mathrm{d}\Gamma_n(\mathfrak{a})$ where $\mathfrak{a}$ is as in condition (OA), p. 127. Exactly as in the proof of Theorem 8.6 we now get:

$$\tau_{A_n}(H_n) = \bigcup_{k=1}^{n} \big[\tau_{\mathfrak{a}}^{k}(h) + \sigma_{\mathrm{p}}(H_{n-k})\big], \tag{11.10}$$

where we make the convention $\sigma_{\mathrm{p}}(H_0) = \{0\}$. Indeed, if we abbreviate $\tau(h) = \tau_{\mathfrak{a}}(h)$ and $\tau(H_n) = \tau_{A_n}(H_n)$, then (11.10) follows by induction from the analogue in the present context of (8.10), namely:

$$\begin{aligned} \tau(H_n) &= \tau(h) + \big[\sigma_{\mathrm{p}}(H_{n-1}) \cup \tau(H_{n-1})\big] \\ &= \big[\tau(h) + \sigma_{\mathrm{p}}(H_{n-1})\big] \cup \big[\tau(h) + \tau(H_{n-1})\big]. \end{aligned} \tag{11.11}$$

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