Boundedness for the general semilinear Duffing equations via the twist theorem ♠

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Abstract

In this paper, we prove the boundedness of all solutions for the periodic equation
\[ x'' + \omega^2 x + \phi(x) = G(x, t) + p(t), \]
where \( \omega \) satisfies the Diophantine condition, \( \phi(x) \in C^1 \) is bounded, \( p(t) \in C^2 \) and \( D_i^j G(x, t) \) is bounded for \( 0 \leq i + j \leq 21 \).

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1. Introduction

There have been many results on the boundedness of solutions for the following semilinear Duffing equations:

\[ x'' + \omega^2 x + \psi(x, t) = 0, \quad \psi(x, t + 2\pi) = \psi(x, t) \]

where \( \omega > 0 \) and \( \psi(x, t) \) is smooth.

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In 1998, Liu [12] proved the Lagrangian stability for the equation

$$x'' + \omega^2 x + \phi(x) = p(t), \quad p \in C^5(\mathbb{R}/2\pi \mathbb{Z})$$

where $\omega > 0$ and the unbounded function $\phi(x) \in C^5$ satisfies:

(i) $\beta x \phi(x) \geq x^2 \phi'(x) > 0$, with some constants $0 < \beta < 1 < a < 2$;

(ii) $x^k \left( \frac{d^k}{dx^k} \int_0^x \phi(s) \, ds \right) \leq C \cdot \int_0^x \phi(s) \, ds$ for $3 \leq k \leq 6$.

In 1999 Ortega [20] proved a variant of Moser’s small twist theorem and applied it to obtain the boundedness of solutions for the equation

$$x'' + n^2 x + h_L(x) = p(t), \quad p \in C^5(\mathbb{R}/2\pi \mathbb{Z})$$

where $n \in \mathbb{N}$ and $h_L(x)$ is of the form

$$h_L(x) = \begin{cases} 
L & \text{if } x \geq 1, \\
Lx & \text{if } -1 \leq x \leq 1, \\
-L & \text{if } x \leq -1,
\end{cases}$$

and satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^t p(t)e^{-int} \, dt \right| < \frac{2L}{\pi}.$$ 

Subsequently, Liu [13] considered the equation

$$x'' + n^2 x + \phi(x) = p(t), \quad p \in C^7(\mathbb{R}/2\pi \mathbb{Z})$$

where $n \in \mathbb{N}$, $\phi(x) \in C^6$ and the limits

$$\phi(\pm \infty) = \lim_{x \to \pm \infty} \phi(x)$$

are finite and

$$\lim_{|x| \to +\infty} x^6 \phi^{(6)}(x) = 0.$$ 

He proved with suitable conditions that each solution of (1.2) is bounded if and only if

$$\left| \int_0^{2\pi} p(t)e^{-int} \, dt \right| < 2(\phi(+\infty) - \phi(-\infty)).$$

Recently Liu and Wang [16] also studied the equation

$$x'' + \omega^2 x + \phi(x) = p(t), \quad p \in C^6(\mathbb{R}/2\pi \mathbb{Z})$$

where $\omega \in \mathbb{R}/\mathbb{Q}$ and $\phi(x) \in C^6$ satisfies (1.3) and (1.4). They proved that if $\phi(\pm \infty) \neq \phi(-\infty)$, then Eq. (1.5) possesses the boundedness.
For more results on the boundedness of (1.1), see [1,8] and references therein.

The study of the boundedness problem for Duffing equations goes back to the early 1960's when Littlewood [11] proposed to study whether or not the Duffing equation

\[ x'' + g(x) = p(t), \quad p(t + 2\pi) = p(t) \]

possesses Lagrangian stability. In 1976, Morris [17] first applied Moser's small twist theorem and obtained the boundedness result for the equation

\[ x'' + x^3 = p(t) \]

with \( p(t + 2\pi) = p(t) \) piecewise continuous. Since then Moser's small twist theorem has been the most important tool in this field. The main idea is as follows.

By means of transformation theory the original system outside of a large disc \( D = \{(x, x') \in \mathbb{R}^2: x^2 + x'^2 \leq r^2\} \) in \((x, x')\)-plane is transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is close to a so-called twist map in \( \mathbb{R}^2 \setminus D \). Then Moser's twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the \((x, x')\)-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space \((x, x', t) \in \mathbb{R}^2 \times \mathbb{R}\), which confines the solutions in the interior and which leads to a bound of these solutions.

For the situation that \( \psi(x, t) \) is bounded in (1.1), we found that although great progresses have been made on the boundedness, all the results stated above focus on a special case, that is, the limits

\[ \lim_{x \to \pm \infty} \psi(x, t) \text{ exist and are finite} \tag{1.6} \]

and \( \psi(x, t) \) satisfies the following growth condition:

\[ \lim_{|x| \to +\infty} x^m \psi^{(m)}(x, t) = 0 \tag{1.7} \]

for some finite \( m \).

The reason that people impose the above growth condition on \( \psi(x, t) \) is related to the fact that the application of KAM theorem requires the estimates of several derivatives of the solutions with respect to initial conditions, which is tedious and difficult.

For the further study of the interplay between the linear equation

\[ x'' + \omega^2 x = 0 \]

and the perturbed system (1.1), we need to consider the general case without the restrictions of (1.6) and (1.7).

In this paper, we consider the situation that \( \psi(x, t) \) is of the form:

\[ \psi(x, t) = \phi(x) - G(x, t) - p(t), \]

where \( \phi(x) \in C^{19}(\mathbb{R}) \), \( p(t) \in C^{23}(\mathbb{R}/2\pi \mathbb{Z}) \) and \( G(x, t) \in C^{21}(\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}) \). Moreover, we assume that \( \phi(x) \) satisfies (1.3) and

\[ \lim_{|x| \to +\infty} x^{19} \phi^{(19)}(x) = 0. \tag{1.8} \]

Moreover, we assume \( G(x, t) \) satisfies

\[ |D_x^i D_t^j G(x, t)| \leq C, \quad 0 \leq i + j \leq 21 \tag{1.9} \]
and

$$|D_j^t \hat{G}| \leq C, \quad 0 \leq j \leq 21$$  \hspace{1cm} (1.10)

for some $C > 0$, where $\hat{G}$ is some function satisfying $\frac{d \hat{G}}{dx} = G$.

Then for the equation:

$$x'' + \omega^2 x + \phi(x) = G(x, t) + p(t),$$  \hspace{1cm} (1.11)

we have the following conclusion:

**Theorem 1.** Assume (1.3), (1.8), (1.9) hold true for $\phi(x)$ and $G(x, t)$, respectively. Let $\omega \in \mathbb{R}^+ \setminus \mathbb{Q}$ satisfy the Diophantine condition:

$$|m \omega + n| \geq \frac{\gamma}{|m|^{\tau}}, \quad \forall (m, n) \neq (0, 0) \in \mathbb{Z}^2,$$  \hspace{1cm} (1.12)

where $1 < \tau < 2, \ \gamma > 0$. Suppose $\phi(-\infty) \neq \phi(+\infty)$, then all solutions of (1.11) are bounded.

**Remark 1.1.** The class of functions $\psi(x, t) = \phi(x) - G_x(x, t) - p(t)$ in Theorem 1 widens drastically to allow some oscillation in $x$. For example, it is obvious that $\psi(x, t) = \arctan x + 10 \cdot \cos x + p(t)$ or $\arctan x + \cos x \cdot p(t) + p(t + 2\pi) = p(t)$ satisfy the conditions of Theorem 1 but not (1.6) and (1.7).

**Remark 1.2.** As mentioned above, without the condition (1.7) we will meet the difficulty to estimate derivatives of solutions. Moreover, it even may happen that the corresponding Poincaré map is not monotone. To overcome these difficulties, we will use the method of Principle Integral to solve the homological equation, where the known function is allowed to be non-smooth on angle variable if it is sufficiently smooth on time variable, see Lemma 3.3.

**Remark 1.3.** We believe that if $\omega \in \mathbb{Q}$ in (1.11), similar results still hold true. We will study this problem in one of our future work.

Another one of most studied semilinear equations is

$$x'' + ax^+ - bx^- = f(x, t),$$  \hspace{1cm} (1.13)

where $x^+ = \max(x, 0), x^- = \max(-x, 0), f(x, t)$ is a smooth $2\pi$-periodic function on $t$, $a$ and $b$ are positive constants ($a \neq b$).

If $f(x, t)$ depends only on $t$, Eq. (1.13) becomes

$$x'' + ax^+ - bx^- = f(t), \quad f(t + 2\pi) = f(t),$$  \hspace{1cm} (1.14)

which had been studied by Fucik [5] and Dancer [3] in their investigations of boundary value problems associated to equations with “jumping nonlinearities”. For recent developments, we refer to [6,7,9] and references therein.

In 1996, Ortega [19] proved the Lagrangian stability for the equation

$$x'' + ax^+ - bx^- = 1 + \gamma h(t)$$  \hspace{1cm} (1.15)

if $|\gamma|$ is sufficiently small and $h \in C^4(S^1)$. 

On the other hand, when \( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q} \), Alonso and Ortega [2] proved that there is a \( 2\pi \)-periodic function \( f(t) \) such that all the solutions of Eq. (1.14) with large initial conditions are unbounded. Moreover for such an \( f(t) \), Eq. (1.14) has periodic solutions.

In 1999, B. Liu [14] removed the smallness assumption on \( |\gamma| \) in Eq. (1.15) when \( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q} \) and obtained the same result. Later X. Wang extended Liu’s result to the equation

\[
x'' + ax^+ - bx^- + \phi(x) = p(t), \quad p(t + 2\pi) = p(t)
\]

under some conditions.

For the case \( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{R} / \mathbb{Q} \), in 2001 Ortega [21] proved that each solution of (1.14) is bounded if \( p(t) \) is smooth and \( \int_0^{2\pi} p(t) \, dt \neq 0 \). Similar result can be found in [22].

Recently, Liu [15] consider the boundedness of solutions for the equation

\[
\left( \phi_p(x') \right)' + a \phi_p(x^+) - b \phi_p(x^-) = f(x, t), \quad f(x, t + 2\pi) = f(x, t),
\]

where

\[
\phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad \frac{\pi p}{a} + \frac{\pi p}{b} = \frac{2\pi}{n}, \quad n \in \mathbb{N},
\]

\( f \in C^{7,6}(\mathbb{R} \times S^1) \) and satisfies that

(i) the following limits exists uniformly in \( t \),

\[
\lim_{x \to \infty} f(x, t) = f_\pm(t),
\]

(ii) the following limits exists uniformly in \( t \),

\[
\lim_{x \to \infty} x^m \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(x, t) = f_{\pm,m,n}(t)
\]

for \( (n, m) = (0, 6), (7, 0) \) and \( (7, 6) \). Moreover, \( f_{\pm,m,n}(t) \equiv 0 \) for \( m = 6, n = 0, 7 \).

In the following, we will remove the restrictions of these two conditions and extend the result to the equation:

\[
x'' + ax^+ - bx^- = f(t) + G_x(x, t), \quad (1.16)
\]

where \( f \) and \( G \) are smooth \( 2\pi \)-periodic functions.

More precisely, we will prove the following theorem.

**Theorem 2.** Assume \( \omega = \frac{1}{2} \left( \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) \) satisfies the Diophantine condition (1.12) and \( f(t) \in C^{23}(\mathbb{R}/2\pi \mathbb{Z}) \).

Suppose \( G \in C^{21}(\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}) \) satisfy (1.9) and \( |f'f(t)| = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt \neq 0 \). Then all the solutions of Eq. (1.16) are bounded.

The remain part of this paper is organized as follows. In Section 2, we introduce action-angle variables and exchange the role of time and angle variables. In Section 3, we construct canonical transformations such that the new Hamiltonian system is closed to an integrable one. In Section 4, we will prove Theorem 1 by Moser’s twist theorem. The sketch for the proof of Theorem 2 will be given in the last section.

In the following, we use \( c \) and \( C \) to denote positive constants without concerning their quantity.
2. Some canonical transformations

In this section, we will state some technical lemmas which will be used in the proof of Theorem 1. Throughout this section, we assume the hypotheses of Theorem 1 hold.

2.1. Action-angle variables

Let \( y = -\omega^{-1}x' \). Then (1.11) is equivalent to the following equations:

\[
\begin{align*}
 x' &= -\omega y, \\
 y' &= \omega x + \omega^{-1} \phi(x) - \omega^{-1} G_x(x, t) - \omega^{-1} p(t),
\end{align*}
\]

which is a planar non-autonomous Hamiltonian system

\[
\begin{align*}
 x' &= -\frac{\partial H}{\partial y}(x, y, t), \\
 y' &= \frac{\partial H}{\partial x}(x, y, t)
\end{align*}
\]

with

\[
H(x, y, t) = \frac{1}{2} \omega(x^2 + y^2) + \frac{1}{\omega} \Phi(x) - \frac{1}{\omega} G(x, t) - \frac{1}{\omega} x p(t),
\]

where \( \Phi(x) = \int_0^x \phi(x) \, dx \).

Under the transformation \((r, \theta) \mapsto (x, y)\) with \( r > 0 \) and \( \theta \) (mod \( 2\pi \)), given by

\[
\begin{align*}
 x &= r^{1/2} \cos \theta, \\
 y &= r^{1/2} \sin \theta,
\end{align*}
\]

the system (2.2) is transformed into another Hamiltonian system,

\[
\begin{align*}
 r' &= -\frac{\partial h}{\partial \theta}, \\
 \theta' &= \frac{\partial h}{\partial r}
\end{align*}
\]

with the Hamiltonian

\[
h(r, \theta, t) = \omega r + \omega^{-1} f_1(r, \theta) - 2\omega^{-1} r^{1/2} \cos \theta p(t) - \omega^{-1} f_2(r, \theta, t),
\]

where \( f_1 = 2\Phi(r^{1/2} \cos \theta) \) and \( f_2 = 2G(r^{1/2} \cos \theta, t) \).

For any function \( f(\cdot, \theta) \), we denote by \([f](\cdot)\) the average value of \( f(\cdot, \theta) \) over \( S^1 \), that is,

\[
[f](\cdot) := \frac{1}{2\pi} \int_0^{2\pi} f(\cdot, \theta) \, d\theta.
\]

Then similar to [13], we obtain from (1.3), (1.4), (1.9) the following conclusions:

**Lemma 2.1.** It holds that:

\[
\begin{align*}
 |f_1(r, \theta)| &\leq C \cdot r^{1/2}, \\
 |[f_1](r)| &\leq C \cdot r^{1/2}.
\end{align*}
\]

Moreover, for \( 1 \leq k \leq 19 \),

\[
\begin{align*}
 \left| \frac{\partial^k f_1}{\partial r^k}(r, \theta) \right| &\leq C \cdot r^{-k+1/2}, \\
 \left| \frac{d^k [f_1]}{dr^k}(r) \right| &\leq C \cdot r^{-k+1/2}.
\end{align*}
\]
Lemma 2.2. The following conclusion holds:

\[
\lim_{r \to +\infty} \sqrt{r} [f_1]'(r) = \frac{1}{\pi} \left\{ \phi(+\infty) - \phi(-\infty) \right\},
\]

(2.8)

\[
\lim_{r \to +\infty} \frac{[f_1](r)}{\sqrt{r}} = \frac{2}{\pi} \left\{ \phi(+\infty) - \phi(-\infty) \right\},
\]

(2.9)

\[
\lim_{r \to +\infty} r^{3/2} [f_1]''(r) = -\frac{1}{2\pi} \left\{ \phi(+\infty) - \phi(-\infty) \right\}.
\]

(2.10)

Hence for \( r \gg 1 \), we have

\[
c \cdot [f_1]'(r) \leq r [f_1]''(r) \leq C \cdot [f_1]'(r).
\]

(2.11)

From (2.7), (2.8) and (2.9), it follows that for \( r \gg 1 \),

\[
\left| D^i_i D^j_j f_2(r, \theta, t) \right| \leq C \cdot r^{-\frac{i}{j}}, \quad 0 \leq i + j \leq 19.
\]

(2.12)

The following technique lemma will be used to refine the estimates on \([f_2](r, t)\).

Lemma 2.3. The following conclusion holds true:

\[
\left| D_i^i D_j^j f_2(r, \theta, t) \right| \leq C \cdot r^{-\frac{i}{j}}, \quad 0 \leq i + j \leq 19.
\]

(2.13)

Lemma 2.4. Assume \( f \in C^1(\mathbb{R} / 2\pi \mathbb{Z}), G(x, t) \in C^1(\mathbb{R}^1 \times \mathbb{S}^1) \) and \( G'(x, t) = g(x, t) \). Suppose there are two positive constants \( \bar{G} \) and \( \bar{g} \) such that \( |G(x, t)| \leq \bar{G}, |g(x, t)| \leq \bar{g} \) for any \((x, t)\). Let \( A(r, \theta) \in C^2(\mathbb{R}^1 \times \mathbb{S}^1) \) be of the form

\[
A(r, \theta) = (r + h(r, \theta))^\frac{1}{2}
\]

with

\[
h, \frac{\partial h}{\partial \theta}, \frac{\partial^2 h}{\partial \theta^2} = O(r^{\frac{1}{2}})
\]

for \( r \gg 1 \).

Then for any constant \( \delta_0 \in \left(0, \frac{1}{10}\right)\) it holds that

\[
\left| \int_0^{2\pi} f(\theta) g(A \cos \theta, t) \, d\theta \right| \leq C \cdot r^{-\delta_0}, \quad r \gg 1,
\]

(2.14)

where \( C \) depends only on \( \bar{G}, \bar{g} \) and \( \|f\|_{C^0} \).
Proof. Let \([0, 2\pi] = I_1 \cup I_2\), where \(I_1 = \{0, \pi - r^{-2\theta_0}\} \cup [\pi - r^{-2\theta_0}, \pi + r^{-2\theta_0}] \cup [2\pi - r^{-2\theta_0}, 2\pi\}\) and \(I_2 = [r^{-2\theta_0}, \pi - r^{-2\theta_0}] \cup [\pi + r^{-2\theta_0}, 2\pi - r^{-2\theta_0}\]. Then
\[
\int_0^{2\pi} f(\theta) g(A \cos \theta, t) \, d\theta = \int_{I_1} f(\theta) g(A \cos \theta, t) \, d\theta + \int_{I_2} f(\theta) g(A \cos \theta, t) \, d\theta.
\]

Obviously, \(|I_1| \leq C \cdot r^{-2\theta_0}\), where \(| \cdot |\) denotes the Lebesgue measure. Then from the boundedness of \(g(x,t)\), it is easy to see that
\[
\left| \int_{I_1} f(\theta) g(A \cos \theta, t) \, d\theta \right| \leq C \cdot r^{-2\theta_0}.
\]

To estimate the integral on \(I_2\), we first estimate the integral on the interval \(I_{21} = [r^{-2\theta_0}, \pi - r^{-2\theta_0}\].

Consider \(D_\theta(A \cos \theta) = A'_\theta \cos \theta - A \sin \theta\). From (2.13), it holds that \(|A \sin \theta| \geq c \cdot r^{1/2 - 2\theta_0}\) and \(A'_\theta \cos \theta = O(1)\) for \(\theta \in I_{21}\), which implies
\[
|D_\theta(A \cos \theta)| \geq c \cdot r^{1/2 - 2\theta_0}. \tag{2.15}
\]

Similarly from the definition of \(A\) and the condition (2.13), we have
\[
D^2_\theta(A \cos \theta) = D^2_\theta(A \cos \theta) - 2D_\theta A \cdot \sin \theta - A \cos \theta = O(r^2). \tag{2.16}
\]

By direct computation, we have
\[
D_\theta(f(\theta)(D_\theta(A \cos \theta))^{-1}) = f'(\theta)(D_\theta(A \cos \theta))^{-1} + f(\theta)(D_\theta(A \cos \theta))^{-2} \cdot (-D^2_\theta(A \cos \theta)).
\]

Thus from (2.15) and (2.16), we obtain the estimate
\[
|D_\theta(f(\theta)D_\theta(A \cos \theta)^{-1})| \leq C \cdot r^{4\theta_0 - 1/2}. \tag{2.17}
\]

By integration by parts, we have that
\[
\int_{I_{21}} f(\theta) g(A \cos \theta, t) \, d\theta = \int_{I_{21}} f(\theta)(D_\theta(A \cos \theta))^{-1} dG(A \cos \theta, t)
\]
\[
= (D_\theta(A \cos \theta))^{-1} f(\theta) G(A \cos \theta, t)|_{\theta = r^{-2\theta_0} - 2\theta_0}^{\theta = \pi - 2\theta_0}
\]
\[
- \int_{I_{21}} G(A \cos \theta, t) D_\theta(f(\theta)D_\theta((A \cos \theta)^{-1}) \, d\theta.
\]

From (2.15) and (2.17), for \(\theta \in I_{21}\) it holds that
\[
|D_\theta(A \cos \theta)^{-1} f(\theta) G(A \cos \theta, t)|_{\theta = r^{-2\theta_0}}, \quad |(r \sin \theta)^{-1} f(\theta) G(A \cos \theta, t)|_{\theta = \pi - r^{-2\theta_0}} \leq C \cdot r^{4\theta_0 - 1/2}
\]
and
\[
|G(A \cos \theta, t) \cdot D_\theta(f(\theta)D_\theta((A \cos \theta)^{-1})| \leq C \cdot r^{4\theta_0 - 1/2}.
\]
Similarly, we can have the same estimate for the other parts of $I_2$. Hence from the fact $0 < \delta_0 < \frac{1}{10}$, we obtain (2.14). The proof of this lemma is completed. \hfill \Box

For $[f_2](r, t)$, we have the following result:

**Corollary 2.1.** The following conclusion holds true:

$$
|D_i^r D_j^t[f_2](r, t)| \leq C \cdot r^{-\delta_1 - \frac{i}{2}}, \quad 0 \leq i + j \leq 19,
$$

where the constant $\delta_1$ is in $(0, \frac{1}{10})$.

**Proof.** From the definition of $f_2$, we have $[f_2](r, t) = \frac{1}{\pi} \int_0^{2\pi} G(r^2 \cos \theta, t) d\theta$. From (1.9) and (1.10), we know that $G$ and $\hat{G}$ are bounded. Thus for $i + j = 0$, (2.18) is deduced from Lemma 2.4 where we set $f \equiv 1$ and $A(r, \theta) = r^2$. For $i + j \geq 1$, it can be easily seen that $\partial^i r^j G$ are the sum of the term like

$$
\frac{\partial^{k+j}}{\partial x^i \partial t^j} G(r^2 \cos \theta, t) (r^2)^{(i_1)} \cdots (r^2)^{(i_k)} \cdot (\cos \theta)^k,
$$

where $i_1 + \cdots + i_k = i$. Thus (2.18) is implied from Lemma 2.4 for the function $\partial^i r^j G(r^2 \cos \theta, t)$ and (1.9). This ends the proof of the lemma. \hfill \Box

2.2. Exchange of the roles of time and angle variables

Observe that

$$
rd\theta - h dt = -(h dt - r d\theta).
$$

This means that if one can solve $r = r(h, t, \theta)$ from Eq. (2.4) as a function of $h, t$ and $\theta$, then

$$
\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h, t, \theta), \quad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h, t, \theta),
$$

i.e., Eq. (2.19) is a Hamiltonian system with Hamiltonian function $r = r(h, t, \theta)$ and now the action, angle and time variables are $h, t$, and $\theta$, respectively. This trick has been used in [10].

From Eq. (2.5) and Lemmas 2.1, 2.2 and 2.3, it follows that

$$
\lim_{r \to +\infty} \frac{h}{r} = \omega > 0
$$

and for $r \gg 1$,

$$
\frac{\partial h}{\partial r} = \omega + \omega^{-1} \frac{\partial}{\partial r} f_1(r, \theta) + \omega^{-1} r^{-\frac{1}{2}} \cos \theta p(t) - \omega^{-1} \frac{\partial}{\partial r} f_2(r, \theta, t) > 0.
$$

By the implicit function theorem, we know that there is a function $R = R(h, t, \theta)$ such that

$$
r(h, t, \theta) = \omega^{-1} h - R(h, t, \theta).
$$

Moreover, for $h \gg 1$,

$$
|R(h, t, \theta)| \leq \omega^{-1} h/2
$$

and $R(h, t, \theta)$ is $C^{19}$ in $h$ and $t$. 
From (2.5), it holds that

$$R = \omega^{-2}f_1(\omega^{-1}h - R, \theta) - 2\omega^{-2}(\omega^{-1}h - R)^{\frac{1}{2}}\cos \theta p(t) + \omega^{-2}f_2(\omega^{-1}h - R, t, \theta).$$

(2.21)

The following two lemmas are similar to [13] and we give the proofs of them in Appendix A for the convenience of readers.

**Lemma 2.5.** Assume $R$ is defined by (2.21) with $|R| \ll h$ for $h \gg 1$. Then it holds that

$$|D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq 19$$

(2.22)

for $h \gg 1$, where $n(i) = -\frac{i}{2}$ for $i \geq 1$ and $n(0) = \frac{1}{2}$.

In (2.21), we denote $R = \omega^{-2}f_1(\omega^{-1}h, \theta) - 2\omega^{-2}(\omega^{-1}h)^{\frac{1}{2}}\cos \theta p(t) + R_1(h, t, \theta)$. Then

$$R_1 = \omega^{-2}\int_0^1 \frac{\partial f_1}{\partial r}(\omega^{-1}h - \tau R, \theta) R d\tau - \omega^{-2}\int_0^1 (\omega^{-1}h - \tau R)^{-\frac{1}{2}} R \cos \theta p(t) d\tau$$

$$+ \omega^{-2}f_2(\omega^{-1}h - R, t, \theta).$$

(2.23)

Then we have the following conclusion:

**Lemma 2.6.** It holds that

$$|D_h^i D_t^j R_1| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq 19.$$

From the definition of $R_1$, we can obtain the following conclusion:

**Lemma 2.7.** For the function $[R_1](h, t)$, we have that

$$|D_h^i D_t^j [R_1]| \leq C \cdot (h^{-i} + h^{-\delta_1} - \frac{1}{2}), \quad 0 \leq i + j \leq 19,$$

where $\delta_1 \in (0, \frac{1}{10})$.

**Proof.** Since the proof is similar to Lemmas 2.5 and 2.6, we only give the sketch of the proof. We rewrite $R_1$ as follows:

$$R_1 = R_{11} + R_{12},$$

where

$$R_{11} = \omega^{-2}\frac{\partial f_1}{\partial r}(\omega^{-1}h, \theta) R - \omega^{-2}(\omega^{-1}h)^{-\frac{1}{2}} R \cos \theta p(t) + \omega^{-2}f_2(\omega^{-1}h - R, t, \theta)$$

and $R_{12} = R_1 - R_{11}$. With the help of Lemma 2.6, it is obvious that $R_{12}$ are of higher order terms in $R_1$. Hence it is sufficient to estimate $R_{11}$.
Similarly, we rewrite $R_{11}$ as follows:

$$R_{11} = R_{111} + R_{112},$$

where

$$R_{111} = \omega^{-2} \frac{\partial f_1}{\partial r}(\omega^{-1}h, \theta)R_0 - \omega^{-2}(\omega^{-1}h)^{-\frac{1}{2}}R_0 \cos \theta p(t) + \omega^{-2} f_2(\omega^{-1}h - R_0, t, \theta)$$

with $R_0 = \omega^{-2} f_1(\omega^{-1}h, \theta) - \omega^{-2}(\omega^{-1}h)^{\frac{1}{2}} \cos \theta p(t)$ and $R_{112} = R_{11} - R_{111}$. Since $R - R_0$ are of higher order terms in $R$, it follows that $R_{112}$ are of higher order terms in $R_{11}$. Thus it is sufficient to estimate $R_{111}$. Now it is easy to see that the first two terms in $R_{111}$, denoted by $R_{1111}$ satisfies

$$|D^i h D^j [(R_{1111})]| \leq C \cdot h^{-i}, \quad 0 \leq i \leq 19.$$

Now we apply Lemma 2.4 on the last term in $R_{111}$, denoted by $R_{2111}$, where we set $f \equiv 1$ and $h(r, \theta) = R_0$. Then we obtain

$$|D^i h D^j [(R_{2111})]| \leq C \cdot h^{-\delta_1 - \frac{i}{2}}, \quad 0 \leq i \leq 19.$$

This ends the proof of the lemma. \(\square\)

From (2.20), (2.21) and Lemma 2.6, we obtain that the Hamiltonian $r(h, t, \theta)$ in (2.20) is of the form:

$$r = \omega^{-1}h + r_1(h, \theta) + r_2(h, t, \theta) + r_3(h, t, \theta), \quad (2.24)$$

where $r_1 = \omega^{-2} f_1(\omega^{-1}h, \theta)$ satisfies

$$|D^i h r_1| \leq C \cdot h^{\frac{1}{2} - i}, \quad 0 \leq i \leq 19, \quad (2.25)$$

$r_2 = -2\omega^{-2}(\omega^{-1}h)^{\frac{1}{2}} \cos \theta p(t)$ satisfies

$$|D^i h D^j \theta D^k r_2| \leq C \cdot h^{\frac{1}{2} - i}, \quad 0 \leq i + j + k \leq 22 \quad (2.26)$$

and $r_3 = R_1(h, t, \theta)$ satisfies

$$|D^i h D^j r_3| \leq C \cdot h^{-\frac{i}{2}}, \quad 0 \leq i \leq 19. \quad (2.27)$$

3. More canonical transformations

In this section, we will make some more canonical transformations such that the Poincaré map of the new system is close to twist map.

**Lemma 3.1.** There exists a canonical transformation $\Phi_1$ of the form:

$$\begin{cases}
    I = I, \\
    t = s + v_1(I, s, \theta)
\end{cases}$$

such that the system with Hamiltonian (2.24) is transformed into the following one
\[ \tilde{r} = \omega^{-1} I + \tilde{r}_1(I) + \tilde{r}_2(I, s, \theta) + \tilde{r}_3(I, s, \theta) \]  
(3.1)

with

\[ |\tilde{r}_1^{(i)}(I)| \geq c \cdot I^{\frac{1}{2} - i}, \quad i = 0, 1, 2, \]
\[ |\tilde{r}_1^{(i)}(I)| \leq C \cdot I^{\frac{1}{2} - i}, \quad 0 \leq i \leq 5, \]
\[ (3.2) \]
\[ \tilde{r}_2 = -2\omega^{-2}(\omega^{-1}I)^{\frac{1}{2}} \cos \theta p(s) \]
satisfying
\[ |D^i_{D^j_s D^k_{\theta}} \tilde{r}_2(I, s, \theta)| \leq C \cdot I^{-\frac{1}{2} - i}, \quad 0 \leq i + j + k \leq 22, \]
\[ (3.3) \]
\[ |D^i_{D^j_s} \tilde{r}_3(I, s, \theta)| \leq C \cdot I^{1-i}, \quad 0 \leq i + j \leq 19. \]
\[ (3.4) \]

More for the function $[\tilde{r}_3](I, s)$ we have that
\[ |D^i_{D^j_s}[\tilde{r}_3]| \leq C \cdot (I^{-i} + I^{-\delta_1 - \frac{1}{2}}), \quad 0 \leq i + j \leq 19. \]
\[ (3.5) \]

**Proof.** We construct the canonical transformation by means of generating function:

\[ \Phi_1: \quad I = l, \quad t = s + \frac{\partial S_1}{\partial I} (l, \theta). \]

We determine $S_1(l, \theta)$ by the equation

\[ \frac{\partial S_1}{\partial \theta} + r_1(l, \theta) - [r_1](l) = 0, \]

that is, $S_1 = -\int_0^\theta (r_1(l, \theta) - [r_1](l)) \, d\theta$.

Setting

\[ \tilde{r}_1(l) = [r_1](l), \quad \tilde{r}_2(l, s, \theta) = r_2(l, s, \theta) \]

and

\[ \tilde{r}_3(l, s, \theta) = r_3 + \int_0^{\tau} \frac{\partial r_2}{\partial l} (l, s + \tau, \theta) \frac{\partial S_1}{\partial l} d\tau, \]

we obtain that the new Hamiltonian is of the form (3.1). Moreover, (3.2) and (3.3) follow directly from Lemmas 2.1, 2.2 and (2.26).

Combining (2.25), (2.26) and (2.27), we can easily prove (3.4).

From the definition of $\tilde{r}_3$ and Lemma 2.7, it is ready to obtain (3.5). This ends the proof of this lemma. \[ \square \]

**Lemma 3.2.** Let $0 < \delta_1 < \frac{1}{10}$ be a constant. Then there exists a canonical transformation $\Phi_2$ of the form:

\[ \Phi_2: \quad \begin{cases} l = \rho + u_2(\rho, \tau, \theta), \\ s = \tau + v_2(\rho, \tau, \theta) \end{cases} \]

such that the system with Hamiltonian (3.1) is transformed into the following one
\[ \tilde{r}(\rho, \tau, \theta) = \omega^{-1}\rho + \tilde{r}_1(\rho) + \tilde{r}_2(\rho, \tau, \theta), \]  

(3.6)

where \( \tilde{r}_1 = \tilde{r}_1 \) and \( \tilde{r}_2 \) satisfies

\[ |D^j_\rho D^i_\tau \tilde{r}_2(\rho, \tau, \theta)| \leq C \cdot \rho^{-\frac{i}{2}}, \quad 0 \leq i + j \leq 19. \]  

(3.7)

Moreover, for the function \( \tilde{r}_2^0(\rho) \), it holds that

\[ |D^j_\rho \tilde{r}_2^0(\rho)| \leq C \cdot (\rho^{-i} + \rho^{-\frac{1}{2} - \delta_1 - \frac{j}{2}}), \quad 0 \leq i \leq 19. \]  

(3.8)

**Proof.** The proof is similar to [16], but for the convenience of readers we still give a detailed argument. We shall look for the required transformation \( \Phi_2 \) by means of a generating function \( S_2(\rho, \tau, \theta) \), so that

\[ \Phi_2: \quad I = \rho + \frac{\partial}{\partial s} S_2(\rho, s, \theta), \quad \tau = s + \frac{\partial}{\partial \rho} S_2(\rho, s, \theta). \]  

(3.9)

Under this transformation, the new Hamiltonian function \( \tilde{r} \) is of the form

\[ \tilde{r} = \omega^{-1}\rho + \tilde{r}_1(\rho) + \tilde{r}^* + \tilde{r}_2 \]

where

\[ \tilde{r}^* = \omega^{-1} \frac{\partial S_2}{\partial s} + \frac{\partial S_2}{\partial \theta} - 2\omega^{-2}(\omega^{-1}h)^{\frac{1}{2}} \cos \theta p(s) \]

and

\[ \tilde{r}_2 = \tilde{r}_3 \left( I(\rho, \tau, \theta), s(\rho, \tau, \theta), \theta \right) + \int_0^1 \tilde{r}_1 \left( \rho + \lambda \frac{\partial S_2}{\partial s} \right) \frac{\partial S_2}{\partial s} d\lambda \]

\[ - \omega^{-\frac{3}{2}} p(s(\rho, \tau, \theta)) \cos \theta \int_0^1 \left( \rho + \lambda \frac{\partial S_2}{\partial s} \right)^{-\frac{1}{2}} \frac{\partial S_2}{\partial s} d\lambda. \]  

(3.10)

We now determine \( S_2 = \rho^{\frac{1}{2}} \tilde{S}_2(s, \theta) \) by the equation \( \tilde{r}^* = 0 \), i.e.,

\[ \omega^{-1} \frac{\partial}{\partial s} \tilde{S}_2(s, \theta) + \frac{\partial}{\partial \theta} \tilde{S}_2(s, \theta) - 2\omega^{-\frac{3}{2}} p(s) \cos \theta = 0. \]  

(3.11)

Write \( p(s) \) and \( \cos \theta \) into Fourier series:

\[ p(s) = \sum_{k=-\infty}^{\infty} p_k e^{iks}, \quad \cos \theta = \frac{e^{-i\theta} + e^{i\theta}}{2}. \]

Then one solution of Eq. (3.11) can be written in the form

\[ \tilde{S}_2(s, \theta) = e^{i\theta} \sum_{k=-\infty}^{\infty} \frac{i\omega^{-\frac{1}{2}}}{k + \omega} p_k e^{iks} + e^{-i\theta} \sum_{k=-\infty}^{\infty} \frac{i\omega^{-\frac{1}{2}}}{k - \omega} p_k e^{iks}. \]  

(3.12)
Since \( \omega \in \mathbb{R}^+ \setminus \mathbb{N} \) and \( p \in C^{23} \), we know that \( |D_i^j S_2(s, \theta)| \leq C \) for \( 0 \leq i \leq 20 \). This together with (3.2), (3.3) and (3.4) implies (3.7). Then (3.8) follows from (3.5) in Lemma 3.1. This ends the proof of this lemma. □

The following result is critical for this paper.

**Lemma 3.3.** Let \( 0 < \delta_1 < \frac{1}{10} \) be a constant. Consider the Hamiltonian
\[
\bar{r}(\rho, \tau, \theta) = \omega^{-1} \rho + \bar{r}_1(\rho) + \mathcal{R}(\rho, \tau, \theta),
\]
where \( \bar{r}_1 \) is defined as in Lemma 3.2 and \( \mathcal{R} \) satisfies
\[
|D_i^j D_k^l \mathcal{R}| \leq C \cdot \rho^{-\varepsilon - \frac{1}{2}}
\]
for \( 0 \leq i + j \leq l \) with \( \varepsilon \geq 0 \).

Then there exists a canonical transformation \( \Phi_3 \) of the form:
\[
\Phi_3: \begin{cases}
\rho = Q + u_3(Q, \zeta, \theta), \\
\tau = \zeta + v_3(Q, \zeta, \theta),
\end{cases}
\]
such that the system with Hamiltonian (3.13) is transformed into the following one
\[
\hat{r}(Q, \zeta, \theta) = \omega^{-1} Q + \hat{r}_1(Q) + \mathcal{R}_1(Q, \zeta, \theta),
\]
where \( \hat{r}_1(Q) = \bar{r}_1(Q) + [\mathcal{R}]_0(Q) \) with \( [\mathcal{R}]_0(Q) = \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_0^{2\pi} \mathcal{R}(Q, \tau, \theta) d\tau d\theta \) and \( \mathcal{R}_1 \) satisfies
\[
|D_i^j D_k^l \mathcal{R}_1| \leq C \cdot Q^{-\varepsilon - \frac{1}{2} - \frac{i}{2}}, \quad 0 \leq i + j \leq l - 3.
\]

**Proof.** We will prove this lemma by means of Principle Integral method instead of Fourier series method. Let \( \Phi_3 \) be of the following form:
\[
\rho = Q + \frac{\partial S_3}{\partial \tau}(Q, \tau, \theta), \quad \zeta = \tau + \frac{\partial S_3}{\partial Q}(Q, \tau, \theta),
\]
where the generating function \( S_3(Q, \tau, \theta) \) satisfies \( S_3 = S_3(Q, \tau + 2\pi, \theta) = S_3(Q, \tau, \theta + 2\pi) \) and will be determined later.

Then the transformed Hamiltonian is
\[
\hat{r} = \omega^{-1} \left( Q + \frac{\partial S_3}{\partial \tau} \right) + \hat{r}_1(Q) + \mathcal{R} \left( Q + \frac{\partial S_3}{\partial \tau}, \tau, \theta \right) + \frac{\partial S_3}{\partial \theta}
\]
\[
= \omega^{-1} Q + \hat{r}_1(Q) + [\mathcal{R}]_0(Q) + \omega^{-1} \frac{\partial S_3}{\partial \tau} + \frac{\partial S_3}{\partial \theta} + R + \mathcal{R}_1,
\]
where
\[
R = \mathcal{R}(Q, \tau, \theta) - [\mathcal{R}]_0(Q)
\]
and

$$\mathcal{R}_1 = \int_{\theta}^{\theta+2\pi} R_1 \left( \frac{\partial S_3}{\partial \tau} \right) d\lambda + \int_{0}^{\theta} \frac{\partial}{\partial \rho} \left( \frac{\partial S_3}{\partial \tau}, \tau, \theta \right) d\lambda. \quad (3.17)$$

Obviously, it holds that

$$\left( \frac{1}{2\pi} \right)^2 \int_{0}^{2\pi} \int_{0}^{2\pi} R(\rho, \tau, \theta) d\tau d\theta = 0. \quad (3.18)$$

Now we determine the periodic function $S_3$ by the following equation

$$\omega^{-1} \frac{\partial S_3}{\partial \tau}(\rho, \tau, \theta) + \frac{\partial S_3}{\partial \theta}(\rho, \tau, \theta) + R(\rho, \tau, \theta) = 0, \quad (3.19)$$

whose characteristic equation is

$$\frac{d\tau}{\omega^{-1}} = \frac{d\theta}{\Omega} = -\frac{dS_3}{R(\rho, \tau, \theta)}.$$ 

Obviously, the characteristic equation possesses two independent Principle Integrals as follows:

$$\tau - \omega^{-1} \theta = c_1$$

and

$$S_3 + \int_{0}^{\theta} R(\rho, \tau - \omega^{-1} \phi, \phi) d\phi = c_2.$$

Thus the solution of (3.19) is of the form:

$$S_3(\rho, \tau, \theta) = -\int_{0}^{\theta} R(\rho, \tau - \omega^{-1} \theta + \omega^{-1} \phi, \phi) d\phi + \Omega(\rho, \tau - \omega^{-1} \theta) \quad (3.20)$$

with $\Omega$ a differential function determined later.

To ensure $S_3$ be $2\pi$-periodic on $\tau$ and $\theta$, $\Omega$ must be $2\pi$-periodic on the second variable, that is $\Omega(\rho, x + 2\pi) = \Omega(\rho, x)$. Then by direct computation, we obtain that $S_3$ is $2\pi$-periodic on $\tau$.

Next we determine $\Omega$ by the periodicity of $S_3$ on $\theta$.

Let $J(\rho, \chi) = \int_{0}^{2\pi} R(\rho, x + \omega^{-1} \phi, \phi) d\phi$. Then we have

$$S_3(\rho, \tau, \theta + 2\pi) = -\int_{0}^{\theta+2\pi} R(\rho, \tau - \omega^{-1}(\theta + 2\pi - \phi), \phi) d\phi + \Omega(\rho, \tau - \omega^{-1}(\theta + 2\pi))$$

$$= J(\rho, \tau - \omega^{-1}(\theta + 2\pi)) - \int_{2\pi}^{2\pi+\theta} R(\rho, \tau - \omega^{-1}(\theta + 2\pi - \phi), \phi) d\phi$$
+ \Omega(\varrho, \tau - \omega^{-1}(\theta + 2\pi)).

On the other hand, from \( R(\varrho, \tau, \phi + 2\pi) = R(\varrho, \tau, \phi) \) we have

\[
\int_{2\pi}^{2\pi+\omega} R(\varrho, \tau - \omega^{-1}(\theta + 2\pi - \phi), \phi) \, d\phi = \int_{0}^{\theta} R(\varrho, \tau - \omega^{-1}(\theta - \phi), \phi) \, d\phi,
\]

which implies that

\[
S_3(\varrho, \tau, \theta + 2\pi) = J(\varrho, \tau - \omega^{-1}(\theta + 2\pi)) - \int_{0}^{\theta} R(\varrho, \tau - \omega^{-1}(\theta - \phi), \phi) \, d\phi + \Omega(\varrho, \tau - \omega^{-1}(\theta + 2\pi)). \tag{3.21}
\]

Setting \( S_3(\varrho, \tau, \theta + 2\pi) = S_3(\varrho, \tau, \theta) \), it follows from (3.20) and (3.21) that

\[
J(\varrho, \tau - \omega^{-1}(\theta + 2\pi)) + \Omega(\varrho, \tau - \omega^{-1}(\theta + 2\pi)) - \Omega(\varrho, \tau - \omega^{-1}(\theta + 2\pi)) = 0,
\]

or equivalently,

\[
J(\varrho, x) = \Omega(\varrho, x + x_0) - \Omega(\varrho, x), \tag{3.22}
\]

where \( x = \tau - \omega^{-1}(\theta + 2\pi) \) and \( x_0 = 2\pi \omega^{-1} \).

From (3.18) and the definition of \( J \), we have

\[
\int_{0}^{2\pi} J(\varrho, x) \, dx = - \int_{0}^{2\pi} R(\varrho, x + \omega^{-1}\phi, \phi) \, dx \, d\phi = - \int_{0}^{1} \int_{0}^{2\pi} R(\varrho, x, \phi) \, dx \, d\phi = 0.
\]

Thus we assume \( J(\varrho, x) = \sum_{0 \neq k \in \mathbb{Z}} J_k(\varrho) e^{ikx} \) and \( \Omega(\varrho, x) = \sum_{0 \neq k \in \mathbb{Z}} \Omega_k(\varrho) e^{ikx} \). Then the homological equation (3.22) implies that

\[
\Omega_k = \frac{J_k}{e^{ikx_0} - 1}, \quad k \neq 0.
\]

The definition of \( J(\varrho, x) \) implies that \( J(\varrho, x) \) is \( C^1 \) on \( x \). Thus it holds that

\[
|J_k| \leq C \cdot \|J(\cdot, x)\|_{C^1} \cdot |k|^{-1}, \quad k \neq 0. \tag{3.23}
\]

From the Diophantine condition (1.12), we have that

\[
|e^{ikx_0} - 1| \geq \frac{1}{\gamma} |k|^{-\tau}, \quad k \neq 0. \tag{3.24}
\]
Combining (3.23) and (3.24), we obtain that
\[ |\Omega_k| \leq C \| J(\cdot, x) \| c_l \cdot |k|^{\tau - l}, \quad k \neq 0, \]
which implies \( \Omega \) is well-defined and \( C^{l-3} \) on \( x \) since \( 1 < \tau < 2 \).

For the definition of \( \Omega \) and (3.14), we have that
\[ \left| \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \Omega \right| \leq C \cdot \rho^{-\varepsilon - \frac{i}{2}}, \quad 0 \leq i + j \leq l - 2, \]
which together with (3.14) and (3.20) implies
\[ \left| \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \tau_{S_3} \right| \leq C \cdot \rho^{-\varepsilon - \frac{i}{2}}, \quad 0 \leq i + j \leq l - 2. \]

Thus we obtain (3.16) from (3.17) and (3.25) and the proof is completed. \( \square \)

By Lemma 3.2 and the repeated use of Lemma 3.3, we have the following result.

**Corollary 3.1.** There exists a canonical transformation \( \Phi_4 \) of the form:
\[ \Phi_4: \begin{cases} \rho = \zeta + u_4(\zeta, \eta, \theta), \\ \tau = \eta + v_4(\zeta, \eta, \theta) \end{cases} \]
such that the system with Hamiltonian (3.6) is transformed into the following one
\[ \tau(\zeta, \eta, \theta) = \omega^{-1} \zeta + \tau_1(\zeta) + \tau_2(\zeta, \eta, \theta), \]
where \( \tau_1 = \tilde{r}_1 + [\tilde{r}_2]_0 \) with \( \tilde{r}_1, [\tilde{r}_2]_0 \) satisfying (3.2), (3.8), and \( \tau_2 \) satisfies
\[ \left| \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \tau_2 \right| \leq C \cdot \zeta^{-2 - \frac{i}{2}} \]
for \( 0 \leq i + j \leq 5. \)

**4. Proof of Theorem 1**

In this section, we first give the expression of the Poincaré map of the Hamiltonian system with the Hamiltonian (3.26). Then we will prove Theorem 1 via Moser’s twist theorem.

**Expression of the Poincaré map.** From Corollary 3.1, it follows that the Hamiltonian system with the Hamiltonian (3.26) is of the form:
\[ \begin{cases} \frac{d\eta}{d\theta} = \omega^{-1} + \tau_1'(\zeta) + \frac{\partial \tau_2}{\partial \zeta}(\zeta, \eta, \theta), \\ \frac{d\zeta}{d\theta} = -\frac{\partial \tau_2}{\partial \eta}(\zeta, \eta, \theta), \end{cases} \]
where \( \tau_1(\zeta) = \tilde{r}_1(\zeta) + [\tilde{r}_2]_0(\zeta) \) satisfying (3.2) and (3.8), \( \tau_2(\zeta, \eta, \theta) \) satisfies (3.27).
Thus the Poincaré map of Eq. (4.1) is of the form:
\[ P: \begin{cases} \eta_1 = \omega^{-1} + \eta + \alpha(\zeta) + F_1(\zeta, \eta), \\ \zeta_1 = \zeta + F_2(\zeta, \eta). \end{cases} \]
Moreover, we have that

$$\alpha(\zeta) = \alpha_1(\zeta) + \alpha_2(\zeta)$$  \hspace{1cm} (4.3)

with

$$|\alpha_1^{(i)}(\zeta)| \geq c \cdot \zeta^{-\frac{1}{2}-i}, \quad i = 0, 1,$$
$$|\alpha_1^{(i)}(\zeta)| \leq C \cdot \zeta^{-\frac{1}{2}-i}, \quad |\alpha_2^{(i)}(\zeta)| \leq C \cdot \zeta^{-\frac{1}{2}-i}, \quad 0 \leq i \leq 4$$  \hspace{1cm} (4.4)

and

$$|D_i^j D_{\eta k} F_k(\zeta, \eta)| \leq C \cdot \zeta^{-3}, \quad 0 \leq i + j \leq 4, \quad k = 1, 2.$$  \hspace{1cm} (4.5)

From (4.4), we see that it is possible that the function $\alpha(\zeta)$ is not monotone. To find a monotone interval for $\alpha(\zeta)$, we consider the interval $[\zeta_0, 2\zeta_0]$ with $\zeta_0 \gg 1$. From (4.3) and (4.4), it follows that the set $\alpha([\frac{3}{4}\zeta_0, \frac{7}{4}\zeta_0])$ covers some interval with length longer than $c \cdot \zeta_0^{-\frac{1}{2}}$. Thus by Mean Value theorem of Differentials, there exists some point $\zeta^* \in \left[\frac{3}{4}\zeta_0, \frac{7}{4}\zeta_0\right]$ such that $|\alpha'(\zeta^*)| \geq c \cdot \zeta_0^{-\frac{1}{2}}$.

On the other hand, (4.4) implies $|\alpha''(\zeta)| \leq C \cdot \zeta^{-2-\delta_1}$. Consequently, for each $\zeta \in [\zeta^*, \zeta^* + \frac{\delta_1}{2}]$, we have

$$|\alpha'(\zeta)| \geq c \cdot \zeta^{-\frac{3}{2}}.$$  \hspace{1cm} (4.6)

Next we make a scale transformation as follows:

$$\alpha(\zeta) - \alpha(\zeta^*) = \zeta_0^{-\frac{3}{2}} \nu, \quad \nu \in [1, 2].$$  \hspace{1cm} (4.7)

Then the Poincaré map $P$ is changed into the following one:

$$\tilde{P}: \begin{cases} 
\eta_1 = \omega^{-1} + \alpha(\zeta^*) + \eta + \zeta_0^{-\frac{3}{2}} \nu + \tilde{F}_1(\nu, \eta), \\
\nu_1 = \nu + \tilde{F}_2(\nu, \eta),
\end{cases}$$  \hspace{1cm} (4.8)

where

$$\tilde{F}_1(\nu, \eta) = F_1(\zeta(\nu), \eta), \quad \tilde{F}_2(\nu, \eta) = \zeta_0^{\frac{3}{2}} \left( \alpha(\zeta(\nu) + F_2(\zeta(\nu), \eta)) - \alpha(\zeta(\nu)) \right)$$  \hspace{1cm} (4.9)

with $\zeta(\nu)$ determined by (4.7).

From (4.4), (4.6) and (4.7), we have that

$$|\zeta^{(i)}(\nu)| \leq C, \quad 0 < i \leq 4,$$  \hspace{1cm} (4.10)

which together with (4.5) and (4.9) implies

$$|D_i^j D_{\eta k} \tilde{F}_1| \leq C \cdot \zeta_0^{-3}, \quad |D_i^j D_{\eta k} \tilde{F}_2| \leq C \cdot \zeta_0^{-2}, \quad 0 \leq i + j \leq 4.$$  \hspace{1cm} (4.11)
Moreover, since the map \( \tilde{P} \) is time 1 map of the Hamiltonian system (3.26), it is area-preserving. Thus it possesses the intersection property in the annulus \([1, 2] \times S^1\), that is, if \( \Gamma \) is an embedded circle in \([1, 2] \times S^1\) homotopic to a circle \( \nu = \) constant then \( \tilde{P}(\Gamma) \cap \Gamma \neq \emptyset \). The proof can be found in [4].

**Proof of Theorem 1 via Moser’s twist theorem.** Until now, we have verified that the mapping \( \tilde{P} \) satisfies all the conditions of Moser’s small twist theorem [18]. Hence there is an invariant curve \( \Gamma \) of \( \tilde{P} \) surrounding \( \nu \equiv 1 \) if \( \xi_0 \gg 1 \). This means that there exist invariant curves of the Poincaré mapping of the system (3.26), which surround the origin \((x, y) = (0, 0)\) and are arbitrarily far from the origin. The statement of Theorem 1 has been proved. \( \square \)

5. The sketch for the proof of Theorem 2

In this section, we will give The sketch for the proof of Theorem 2.

**Step 1. Action-angle variables.**

Introducing a new variable \( y \) as \( x' = -y \), then Eq. (1.16) is equivalent to the following system:

\[
\begin{align*}
  x' &= -y, \\
  y' &= ax^+ - bx^- - f(t) - G(x, t),
\end{align*}
\]

which is a Hamiltonian system with the Hamiltonian function

\[
H(x, y, t) = \frac{1}{2}y^2 + \frac{1}{2}ax^+ + \frac{1}{2}bx^- - f(t)x - G(x, t).
\]

Denote by \( C(t) \) the solution of

\[
x'' + ax^+ - bx^- = 0
\]

with the initial condition \( x(0) = 1, x'(0) = 0 \). The derivative of \( C(t) \) will be denoted by \(-S(t)\). Then \( C(t) \in C^2(R/2\omega \pi Z) \). Moreover, \( C(t) \) is even and can be given by

\[
C(t) = \begin{cases} 
  \cos \sqrt{a}t, & 0 \leq t \leq \frac{\pi}{\sqrt{a}} \\
  -\sqrt{\frac{a}{b}} \sin \sqrt{b}(t - \frac{\pi}{2\sqrt{a}}), & \frac{\pi}{2\sqrt{a}} < |t| \leq \omega \pi.
\end{cases}
\]

Notice that if \( a = b \), then \( C(t) = \cos \sqrt{a}t \).

Hence under the transformation

\[
\begin{align*}
  x &= \lambda r^{\frac{1}{2}} C(\omega \theta), \\
  y &= \lambda r^{\frac{1}{2}} S(\omega \theta),
\end{align*}
\]

where \( \lambda = \sqrt{\omega^{-1} a^{-1}} \), Eq. (5.1) is changed into another Hamiltonian system

\[
\begin{align*}
  r' &= -\frac{\partial h}{\partial \theta}(r, \theta, t), \\
  \theta' &= \frac{\partial h}{\partial r}(r, \theta, t)
\end{align*}
\]

where

\[
h(r, \theta, t) = \omega^{-1}r - 2\lambda r^{\frac{1}{2}} C(\omega \theta) f(t) - 2G(\lambda r^{\frac{1}{2}} C(\omega \theta), t).
\]
Step 2. Exchange the roles of time and angle variables.

Observe that
\[ r \, d\theta - h \, dt = -(h \, dt - r \, d\theta). \]

This means that if one can solve \( r = r(h, t, \theta) \) from Eq. (5.5) as a function of \( h, t \) and \( \theta \), then
\[ \frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h, t, \theta), \quad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h, t, \theta), \]
(5.6)

that is, Eq. (5.6) is a Hamiltonian system with Hamiltonian function \( r = r(h, t, \theta) \) and now the action, angle and time variables are \( h, t, \) and \( \theta \), respectively.

From (5.5), we have that if we denote
\[ r = \omega h + 2\omega^\frac{3}{2} \lambda f(t)C(\omega \theta) + \tilde{R}(h, t, \theta), \]
(5.7)

then \( \tilde{R} \) satisfies
\[ |D_h^i D_t^j \tilde{R}| \leq C \cdot h^{-\frac{1}{2}}, \quad 0 \leq i + j \leq 21. \]
(5.8)

Step 3. Canonical transformations.

There exists a canonical transformation \( \tilde{\Phi}_1 \) of the form:
\[ \tilde{\Phi}_1: \quad h = \rho + \tilde{U}_1(\rho, t, \theta), \quad \tau = t + \tilde{V}_1(\rho, t, \theta) \]

where the functions \( \tilde{U}_1, \tilde{V}_1 \) are periodic in \( t, \theta \). Under this transformation, the Hamiltonian system with Hamiltonian (5.7) is changed into the form:
\[ H(\rho, \tau, \theta) = \omega \rho + 2\omega^\frac{3}{2} \lambda [f] \rho^\frac{1}{2} C(\omega \theta) + \tilde{R}_1(\rho, \tau, \theta). \]
(5.9)

Moreover, the new perturbation \( \tilde{R}_1 \) satisfies
\[ \left| \frac{\partial^{i+j}}{\partial \rho^i \partial \tau^j} \tilde{R}_1 \right| \leq C \cdot \rho^{-\frac{1}{2}}, \quad 0 \leq i + j \leq 19. \]

Then we can construct another canonical transformation \( \tilde{\Phi}_2 \) of the form:
\[ \tilde{\Phi}_2: \quad \rho = I, \quad \tau = s + \tilde{T}(\rho, \theta) \]

with \( \tilde{T}(\rho, \theta + 2\pi) = \tilde{T}(\rho, \theta) \), such that the transformed system with Hamiltonian (5.9) is of the form:
\[ \frac{dI}{d\theta} = -\frac{\partial \tilde{H}}{\partial s}(I, s, \theta), \quad \frac{ds}{d\theta} = \frac{\partial \tilde{H}}{\partial I}(I, s, \theta) \]
(5.10)

with \( \tilde{H}(I, s, \theta) = \omega I + c^* I^\frac{1}{2} + \tilde{R}_2(I, s, \theta) \) and \( c^* \neq 0 \), where we use the fact that \([f] \neq 0\).

Moreover, the new perturbation \( \tilde{R}_2 \) satisfies
\[ \left| \frac{\partial^{k+l}}{\partial I^k \partial s^l} \tilde{R}_2(I, s, \theta) \right| \leq C \cdot I^{-k}, \quad 0 \leq i + j \leq 19. \]
(5.11)
Now the situation is similar to the system with Hamiltonian (3.6), we omit the remain part of the proof.

**Appendix A. Proof of Lemmas 2.5 and 2.6**

**Proof of Lemma 2.5.**

(i) \( i + j = 0 \). The proof for this case can be easily obtained from Lemmas 2.1 and 2.3.

(ii) \( i + j = 1 \). It is clear that for \( h \gg 1 \),

\[
\left| \omega^{-2} \frac{\partial f_1}{\partial r} (\omega^{-1} h - R, \theta) \right| + \left| \frac{\omega^2}{2} (\omega^{-1} h - R)^{-\frac{1}{2}} \cos \theta p(t) \right| + \left| \omega^{-2} \frac{\partial f_2}{\partial r} (\omega^{-1} h - R, \theta) \right| \leq \frac{1}{2}.
\]

Define

\[
\Delta(h, t, \theta) = 1 + \omega^{-2} \frac{\partial f_1}{\partial r} (\omega^{-1} h - R, \theta) - \frac{\omega^{-2}}{2} (\omega^{-1} h - R)^{-\frac{1}{2}} \cos \theta p(t) + \omega^{-2} \frac{\partial f_2}{\partial r} (\omega^{-1} h - R, t, \theta),
\]

\[
g_1 = \omega^{-3} \frac{\partial f_1}{\partial r} (\omega^{-1} h - R, \theta) - \frac{\omega^{-3}}{2} (\omega^{-1} h - R)^{-\frac{1}{2}} \cos \theta p(t) + \omega^{-3} \frac{\partial f_2}{\partial r} (\omega^{-1} h - R, t, \theta),
\]

\[
g_2 = -\omega^{-2} (\omega^{-1} h - R)^{\frac{1}{2}} \cos \theta p'(t) + \omega^{-2} \frac{\partial f_2}{\partial t} (\omega^{-1} h - R, t, \theta).
\]

Then it follows that

\[
\Delta \cdot \frac{\partial R}{\partial h} = g_1, \quad \Delta \cdot \frac{\partial R}{\partial t} = g_2.
\]

(A.1)

From Lemmas 2.1 and 2.3, we have \(|g_1| \leq C \cdot h^{-\frac{1}{2}}\) and \(|g_2| \leq C \cdot h^{\frac{1}{2}}\). Thus the proof for this case is completed.

(iii) \( i + j = 2 \). Lemmas 2.1 and 2.3 imply that

\[
\left| \frac{\partial \Delta}{\partial t} \right| \leq C \cdot h^{-\frac{1}{2}}, \quad \left| \frac{\partial \Delta}{\partial h} \right| \leq C \cdot h^{-1}, \quad \left| \frac{\partial g_1}{\partial t} \right| \leq C \cdot h^{-1},
\]

\[
\left| \frac{\partial g_2}{\partial h} \right| \leq C \cdot h^{-\frac{1}{2}}, \quad \left| \frac{\partial g_2}{\partial t} \right| \leq C \cdot h^{\frac{1}{2}}.
\]

From the second equation of (A.1), we obtain

\[
\Delta \frac{\partial^2 R}{\partial t^2} + \frac{\partial \Delta}{\partial t} \cdot \frac{\partial R}{\partial t} = \frac{\partial g_2}{\partial t}
\]

and

\[
\Delta \frac{\partial^2 R}{\partial t \partial h} + \frac{\partial \Delta}{\partial h} \cdot \frac{\partial R}{\partial h} = \frac{\partial g_2}{\partial h}.
\]

The above inequalities and equations imply that

\[
\left| \frac{\partial^2 R}{\partial t^2} \right| \leq C \cdot h^{\frac{1}{2}}, \quad \left| \frac{\partial^2 R}{\partial h \partial t} \right| \leq C \cdot h^{-\frac{1}{2}}.
\]
From the first equation of (A.1), we know that
\[ \Delta \frac{\partial^2 R}{\partial h^2} + \frac{\partial \Delta}{\partial h} \frac{\partial R}{\partial h} = \frac{\partial g_1}{\partial h}, \]
which implies \( |\frac{\partial^2 R}{\partial h^2}| \leq C \cdot h^{-1}. \) Thus we complete the proof for this case.

In general, if
\[ |D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq m, \]
then it holds that
\[ |D_h^i D_t^j \Delta| \leq C \cdot h^{-\frac{1}{2}+n(i)}, \quad |D_h^i D_t^j g_1| \leq C \cdot h^{-\frac{1}{2}}, \quad |D_h^i D_t^j g_2| \leq C \cdot h^{-\frac{1}{2}}. \]
Consequently, we obtain
\[ |D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq m + 1. \]
The proof is completed. □

**Proof of Lemma 2.6.** The lemma is easily followed from the following claim:

**Claim.**
\[ |D_h^i D_t^j \partial f_1 | (\omega^{-1} h - \tau R, \theta) \leq C \cdot h^{-\frac{1}{2}+n(i)}, \]
\[ |D_h^i D_t^j (\omega^{-1} h - \tau R)^{-\frac{1}{2}} R \cos \theta p(t) \leq C \cdot h^{-\frac{1}{2}+\frac{1}{2}}, \]
\[ |D_h^i D_t^j f_2 (\omega^{-1} h - \tau R, \theta) \leq C \cdot h^{-\frac{1}{2}} \] (A.2)
for \( 0 \leq i + j \leq 19. \)

**Proof of Claim.** We only prove the first inequality of (A.2) and the proof for two others are similar.

(i) \( i + j = 0. \) The proof for this case can be obtained directly from Lemmas 2.1 and 2.3.

(ii) \( i > 0, j = 0. \) We have the following equality:
\[ D_h^i \frac{\partial f_1}{\partial r} (\omega^{-1} h - \tau R, \theta) = \sum \frac{\partial^{k+1} f_1}{\partial r^{k+1}} (u, \theta) \cdot \frac{\partial^{i_1} u}{\partial h^{i_1}} \cdots \frac{\partial^{i_k} u}{\partial h^{i_k}} \]
with \( 0 < k \leq i, i_1, \ldots, i_k > 0, \) and \( i_1 + \cdots + i_k = i \) and \( u = \omega^{-1} h - \tau R. \) Assume there are \( l \leq k \) numbers in \( \{ i_1, \ldots, i_k \} \) which is equal to 1. Then we obtain
\[ |D_h^i \frac{\partial f_1}{\partial r} (u, \theta) | \leq C \cdot h^{-\frac{k+1}{2}} \cdot h^{-\frac{i_1+\cdots+i_k}{2}} \leq C \cdot h^{-\frac{i+1}{2}}. \]

(iii) \( i = 0, j > 0. \) By direct computation, we have
\[ D_h^i \frac{\partial f_1}{\partial r} (\omega^{-1} h - \tau R, \theta) = \sum \frac{\partial^{k+1} f_1}{\partial r^{k+1}} (u, \theta) \cdot \frac{\partial^{j_1} u}{\partial t^{j_1}} \cdots \frac{\partial^{j_k} u}{\partial t^{j_k}} \]
with \( k \leq j, j_1, \ldots, j_k > 0, j_1 + \cdots + j_k = j \). It follows that

\[
\left| D_t^i \frac{\partial f_1}{\partial r}(u, \theta) \right| \leq C \cdot h^{-\frac{k+1}{2}} \cdot h^j \leq C \cdot h^{-\frac{1}{2}}.
\]

(iv) \( i > 0, j > 0 \). By direct computation, we have

\[
D^i_t D^j_r \frac{\partial f_1}{\partial r}(u, \theta) = \sum \frac{\partial^{k_1+k_2+1} f_1}{\partial r^{k_1+k_2+1}}(u, \theta) \cdot \frac{\partial^{i_1} u}{\partial h^{i_1}} \cdot \frac{\partial^{i_1+j_1} u}{\partial h^{i_1} \partial t^{j_1}} \cdots \frac{\partial^{j_2+k_2} u}{\partial h^{j_2} \partial t^{j_2}},
\]

where \( u = \omega^{-1} h - \tau \mathcal{R} \) and

\[
0 \leq k_1 \leq i, \quad 0 \leq k_2 \leq j, \quad i_1, \ldots, i_{k_1}, \quad j_1, \ldots, j_{k_2} > 0, \quad l_1, \ldots, l_{k_2} \geq 0,
\]

\[
i_1 + \cdots + i_{k_1} + l_1 + \cdots + l_{k_2} = i, \quad j_1 + j_{k_2} = j.
\]

Assume that there are \( m(\leq k_1) \) numbers in \( \{i_1, \ldots, i_{k_1}\} \) which is equal to 1. Then

\[
\left| D^i_t D^j_r \frac{\partial f_1}{\partial r} \right| \leq C \cdot h^{-\frac{k_1+k_2+1}{2}} \cdot h^{-\frac{i_1+j_1+j_2+k_2-m}{2}} \leq C \cdot h^{-\frac{i+1}{2}}.
\]

This ends the proof of the claim. \( \square \)

References