Computers and Mathematics with Applications 56 (2008) 2356–2359

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Refinements of Aczél, Popoviciu and Bellman's inequalities

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ARTICLE INFO

Article history:
Received 15 January 2008
Received in revised form 22 April 2008
Accepted 27 May 2008

Keywords:
General inequalities
Discrete multivariate inequalities
Aczél's inequality
Popoviciu's inequality
Bellman's inequality

ABSTRACT

In this paper, splitting finite sums, refinements of the inequalities of Aczél, Popoviciu and Bellman are obtained.

1. Introduction

In 1956, Aczél [1,2] published the following result.

Theorem 1. *Let n be a positive integer and let $A, B, a_k, b_k, (1 \le k \le n)$ be real numbers such that $A^2 \ge \sum_{k=1}^n a_k^2$ or $B^2 \ge \sum_{k=1}^n b_k^2$. Then*

$$\left(A^2 - \sum_{k=1}^n a_k^2\right)\left(B^2 - \sum_{k=1}^n b_k^2\right) \le \left(AB - \sum_{k=1}^n a_k b_k\right)^2$$

with equality if and only if the sequences $A, a_1, \ldots, a_n$ and $B, b_1, \ldots, b_n$ are proportional.

Later in 1959 Popoviciu [2,3] gave a generalization of the preceding inequality.

Theorem 2. *Let n, p be positive integers and let $A, B, a_k, b_k, (1 \le k \le n)$ be real numbers such that $A^p \ge \sum_{k=1}^n a_k^p$ or $B^p \ge \sum_{k=1}^n b_k^p$. Then,*

$$\left(A^p - \sum_{k=1}^n a_k^p\right)\left(B^p - \sum_{k=1}^n b_k^p\right) \le \left(AB - \sum_{k=1}^n a_k b_k\right)^p.$$

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doi:10.1016/j.camwa.2008.05.013

A related result due to Bellman [2,4] that appeared in 1956 is stated in:

Theorem 3. *Let n, p be positive integers and let A, B, a_k, b_k, $(1 \leq k \leq n)$ be nonnegative real numbers such that $A^p \geq \sum_{k=1}^n a_k^p$ and $B^p \geq \sum_{k=1}^n b_k^p$. Then,*

$$\left(A^p - \sum_{k=1}^n a_k^p\right)^{1/p} + \left(B^p - \sum_{k=1}^n b_k^p\right)^{1/p} \leq \left((A+B)^p - \sum_{k=1}^n (a_k+b_k)^p\right)^{1/p}.$$

Our goal in this paper is to split conveniently the sums that appear in the above inequalities, and to apply the inequalities again, to obtain refinements of the preceding results.

2. Main results

In the following refinements of Popoviciu's and Aczél's inequalities are given. The first is stated and proved in the following.

Theorem 4. *Let n, p be positive integers and let A, B, a_k, b_k, $(1 \leq k \leq n)$ be real numbers such that $A^p \geq \sum_{k=1}^n a_k^p$ or $B^p \geq \sum_{k=1}^n b_k^p$. Then, for $1 \leq j < n$*

$$\left(A^p - \sum_{k=1}^n a_k^p\right)\left(B^p - \sum_{k=1}^n b_k^p\right) \leq R(A, B, a_k, b_k) \leq \left(AB - \sum_{k=1}^n a_k b_k\right)^p,$$

where

$$R(A, B, a_k, b_k) = \left(\sqrt[p]{A^p - \sum_{k=1}^j a_k^p}\,\sqrt[p]{B^p - \sum_{k=1}^j b_k^p} - \sum_{k=j+1}^n a_k b_k\right)^p.$$

Proof. First, we split the left-hand side of the preceding inequality as follows

$$\begin{aligned}\left(A^p - \sum_{k=1}^n a_k^p\right)\left(B^p - \sum_{k=1}^n b_k^p\right) &= \left(A^p - \sum_{k=1}^j a_k^p - \sum_{k=j+1}^n a_k^p\right)\left(B^p - \sum_{k=1}^j b_k^p - \sum_{k=j+1}^n b_k^p\right)\\ &= \left(M^p - \sum_{k=j+1}^n a_k^p\right)\left(N^p - \sum_{k=j+1}^n b_k^p\right),\end{aligned}$$

where $M = \sqrt[p]{A^p - \sum_{k=1}^j a_k^p}$ and $N = \sqrt[p]{B^p - \sum_{k=1}^j b_k^p}$. From the above expression and the hypothesis immediately we obtain $M^p \geq \sum_{k=j+1}^n a_k^p$ or $N^p \geq \sum_{k=j+1}^n b_k^p$. Therefore, applying generalized Aczél's inequality, we get

$$\left(M^p - \sum_{k=j+1}^n a_k^p\right)\left(N^p - \sum_{k=j+1}^n b_k^p\right) \leq \left(MN - \sum_{k=j+1}^n a_k b_k\right)^p.$$

On the other hand, applying generalized Aczél's inequality again it follows that $MN \leq AB - \sum_{k=1}^j a_k b_k$ and we have

$$\begin{aligned}\left(A^p - \sum_{k=1}^n a_k^p\right)\left(B^p - \sum_{k=1}^n b_k^p\right) &= \left(M^p - \sum_{k=j+1}^n a_k^p\right)\left(N^p - \sum_{k=j+1}^n b_k^p\right)\\ &\leq \left(MN - \sum_{k=j+1}^n a_k b_k\right)^p \leq \left(AB - \sum_{k=1}^j a_k b_k - \sum_{k=j+1}^n a_k b_k\right)^p\\ &= \left(AB - \sum_{k=1}^n a_k b_k\right)^p\end{aligned}$$

and the proof is complete. □

Setting $p = 2$ in the preceding result we get a refinement of Aczél's inequality.

Corollary 5. *Let n be a positive integer and let A, B, a_k, b_k, $(1 \leq k \leq n)$ be real numbers such that $A^2 \geq \sum_{k=1}^n a_k^2$ or $B^2 \geq \sum_{k=1}^n b_k^2$. Then, for $1 \leq j < n$*

$$\left(A^2-\sum_{k=1}^{n}a_k^2\right)\left(B^2-\sum_{k=1}^{n}b_k^2\right)\le R(A,B,a_k,b_k)\le\left(AB-\sum_{k=1}^{n}a_kb_k\right)^2,$$

where

$$R(A,B,a_k,b_k)=\left(\sqrt{A^2-\sum_{k=1}^{j}a_k^2}\sqrt{B^2-\sum_{k=1}^{j}b_k^2}-\sum_{k=j+1}^{n}a_kb_k\right)^2.$$

Likewise, applying the same procedure we get the following refinement of Bellman's inequality.

Theorem 6. *Let n, p be positive integers and let A, B, a_k, b_k, $(1\le k\le n)$ be nonnegative real numbers such that $A^p\ge\sum_{k=1}^{n}a_k^p$ and $B^p\ge\sum_{k=1}^{n}b_k^p$. Then, for $1\le j<n$*

$$\left(A^p-\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(B^p-\sum_{k=1}^{n}b_k^p\right)^{1/p}\le R(A,B,a_k,b_k)\le\left((A+B)^p-\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p}$$

where

$$R(A,B,a_k,b_k)=\left[\left(\sqrt[p]{A^p-\sum_{k=1}^{j}a_k^p}+\sqrt[p]{B^p-\sum_{k=1}^{j}b_k^p}\right)^p-\sum_{k=j+1}^{n}(a_k+b_k)^p\right]^{1/p}.$$

Proof. We have

$$\begin{aligned}\left(A^p-\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(B^p-\sum_{k=1}^{n}b_k^p\right)^{1/p}&=\left(A^p-\sum_{k=1}^{j}a_k^p-\sum_{k=j+1}^{n}a_k^p\right)^{1/p}+\left(B^p-\sum_{k=1}^{j}b_k^p-\sum_{k=j+1}^{n}b_k^p\right)^{1/p}\\&=\left(M^p-\sum_{k=j+1}^{n}a_k^p\right)^{1/p}+\left(N^p-\sum_{k=j+1}^{n}b_k^p\right)^{1/p}.\end{aligned}$$

Since $M^p=A^p-\sum_{k=1}^{j}a_k^p$ and $N^p=B^p-\sum_{k=1}^{j}b_k^p$, then from the hypothesis we have $M^p\ge\sum_{k=j+1}^{n}a_k^p$ and $N^p\ge\sum_{k=j+1}^{n}b_k^p$. Applying Bellman's inequality yields

$$\left(M^p-\sum_{k=j+1}^{n}a_k^p\right)^{1/p}+\left(N^p-\sum_{k=j+1}^{n}b_k^p\right)^{1/p}\le\left((M+N)^p-\sum_{k=j+1}^{n}(a_k+b_k)^p\right)^{1/p}.$$

Applying Bellman's inequality again, we get

$$M+N=\sqrt[p]{A^p-\sum_{k=1}^{j}a_k^p}+\sqrt[p]{B^p-\sum_{k=1}^{j}b_k^p}\le\left((A+B)^p-\sum_{k=1}^{j}(a_k+b_k)^p\right)^{1/p}.$$

Now, combining the preceding expressions, we obtain

$$\begin{aligned}\left(A^p-\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(B^p-\sum_{k=1}^{n}b_k^p\right)^{1/p}&=\left(M^p-\sum_{k=j+1}^{n}a_k^p\right)^{1/p}+\left(N^p-\sum_{k=j+1}^{n}b_k^p\right)^{1/p}\\&\le\left((M+N)^p-\sum_{k=j+1}^{n}(a_k+b_k)^p\right)^{1/p}\\&\le\left[(A+B)^p-\sum_{k=1}^{j}(a_k+b_k)^p-\sum_{k=j+1}^{n}(a_k+b_k)^p\right]^{1/p}\\&=\left((A+B)^p-\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p}\end{aligned}$$

and this completes the proof. □

Acknowledgement

The authors thank the Ministry of Education of Spain that has partially supported this research by grant MTM2006-03040.

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