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Hyperaffine planes over hyperrings ☆

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Abstract

The aim of this paper is to construct a geometric structure over both an hyperring and a multiplicative hyperring. In order to accomplish this goal we will need to define a particular class of planar hyperrings. Moreover, the notions of hyperaffine plane, affine map, translation and homothety are given. Finally, in the first case an hyperaffine plane having all homotheties centered in 0 is obtained; in the second case an hyperaffine translation plane is built.

1. Introduction

First of all we will recall some algebraic definitions that will be used in the paper.

An *hyperring* [2] $(A, \oplus, \cdot)$ is a set A with an hyperoperation $\oplus$ and a product $\cdot$ such that the following properties hold:

(i) $\forall a, b, c \in A$: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(ii) $\forall a, b \in A$: $a \oplus b = b \oplus a$,

(iii) $\exists 0 \in A / \forall a \in A$: $0 \oplus a = a \oplus 0 = a$,

(iv) $\forall a \in A \exists! \, a' \in A$: $a \oplus a' \ni 0$ $(a' = -a)$,

(v) $\forall a, b, c \in A / a \in b \oplus c \Rightarrow c \in a - b$ $(c \in a \oplus b')$,

(vi) $\forall a, b, c \in A$: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(vii) $\forall a, b, c \in A$: $(a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$,

(viii) $\forall a, b, c \in A$: $a \cdot (b \oplus c) = a \cdot b \oplus a \cdot c$,

(ix) $\forall a \in A$: $a \cdot 0 = 0 \cdot a = 0$,

We recall that $(A, \oplus)$ satisfying (i)–(v) is called canonical hypergroup.

Let us observe that axiom (v) is equivalent to (v)′ and also to (v)″:

(v)′ $\forall a, b \in A$: $-(a \oplus b) = -a - b$,

(v)″ $\forall a, b, c, d \in A$: $(a \oplus b) \cap (c \oplus d) \neq \emptyset \Rightarrow (c - a) \cap (b - d) \neq \emptyset$.

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An hyperring A is called hyperfield if $(A^*, \cdot)$ is a group, where $A^* = A \backslash \{0\}$.

We will consider a particular class of hyperrings satisfying the following properties:

(x) $\forall a, b, c, d \in A / (a \oplus b) \cap (c \oplus d) \neq \emptyset \Rightarrow (a \oplus b) \subseteq (c \oplus d)$ or $(c \oplus d) \subseteq (a \oplus b)$

(xi) $\forall a, b \in A / (a \oplus b) \ni a \Rightarrow a \oplus b = \{a\}$.

If the canonical hypergroup $(A, \oplus)$ satisfies (x) and (xi) then it is called strongly canonical.

A *multiplicative hyperring* [4] $(A, +, \circ)$ is an abelian group $(A, +)$ together with an hyperproduct satisfying the following properties:

(i) $\forall a, b, c \in A$: $a \circ (b \circ c) = (a \circ b) \circ c$;

(ii) $\forall a, b, c \in A$: $(a + b) \circ c \subseteq a \circ c + b \circ c$;

(iii) $\forall a, b, c \in A$: $a \circ (b + c) \subseteq a \circ b + a \circ c$;

(iv) $\forall a, b \in A$: $(-a) \circ b = a \circ (-b) = -(a \circ b)$.

If a multiplicative hyperring satisfies, instead of properties (ii) and (iii), the following:

(ii)′ $\forall a, b, c \in A$: $(a + b) \circ c = a \circ c + b \circ c$,

(iii)′ $\forall a, b, c \in A$: $a \circ (b + c) = a \circ b + a \circ c$,

then $(A, +, \circ)$ is called *strongly distributive*. Moreover, $(A, +, \circ)$ is *strongly left (right) distributive* if (ii)′ ((iii)′) holds.

2. Hyperaffine planes: basic definitions and properties

Let π be a non-empty set whose elements will be called *points* and let R be a family of subsets of π whose elements will be called *lines* such that the following properties hold: (i) $\forall P, P' \in \pi, \exists r \in R$ such that $P, P' \in r$; (ii) $\forall r \in R$ and $\forall P \in \pi$, $P \notin r$, $\exists s \in R$ such that $P \in s$ and $s \cap r = \emptyset$ or $s = r$ almost always; (iii) $\exists P, P', P'' \in \pi$ such that $P, P', P'' \notin s$ $\forall s \in R$. The pair (π, R) satisfying the previous three conditions will be called *hyperaffine plane*. In π it is possible to define the relation: $r \parallel s \Leftrightarrow r = s$ almost always or $r \cap s = \emptyset$; we observe that this relation is in general non-transitive since we do not request a unique line in (ii).

If $(\pi, \mathcal{R})$ and $(\pi', \mathcal{R}')$ are two hyperaffine planes, an *affine map* from $(\pi, \mathcal{R})$ to $(\pi', \mathcal{R}')$ is a triple $(\varphi, \varphi^*, \varphi^{-1*})$ where

$\varphi : \pi \to \pi'$ is a bijection,

$\varphi^* : \mathcal{R} \to \mathcal{P}(\mathcal{R}')$,

$\varphi^{-1*} : \mathcal{R}' \to \mathcal{P}(\mathcal{R})$,

such that the following conditions are verified:

(1) $r'_1, r'_2 \in \varphi^*(r) \Rightarrow r'_1 \parallel r'_2$,

(1′) $r_1, r_2 \in \varphi^{-1*}(r') \Rightarrow r_1 \parallel r_2$,

(2) $P \in r \Rightarrow \varphi(P) \in \bigcup \{r' : r' \in \varphi^*(r)\}$,

(2′) $P' \in r' \Rightarrow \varphi^{-1}(P') \in \bigcup \{r : r \in \varphi^{-1*}(r')\}$,

(3) $r' \in \varphi^*(r) \Leftrightarrow r \in \varphi^{-1*}(r')$,

(4) $r \parallel s \Rightarrow [r' \parallel s' : \forall r' \in \varphi^*(r), \forall s' \in \varphi^*(s)]$,

(4′) $r' \parallel s' \Rightarrow [r \parallel s : \forall r \in \varphi^{-1*}(r'), \forall s \in \varphi^{-1*}(s')]$.

If $\mathscr{A}$ is the set of all affine maps in an hyperaffine plane (π, R) and $(\varphi, \varphi^*, \varphi^{-1*}), (\psi, \psi^*, \psi^{-1*}) \in \mathscr{A}$, then we can define a composition between $(\varphi, \varphi^*, \varphi^{-1*})$ and $(\psi, \psi^*, \psi^{-1*})$ as $(\varphi, \varphi^*, \varphi^{-1*}) \cdot (\psi, \psi^*, \psi^{-1*}) = (\varphi \circ \psi, (\varphi \circ \psi)^*, (\varphi \circ \psi)^{-1*})$, where $(\varphi \circ \psi)^*(r) = \bigcup \varphi^*(r'_i), r'_i \in \psi^*(r)$, and, similarly, $(\varphi \circ \psi)^{-1*}(s) = \bigcup \psi^{-1*}(s_i)$ with $s_i \in \varphi^{-1*}(s)$.

Proposition 2.1. *The set $\mathscr{A}$ is a group with respect to the composition of maps.*

Proof. First of all if $(\varphi, \varphi^*, \varphi^{-1*})$ and $(\psi, \psi^*, \psi^{-1*})$ are two affine maps of $(\pi, \mathscr{R})$, then $(\varphi, \varphi^*, \varphi^{-1*}) \cdot (\psi, \psi^*, \psi^{-1*}) \in \mathscr{A}$; in fact, $\varphi \circ \psi$ is a bijection of π and (1), (1′) follow from (4), (4′). As for property (2), $\forall P \in \pi$, $P \in r$, $(\varphi \circ \psi)(P) \in r''$, where $r'' \in (\varphi \circ \psi)^*(r)$ as $\psi(P) \in r'$, for some $r' \in \psi^*(r)$, thus $r'' \in (\varphi \circ \psi)^*(r)$ exists such that $\varphi(\psi(P)) \in r''$; similarly for property (2′). Property (3) holds since both φ and ψ satisfy it; in fact, $r'' \in (\varphi \circ \psi)^*(r) = \bigcup \varphi^*(r'_i)$, $r'_i \in \psi^*(r) \Leftrightarrow r \in \psi^{-1*}(r'_i)$ and $r'_i \in \varphi^{-1*}(r'') \Leftrightarrow r \in (\varphi \circ \psi)^{-1*}(r'')$. Similarly, properties (4) and (4′) are verified for $(\varphi, \varphi^*, \varphi^{-1*}) \cdot (\psi, \psi^*, \psi^{-1*})$ because they hold for $(\varphi, \varphi^*, \varphi^{-1*})$ and $(\psi, \psi^*, \psi^{-1*})$. The associative law being obvious, we observe that, by defining $\mathrm{id}^*(r) = \{r\}$, $(\mathrm{id}, \mathrm{id}^*, \mathrm{id}^{-1*})$ is the unity of $\mathscr{A}$; moreover, by definition of $(\varphi, \varphi^*, \varphi^{-1*})$, $(\varphi, \varphi^*, \varphi^{-1*})^{-1} = (\varphi^{-1}, \varphi^{-1*}, \varphi^*) \in \mathscr{A}$ and thus $(\mathscr{A}, \cdot)$ is a group.

An element of $\mathscr{A}$, $(\tau, \tau^*, \tau^{-1*})$, is called *translation* if either is the unity of $\mathscr{A}$ or is an affine map such that

(1) $\tau(P) \neq P, \forall P \in \pi$,

(2) $r \parallel r', \forall r \in \mathscr{R}, \forall r' \in \tau^*(r)$.

An hyperaffine plane $(\pi, \mathscr{R})$ is called *translation plane* if, and only if, $\forall P, Q \in \pi$, a translation $(\tau, \tau^*, \tau^{-1*})$ exists such that $\tau(P) = Q$.

An element of $\mathscr{A}$, $(\omega, \omega^*, \omega^{-1*})$, is an *homothety centered in O* (where $O \in \pi$) if, and only if, either is the unity of $\mathscr{A}$ or is an affine map such that

(i) $\omega(O) = O$,

(ii) $\omega(P) \neq P \ \forall P \in \pi, P \neq O$,

(iii) $r \parallel r'; \forall r \in \mathscr{R}, \forall r' \in \omega^*(r)$. □

Proposition 2.2. *The set $\mathscr{T}$ of the translations of the hyperaffine plane π is a normal subgroup of $\mathscr{A}$ whenever the product of two translations is a translation.*

Proof. First we observe that if $(\tau, \tau^*, \tau^{-1*}) \in \mathscr{T}$, then $(\tau, \tau^*, \tau^{-1*})^{-1} = (\tau^{-1}, \tau^{-1*}, \tau^*) \in \mathscr{T}$; in fact, $\forall P \in \pi \Rightarrow \tau^{-1}(P) \neq P$. Moreover, if $r' \in \mathscr{R}$, $\tau^{-1*}(r') = \{r_i: r_i \parallel r_j\}$ and, by property (3), $\tau^*(r_i) \ni r'$; since $(\tau, \tau^*, \tau^{-1*})$ is a translation, the lines contained in $\tau^*(r_i)$ are all parallel to r_i, thus r' is parallel to every r_i.

Let now $(\tau, \tau^*, \tau^{-1*}) \in \mathscr{T}$ and $(\varphi, \varphi^*, \varphi^{-1*}) \in \mathscr{A}$, we want to prove that $(\varphi, \varphi^*, \varphi^{-1*})^{-1} \cdot (\tau, \tau^*, \tau^{-1*}) \cdot (\varphi, \varphi^*, \varphi^{-1*}) \in \mathscr{T}$. First of all $\varphi^{-1} \circ \tau \circ \varphi$ has no fixed points; in fact, if $P = (\varphi^{-1} \circ \tau \circ \varphi)(P)$, then $\varphi(P) = \tau(\varphi(P))$ that is $\tau = \mathrm{id}$ and $\varphi^{-1} \circ \tau \circ \varphi = \mathrm{id}$. Let us now prove that all lines contained in $(\varphi^{-1} \circ \tau \circ \varphi)^*(r)$ are parallel to r; if $\varphi^*(r) = \{r_i: r_i \parallel r_j\}$ then $(\tau \circ \varphi)^*(r) = \bigcup \tau(r_i) = \{s_{ki}: s_{ki} \parallel r_i\}$ and

$(\varphi^{-1} \circ \tau \circ \varphi)^*(r) = \bigcup \varphi^{-1*}(s_{ki})$. From $r_i \parallel s_{ki}$ it follows that if $l \in \varphi^{-1*}(s_{ki})$ and $l' \in \varphi^{-1*}(r_i)$ then $l \parallel l'$, but $\varphi^{-1*}(r_i) \ni r$ and this implies $l \parallel r \; \forall l \in \varphi^{-1*}(s_{ki})$. Thus, $\mathscr{T}$ is a normal subgroup of $\mathscr{A}$. $\square$

3. Hyperaffine planes over a multiplicative hyperring

Let A be a strongly right distributive multiplicative hyperring such that: (α) $\forall a \in A$, $Aa = A$, (β) $\forall a \in A$, $\mathscr{P} = \{ax: x \in A\}$ is a partition of A, (γ) $\exists c, d \in A^* / \forall a \in A$: $c \in a0$, $0 \notin ad$; such hyperring will be called *planar*.

In $A \times A = A^2$ we consider the family $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$, where

$$r \in \mathscr{R}_1 \Leftrightarrow [\exists k \in A \text{ such that } r = \{(k, y); y \in A\}],$$

$$r \in \mathscr{R}_2 \Leftrightarrow [\exists h \in A \text{ such that } r = \{(x, h); x \in A\}],$$

$$r \in \mathscr{R}_3 \Leftrightarrow [\exists a, b \in A \text{ such that } r = \{(x, y) \in A^2; y \in ax + b\}].$$

The elements of A^2 will be called points and the elements of $\mathscr{R}$ lines.

It is now possible to prove the following proposition.

Proposition 3.1. *(A^2, R) is an hyperaffine plane.*

Proof. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two elements of A^2; if $x_1 = x_2 = k$, then P_1 and P_2 belong to the line $r = \{(x, y)/x = k\}$. Similarly, if $y_1 = y_2 = h$, the line $r = \{(x, y)/y = h\}$ contains P_1 and P_2. We must prove now that if $x_1 \neq x_2$ and $y_1 \neq y_2$, then a line r exists containing both points. To do that let us consider $x_1 - x_2, y_1 - y_2 \in A$, by property ($\alpha$) an element $a \in A$ exists such that $y_1 - y_2 \in a(x_1 - x_2)$; i.e. $y_1 - y_2 \in ax_1 - ax_2$ or $y_1 - y_2 = s - t$ with $s \in ax_1$ and $t \in ax_2$. If we consider the line $r \in R_3$, $r = \{(x, y)/y \in ax + y_2 - t\}$, then $P_1 \in r$ since $y_1 \in ax_1 + y_2 - t$ and $P_2 \in r$ since $0 = t - t \in ax_2 - t$ from which $y_2 \in ax_2 + y_2 - t$. Let us notice that the line r passing through P_1 and P_2 is not necessarily unique, since an element $c \neq a$ could exist with the condition $y_1 - y_2 \in c(x_1 - x_2)$. In this way we have proved the first axiom for an hyperaffine plane; let us consider the second one. Let $P_1 = (x_1, y_1) \in A^2$ and $r \in R_1$ such that $P_1 \notin r$; then $s \in R_1$ exists, $s = \{(x, y): x = x_1\}$, such that $P_1 \in s$ and $s \parallel r$. Similarly, we can proceed if $r \in R_2$: thus, let us consider $r \in R_3$, $r = \{(x, y): y \in ax + b\}$, and $P_1 \notin r$, that is $y_1 \notin ax_1 + b$. If $b' \in y_1 - ax_1$ and $a' = a$, then the line $s = \{(x, y): y \in a'x + b'\}$ contains P_1 by definition of b'; we must now prove that $s \parallel r$. To do that let $P = (z, w) \in r \cap s$; then $z \in (aw + b) \cap (aw + b')$, i.e. $z = e + b = f + b'$, $e, f \in aw$. Thus, $b = f - e + b' \in aw - aw + b' = a0 + b' \subseteq a0 + y_1 - ax_1 = y_1 - ax_1$ and this is impossible since $y_1 \notin ax_1 + b$; from this we have the required condition $s \parallel r$. In order to complete our proof we need to prove that three non-collinear points exist; let $P_1 = (0, 0)$, and $P_2 = (0, c)$ and $P_3 = (d, 0)$ where c and d are the elements considered in property (γ). We observe that a line passing through P_1, P_2 and P_3 should be of the following type

$r = \{(x, y)\colon\ y \in ax + b\}$; if $P_1 \in r$ then $0 \in a0 + b$, i.e. $b \in -a0$, $P_2 \in r$ implies $c \in a0 + b = a0$: finally $P_3 \in r$ would imply $0 \in ad + b = ad$ which is impossible by the hypothesis (γ). Thus, the point P_3 is not collinear with P_1 and P_2. □

In order to understand better the structure of such a plane, let us characterize the conditions for parallel lines through their equations.

Proposition 3.2 *In* (A^2, R), *if* $r, s \in R$, $r \neq s$, *then* (i) $r, s \in R_1$: $r \cap s = \emptyset$; (ii) $r, s \in R_2$: $r \cap s = \emptyset$; (iii) $r \in R_1, s \in R_2$: $|r \cap s| = 1$; (iv) $r \in R_1, s \in R_3$: $|r \cap s| \geqslant 1$; (v) $r \in R_2$, $s \in R_3$: $|r \cap s| \geqslant 1$; (vi) $r, s \in R_3, r = \{(x, y)\colon\ y \in ax + b\}$, $s = \{(x, y)\colon\ y \in a'x + b'\}$: $r \cap s = \emptyset \Leftrightarrow a = a'$.

Proof. (i), (ii) and (iii) are obvious. To prove (iv) let $r = \{(x, y)\colon x = k\}$ and $s = \{(x, y)\colon y \in ax + b\}$; then, $\forall h \in ak + b$, the point $(k, h) \in r \cap s$. Now let $r = \{(x, y)\colon y = h\}$ and $s = \{(x, y)\colon y \in ax + b\}$: by property ($\beta$) an element $k \in A$ exists such that $h - b \in ak$, thus $(k, h) \in r \cap s$ and (v) is proved. Finally, let $r = \{(x, y)\colon y \in ax + b\}$ and $s = \{(x, y)\colon y \in a'x + b'\}$. If $a = a'$ then the condition $r \neq s$ implies that a point $P = (x_1, y_1)$ exists such that $P \in r$, $P \notin s$; from $P \in r$ we get $y_1 \in ax_1 + b$ while from $P \notin s$ $y_1 \notin ax_1 + b'$ follows, i.e. $b' \notin y_1 - ax_1$. If $Q = (x', y') \in r \cap s$, then $y' = e + b = f + b'$ with $e, f \in ax'$ and this implies that $b' = e + b - f \in ax' + b - ax' = a0 + b \subseteq a0 + y_1 - ax_1 = y_1 - ax_1$ and this is absurd. Conversely, if $a \neq a'$, by property (β), $z \in A$ exists such that $b' - b \in (a - a')z \subseteq az - a'z$, i.e. $b' - b = u - v$, $u \in az$, $v \in a'z$; from this we have $u + b = v + b'$, thus $w \in A$ exists such that $w \in (az + b) \cap (a'z + b')$ and $(z, w) \in r \cap s$. □

We observe that, as a consequence of the previous result, the parallelism relation in (A^2, R) is transitive; thus, the set of translations is a group.

Proposition 3.3. *In* (A^2, R), *if* $r, s \in R_3$, $r = \{(x, y)\colon y \in ax + b\}$, $s = \{(x, y)\colon y \in ax + c\}$, *then* $r = s \Leftrightarrow b - c \in a0$.

Proof. Let $r = s$; then given $P = (u, v)$ $[v \in au + b \Leftrightarrow v \in au + c] \Rightarrow b - c \in a0$ results. Similarly, if $b - c \in a0$ and $P = (u, v)$ belongs to r, then $v \in au + b \subseteq au + a0 + c = au + c$, i.e., P belongs to s. If $P \in s$, then $v \in au + c$ with $c \in b - a0$; thus $v \in au - a0 + b = au + b$. □

Proposition 3.4. *In* (A^2, R), *if* $r \in R_3$, $r = \{(x\ y)\colon\ y \in ax + b\}$, *and* $P \notin r$, $P = (x_1, y_1)$, *a unique line* $s \ni P$ *belonging to* R_3 *exists, such that* $r \parallel s$.

Proof. From Proposition 3.2, the line s will be such that $s = \{(x, y)\colon\ y \in ax + c\}$; since $P \in s$, then $c \in y_1 - ax_1$. If $s' = \{(x, y)\colon\ y \in ax + d\}$, $s' \parallel r$, contains P, then $d \in y_1 - ax_1$, thus $c - d \in a0$; by Proposition 3.3, $s = s'$. □

Proposition 3.5. *The hyperaffine plane (A^2, R) is a translation plane.*

Proof. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two different points of A^2 and $\tau: A^2 \to A^2$ such that $\tau((x, y)) = (x', y')$ where $x' = x + x_2 - x_1$ and $y' = y + y_2 - y_1$. Trivially, τ is a bijection without fixed points; moreover, $\tau(P) = Q$. We want to prove that τ induces a translation over (π, R). Let $r \in R_1$, $r = \{(x, y)\colon x = k\}$ and $s \in R_2$, $s = \{(x, y)\colon y = h\}$; then we define $\tau^*(r) = \{(x, y)\colon x = k + x_2 - x_1\}$ and $\tau^*(s) = \{(x, y)\colon y = h + y_2 - y_1\}$. Finally, if $t \in R_3$, $t = \{(x, y)\colon y \in ax + b\}$, we define $\tau^*(t) = \{t_i = \{(x, y)\colon y \in ax + b + \alpha_i + y_2 - y_1\},\ \alpha_i \in a(x_1 - x_2)\}$; we observe that $|\tau^*(t)| = 1$ since (see Proposition 3.3) $\alpha_i - \alpha_j \in a0$. In such a way we obtain that if $(\tau, \tau^*, \tau^{-1*})$ is an affine map, it is a translation. Let us prove the characterizing properties: (1) and (1′) are true by definition. As (2) is trivially satisfied for $P \in r$ or $P \in s$, we prove it for $T = (u, v) \in t$; in that case $\tau(T) = (u', v') = (u + x_2 - x_1,\ v + y_2 - y_1)$ and $v' = v + y_2 - y_1 \in au + b + y_2 - y_1 = a(u' - x_2 + x_1) + b + y_2 - y_1 = au' + a(x_1 - x_2) + b + y_2 - y_1$ which implies $\tau(T) \in t'$ where $t' = \tau^*(t)$, $t' = \{(x, y)\colon y \in ax + b + \alpha_i + y_2 - y_1\}$, $\forall \alpha_i \in a(x_1 - x_2)$.

Let us prove property (3). First we observe that $\tau^{-1}((x, y)) = (x - x_2 + x_1, y - y_2 + y_1)$; moreover, $\tau^{-1*}(r) = \{(x, y)\colon x = k - x_2 + x_1\}$, $\tau^{-1*}(s) = \{(x, y)\colon y = h - y_2 + y_1\}$, $\tau^{-1*}(t) = t'' = \{(x, y)\colon y \in ax + b + \beta_h + y_1 - y_2\}, \forall \beta_h \in a(x_2 - x_1)$ (Proposition 3.3). Thus, in order to prove property (3) we must verify that $t = \tau^{-1*}(t')$, $t' = \tau^*(t) = \{(x, y)\colon y \in ax + b + \alpha_i + y_2 - y_1\}$, $\alpha_i \in a(x_1 - x_2)$. For t' we have $\tau^{-1*}(t') = \{(x, y)\colon y \in ax + b + \alpha_i + y_2 - y_1 + \beta_h + y_1 - y_2\}$ and $(b + \alpha_i + y_2 - y_1 + \beta_h + y_1 - y_2) - b = \alpha_i + \beta_h \in a0$; thus $t = \tau^{-1*}(t')$. The inverse implication is proved exactly in the same way.

Property (2′) is obviously true since it is the exact analogous of property (2) with respect to $(\tau^{-1}, \tau^{-1*}, \tau^*)$ since τ^{-1} and τ act in the same way. Similarly, the proof for property (4) is the same as that for property (4′).

Property (4) is trivial for the lines belonging to R_1 and R_2; let us consider two parallel lines $t_1, t_2 \in R_3$, then $t_1 = \{(x, y)\colon y \in ax + b\}$ and $t_2 = \{(x, y)\colon y \in ax + c\}$. Since $\tau^*(t_1) = t_1' = \{(x, y)\colon y \in ax + b + \alpha + y_2 - y_1\}$, $\alpha \in a(x_1 - x_2)$, and $\tau^*(t_2) = t_2' = \{(x, y)\colon y \in ax + c + \beta + y_2 - y_1\}$, $\beta \in a(x_1 - x_2)$, by Proposition 3.2 we have $t_1' \parallel t_2'$. □

Observation. The pair (A^2, R'), where $R_1' = R_1$, $R_2' = R_2$, $R_3' = \{\{x, y) \in A^2/x \in ay + b, a, b \in A\}\}$, is an hyperaffine plane; moreover, the following proposition holds.

Proposition 3.6. *If A is a planar hyperfield such that, $\forall x, y \in A$, $y \in ax, a \neq 0$, $\Leftrightarrow x \in a^{-1}y$, then there exists an affine map between (A^2, R) and (A^2, R').*

Proof. Let $\alpha: A^2 \to A^2$ such that $\alpha((u, v)) = (v, u)$ and $\alpha^*: R \to \mathscr{P}(R)$ defined as follows: if $r \in R_1$, $r = \{(x, y)\colon x = k\}$, $\alpha^*(r) = \{(x, y)\colon y = k\} \in R_2$; if $s \in R_2$, $s = \{(x, y)\colon y = h\}$, $\alpha^*(s) = \{(x, y)\colon x = h\} \in R_1$. Finally, if $t \in R_3$, $t = \{(x, y)\colon y \in ax + b\}$,

$\alpha^*(t) = \{\{(x, y): x \in a^{-1}y + c\}, c \in (-a^{-1}b)\}$. The map $(\alpha, \alpha^*, \alpha^{-1*})$ is trivially an affine map. □

4. Hyperaffine planes over hyperrings

In this section we consider the case of hyperrings [2].

If A is an hyperring such that: (i) $\forall a, b, c, d \in A$, $a \neq b$, $c \neq d$, $\exists z \in A/(a-b)z \cap (c-d) \neq \emptyset$; (ii) $\forall a, b, c, d \in A$, $a \neq b$, $c \neq d$, $\exists z \in A/z(a-b) \cap (c-d) \neq \emptyset$; (iii) $\forall w \in A, \mathscr{P} = \{\{w \oplus y\}: y \in A\}$ is a partition of A; (iv) $au \oplus b = au \oplus c$ for some $u \Rightarrow az \oplus b = az \oplus c$ for all $z \in A$ such that $0, b, c$ do not belong to one of such sets, then A will be called *planar hyperring*.

In $A^2 = A \times A$ let us consider the following family $\mathscr{R}$ of subsets of A^2: $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$ where

$$r \in \mathscr{R}_1 \Leftrightarrow [\exists k \in A \text{ such that } r = \{(k, y): y \in A\}],$$

$$r \in \mathscr{R}_2 \Leftrightarrow [\exists a, b \in A \text{ such that } r = \{(x, y) \in A^2; y \in ax \oplus b\}].$$

Then it is possible to prove that:

Proposition 4.1. *For the intersection of two distinct lines the following hold:* (1) $r, r' \in R_1, r \neq r' \Rightarrow r \cap r' = \emptyset$; (2) $r \in R_1, r' \in R_2 \Rightarrow r \cap r' \neq \emptyset$; (3) $r, r' \in R_2$, $r = \{(x, y): y \in ax \oplus b\}, r' = \{(x, y): y \in cx \oplus d\} \Rightarrow (r \cap r' = \emptyset \Rightarrow a = c)$.

Proof. Let $r = \{(x, y): x = k\}$ and $r' = \{(x, y): x = h\}$; then if $h \neq k$ obviously $r \cap r' = \emptyset$. If $r = \{(x, y): x = k\}$ and $r' = \{(x, y): y \in ax \oplus b\}$ the point $P = (k, u)$ belongs to $r \cap r'$ $\forall u \in ak \oplus b$. Finally, if $r = \{(x, y): y \in ax \oplus b\}$ and $r' = \{(x, y): y \in cx \oplus d\}$ let us suppose $a \neq c$, $b \neq d$; by property (i) an element $z \in A$ exists such that $(a-c)z \cap (d-b) \neq \emptyset$, i.e. $(az - cz) \cap (d-b) \neq \emptyset$. Then we have (see [3]) $(az \oplus b) \cap (cz \oplus d) \neq \emptyset$, i.e. $\exists w \in (az \oplus b) \cap (cz \oplus d)$ and $P = (z, w) \in r \cap r'$. We observe that if $a \neq c$ and $b = d$, the point $(0, b) \in r \cap r'$.

If $a = c$, let $P = (u, v) \in r \cap r'$; then $v \in au \oplus b$ and $v \in au \oplus d$ from which, by (iii), $au \oplus b = au \oplus d$. Thus, $\forall Q = (z, w)$ belonging to r, i.e. $w \in az \oplus b$, such that $0, b, d \notin az \oplus b$, by property (iv), $w \in az \oplus d$; therefore $Q \in r'$. Similarly for a point $Q' \in r'$ which implies $r = r'$ almost always.

As for the case of multiplicative hyperrings, the previous result implies that the parallelism induced by the previous result is an equivalence.

Proposition 4.2. (A^2, R) *is an hyperaffine plane.*

Proof. First of all let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points. If $x_1 = x_2 = k$, the line $r = \{(x, y): x = k\}$ contains P_1 and P_2; if $y_1 = y_2 = h$ the two points belong to $r = \{(x, y): y = h\}$. Let us now suppose $x_1 \neq x_2$ and $y_1 \neq y_2$; then by (ii) an element

$a \in A$ exists such that $(y_1 - y_2) \cap a(x_1 - x_2) \neq \emptyset$, i.e. $(y_1 - y_2) \cap (ax_1 - ax_2) \neq \emptyset$ which implies (see [3]) $(y_1 - ax_1) \cap (y_2 - ax_2) \neq \emptyset$. Let $b \in (y_1 - ax_1) \cap (y_2 - ax_2)$; then $y_1 \in ax_1 \oplus b$ and $y_2 \in ax_2 \oplus b$, thus $P_1, P_2 \in r = \{(x, y): y \in ax \oplus b\}$. From above, the first axiom for hyperaffine planes follows.

Let $r = \{(x, y): x = k\} \in R_1$ and $P_1 = (x_1, y_1) \notin r$; the line $s = \{(x, y): x = x_1\}$ contains P_1 and $r \parallel s$. For $r = \{(x, y): y \in ax \oplus b\} \in R_2$ and $P_1 \notin r$, then the line $s = \{(x, y): y \in ax \oplus d\}$ such that $d \in y_1 - ax_1$ contains P_1 since $d \in y_1 - ax_1$ implies $y_1 \in ax_1 \oplus d$; moreover, $r \parallel s$ by Proposition 4.1. We observe that the line s is not unique; in fact, if $s' = \{(x, y): y \in ax \oplus c\}$ is parallel to r and $P_1 \in s'$, then $c \in y_1 - ax_1$ therefore, by property (iii), $ax_1 \oplus d = ax_1 \oplus c$ and, by property (iv), $au \oplus d = au \oplus c$, i.e. $s = s'$, almost always. Thus, we have proved the second axiom.

In order to prove the third axiom, let $P_1 = (0, 0)$, $P_2 = (0, c)$ and $P_3 = (c, 0)$, $c \neq 0$; trivially any line in R_1 cannot contain all the three points. If $r = \{(x, y): y \in ax \oplus b\} \in R_2$ is a line containing P_i, $i = 1, 2, 3$, then $0 \in a0 \oplus b$, $c \in a0 \oplus b$ and $0 \in ac \oplus b$; the first relation implies $b = 0$ and this, together with the second relation, implies $c = 0$. Since this is impossible we have proved that three non-collinear points exist.

Proposition 4.3. *If A is a planar hyperfield then, for every pair of points P and Q collinear with $O = (0, 0)$ in the hyperaffine plane (A^2, R), an homothety φ centered in O exists such that $\varphi(P) = Q$.*

Proof. First of all we need to know what kind of line can contain the three distinct points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $O = (x_3, y_3) = (0, 0)$; if $x_1 = x_2 = 0$ then $y_1 \neq 0$ and $y_2 \neq 0$ and $r = \{(x, y): x = 0\}$ in R_1 contains the three points. We observe that no line $s = \{(x, y): y \in ax \oplus b\}$ in R_2 can contain P, Q and O since $O \in s$ implies $b = 0$, i.e. $s = \{(x, y): y = ax\}$ and then $P = Q = O$. Similarly, if $y_1 = y_2 = 0$, then $x_1 \neq 0, x_2 \neq 0$ and the unique line containing the three points is $s = \{(x, y): y = 0\} \in R_2$. Finally, if $x_1 \neq 0$ and $y_1 \neq 0$ we can prove that $x_2 \neq 0$; in fact, $x_2 = 0$ would imply that no line in R_1 can obviously contain P, Q and O. The same is also true for any line $s = \{(x, y): y \in ax \oplus b\} \in R_2$ since $O \in s$ implies $b = 0$ and $Q \in s$ implies $y_2 = 0$, i.e. $Q = O$; similarly, if $x_1 \neq 0$ and $y_1 \neq 0$, y_2 must be different from zero. Under these hypothesis a line containing the three points is $s = \{(x, y): y = ax\}$, where $a = y_1 x_1^{-1} = y_2 x_2^{-1}$; moreover, s is unique.

In the three considered cases we have to define the required homothety; if x_1, x_2, y_1, y_2 are all different from zero, we can consider the map $\varphi: A^2 \to A^2$ such that $\varphi((x, y)) = (x'', y'')$ where $x'' = xc$ and $y'' = yc$ with $c = y_1^{-1} y_2 = x_1^{-1} x_2$. This map is a bijection, $\varphi(O) = O$ and $\varphi(P) = Q$; let us study the induced map φ^*. If $R \in r$, $r = \{(x, y): x = k\} \in R_1$ and $R = (k, y)$, then $\varphi(R) = (kc, yc)$; from this we obtain $\varphi^*(r) = \{(x, y): x = kc\} \in R_1$. Now we recall that a line s passing through $O = (0, 0)$ is of the form $s = \{(x, y): y = ax\}$; then a point S belongs to s if and only if $S = (x, ax)$, thus $\varphi(S) = (xc, axc)$ i.e. $\varphi^*(s) = s$. Finally, let $T = (x, y)$, $T \in t = \{(x, y): y \in ax \oplus b\}$, $t \in R_2$; then $\varphi(T) = (x'', y'')$ where $y'' = yc \in (ax \oplus b)c = axc \oplus bc = ax'' \oplus bc$. Thus,

$\varphi^*(t) = t'$ with $t' = \{(x, y)\colon y \in ax \oplus bc\} \in R_2$ and $t' \parallel t$ by Proposition 4.1. All that has been proved implies that φ^* maps each line r exactly in one line r'; moreover, either $r = r'$ or $r \parallel r'$.

In order to prove that $(\varphi, \varphi^*, \varphi^{-1*})$ is an affine map we must study the maps φ^{-1} and φ^{-1*}; easily we obtain $\varphi^{-1}((x, y)) = (xc^{-1}, yc^{-1})$, thus it is obvious that φ^{-1} and φ^{-1*} work exactly as φ and φ^*.

Let us consider the six conditions for an affine map; (1), (1′), (2) and (2′) are trivial, while (3) becomes $\varphi^{-1*}(\varphi^*(r)) = r$ and $\varphi^*(\varphi^{-1*}(r')) = r'$. In order to prove (4), if $r, s \in R_1$, $r \parallel s$, obviously $\varphi^*(r) \parallel \varphi^*(s)$; if $r, s \in R_2$, $r \parallel s$, $r = \{(x, y)\colon y \in ax \oplus b\}$ and $s = \{(x, y)\colon y \in ax \oplus d\}$, thus $\varphi^*(r) = \{(x, y)\colon y \in ax \oplus bc\}$ and $\varphi^*(s) = \{(x, y)\colon y \in ax \oplus dc\}$, i.e. $\varphi^*(r) \parallel \varphi^*(s)$ by Proposition 4.1. Similarly, we can prove (4′).

Thus, we have proved that $(\varphi, \varphi^*, \varphi^{-1*})$ is an homothety centered in O mapping P in Q; we observe that it is also a classical homothety.

We must now consider the cases $x_1 = x_2 = 0$ and $y_1 = y_2 = 0$. If $P = (0, y_1)$, $Q = (0, y_2)$, $O = (0, 0)$, we define $\varphi\colon A^2 \to A^2$ such that $\varphi((x, y)) = (x'', y'')$ where $x'' = xc$ and $y'' = yc$ with $c = y_1^{-1}y_2$; if $P = (x_1, 0)$, $Q = (x_2, 0)$, $O = (0, 0)$, we define $\varphi\colon A^2 \to A^2$ such that $\varphi((x, y)) = (x'', y'')$ where $x'' = xc$ and $y'' = yc$ with $c = x_1^{-1}x_2$. Under these conditions we can repeat the previous arguments and complete the proof. □

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