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Self-improving properties of inequalities of Poincaré type on measure spaces and applications

Seng-Kee Chua [a,*], Richard L. Wheeden [b]

[a] *Department of Mathematics, National University of Singapore, 10, Kent Ridge Crescent, Singapore 119260, Singapore*
[b] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*

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Abstract

We show that the self-improving nature of Poincaré estimates persists for domains in rather general measure spaces. We consider both weak type and strong type inequalities, extending techniques of B. Franchi, C. Pérez and R. Wheeden. As an application in spaces of homogeneous type, we derive global Poincaré estimates for a class of domains with rough boundaries that we call ϕ-John domains, and we show that such domains have the requisite properties. This class includes John (or Boman) domains as well as s-John domains. Further applications appear in a companion paper.

Keywords: Global Poincaré estimates; Power type weights; Quasimetric spaces; s-John domains

1. Introduction

The self-improving nature of Poincaré inequalities over balls is an interesting and powerful property observed initially by Saloff-Coste [22] in the Euclidean case. It has been extensively studied recently in more general settings: see for example Hajłasz and Koskela [15], and Franchi, Pérez and Wheeden [12,13]. There have also been studies regarding s-John domains (see Definition 1.5), $s \geqslant 1$, including the validity of global Poincaré estimates over these domains, such as Hajłasz and Koskela [14], and Kilpeläinen and Malý [17]. The main goals of this and our closely

* Corresponding author.
E-mail addresses: matcsk@nus.edu.sg (S.-K. Chua), wheeden@math.rutgers.edu (R.L. Wheeden).

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related companion paper [8] are to extend the self-improving techniques in [13] and to derive global Poincaré estimates on s-John domains in spaces of homogeneous type. While the abstract and more general theory is discussed in this paper, further specific applications of these results appear in [8].

The notion of an s-John domain was introduced by Smith and Stegenga [25], while the terminology John domain was used earlier first by Martio and Sarvas [20]. In spaces of homogeneous type with the segment (geodesic) property, John domains are the same as Boman domains (see [4]); in general, they are the same as s-John domains in case $s = 1$. When $s > 1$, the notion of an s-John domain is a generalization of that of a John domain, a weakening of requirements relative to the case $s = 1$ in order to accommodate domains with rougher boundaries. It is easy to see that bounded Lipschitz domains (including all bounded domains with smooth boundaries) and bounded domains which satisfy the cone condition are John domains. There have been many studies concerning John domains: see for example [1,3,6] and references listed in those papers. Some examples of s-John domains in case $s > 1$ are given in [14].

An example of a (global) Poincaré estimate for s-John domains is given in the following result stated in [17, Theorem 2.3].

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is an s-John domain. Let a, b, p, q be real numbers which satisfy $a \geqslant 0, b \geqslant 1 - n, 1 \leqslant p < q < \infty, \frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{n}$ and*

$$\frac{1}{q} \geqslant \frac{s(n+b-1) - p + 1}{(n+a)p}. \tag{1.1}$$

Then there is a constant $C = C(n, a, b, p, q, \Omega) > 0$ such that

$$\|f - f_{\Omega,\rho^a dx}\|_{L^q_{\rho^a dx}(\Omega)} \leqslant C \|\nabla f\|_{L^p_{\rho^b dx}(\Omega)} \quad \text{for all } f \in C^1(\Omega), \tag{1.2}$$

where $\rho(x) = \operatorname{dist}(x, \Omega^{\mathrm{c}})$ and $f_{\Omega,\rho^a dx} = \int_\Omega f(x)\rho(x)^a\,dx / \int_\Omega \rho(x)^a\,dx$.

The assumption that $f \in C^1(\Omega)$ in Theorem 1.1 does not automatically imply that the norm on the right-hand side of (1.2) or the average $f_{\Omega,\rho^a dx}$ on the left-hand side is finite. However, as we shall see in Theorem 1.13, (1.2) holds under the weaker hypothesis $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ provided the average on the left-hand side is replaced by the average $|B'|^{-1} \int_{B'} f(x)\,dx$ over a "central" ball $B' \subset \Omega$, which is always finite for such f. If $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and the right-hand side of (1.2) is finite, it follows that $f \in L^q_{\rho^a dx}(\Omega)$, and then $f_{\Omega,\rho^a dx}$ is finite and it is possible to replace the average over the central ball by this average in (1.2).

The inequality (1.2) was also proved by Hajłasz and Koskela [14] except that when $p > 1$, they required strict inequality in (1.1). The necessity of the condition $1/q \geqslant 1/p - 1/n$ is easy to see as usual by considering Lipschitz functions that vanish outside balls in Ω. Condition (1.1) is also sharp as can be seen by considering mushroom-like domains; see [14] for details. On the other hand, for special s-John domains such as s-cusp domains, condition (1.1) can be relaxed; see [8] for some results of this type.

We will use an approach which is different from those in [14] and [17] to prove Theorem 1.1. Our approach is a modification of one used in [13]. Actually, the Poincaré inequality (1.2) is just one consequence of our main results. In this and our companion paper [8], we will extend the techniques in [13] and use the outcome to derive global Poincaré inequalities on s-John domains

Ω (including 1-John domains) in spaces of homogeneous type and for measures which are doubling or just δ-doubling on Ω (see Definition 1.6). The notions of δ-doubling and doubling on Ω are equivalent on 1-John domains (see Proposition 2.2(3)). We note that power type weights of the form $\operatorname{dist}(x, \Omega_0)^a$, with $a \geqslant 0$ and $\Omega_0 \subset \partial\Omega$, are examples of δ-doubling measures. We are also able to prove Theorem 1.1 without the assumption $b \geqslant 1 - n$. The chief geometric contribution of the paper is the construction of suitable chains of balls in s-John domains, as well as in still more general domains which we call ϕ-John domains, in homogeneous spaces. This allows us to clarify some details that we were unable to follow in the Euclidean case given in [17, p. 378, line 4].[1]

Our first theorem is a very general one that applies to any measure space with certain properties and extends [13, Theorem 3], where the underlining space is restricted to being a homogeneous space that satisfies chain/segment conditions. The result yields weak type estimates for an individual function f. Here we assume the existence of a local estimate of Poincaré type for f and derive improved estimates for the same f. By "improved," we mean that the order of (weak) integrability of f is changed, generally with a different measure, and a global estimate on all of Ω is obtained. In this sense, the initial Poincaré estimate is "self-improving." Some strong type results follow as corollaries, still for a particular f. As in [12] and [13], sharper strong type estimates can be obtained by allowing f to vary in the initial hypothesis. A general result of this sort is studied below in Theorem 1.10.

Theorem 1.2. *Let σ and μ be measures on a σ-algebra Σ of subsets of X. Let Ω be a measurable subset of X and f be a fixed measurable function which satisfy the following assumptions for some constants $0 < p_0, q < \infty, 0 < \theta < 1, C_\sigma > 0, 0 < \theta_1 < \theta_2 < 1,$ $0 < A_1, A_2 < \infty$ and $\wp \geqslant 1$:*

(1) *For each $x \in \Omega$, there is a sequence of measurable sets $\{Q_i^x\}_{i=1}^\infty$, depending on x, and a fixed set $B' \subset X$ such that $Q_1^x = B'$,*

$$0 < \sigma\big(Q_i^x \cup Q_{i+1}^x\big) \leqslant C_\sigma \sigma\big(Q_i^x \cap Q_{i+1}^x\big) < \infty, \quad i = 1, 2, \ldots, \tag{1.3}$$

and

$$\left(\frac{1}{\sigma(Q_i^x)} \int_{Q_i^x} |f - f_{Q_i^x}|^{p_0}\, d\sigma\right)^{\frac{1}{p_0}} \leqslant a\big(Q_i^x\big), \tag{1.4}$$

where $\{f_{Q_i^x}\}$ is a sequence of constants that converges to $f(x)$ and $\{a(Q_i^x)\}$ is a sequence of nonnegative numbers.

(2) *For each $x \in \Omega$, there is a sequence $\{B_j^x\}_{j=1}^\infty$ of measurable sets and a sequence $\{\mu^*(B_j^x)\}$ of positive numbers such that*

$$\mu(\Omega) \leqslant \wp\mu^*\big(B_1^x\big) \quad \textit{and} \quad A_1\theta_1^k \leqslant \frac{\mu^*(B_{j+k}^x)}{\mu^*(B_j^x)} \leqslant A_2\theta_2^k, \quad j, k \in \mathbb{N}. \tag{1.5}$$

[1] After we discovered a suitable argument to address this difficulty, the authors of [17] sent us a corrected argument [18] similar to ours.

(3) *Let $\mathfrak{F} = \{B_j^x\}_{x\in\Omega, j\in\mathbb{N}}$. Assume for any $B_j^x \in \mathfrak{F}$, there is $\mathcal{C}(B_j^x) \subset \{Q_l^x\}_{l\in\mathbb{N}}$ such that for each $x \in \Omega$, $\bigcup_{j\in\mathbb{N}} \mathcal{C}(B_j^x) = \{Q_l^x\}_{l\in\mathbb{N}}$, $\mathcal{C}(B_i^x) \cap \mathcal{C}(B_j^x) = \emptyset$ when $i \neq j$. Further, for any countable subcollection I of pairwise disjoint sets $\{B_\alpha\}$ in $\mathfrak{F}$, let*

$$A(B_\alpha) = \sum_{Q\in\mathcal{C}(B_\alpha)} a(Q)$$

and assume that

$$\sum_{B_\alpha\in I} \big(A(B_\alpha)^q \mu^*(B_\alpha)\big)^\theta \leqslant \big(C_0^q \mu(\Omega)\big)^\theta. \tag{1.6}$$

(4) *Suppose the collection $\mathfrak{F}$ is a cover of Vitali type of subsets of Ω with respect to (μ, μ^*), i.e., given any measurable set $E \subset \Omega$ and a collection $\mathcal{B}_E = \{B_{i(x)}^x\colon x \in E\}$, there is a countable pairwise disjoint collection $\mathcal{B}'_E \subset \mathcal{B}_E$ such that*

$$\mu(E) \leqslant V_\mu \sum_{B_\alpha\in\mathcal{B}'_E} \mu^*(B_\alpha), \quad V_\mu \geqslant 1.$$

Then

$$\sup_{t>0} t\mu\big\{x \in \Omega\colon \big|f(x) - f_{B'}\big| > t\big\}^{\frac{1}{q}} \leqslant CC_0\big[\wp V_\mu \mu(\Omega)\big]^{\frac{1}{q}}, \tag{1.7}$$

where C depends on C_σ, $p_0, q, \theta, \theta_1, \theta_2$, A_1 and A_2.

Remark 1.3. 1. By using standard interpolation techniques, the weak L^q estimate (1.7) implies the following strong type inequality for any q_0 with $0 < q_0 < q$:

$$\|f - f_{B'}\|_{L_\mu^{q_0}(\Omega)} \leqslant C(q, q_0)CC_0(\wp V_\mu)^{\frac{1}{q}} \mu(\Omega)^{\frac{1}{q_0}} \tag{1.8}$$

for the same constants C and C_0 as in (1.7). Moreover, if $q_0 \geqslant 1$ in (1.8), it is possible to replace $f_{B'}$ by $f_{\mathcal{D},\mu} = \int_{\mathcal{D}} f\, d\mu/\mu(\mathcal{D})$ for any $\mathcal{D} \subset \Omega$ with $\mu(\mathcal{D}) > 0$, obtaining the estimate

$$\|f - f_{\mathcal{D},\mu}\|_{L_\mu^{q_0}(\Omega)} \leqslant C(q, q_0)CC_0\left(\frac{\mu(\Omega)}{\mu(\mathcal{D})}\right)^{\frac{1}{q_0}} (\wp V_\mu)^{\frac{1}{q}} \mu(\Omega)^{\frac{1}{q_0}}.$$

Note that $f_{\mathcal{D},\mu}$ is well defined as $f \in L_\mu^{q_0}(\Omega)$, $q_0 \geqslant 1$, by (1.8). In fact,

$$\|f - f_{\mathcal{D},\mu}\|_{L_\mu^{q_0}(\Omega)} \leqslant \|f - f_{B'}\|_{L_\mu^{q_0}(\Omega)} + \mu(\Omega)^{\frac{1}{q_0}} |f_{B'} - f_{\mathcal{D},\mu}|,$$

and using $q_0 \geqslant 1$ and (1.8) gives

$$\mu(\Omega)^{\frac{1}{q_0}}|f_{B'} - f_{\mathcal{D},\mu}| \leqslant \left(\frac{\mu(\Omega)}{\mu(\mathcal{D})}\right)^{\frac{1}{q_0}} \|f - f_{B'}\|_{L^{q_0}_\mu(\mathcal{D})}$$
$$\leqslant \left(\frac{\mu(\Omega)}{\mu(\mathcal{D})}\right)^{\frac{1}{q_0}} \|f - f_{B'}\|_{L^{q_0}_\mu(\Omega)}.$$

2. In (1.7), when $q > 1$, $f_{B'}$ can also be replaced by $f_{\mathcal{D},\mu}$ for any $\mathcal{D} \subset \Omega$ with $\mu(\mathcal{D}) > 0$ and with a different constant on the right-hand side of (1.7). To see this, note that when $|f(x) - f_{\mathcal{D},\mu}| > t$, then either $|f(x) - f_{B'}| > t/2$ or $|f_{\mathcal{D},\mu} - f_{B'}| > t/2$. However,

$$|f_{\mathcal{D},\mu} - f_{B'}| \leqslant \frac{1}{\mu(\mathcal{D})}\|f - f_{B'}\|_{L^1_\mu(\Omega)} \leqslant C(q)CC_0\frac{\mu(\Omega)}{\mu(\mathcal{D})}(\wp V_\mu)^{\frac{1}{q}}$$

by (1.8) with $q_0 = 1$ (valid since (1.7) for any $q > 1$ implies (1.8) for $q_0 = 1$ by the previous part of this remark). Thus

$$\bigl\{x \in \Omega\colon \bigl|f(x) - f_{\mathcal{D},\mu}\bigr| > t\bigr\}$$
$$\subset \left\{x \in \Omega\colon \bigl|f(x) - f_{B'}\bigr| > \frac{t}{2}\right\} \cup \left\{x \in \Omega\colon t < 2C(q)CC_0(\wp V_\mu)^{\frac{1}{q}}\frac{\mu(\Omega)}{\mu(\mathcal{D})}\right\}.$$

The last set on the right is empty if $t \geqslant 2C(q)CC_0(\wp V_\mu)^{\frac{1}{q}}\mu(\Omega)/\mu(\mathcal{D})$, and for such t we immediately obtain from (1.7) that

$$\mu\bigl\{x \in \Omega\colon \bigl|f(x) - f_{\mathcal{D},\mu}\bigr| > t\bigr\} \leqslant (CC_0)^q \wp V_\mu \mu(\Omega)\left(\frac{2}{t}\right)^q.$$

On the other hand, if $t < 2C(q)CC_0(\wp V_\mu)^{\frac{1}{q}}(\mu(\Omega)/\mu(\mathcal{D}))$ then

$$\mu\bigl\{x \in \Omega\colon \bigl|f(x) - f_{\mathcal{D},\mu}\bigr| > t\bigr\} \leqslant \mu(\Omega) \leqslant \frac{(2C(q)CC_0)^q}{t^q}\wp V_\mu\left(\frac{\mu(\Omega)}{\mu(\mathcal{D})}\right)^q\mu(\Omega).$$

We will discuss applications of Theorem 1.2 to quasimetric spaces, including results for s-John domains and, more generally, for ϕ-John domains in these spaces. We now list definitions and terminology we will need.

Definition 1.4. A pair $\langle H, d\rangle$ is a quasimetric space if d is a quasimetric on the set H, that is, if there exists a constant κ such that for all $x, y, z \in H$,

(1) $d(y,x) = d(x,y) > 0$ if $x \neq y$, $d(x,x) = 0$ and
(2) $d(x,y) \leqslant \kappa[d(x,z) + d(y,z)]$.

For a quasimetric space $\langle H, d\rangle$, any $x \in H$ and $r > 0$, we write

$$B(x,r) = \bigl\{y \in H\colon d(x,y) < r\bigr\}$$

and call $B(x,r)$ the ball with center x and radius r. If $B = B(x,r)$ is a ball and c is a positive constant, we use cB to denote $B(x, cr)$. If B is a ball, we use $r(B)$ and x_B to denote the radius and center of B.

Definition 1.5. Let $\langle H, d\rangle$ be a quasimetric space. Fix $\Omega \subset H$, and for $x \in H$, set

$$d(x) = \operatorname{dist}\bigl(x, \Omega^{c}\bigr) = \inf_{y\in\Omega^{c}} d(x, y).$$

Let ϕ be a strictly increasing function on $[0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. We say that Ω is a ϕ-John domain with central point (or 'center') $x' \in \Omega$ if for all $x \in \Omega$ with $x \neq x'$, there is a curve $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$, $\gamma(l) = x'$,

$$d\bigl(\gamma(b), \gamma(a)\bigr) \leqslant b - a \quad \text{for all } [a, b] \subset [0, l], \quad \text{and} \tag{1.9}$$

$$d\bigl(\gamma(t)\bigr) > \phi(t) \quad \text{for all } t \in [0, l]. \tag{1.10}$$

If Ω is a ϕ-John domain for the function $\phi = \phi_s$ defined by $\phi_s(t) = c_s t^s$ for $t \leqslant 1$ and $\phi_s(t) = c_s t$ for $t > 1$, with $s \geqslant 1$, we say Ω is an s-John domain. We may assume that $0 < c_s < 1$. This definition is essentially the same as those in [25] and [14], where the authors instead assume that $\phi_s(t) = c_0 t^s$ for some $c_0 > 0$ and all $t \geqslant 0$. For any $M > 1$, we will write $\mathcal{J}_M(t) = t/M$. As M varies, the class of $\mathcal{J}_M$-John domains is the same as the class of 1-John domains. If Ω is a $\mathcal{J}_M$-John domain for some M, then we will refer to M as the 1-John constant of Ω.

Note that (1.10) implies that $d(x) > 0$ for all $x \in \Omega$. Another useful inequality is

$$d\bigl(\gamma(t)\bigr) > \phi\bigl(d\bigl(\gamma(t), \gamma(0)\bigr)\bigr) \quad \text{for all } t; \tag{1.11}$$

in fact $d(\gamma(t), \gamma(0)) \leqslant t$ by (1.9), and then $\phi(d(\gamma(t), \gamma(0))) \leqslant \phi(t) < d(\gamma(t))$ by (1.10).

Definition 1.6. Let $\langle H, d\rangle$ be a quasimetric space. Given $\Omega \subset H$ and $\delta > 0$, we say that a ball $B(x,r)$ is a δ-ball if $x \in \Omega$ and $0 < r \leqslant \delta d(x)$. Balls of the form $B(x,r)$ with $x \in \Omega$ and $r = \delta d(x)$ will be called δ-Whitney balls.

Some useful properties of δ-balls are listed in Observation 2.1 in the next section. See also [24], where such balls play a role in proving regularity of solutions of subelliptic equations.

For technical reasons (see, e.g., the proof of Observation 2.1), whenever we consider δ-balls, we will always assume that $0 < \delta < 1/(2\kappa^2)$ where κ is the quasimetric constant in Definition 1.4. We note now that the weaker restriction $0 < \delta < 1/\kappa$ guarantees that every δ-ball is contained in Ω. In fact, let $x \in \Omega$ and $B(x,r)$ be a δ-ball with $\kappa\delta < 1$. If $y \in B(x,r)$, then

$$d(x) \leqslant \kappa\bigl[d(x, y) + d(y)\bigr] < \kappa\bigl[r + d(y)\bigr] \leqslant \kappa\bigl[\delta d(x) + d(y)\bigr].$$

Hence, $d(y) > [(1/\kappa) - \delta]d(x)$. In particular, $d(y) > 0$ and therefore $y \in \Omega$.

We next define what we mean by δ-doubling and doubling.

Definition 1.7. Let $\langle H, d\rangle$ be a quasimetric space. A nonnegative finite functional σ defined on balls in H, i.e., $\sigma : \{B\colon\ B \text{ is a ball in } H\} \to [0,\infty)$, will be called a ball set function (or a set function on balls). In practice, given $\Omega \subset H$, we will only consider balls B with $x_B \in \Omega$ and

$r(B) \leqslant \operatorname{diam}(\Omega)$, where $\operatorname{diam}(\Omega)$ is defined using the quasimetric d. Given $\Omega \subset H$, $0 < \delta < 1/(2\kappa^2)$, and a ball set function σ, we say that σ is δ-doubling on Ω if there is a positive constant D_σ such that for all δ-balls B in Ω,

$$\sigma(2^k B) \leqslant (D_\sigma)^k \sigma(B) \quad \text{for all } k \in \mathbb{N}.$$

If this inequality holds for all balls with center in Ω and $r(B) \leqslant \operatorname{diam}(\Omega)$, we say that σ is doubling on Ω. If σ is also a measure on Ω, we say that σ is a δ-doubling measure or doubling measure on Ω, respectively. Furthermore, in case σ is a ball set function or measure and there is a constant C such that $\sigma(2B) \leqslant C\sigma(B)$ for all balls $B \subset H$, we say simply that σ is doubling instead of doubling on H.

Some properties of δ-doubling ball set functions are given in Proposition 2.2.

We say that a collection of balls (or cubes in the usual Euclidean case) has *bounded intercepts* if there exists a constant N such that each ball in the collection intersects at most N other balls in the collection. Such a collection also has bounded overlaps in the pointwise sense since no point belongs to more than $N+1$ balls in the collection.

For a ϕ-John domain Ω in $\langle H, d\rangle$, we will derive in Proposition 2.6 useful properties of chains of δ-balls associated with ϕ-John curves that connect points of Ω to a central point x'. In order to state our result about ϕ-John domains, we need to describe some of these properties now. Given $\delta < 1/(2\kappa^2)$ and $1 \leqslant \tau < 1/(2\delta\kappa^2)$, we first associate with each $x \in \Omega$ the sequence of balls $\{B(x, 2^{N_x+1-j}\tau\delta d(x))\}_{j=1}^{\infty}$ where N_x is chosen such that

$$2^{N_x-1}\tau\delta d(x) < \operatorname{diam}(\Omega) \leqslant 2^{N_x}\tau\delta d(x).$$

Next, by Proposition 2.6(c), there is a sequence of δ-balls $\{Q_i^x\}_{i=1}^{\infty}$ with centers along the curve γ from x to x' guaranteed by the ϕ-John condition such that $Q_1^x = B(x', \delta d(x'))$ and $\{Q_i^x\}$ has the intersection property

$$Q_i^x \cap Q_{i+1}^x \text{ contains a } \delta\text{-ball } Q_i' \text{ with } Q_i^x \cup Q_{i+1}^x \subset N Q_i'$$

for some positive constant N independent of x and i. Moreover, for large i, Q_i^x is centered at x; in fact, there exists $K_x \in \mathbb{N}$ such that $Q_i^x = B(x, 2^{K_x-i}\delta d(x))$ for $i \geqslant K_x$. We associate with each ball $B = B(x,r) = B(x, 2^{N_x+1-j}\tau\delta d(x))$, $j \geqslant 1$, the special subcollection of $\{Q_i^x\}$ defined by

$$\begin{aligned} &\mathcal{C}(B) = \mathcal{C}_\phi(B) = \{Q_i^x\colon\ \tau Q_i^x \subset B(x,r) \text{ and } \tau Q_i^x \not\subset B(x, r/2)\} \quad \text{when } 1 \leqslant j \leqslant N_x; \\ &\mathcal{C}(B) = \tau^{-1}B, \quad \text{otherwise, i.e., when } r \leqslant \tau\delta d(x). \end{aligned} \tag{1.12}$$

In case Ω satisfies the nonempty annuli property (see Remark 1.9(2)), then the second case above can be included with the first case by dropping the restriction $j \leqslant N_x$, i.e., $\tau^{-1}B$ is the only Q_i^x such that $\tau Q_i^x \subset B(x,r)$ and $\tau Q_i^x \not\subset B(x,r/2)$ when $r \leqslant \tau\delta d(x)$. In general, Proposition 2.6 will imply that each $\mathcal{C}(B)$ has the bounded intercept property, i.e., the balls in $\mathcal{C}(B)$ have bounded intercepts with bound independent of B. Each ball Q_i^x in $\mathcal{C}(B)$ satisfies $\tau Q_i^x \subset B$ by definition and, as we shall see, also has the important property

$$r(Q_i^x) \geqslant \delta\phi(r(B)/(4\kappa)) \quad \text{if } Q_i^x \in \mathcal{C}(B).$$

The intersection property and the bounded intercept property above are somewhat opposite in nature; the first shows substantial overlap of consecutive balls while the latter limits the overlap of balls in a chain. Both properties are important for our self-improving method of deriving Poincaré estimates. The intersection property is useful for making connections from distant balls to the central ball, while the bounded intercept property helps in finding conditions on weights so that weighted Poincaré estimates hold. The bounded intercept property will allow us to partition the balls $\{Q_i^x\}$ in each $\mathcal{C}(B)$ into a finite number of subfamilies so that the balls in each subfamily are disjoint, and the number of such subfamiles can be taken to be at most the bounded intercept constant (and so independent of the chain); see the proof of Lemma 2.5 for details. The subfamilies of disjoint balls in a chain $\{Q_i^x\}_{i\in\mathbb{N}}$ have an extra property which is useful for verifying weight conditions, namely, for any $\varepsilon > 0$, the number of disjoint balls in $\{Q_i^x\}$ with radius between ε and 2ε is at most $(2/\varepsilon)\phi^{-1}(2\varepsilon/\delta)$.

We now state a result for ϕ-John domains.

Theorem 1.8.

(a) *Let ϕ be a strictly increasing function on $[0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. Let Ω be a ϕ-John domain with central point x' in a quasimetric space $\langle H, d\rangle$, and let $0 < \delta < 1/(2\kappa^2)$, $1 \leqslant \tau < 1/(2\delta\kappa^2)$, $0 < p_0 < \infty$, $0 < \theta < 1$ and $0 < q < \infty$. Let σ and μ be measures on H and σ be δ-doubling on Ω. Suppose f is a fixed function on Ω and $a(B)$ is a ball set function such that*

$$\left(\frac{1}{\sigma(B)}\int_B |f - f_B|^{p_0}\,d\sigma\right)^{\frac{1}{p_0}} \leqslant a(B) \tag{1.13}$$

for each δ-ball B in Ω and some constant f_B, and suppose $f_{B(x,r)} \to f(x)$ as $r \to 0$ for μ-almost all $x \in \Omega$. Let μ^* be a ball set function such that $\mu(B) \leqslant \mu^*(B)$ for all balls B and there are constants $A_1, A_2, D_1, D_2 > 0$ for which

$$A_1\left(\frac{r(B)}{r(\tilde{B})}\right)^{D_1} \leqslant \frac{\mu^*(B)}{\mu^*(\tilde{B})} \leqslant A_2\left(\frac{r(B)}{r(\tilde{B})}\right)^{D_2} \tag{1.14}$$

for all concentric balls $B \subset \tilde{B}$ centered in Ω with $r(\tilde{B}) < 2\,\mathrm{diam}(\Omega)$. Moreover, assume that

$$\sum_{B_j\in I} A(B_j)^{\theta q}\mu^*(B_j)^{\theta} = \sum_{B_j\in I}\left[\sum_{Q\in\mathcal{C}_\phi(B_j)} a(Q)\right]^{\theta q}\mu^*(B_j)^{\theta} \leqslant C_0^{\theta q}\mu(\Omega)^{\theta} \tag{1.15}$$

for all pairwise disjoint collections $I = \{B_j\}$ of balls $B_j = B(x_j, 2^{N_{x_j}+1-i_j}\tau\delta d(x_j))$ with $x_j \in \Omega$ and $r(B_j) < 2\,\mathrm{diam}(\Omega)$. Then

$$\sup_{t>0} t\mu\bigl\{x \in \Omega\colon \bigl|f(x) - f_{B'}\bigr| > t\bigr\}^{\frac{1}{q}} \leqslant CC_0\left(\frac{\mu(\Omega)}{\mu(B')}\right)^{\frac{1}{q}}\mu(\Omega)^{\frac{1}{q}}, \tag{1.16}$$

where $B' = B(x', \delta d(x'))$. Here C depends on $\kappa, \tau, D_\sigma, p_0, \theta, q$ and the constants in (1.14) but is independent of f.

(b) *Suppose Ω is a* 1*-John domain,* $1 \leqslant \tau < 1/(2\delta\kappa^2)$, (1.13) *holds with $\sigma = \mu$ for some $p_0 = q \geqslant 1$, and*

$$\sum_{\tau B_j \in I} a(B_j)^q \mu(B_j) \leqslant \left(C_0'\right)^q \mu(\Omega) \tag{1.17}$$

for all collections I of disjoint $\tau\delta$-balls in Ω. Then the strong type inequality

$$\|f - f_{B'}\|_{L^q_\mu(\Omega)} \leqslant C C_0' \mu(\Omega)^{\frac{1}{q}} \tag{1.18}$$

holds with C depending on $D_\mu, \tau, \kappa, \delta, q$ and the 1*-John constant, but not on f. The existence of μ^* is not needed for this part of the theorem and* (1.14) *is not required.*

Remark 1.9. 1. In Theorem 1.8, the choice of $\mathcal{C}(B)$ does not have to be $\mathcal{C}_\phi(B)$; any way of partitioning the chains of balls $\{Q_i^x\}_{i\in\mathbb{N}}$ such that (1.15) holds is sufficient.

2. If $B(x,r) \setminus B(x,r') \neq \emptyset$ for all $0 < r' < r$, $x \in H$, we say the quasimetric satisfies the *nonempty annuli property* in H. Similarly, we say that a set $\Omega \subset H$ has the nonempty annuli property if $(\Omega \cap B(x,r)) \setminus B(x,r') \neq \emptyset$ for all $0 < r' < r$ and $x \in \Omega$ for which Ω is not a subset of $B(x,r')$. A doubling measure on Ω satisfies a reverse condition of the same type provided Ω has the nonempty annuli property; see Proposition 2.3. Thus, at least in case μ^* is a doubling measure, the first inequality in (1.14) implies the second one.

We now turn to the question of obtaining sharper strong type estimates than in (1.8), beginning with a strong type analogue of Theorem 1.2.

Given $\omega > 0$ and a nonnegative function g, the truncation $\tau_\omega g$ is defined by

$$\tau_\omega g(x) = \min\{g(x), 2\omega\} - \min\{g(x), \omega\} = \begin{cases} \omega & \text{if } g(x) \geqslant 2\omega, \\ g(x) - \omega & \text{if } \omega \leqslant g(x) < 2\omega, \\ 0 & \text{if } g(x) < \omega. \end{cases}$$

Let f be a fixed measurable function on Ω and B' be a fixed measurable set in Ω. Set $f_{B',\sigma} = \int_{B'} f\,d\sigma/\sigma(B')$. For each function $\tau_\omega|f - f_{B',\sigma}|$, $\omega > 0$, and each $x \in \Omega$, we assume the existence of sequences $\{B_i^x\}$, $\{Q_i^x\}$ and $\{a(Q_i^x)\}$ with properties as in Theorem 1.2, but as there, these sequences as well as $\mathfrak{F}$ and the collections $\mathcal{C}(B)$ may depend on $\tau_\omega|f - f_{B',\sigma}|$. For easy reference, we will denote $\tilde{f} = |f - f_{B',\sigma}|$ and write $b(Q_i^x, \tau_\omega\tilde{f})$ instead of $a(Q_i^x)$ and $\mathfrak{F}(\tau_\omega\tilde{f})$ instead of $\mathfrak{F}$, but we do not adopt new notation to indicate that $\{B_i^x\}$ and $\{Q_i^x\}$ may vary with ω. In case Q is a ball, a typical example of $b(Q,g)$ is

$$b(Q,g) = b_Y(Q,g) = r(Q)^\beta \left(\frac{1}{w(Q)} \int_Q |Yf|^p\,dw\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

where Y is a vector field (see (1.23) below).

Given f and setting $\tilde{f} = |f - f_{B',\sigma}|$, the analogue of (1.4) that we now assume is

$$\frac{1}{\sigma(Q_i^x)^{1/p_0}}\big\|\tau_\omega\tilde f-(\tau_\omega\tilde f)_{Q_i^x,\sigma}\big\|_{L^{p_0}_\sigma(Q_i^x)}\leqslant b\big(Q_i^x,\tau_\omega\tilde f\big),\quad (\tau_\omega\tilde f)_{Q_i^x,\sigma}=\frac{1}{\sigma(Q_i^x)}\int_{Q_i^x}\tau_\omega\tilde f\,d\sigma, \tag{1.19}$$

for all $\omega>0$.

We also assume an analogue of (1.6): for some constants $q>0$ and $0<\theta<1$,

$$\begin{aligned}\sum_{B_\alpha\in I}\big(A(B_\alpha,\tau_\omega\tilde f)^q\mu^*(B_\alpha)\big)^\theta&=\sum_{B_\alpha\in I}\bigg(\bigg(\sum_{Q\in\mathcal{C}(B_\alpha)}b(Q,\tau_\omega\tilde f)\bigg)^q\mu^*(B_\alpha)\bigg)^\theta\\&\leqslant\big(h(\Omega,\tau_\omega\tilde f)^q\mu(\Omega)\big)^\theta\end{aligned}\tag{1.20}$$

for every disjoint subcollection I of $\mathfrak{F}(\tau_\omega\tilde f)$ and all $\omega>0$. Here $h(\Omega,\cdot)$ is a constant which is assumed to satisfy

$$h^*(\Omega,f)^q:=\sup_{\omega>0}\sum_{k=1}^{\infty}h(\Omega,\tau_{2^k\omega}\tilde f)^q<\infty. \tag{1.21}$$

The following theorem is a simple abstract extension of both [13, Corollary 3] and [12, Theorem 3.1].

Theorem 1.10. *Let σ and μ be measures on a σ-algebra of subsets of X, Ω be a measurable set, and f be a fixed measurable function. Suppose for each $\tau_\omega|f-f_{B',\sigma}|$, $\omega>0$, there are sets $\{Q_i^x\}$ and $\{B_i^x\}$ (possibly depending on ω and f in addition to x, but with $Q_1^x=B'$ for all x) satisfying the conditions of Theorem* 1.2, *but now assuming* (1.19) *instead of* (1.4), *and* (1.20) *for all $\omega>0$ instead of* (1.6). *If* (1.21) *holds, then the strong type Poincaré inequality*

$$\frac{1}{\mu(\Omega)}\|f-f_{B',\sigma}\|^q_{L^q_\mu(\Omega)}\leqslant C\wp V_\mu h^*(\Omega,f)^q+\bigg(\frac{8}{\sigma(B')}\|f-f_{B',\sigma}\|_{L^1_\sigma(B')}\bigg)^q \tag{1.22}$$

holds with C as in Theorem 1.2, *i.e., C depends only on $p_0,q,\theta,A_1,A_2,\theta_1,\theta_2$ and C_σ.*

Remark 1.11. 1. In [12] and [13], the function f in the hypothesis of Theorem 1.10 is not fixed but allowed to vary over a collection $\mathcal{F}$ of functions in $L^1_{\text{loc}}(\Omega)$ that is large enough to include truncations. The class $\mathcal{F}$ is assumed to satisfy

- $f\in\mathcal{F}\Longrightarrow f+c\in\mathcal{F}$ for $c\in\mathbb{R}$,
- $f\in\mathcal{F}\Longrightarrow|f|\in\mathcal{F}$,
- $f\in\mathcal{F}\Longrightarrow\tau_\omega(|f|)\in\mathcal{F}$ for all $\omega>0$.

In applications, the main examples of $\mathcal{F}$ are the Lipschitz class or Sobolev classes, although our results are not restricted to these spaces. As already mentioned, typical examples of b are functionals associated with the right-hand sides of Poincaré inequalities, namely, assuming Q is a ball,

$$b(Q,f)=b_Y(Q,f)=r(Q)^{\beta}\left(\frac{1}{w(Q)}\int\limits_Q |Yf|^p\,dw\right)^{\frac{1}{p}},\quad 1\leqslant p<\infty, \tag{1.23}$$

where Y is a differential operator with $Y1=0$, i.e., with no zero order term. In particular, in Euclidean space with the usual Euclidean metric, Y could be ∇^m or some other combination of partial derivatives, and then in case all the derivatives are of first order and the measure w is absolutely continuous with respect to Lebesgue measure, $\mathcal{F}$ can be chosen to be the Lipschitz class since such functions are differentiable almost everywhere by the Rademacher–Stepanov theorem. Conditions (1.20) and (1.21) are similar to stability properties of the functional b_Y under truncation that were introduced in [19,21] and exploited in many papers such as [11,12] and [13].

For a general functional b, if the estimate

$$\frac{1}{\sigma(B)^{1/p_0}}\|f-f_{B,\sigma}\|_{L^{p_0}_\sigma(B)}\leqslant b(B,f) \tag{1.24}$$

holds for a function f, then the estimate remains true if f is replaced on the left-hand side by $f+c$ for any constant c. Thus, we can assume that b is translation invariant, i.e., $b(B,f+c)=b(B,f)$, even if b does not arise from a differentiation operator.

2. If $p_0\geqslant 1$ in Theorem 1.10, then by applying Hölder's inequality and (1.24), we obtain

$$\frac{1}{\sigma(B')}\|f-f_{B',\sigma}\|_{L^1_\sigma(B')}\leqslant b(B',f),$$

and then the conclusion of Theorem 1.10 yields

$$\frac{1}{\mu(\Omega)^{1/q}}\|f-f_{B',\sigma}\|_{L^q_\mu(\Omega)}\leqslant C\big\{(\wp V_\mu)^{1/q}h^*(\Omega,f)+b(B',f)\big\}.$$

One consequence of Theorem 1.10 is the following result which contains Theorem 1.1 as a special case. Theorem 1.1 is included in the case $\Omega_0=\Omega^c$. We do not require the condition $b\geqslant 1-n$ and we consider more general types of distance weights than those in Theorem 1.1. Moreover, we include the case $p=q=1$. However, since the details of its proof are quite technical, we will only discuss its proof in [8].

In order to state the result, we need more terminology. Given an s-John domain with central point x' and a number $M>1$, we distinguish two types of points x depending on whether or not x can be connected to x' by a curve satisfying the $\mathcal{J}_M$-John condition:

Definition 1.12. Let $M>1$ and Ω be an s-John domain with central point x'. Let Ω_g^M be the set of points x in Ω such that there is $\gamma_x:[0,l_x]\to\Omega$ with $\gamma_x(0)=x$, $\gamma_x(l_x)=x'$, $d(\gamma_x(t_1),\gamma_x(t_2))\leqslant|t_1-t_2|$ for $t_1,t_2\in[0,l_x]$, and

$$d\big(\gamma_x(t)\big)>\mathcal{J}_M(t)\quad\text{for all } t\in[0,l_x].$$

We will say points in Ω_g^M are M-good points of Ω and points in $\Omega \setminus \Omega_g^M = \Omega_\flat^M$ are M-bad points of Ω. Note that if $\Omega_g^M = \Omega$, then Ω is a 1-John domain.

For further discussion concerning M-good and bad points, see [8].

Theorem 1.13. *(See [8, Theorem 1.12].) Suppose that $s \geqslant 1$ and $\Omega \subset \mathbb{R}^n$ is an s-John domain with respect to ordinary Euclidean distance d_E. Let $0 < \delta < 1/2$ and $B' = B(x', \delta d_E(x'))$ be the δ-Whitney ball centered at the central point x' of Ω. Suppose $\varepsilon > 0$, $M > 1$ and Ω_0 satisfies*

$$\partial\Omega \cap \left(\bigcup_{x \in \Omega_\flat^M} B(x, \varepsilon) \right) \subset \Omega_0 \subset \Omega^{\mathrm{c}}, \tag{1.25}$$

and set $\rho(x) = d_E(x, \Omega_0)$ and $\rho(\Omega) = \sup_{x\in\Omega} \rho(x)$. Let $a \geqslant 0$, $b \in \mathbb{R}$, $1 \leqslant p, q < \infty$ satisfy $1/q \geqslant 1/p - 1/n$ and

$$\frac{s(n+b-1) - p + 1}{(n+a)p} \leqslant \min\left\{\frac{1}{q}, \frac{1}{p}\right\}$$
$$\text{with strict inequality if } (p,q) \text{ satisfies } p > 1 \text{ and } q \leqslant p. \tag{1.26}$$

Then there is a constant C depending on $n, p, q, a, b, \operatorname{diam}(\Omega), \rho(\Omega), M, \varepsilon, \delta, c_s$ and s such that

$$\left\| f - C(\Omega, f)\right\|_{L^q_{\rho^a\,dx}(\Omega)} \leqslant C \|\nabla f\|_{L^p_{\rho^b\,dx}(\Omega)} \tag{1.27}$$

for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, i.e., for all locally Lipschitz continuous functions f on Ω, where $C(\Omega, f)$ can be chosen to be either

$$\frac{1}{|B'|}\int_{B'} f\,dx \quad \text{or} \quad f_{\mathcal{D},\rho^a\,dx} = \frac{1}{|\mathcal{D}|_{\rho^a\,dx}}\int_{\mathcal{D}} f\rho^a\,dx$$

for any $\mathcal{D} \subset \Omega$ with $|\mathcal{D}|_{\rho^a\,dx} > 0$. In case $C(\Omega, f) = f_{\mathcal{D},\rho^a\,dx}$, the constant C also depends on the ratio $|\Omega|_{\rho^a\,dx}/|\mathcal{D}|_{\rho^a\,dx}$.

Remark 1.14. 1. Note that $f_{\mathcal{D},\rho^a\,dx}$ is well defined whenever

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \|\nabla f\|_{L^p_{\rho^b\,dx}(\Omega)} < \infty.$$

This follows as usual by applying (1.27) with $C(\Omega, f)$ chosen to be $|B'|^{-1}\int_{B'} f\,dx$.

2. When $p > 1$, (1.26) with strict inequality implies that there exists $q_0 > p$ such that

$$\frac{1}{q_0} \geqslant \frac{s(n+b-1) - p + 1}{(n+a)p}.$$

Note that we can have $p = q = 1$ in (1.27) provided $n + a \geqslant s(n+b-1)$. The case $p = q = 1$ is also considered in [14] except that $b \geqslant 1 - n$ is assumed there.

3. Condition (1.25) involving $\Omega_\flat^M$ clearly holds when $\Omega_0 = \partial\Omega$.

4. As mentioned earlier, the q range in Theorem 1.13 can be enlarged for special s-John domains. Some results of this type are given in [8].

2. Preliminaries

In general, we will not attempt to give very precise values for constants which arise in the proofs, although we will keep track of important parameters on which constants depend. We will consistently use the notation

$$\lambda = \kappa + 2\kappa^2$$

in our computations. The constant λ arises naturally in Observation 2.1 and Proposition 2.2. For simplicity, we often use λ in estimates in which better constants could be chosen.

We now list several useful geometric facts which require only that d be a quasimetric.

Observation 2.1.

(1) *If $z \in B(x,r)$, then*

$$B(z,r) \subset 2\kappa B(x,r) \subset \lambda B(z,r).$$

(2) *Let B_1 and B_2 be balls with $B_1 \cap B_2 \neq \emptyset$. Then*

(a) $B_2 \subset \lambda \max\{\frac{r(B_2)}{r(B_1)}, 1\} B_1$.

(b) *If in addition both B_1 and B_2 are δ-balls with $\delta < 1/(2\kappa^2)$, then*

$$\lambda^{-1} d(x_{B_2}) \leqslant d(x_{B_1}) \leqslant \lambda d(x_{B_2}).$$

Thus if B_1 and B_2 are intersecting δ-Whitney balls, then

$$\lambda^{-1} \leqslant \frac{r(B_2)}{r(B_1)} \leqslant \lambda \quad \textit{and} \quad \lambda^{-2} B_1 \subset B_2 \subset \lambda^2 B_1.$$

(c) *If $\delta < 1/(2\kappa^2)$ and z is in a δ-ball $B(x,r)$, then*

$$\frac{1}{2\kappa} \leqslant \frac{d(x)}{d(z)} \leqslant 2\kappa.$$

Proof. Let $z \in B(x,r)$. Then for any $y \in B(z,r)$,

$$d(y,x) \leqslant \kappa\big[d(y,z) + d(z,x)\big] < 2\kappa r,$$

and so $B(z,r) \subset 2\kappa B(x,r)$. On the other hand, if $y \in 2\kappa B(x,r)$ then

$$d(y,z) \leqslant \kappa\big[d(y,x) + d(x,z)\big] < \kappa(2\kappa r + r) = \lambda r.$$

This proves (1).

Next, for (2), let $z \in B_1 \cap B_2$. If $u \in B_2$,

$$\begin{aligned} d(u,x_{B_1}) &\leqslant \kappa\big(d(z,x_{B_1})+d(u,z)\big)\\ &\leqslant \kappa\big(d(z,x_{B_1})+\kappa\big[d(u,x_{B_2})+d(z,x_{B_2})\big]\big)\\ &< \lambda \max\big\{r(B_1),r(B_2)\big\} = \lambda \max\left\{\frac{r(B_2)}{r(B_1)},1\right\} r(B_1) \end{aligned}$$

and 2(a) is now clear. If in addition B_1 and B_2 are both δ-balls and $z \in B_1 \cap B_2$, then

$$\begin{aligned} d(x_{B_1}) &\leqslant \kappa\big(d(x_{B_2})+d(x_{B_1},x_{B_2})\big) \leqslant \kappa\big(d(x_{B_2})+\kappa\big[d(x_{B_1},z)+d(x_{B_2},z)\big]\big)\\ &< \kappa\big(d(x_{B_2})+\kappa\big[r(B_1)+r(B_2)\big]\big). \end{aligned}$$

Since $r(B_1)+r(B_2) \leqslant \delta(d(x_{B_1})+d(x_{B_2}))$, a simple computation based on our assumption that $\delta\kappa^2 < 1/2$ and the definition of λ then gives

$$d(x_{B_1}) < \frac{\kappa+\kappa^2\delta}{1-\kappa^2\delta} d(x_{B_2}) < \lambda d(x_{B_2}).$$

By interchanging the roles of B_1 and B_2, we also have $d(x_{B_2}) < \lambda d(x_{B_1})$, which proves the first part of 2(b). The remaining part of 2(b) follows by combining the fact that $\lambda^{-1} \leqslant r(B_1)/r(B_2) \leqslant \lambda$ with $B_2 \subset \lambda \max\{r(B_2)/r(B_1), 1\} B_1$.

Finally, part (c) can be proved by using the quasi-triangle property and the fact that $\delta < 1/(2\kappa^2) < 1$. □

We now list three facts about δ-doubling set functions on balls.

Proposition 2.2.

(1) *If* $0 < \delta_1, \delta_2 < 1/\kappa$ *and* σ *is* δ_1*-doubling on* Ω*, then* σ *is also* δ_2*-doubling on* Ω*.*
(2) *Let* σ *be a measure on* Ω*. If* σ *is* δ*-doubling on* Ω *and* $\sigma|_\Omega$ *is defined by* $\sigma|_\Omega(B) = \sigma(B \cap \Omega)$ *for balls* $B \subset H$*, then* $\sigma|_\Omega$ *is also* δ*-doubling since* $\sigma|_\Omega$ *and* σ *are the same on* δ*-balls.*
(3) *If* Ω *is a* 1*-John domain, then the notions of* δ*-doubling on* Ω *and doubling on* Ω *are equivalent.*

Parts (1) and (2) are just easy observations, while part (3) holds since any ball with center in a 1-John domain Ω and with radius less than $\operatorname{diam}(\Omega)$ must contain a δ-ball of comparable radius; a detailed proof can be found in [8, Proposition 2.2].

We next state a property of δ-doubling measures which extends a fact from [26, p. 269] for doubling measures. The proofs are similar.

Proposition 2.3. *Let* Ω *be a domain in a quasimetric space and suppose that* Ω *satisfies the nonempty annuli property. If* μ *is a* δ*-doubling measure on* Ω*,* $0 < \delta < 1/(2\kappa^2)$*, then there exist positive constants* A_1, A_2, D_1, D_2 *depending only on* δ *and the doubling constant of* μ *such that*

$$A_1\left(\frac{r(B)}{r(\tilde{B})}\right)^{D_1} \leqslant \frac{\mu(B)}{\mu(\tilde{B})} \leqslant A_2\left(\frac{r(B)}{r(\tilde{B})}\right)^{D_2} \tag{2.1}$$

for all concentric balls $B, \tilde{B}$ having center in Ω with $B \subset \tilde{B}$ such that B is a δ-ball and $r(\tilde{B}) \leqslant$ $\operatorname{diam}(\Omega)$. *Moreover, if μ is doubling on Ω, then* (2.1) *holds for all concentric balls $B, \tilde{B}$ having center in Ω with $B \subset \tilde{B}$ and $r(\tilde{B}) \leqslant \operatorname{diam}(\Omega)$.*

The next proposition gives a simple extension of facts from [23, p. 843] concerning the existence of a collection of balls which furnish a crude notion of dyadic cubes in a quasimetric space $\langle H, d\rangle$.

Proposition 2.4. *Let $\lambda = \kappa + 2\kappa^2$ where κ is the quasimetric constant. For each $k \in \mathbb{Z}$, there is a collection $\mathcal{C}_k = \{B_i^k = B(x_i^k, \lambda^k)\}_i$ of balls in H such that*:

(1) *$H = \bigcup_i B_i^k$ for each k, and every ball in H of radius λ^{k-1} is inside at least one B_i^k.*
(2) *The balls $\hat{B}_i^k = B(x_i^k, \lambda^{k-1})$ are disjoint in i for each k, i.e., for every k, $\hat{B}_j^k \cap \hat{B}_i^k = \emptyset$ if $i \neq j$.*
(3) *If $\Omega \subset H$, $0 < \delta < 1/(2\kappa^2)$ and there is a δ-doubling measure on Ω, then for each k, the subcollection of $\{B_i^k\}_i$ consisting of δ-balls (in Ω) has bounded intercepts with bound depending only on κ and the doubling constant.*

Proof. For parts (1) and (2), see [23, pp. 843, 844]. For a doubling measure on H, part (3) is proved in [23, p. 844] by using a standard volume argument, and a similar argument works for a δ-doubling measure. □

Next we will prove a lemma about the bounded intercept property.

Lemma 2.5. *Let $\langle H, d\rangle$ be a quasimetric space, $0 < \delta < 1/(2\kappa^2)$ and $M, N \geqslant 1$. Suppose $\Omega \subset H$ and there is a δ-doubling measure μ on Ω with doubling constant D_μ. If $\mathcal{F} = \{B_i\}$ is a family of δ-balls in Ω with bounded intercepts such that $M^{-1} \leqslant r(B_i)/r(B_j) \leqslant M$ for all $B_i, B_j \in \mathcal{F}$ which satisfy $NB_i \cap NB_j \neq \emptyset$, then the family $N\mathcal{F} = \{NB_i\}_{B_i \in \mathcal{F}}$ also has bounded intercepts with bound $C(M, N, \kappa, D_\mu)$ times the bound for $\mathcal{F}$.*

Proof. We first show the following general fact about the bounded intercept property: if $\{S_i\}$ is a collection of sets with intercept constant K, then $\{S_i\}$ can be partitioned into at most K disjoint subfamilies such that the sets in each subfamily are disjoint. In fact, suppose that there were more than K disjoint families of disjoint sets S_i such that each family is maximal, i.e., any S_i outside a given family must intersect at least one set in that family. Consider any S_i not in one of the first K such families. Then S_i must intersect at least one set in each of the first K families (as well as intersecting itself), which contradicts the fact that the intercept constant is K.

We can now prove the lemma. Fix a ball $B_0 \in \mathcal{F}$. First note that if $B_i \in \mathcal{F}$ and NB_i intersects NB_0, then $M^{-1} \leqslant r(B_i)/r(B_0) \leqslant M$, and so $B_i \subset C_1 B_0 = C(M, N, \kappa) B_0$. By a standard volume argument, there can be at most $C_2 = C(M, N, \kappa, D_\mu)$ disjoint balls B_i in $C_1 B_0$ such that $M^{-1} \leqslant r(B_i)/r(B_0) \leqslant M$. With K as above, since $\mathcal{F}$ can be partitioned into K subfamilies of disjoint balls B_i, it follows that there are at most $C_2 K$ balls B_i from $\mathcal{F}$ which satisfy $NB_i \cap NB_0 \neq \emptyset$. Thus, the family $\{NB_i\}_{B_i \in \mathcal{F}}$ has bounded intercepts with intercept constant $C_2 K$, completing the proof. □

We next use Proposition 2.4 to find a decomposition of Whitney type of a ϕ-John domain in a quasimetric space, and to construct the chains of Whitney balls mentioned in the introduction.

In the case of 1-John domains, the properties in parts (a), (b) and (c) below include the features that define Boman domains, so the next proposition can be thought of as an analogue of these properties for general ϕ-John domains. Parts (d) and (e) below are not needed in this paper but are included since they play an important role in the proofs of results in [8]. Adaptations of Whitney-type lemmas to various geometric situations already exist in the literature: see, e.g., [9, Theorem 1.3, p. 70] for general spaces of homogeneous type, and see [10, Theorem 5.4] for balls in metric spaces with the geodesic (or segment) property. The version below is especially adapted to ϕ-John domains.

Proposition 2.6. *Let $\langle H, d\rangle$ be a quasimetric space and $0 < \delta < 1/(2\kappa^2)$. Suppose $\Omega \subset H$, there is a δ-doubling measure μ on Ω with doubling constant D_μ, and $d(x) = d(x, \Omega^c) > 0$ for all $x \in \Omega$. Then there exists a covering $W = \{B_i\}$ of Ω by δ-balls B_i such that:*

(a) *$r(B_i) \leqslant \delta d(x_{B_i}) \leqslant \lambda^2 r(B_i)$, where x_{B_i} is the center of B_i.*
(b) *For every $\tau \geqslant 1$ which satisfies $\tau\delta < 1/(2\kappa^2)$, there is a constant K depending only on τ, κ and D_μ so that the balls $\{\tau B_i\colon B_i \in W\}$ have bounded intercepts with bound K; in particular, the balls $\{\tau B_i\colon B_i \in W\}$ also have pointwise bounded overlaps with overlap constant K.*
(c) *Let $x' \in \Omega$ and ϕ be a strictly increasing function on $[0,\infty)$ which satisfies $\phi(0) = 0$ and $\phi(t) < t$ for all t. Then for each $x \in \Omega$ for which there is a curve $\gamma : [0,l] \to \Omega$ satisfying $\gamma(0) = x$, $\gamma(l) = x'$ and the ϕ-John properties* (1.9) *and* (1.10)*, there exists a finite chain of δ-balls $\{B_i\}_{i=0}^L \subset W$, depending on x and with $L = L_x$, such that $x \in B_0$, $x' \in B_L$, B_L is independent of x and satisfies $\lambda^{-2}B(x', \delta d(x')) \subset B_L \subset B(x', \delta d(x'))$, $B_i \cap B_{i+1}$ contains a δ-ball B_i' with $B_i \cup B_{i+1} \subset \lambda^4 B_i'$ for all i, and*

$$B_0 \subset \frac{\lambda^2\phi^{-1}(2\kappa\lambda^2 r(B_i)/\delta)}{r(B_i)} B_i \quad \text{for all } i. \tag{2.2}$$

Furthermore, there is a finite chain of δ-Whitney balls $\{Q_i\}_{i=0}^L$ depending on x with bounded intercepts and centers on γ such that $Q_0 = B(x, \delta d(x))$, $Q_L = B(x', \delta d(x'))$, $\frac{1}{\lambda^2} Q_i \subset B_i \subset Q_i$, and $Q_i \cap Q_{i+1}$ contains a δ-ball Q_i' with $Q_i \cup Q_{i+1} \subset \lambda^6 Q_i'$.
(d) *Let x and $\{Q_i\}$ be as in* (c)*. If $Q_i \not\subset B(x, r)$, then $r(Q_i) \geqslant \delta\phi(r/(2\kappa))$.*
(e) *Let x, γ and $\{Q_i\}$ be as in* (c)*. For all $\varepsilon > 0$, the number of disjoint Q_i having radius between ε and 2ε is at most $2\phi^{-1}(2\varepsilon/\delta)/\varepsilon$. In particular, if $\phi = \mathcal{J}_M$, the number of disjoint Q_i with radius between $\delta\varepsilon/(4\kappa^2 M)$ and $4\kappa\varepsilon$ is at most a constant depending only on δ, κ and M.*

Proof. For each $k \in \mathbb{Z}$, let $\mathcal{C}_k$ be a collection of balls of radius λ^k as in Proposition 2.4, and for fixed δ satisfying $0 < \delta < 1/(2\kappa^2)$, let $\{B_i^k = B(x_i^k, \lambda^k)\}_i$ be the subcollection of $\mathcal{C}_k$ with

$$2\kappa\lambda^k \leqslant \delta d\left(x_i^k\right) < 2\kappa\lambda^{k+2}, \tag{2.3}$$

or equivalently, with

$$2\kappa r\left(B_i^k\right) \leqslant \delta d\left(x_i^k\right) < 2\kappa\lambda^2 r\left(B_i^k\right).$$

Let

$$\tilde{W} = \{B_i^k\}_{i,k}$$

be the collection of all B_i^k as both i and k vary.

First note that if $B_i^k \in \tilde{W}$, then $2\kappa B_i^k \subset \Omega$ since $2\kappa B_i^k$ is a δ-ball. Next, let us show that $\tilde{W}$ covers Ω. Fix any $x \in \Omega$. Using the fact that $d(x) > 0$, choose $k \in \mathbb{Z}$ such that $\lambda^{k+1} \leqslant \delta d(x) < \lambda^{k+2}$. By Proposition 2.4, we may select a ball $\tilde{B} \in \mathcal{C}_k$ (so that $r(\tilde{B}) = \lambda^k$) with $B(x, \lambda^{k-1}) \subset \tilde{B}$. Let us show that $\tilde{B} \in \tilde{W}$. We have

$$d(x_{\tilde{B}}) \leqslant \kappa\big[d(x) + d(x, x_{\tilde{B}})\big] < \kappa\left[\frac{\lambda^{k+2}}{\delta} + \lambda^k\right],$$

and hence

$$\delta d(x_{\tilde{B}}) < \kappa\big[\lambda^{k+2} + \delta\lambda^k\big] \leqslant 2\kappa\lambda^{k+2}.$$

Moreover,

$$\kappa d(x_{\tilde{B}}) \geqslant d(x) - \kappa d(x, x_{\tilde{B}}) > d(x) - \kappa\lambda^k,$$

and thus

$$\delta d(x_{\tilde{B}}) > \frac{1}{\kappa}\lambda^{k+1} - \delta\lambda^k > 2\kappa\lambda^k.$$

Combining estimates shows that $\tilde{B} \in \tilde{W} \cap \mathcal{C}_k$ as desired. In particular, it follows that $\tilde{W}$ covers Ω. Since $x \in \tilde{B}$, we have by Observation 2.1(1) that

$$B\big(x, \lambda^k\big) \subset 2\kappa\tilde{B} \subset B\big(x, \lambda^{k+1}\big). \tag{2.4}$$

Recalling that $\lambda^{k+1} \leqslant \delta d(x) < \lambda^{k+2}$ and $B(x, \lambda^{k-1}) \subset \tilde{B}$, and setting $\delta' = \delta/\lambda^3$, we obtain

$$B\big(x, \delta' d(x)\big) \subset \tilde{B} \quad \text{and} \quad B\big(x, \lambda\delta' d(x)\big) \subset 2\kappa\tilde{B} \subset B\big(x, \delta d(x)\big). \tag{2.5}$$

Next, if $z \in B(x, \delta d(x))$, then

$$d(z, x_{\tilde{B}}) \leqslant \kappa\big(d(z, x) + d(x, x_{\tilde{B}})\big) < \kappa\big(\delta d(x) + \lambda^k\big) < 2\kappa\lambda^{k+2}.$$

In summary, given $x \in \Omega$ there exists $\tilde{B} \in \tilde{W}$ such that

$$B\big(x, \delta' d(x)\big) \subset \tilde{B} \quad \text{and} \quad B\big(x, \lambda\delta' d(x)\big) \subset 2\kappa\tilde{B} \subset B\big(x, \delta d(x)\big) \subset 2\kappa\lambda^2\tilde{B}, \quad \delta' = \frac{\delta}{\lambda^3}. \tag{2.6}$$

We now define

$$W = \{B_j = 2\kappa\tilde{B}_j\colon\ \tilde{B}_j \in \tilde{W}\}.$$

By (2.3) we have for any $B_j \in W$ that

$$r(B_j) \leqslant \delta d(x_{B_j}) < \lambda^2 r(B_j),$$

which implies B_j is a δ-ball and also proves (a).

Next, we will show (b). Let $1 \leqslant \tau < 1/(2\kappa^2\delta)$. If $\tau B_i \cap \tau B_j \neq \emptyset$ and $B_i, B_j \in W$, then since $\tau B_i, \tau B_j$ are $\tau\delta$-balls with $\tau\delta < 1/(2\kappa^2)$, part (2b) of Observation 2.1 (applied with δ there replaced by $\tau\delta$) gives

$$\lambda^{-1} \leqslant \frac{d(x_{B_i})}{d(x_{B_j})} \leqslant \lambda. \tag{2.7}$$

But by (2.3),

$$\delta d(x_{B_i})/\lambda^2 \leqslant r(B_i) \leqslant \delta d(x_{B_i}),$$

and similarly for B_j, and combining this with (2.7) we obtain

$$\lambda^{-3} \leqslant \frac{r(B_i)}{r(B_j)} \leqslant \lambda^3. \tag{2.8}$$

Now, to complete the proof of (b), fix B_{i_0} in W and partition those τB_j with $\tau B_{i_0} \cap \tau B_j \neq \emptyset$ into classes such that the balls in each class have equal radii. Since $r(B_j)$ has the form $2\kappa\lambda^{k(j)}$ for some $k(j)$, i.e., $r(B_j)$ is a fixed multiple of a power of a fixed number, (2.8) implies that the number of classes is at most 7. It is thus enough to show that for each such class $G = \{\tau B_j\}$, the collection $G \cup \{\tau B_{i_0}\}$ (i.e., the collection G with τB_{i_0} adjoined) has bounded intercepts with bound $C(\kappa, \tau, D_\mu)$. A typical class G has the form $\tau 2\kappa \tilde{\mathcal{F}}_k$ for some k, where

$$\tilde{\mathcal{F}}_k = \{\tilde{B}_j \in \tilde{W} \cap \mathcal{C}_k\colon\ B_j = 2\kappa \tilde{B}_j \text{ satisfies } \tau B_j \cap \tau B_{i_0} \neq \emptyset\}.$$

Let $\mathcal{F}_k$ be $\tilde{\mathcal{F}}_k$ with B_{i_0} adjoined: $\mathcal{F}_k = \tilde{\mathcal{F}}_k \cup \{B_{i_0}\}$. By Proposition 2.4(3), the balls in $\mathcal{C}_k$ have bounded intercepts uniformly in k, and so the same is true for the balls in $\mathcal{F}_k$. Applying Lemma 2.5 to $\mathcal{F}_k$ (recall by (2.8) that $r(B_{i_0}) \sim r(\tilde{B}_j)$ for all $\tilde{B}_j \in \tilde{\mathcal{F}}_k$, with constants uniform in k, j, i_0) and choosing $N = \tau 2\kappa$ there, we see that the balls in $N\mathcal{F}_k$ also have bounded intercepts with bound $C(\kappa, \tau, D_\mu)$, and we are done since every ball in $G \cup \{\tau B_{i_0}\}$ is contained in a ball in $N\mathcal{F}_k$.

We will now prove (c). Fix a point $x \in \Omega$, and let $\gamma(t), t \in [0, l]$, be a curve connecting x and x' as in (c), i.e., satisfying conditions guaranteed by the ϕ-John properties. With $\delta' = \delta/\lambda^3$, we begin by constructing a special sequence of δ'-Whitney balls centered along γ. For $t \in [0, l]$, let

$$\mathcal{R}_{\gamma(t)} = B\big(\gamma(t), \delta' d\big(\gamma(t)\big)\big).$$

Use (2.5) to pick $\tilde{B}_0 \in \tilde{W}$ containing $\mathcal{R}_{\gamma(0)}$, and let

$$t_1 = \sup\big\{t \in [0, l]\colon\ \gamma[0, t] \subset \tilde{B}_0\big\}.$$

Note that $t_1 > 0$ since $\gamma(t) \in \mathcal{R}_{\gamma(0)}$ for all $t < \delta' d(\gamma(0))$ due to the fact that $d(\gamma(0), \gamma(t)) \leqslant t$ by (1.9). A similar argument shows that $\mathcal{R}_{\gamma(t_1)}$ intersects $\tilde{B}_0$. Use (2.5) again to choose a ball $\tilde{B}_1 \in \tilde{W}$ containing $\mathcal{R}_{\gamma(t_1)}$. Then clearly $\tilde{B}_0$ intersects $\tilde{B}_1$. If $t_1 = l$, we stop the construction process. If $t_1 < l$, we define

$$t_2 = \sup\{t \in [t_1, l]\colon\ \gamma[t_1, t] \subset \tilde{B}_1\}$$

and choose $\tilde{B}_2 \in \tilde{W}$ containing $\mathcal{R}_{\gamma(t_2)}$. Again, $t_1 < t_2 \leqslant l$ and $\tilde{B}_1 \cap \tilde{B}_2 \neq \emptyset$. In general, if $0 = t_0 < t_1 < \cdots < t_k$ and $\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_k$ with $\tilde{B}_i \cap \tilde{B}_{i+1} \neq \emptyset$ have been constructed and if $t_k < l$, we continue by defining

$$t_{k+1} = \sup\{t \in [t_k, l]\colon\ \gamma[t_k, t] \subset \tilde{B}_k\}$$

and using (2.5) to pick $\tilde{B}_{k+1} \in \tilde{W}$ containing $\mathcal{R}_{\gamma(t_{k+1})}$. As usual, by (1.9), we have $t_k < t_{k+1} \leqslant l$ and $\tilde{B}_k \cap \tilde{B}_{k+1} \neq \emptyset$. We stop the construction if $t_{k+1} = l$.

Let us show that the process must end after a finite number of steps, i.e., that there is a positive integer $L = L_x$ such that $t_L = l$. To see this, it is enough to show that

$$t_{i+1} - t_i \geqslant \delta' \phi(t_1) \quad \text{if } i \geqslant 1 \text{ and } t_{i+1} < l$$

since the quantity $\delta' \phi(t_1)$ is a fixed positive number. Fix $i \geqslant 1$ such that $t_i < l$. For all $t \in [t_i, \min\{l, t_i + \delta' \phi(t_1)\}]$, (1.9) implies

$$d(\gamma(t), \gamma(t_i)) \leqslant t - t_i \leqslant \delta' \phi(t_1) < \delta' d(\gamma(t_i)) = r(\mathcal{R}_{\gamma(t_i)})$$

since the monotonicity of ϕ and (1.4) give $\phi(t_1) \leqslant \phi(t_i) < d(\gamma(t_i))$. It follows that $\gamma(t) \in \mathcal{R}_{\gamma(t_i)} \subset \tilde{B}_i$ for all such t, and consequently that $t_{i+1} \geqslant \min\{l, t_i + \delta' \phi(t_1)\}$. In particular, if $t_{i+1} < l$ then $t_{i+1} - t_i \geqslant \delta' \phi(t_1)$ as desired.

For each $\tilde{B}_i$ constructed above, let $B_i = 2\kappa \tilde{B}_i$. When $i = L$, we have $t_L = l$ by construction, and consequently

$$\mathcal{R}_{\gamma(t_L)} = B(\gamma(l), \delta' d(\gamma(l))) = B\left(x', \frac{\delta}{\lambda^3} d(x')\right)$$

is fixed independent of x. Thus $\tilde{B}_L$ is also independent of x, and so the same is true for B_L $(= 2\kappa \tilde{B}_L)$. By (2.5),

$$\lambda^{-2} B(x', \delta d(x')) \subset B_L \subset B(x', \delta d(x')).$$

For any i, since $\tilde{B}_i \cap \tilde{B}_{i+1} \neq \emptyset$, it follows from (2.8) that

$$\lambda^{-3} \leqslant \frac{r(B_i)}{r(B_{i+1})} \leqslant \lambda^3. \tag{2.9}$$

Let us show that $B_i \cap B_{i+1}$ contains a ball of radius $\min\{r(\tilde{B}_i), r(\tilde{B}_{i+1})\} := \lambda^k$. To this end, recall that for any $z \in \tilde{B}_i \cap \tilde{B}_{i+1}$, Observation 2.1(1) implies that

$$B(z, r(\tilde{B}_j)) \subset 2\kappa \tilde{B}_j \subset B(z, \lambda r(\tilde{B}_j)) \quad \text{for } j = i, i+1.$$

It is now clear by (2.9) that

$$B(z, \lambda^k) \subset B_i \cap B_{i+1} \quad \text{and} \quad B_i \cup B_{i+1} \subset B(z, \lambda^{k+4}).$$

Note that $B(z, \lambda^k)$ is a δ-ball since by Observation 2.1(2c) and (2.3),

$$\delta d(z) \geqslant \delta d(x_{B_i})/(2\kappa) \geqslant r(B_i) \geqslant \lambda^k.$$

Hence, except for (2.2), the first part of (c) is proved.

Let us now prove (2.2). Fix i and first suppose that $B_0 \cap B_i \neq \emptyset$. By (2.8), we have $r(B_0) \leqslant \lambda^3 r(B_i)$, and then $B_0 \subset \lambda^4 B_i$ by Observation 2.1(2a) again. Since $t < \phi^{-1}(t)$ for all t, it follows that

$$B_0 \subset \lambda^2 \frac{\phi^{-1}(\lambda^2 r(B_i))}{r(B_i)} B_i.$$

Next, suppose that $B_0 \cap B_i = \emptyset$. Due to the construction of B_i, there is a point $\xi \in \tilde{B}_i \cap \gamma[0, l]$. Since $\xi \notin B_0$ and $x \in \tilde{B}_0$, $\tilde{B}_0 = B_0/(2\kappa)$, the quasi-triangle inequality gives $d(\xi, x) \geqslant r(B_0)/(2\kappa)$. Similarly, since $x \notin B_i$ and $\xi \in \tilde{B}_i$, we have $d(\xi, x) \geqslant r(B_i)/(2\kappa)$. Hence,

$$d(\xi, x) \geqslant \frac{1}{2\kappa} \max\{r(B_0), r(B_i)\}.$$

We can use this to show that

$$B_0 \subset \frac{\lambda^2 d(\xi, x)}{r(B_i)} B_i.$$

In fact, if $z \in B_0$ then

$$\begin{aligned} d(z, x_{B_i}) &\leqslant \kappa\big[d(z, x) + d(x, x_{B_i})\big] \\ &\leqslant \kappa\big[\kappa\{d(z, x_{B_0}) + d(x, x_{B_0})\} + \kappa\{d(x_{B_i}, \xi) + d(\xi, x)\}\big] \\ &< \kappa\big[2\kappa r(B_0) + \kappa r(\tilde{B}_i) + \kappa d(\xi, x)\big], \end{aligned}$$

and thus by the previous estimate for the size of $d(\xi, x)$,

$$d(z, x_{B_i}) < (4\kappa^3 + 2\kappa^2) d(\xi, x) < \lambda^2 d(\xi, x)$$

as desired. To complete the proof of (2.2), we now estimate $d(\xi, x)$ in terms of $\phi^{-1}(r(B_i))$. Since $\xi \in \gamma[0, l]$ and $x = \gamma(0)$, (1.11) implies that

$$\phi\big(d(\xi, x)\big) < d(\xi).$$

But since $\xi \in \tilde{B}_i$ and $\tilde{B}_i$ is a δ-ball, Observation 2.1(2c) and (2.3) give

$$d(\xi) \leqslant 2\kappa d(x_{B_i}) \leqslant 2\kappa \frac{\lambda^2}{\delta} r(B_i).$$

Combining estimates, we obtain $\phi(d(\xi,x)) < (2\kappa\lambda^2/\delta)r(B_i)$, so that

$$B_0 \subset \frac{\lambda^2 d(\xi,x)}{r(B_i)} B_i \subset \frac{\lambda^2\phi^{-1}(2\kappa\lambda^2 r(B_i)/\delta)}{r(B_i)} B_i,$$

which proves (2.2) in all cases.

To prove the last statement in (c), we return to the δ'-Whitney balls $\{\mathcal{R}_{\gamma(t_i)}\}_{i=0}^L$ centered on the ϕ-John curve γ from x to x', and define balls $\mathcal{Q}_i$ by

$$\mathcal{Q}_i = \lambda^3 \mathcal{R}_{\gamma(t_i)}.$$

Then $\mathcal{Q}_i$ has center on γ and is a δ-Whitney ball since $r(\mathcal{Q}_i) = \lambda^3\delta' d(\gamma(t_i)) = \delta d(\gamma(t_i))$. In particular, since $t_0 = 0$ and $t_L = l$ by construction, and so $\gamma(t_0) = x$ and $\gamma(t_L) = x'$, it follows that

$$\mathcal{Q}_0 = B\big(x, \delta d(x)\big) \quad \text{and} \quad \mathcal{Q}_L = B\big(x', \delta d(x')\big).$$

For the ball B_i that is associated with $\mathcal{R}_{\gamma(t_i)}$ in the construction, (2.6) gives

$$\mathcal{Q}_i/\lambda^2 \subset B_i \subset \mathcal{Q}_i \subset \lambda^2 B_i. \tag{2.10}$$

Also recall that there is a δ-ball $B(z,\lambda^k) \subset B_i \cap B_{i+1}$ with

$$\lambda^k = \min\big\{r(B_i), r(B_{i+1}\big\}/(2\kappa) \geqslant \frac{1}{2\kappa\lambda^3}\max\big\{r(B_i), r(B_{i+1})\big\},$$

using the formula $r(B_i) = 2\kappa r(\tilde{B}_i)$ and (2.9). However, since $z \in B_i \subset \frac{1}{2\kappa}(\lambda^2 B_i)$, Observation 2.1(1) gives

$$\lambda^2 B_i \subset \lambda B\left(z, \frac{\lambda^2}{2\kappa} r(B_i)\right) \subset \lambda^6 B\big(z, \lambda^k\big).$$

Hence, $\mathcal{Q}_i \subset \lambda^6 B(z,\lambda^k)$, and the same is true for $\mathcal{Q}_{i+1}$.

Finally, to complete the proof of (c), recall that W has bounded intercepts and hence so does $\{\lambda^{-2}\mathcal{Q}_i\}$. If $\mathcal{Q}_i$ intersects $\mathcal{Q}_j$ then $\lambda^{-1} \leqslant r(\mathcal{Q}_i)/r(\mathcal{Q}_j) \leqslant \lambda$ by Observation 2.1(2b) since $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are δ-Whitney balls, and therefore the family $\{\mathcal{Q}_i\}$ has bounded intercepts with bound $C(\kappa, D_\mu)$ by Lemma 2.5.

To verify part (d), note that the hypothesis $\mathcal{Q}_i \not\subset B(x,r)$ implies there exists $z \in \mathcal{Q}_i$ such that $d(z,x) \geqslant r$. Let $x_i = \gamma(t_i)$ be the center of $\mathcal{Q}_i$ and $r_i = r(\mathcal{Q}_i)$. Then by the triangle inequality and the fact that $d(x_i,x) = d(\gamma(t_i),\gamma(0)) \leqslant t_i$,

$$d(z,x) \leqslant \kappa\big(d(z,x_i) + d(x_i,x)\big) \leqslant \kappa(r_i + t_i).$$

It is now clear that $t_i \geqslant \frac{r-\kappa r_i}{\kappa}$. If $r_i < r/(2\kappa)$, then $t_i \geqslant r/(2\kappa)$ and

$$r_i = \delta d\big(\gamma(t_i)\big) > \delta\phi(t_i) \geqslant \delta\phi\big(r/(2\kappa)\big).$$

On the other hand, if $r_i \geqslant r/(2\kappa)$, the desired estimate $r_i \geqslant \delta\phi(r/(2\kappa))$ follows easily since $t > \phi(t)$ for all t and $\delta < 1$. This completes the proof of (d).

To prove part (e), we will again use the estimate $r(\mathcal{Q}_i) = \delta d(\gamma(t_i)) > \delta\phi(t_i)$, which follows from the ϕ-John condition. Thus if $r(\mathcal{Q}_i) \leqslant 2\varepsilon$, then

$$2\varepsilon \geqslant \delta\phi(t_i) \quad \text{and hence} \quad t_i \leqslant \phi^{-1}(2\varepsilon/\delta).$$

We now fix ε and estimate the number I of *disjoint* balls $\mathcal{Q}_i$ with $\varepsilon \leqslant r(\mathcal{Q}_i) \leqslant 2\varepsilon$. Denote these $\mathcal{F}_1, \dots, \mathcal{F}_I$ with centers $\gamma(s_1), \dots, \gamma(s_I)$ respectively, in the order with $s_i < s_{i+1}$. Suppose for the moment that $I \geqslant 2$. Since $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are disjoint and have radii at least ε, then $d(\gamma(s_{i+1}), \gamma(s_i)) \geqslant \varepsilon$, and consequently by (1.9) we have $s_{i+1} - s_i \geqslant \varepsilon$. Therefore,

$$s_I \geqslant \sum_{i=1}^{I-1} (s_{i+1} - s_i) \geqslant (I-1)\varepsilon.$$

Using this together with our earlier estimate $s_I \leqslant \phi^{-1}(2\varepsilon/\delta)$ (valid since $r(\mathcal{F}_I) \leqslant 2\varepsilon$) gives

$$I - 1 \leqslant \frac{1}{\varepsilon}\phi^{-1}\left(\frac{2\varepsilon}{\delta}\right) \quad \text{if } I \geqslant 2.$$

Thus, for any $I \geqslant 1$,

$$I \leqslant 1 + \frac{1}{\varepsilon}\phi^{-1}\left(\frac{2\varepsilon}{\delta}\right) \leqslant \frac{2}{\varepsilon}\phi^{-1}\left(\frac{2\varepsilon}{\delta}\right)$$

since $\phi^{-1}(t) \geqslant t$ for all t. This completes the proof of Proposition 2.6. □

Now we consider the Hardy–Littlewood maximal function with respect to a doubling measure w on a domain Ω.

Definition 2.7. Fix a domain Ω in a homogeneous space $\langle H, d\rangle$. Given a doubling measure w on Ω and a function f on Ω, let

$$M_w^\Omega f(x) = \sup \frac{1}{w(B)} \int_{B\cap\Omega} |f|\,dw, \quad x \in \Omega,$$

where the supremum is taken over all balls B with center x.

A proof based on the Vitali covering lemma and similar to the proof of weak type (1, 1) for the usual Hardy–Littlewood maximal function shows that there is a constant C_1 depending only on κ and the doubling constant of w such that

$$\sup_{t>0} t\,w\{x \in \Omega\colon\, M_w^\Omega f(x) > t\} \leqslant C_1 \|f\|_{L^1_w(\Omega)}.$$

On the other hand, it is obvious that $\|M_w^\Omega f\|_{L_w^\infty(\Omega)} \leqslant \|f\|_{L_w^\infty(\Omega)}$, and a standard interpolation argument then gives

$$\left\|M_w^\Omega f\right\|_{L_w^p(\Omega)} \leqslant C_2\|f\|_{L_w^p(\Omega)}, \quad 1 < p < \infty,$$

where C_2 depends only on p, κ and the doubling constant of w.

Next, we state a lemma similar to [5, Lemma 2.5], which extends [16, Lemma 4] and [2, Lemma 4.2].

Lemma 2.8. *Let Ω be a domain in a quasimetric space, and let w be a doubling measure on Ω. If $\{Q_\alpha\}_{\alpha\in I}$ is an arbitrary family of balls with center in Ω and $\{a_\alpha\}_{\alpha\in I}$ is a family of nonnegative numbers, then for $1 \leqslant p < \infty$ and $N \geqslant 1$,*

$$\left\|\sum_\alpha a_\alpha \chi_{NQ_\alpha}\right\|_{L_w^p(\Omega)} \leqslant C(D_w, p, N)\left\|\sum_\alpha a_\alpha \chi_{Q_\alpha}\right\|_{L_w^p(\Omega)}.$$

Sketch of the proof. This can be proved by almost exactly the same approach as in [5], except that in case $1 < p < \infty$, we now use M_w^Ω instead of the usual weighted Hardy–Littlewood maximal function, bearing in mind that if $N \geqslant 1$ and B is any ball with center in Ω, then

$$\frac{1}{w(NB)} \int\limits_{NB\cap\Omega} |f|\,dw \leqslant C(\kappa) M_w^\Omega f(x), \quad x \in B\cap\Omega.$$

The case $p = 1$ follows easily from the fact that w is doubling on Ω, by using $w(NQ_\alpha) \leqslant C(N, D_w)w(Q_\alpha)$ since the Q_α are balls with center in Ω. □

Next, by using Lemma 2.8 (see also [16]) and checking through the proof of [5, Theorem 1.5], we obtain the following result.

Theorem 2.9. *Let Ω be a domain in a quasimetric space with quasimetric constant κ (the nonempty annuli property is not required), and let δ satisfy $0 < \delta < 1/(2\kappa^2)$. Suppose Ω is covered by a countable collection W of δ-balls such that for some $N \geqslant 1$,*

(i) $\sum_{B\in W} \chi_B \leqslant N\chi_\Omega$.
(ii) *There is a central ball $B_0 \in W$ that can be connected with every ball $B \in W$ by a finite chain of balls $B_0, B_1, \ldots, B_{k(B)} = B$ from W such that $B \subset NB_j$ for all j and each $B_j \cap B_{j+1}$ contains a ball B_j' with $B_j \cup B_{j+1} \subset NB_j'$.*

(*Domains satisfying* (i) *and* (ii) *are often called Boman chain domains.*)

Let f be a function on Ω and f_B be an associated constant for every $B \in W$. If w is a δ-doubling measure on Ω and $1 \leqslant q < \infty$, then

$$\|f - f_{B_0}\|_{L_w^q(\Omega)}^q \leqslant C \sum_{B\in W} \|f - f_B\|_{L_w^q(B)}^q \tag{2.11}$$

where C depends only on κ, q, N and the doubling constant of w.

Remark. It is easy to see from parts (a)–(c) of Proposition 2.6 with $\phi = \mathcal{J}_M$ that 1-John domains satisfy the Boman chain condition. The converse is also true for certain metric spaces; see [4,7].

3. Proofs of the main theorems

First, we will prove a useful lemma concerning sums and chains of balls.

Lemma 3.1. *Let $\langle X, \Sigma, \sigma\rangle$ be a measure space and $0 < p_0 < \infty$. Suppose $\{Q_i\}_{i\in\mathbb{N}}$ is an admissible chain of measurable sets, i.e., (1.3) holds. Then for any sequence $\{h_i\}_{i\in\mathbb{N}}$ of constants and any measurable function f on X,*

$$\sum_{i=1}^{\infty} |h_i - h_{i+1}| \leqslant \sum_{i=1}^{\infty} \frac{\max\{2^{\frac{1}{p_0}}, 2\} C_\sigma^{1/p_0}}{\sigma(Q_i)^{1/p_0}} \|f - h_i\|_{L_\sigma^{p_0}(Q_i)}. \tag{3.1}$$

Proof. When $p_0 < 1$, we will use the facts that if x, y are nonnegative real numbers then $(x + y)^{p_0} \leqslant x^{p_0} + y^{p_0}$ and $(x + y)^{1/p_0} \leqslant 2^{(1/p_0)-1}(x^{1/p_0} + y^{1/p_0})$. We have

$$\begin{aligned}
\sum |h_i - h_{i+1}| &= \sum \frac{1}{\sigma(Q_i \cap Q_{i+1})^{1/p_0}} \|h_i - h_{i+1}\|_{L_\sigma^{p_0}(Q_i\cap Q_{i+1})} \\
&\leqslant \max\bigl\{2^{\frac{1}{p_0}-1}, 1\bigr\} \sum \biggl(\frac{1}{\sigma(Q_i \cap Q_{i+1})^{1/p_0}} \|h_i - f\|_{L_\sigma^{p_0}(Q_i\cap Q_{i+1})} \\
&\quad + \frac{1}{\sigma(Q_i \cap Q_{i+1})^{1/p_0}} \|h_{i+1} - f\|_{L_\sigma^{p_0}(Q_i\cap Q_{i+1})} \biggr) \\
&\leqslant \max\bigl\{2^{\frac{1}{p_0}-1}, 1\bigr\} C_\sigma^{1/p_0} \sum \biggl(\frac{1}{\sigma(Q_i)^{1/p_0}} \|h_i - f\|_{L_\sigma^{p_0}(Q_i)} \\
&\quad + \frac{1}{\sigma(Q_{i+1})^{1/p_0}} \|h_{i+1} - f\|_{L_\sigma^{p_0}(Q_{i+1})} \biggr) \\
&\quad \bigl(\text{since } \sigma(Q_i), \sigma(Q_{i+1}) \leqslant C_\sigma \sigma(Q_i \cap Q_{i+1})\bigr) \\
&\leqslant \sum \frac{\max\{2^{\frac{1}{p_0}}, 2\} C_\sigma^{1/p_0}}{\sigma(Q_i)^{1/p_0}} \|h_i - f\|_{L_\sigma^{p_0}(Q_i)}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. By hypothesis, for each $x \in \Omega$, there exist sequences of measurable sets $\{Q_i^x\}_{i\in\mathbb{N}}, \{B_i^x\}_{i\in\mathbb{N}}$ that satisfy assumptions (1)–(4) in the theorem. Since the collection $\{Q_i^x\}_{i\in\mathbb{N}}$ satisfies the hypotheses in Lemma 3.1, $Q_1^x = B'$ and we have assumed that $f_{Q_i^x} \to f(x)$ as $i \to \infty$, it follows that

$$\bigl|f(x) - f_{B'}\bigr| \leqslant \sum_{i\in\mathbb{N}} |f_{Q_i^x} - f_{Q_{i+1}^x}| \leqslant C(p_0, C_\sigma) \sum_{i\in\mathbb{N}} a\bigl(Q_i^x\bigr) \tag{3.2}$$

by Lemma 3.1 and (1.4).

Modifying an idea from [13], letting $p = q\theta$, and recalling that by definition $A(B_j^x) = \sum_{Q_i^x \in \mathcal{C}(B_j^x)} a(Q_i^x)$, we have for any $J \in \mathbb{N}$ that

$$\sum_{i\in\mathbb{N}} a\big(Q_i^x\big) = \sum_{j\in\mathbb{N}} \sum_{Q_i^x\in\mathcal{C}(B_j^x)} a\big(Q_i^x\big) = \sum_{j\in\mathbb{N}} A\big(B_j^x\big)$$
$$= \sum_{j\leqslant J} A\big(B_j^x\big) + \sum_{j>J} A\big(B_j^x\big)$$
$$= \sum_{j\leqslant J} \big(A\big(B_j^x\big)\mu^*\big(B_j^x\big)^{\frac{1}{q}}\big)\mu^*(B_j)^{-\frac{1}{q}} + \sum_{j>J}\big(A\big(B_j^x\big)\mu^*\big(B_j^x\big)^{\frac{1}{q}-\frac{1}{p}}\big)\mu^*\big(B_j^x\big)^{\frac{1}{p}-\frac{1}{q}}.$$

For simplicity, we will now write Q_i instead of Q_i^x and B_j instead of B_j^x. Let

$$S(x) = \sup_{j\in\mathbb{N}} A(B_j)\mu^*(B_j)^{\frac{1}{q}-\frac{1}{p}}.$$

Then by (3.2), the estimates above, and the one-term version of (1.6), we have

$$\big|f(x) - f_{B'}\big| \leqslant C(p_0, C_\sigma)\Big[C_0\mu(\Omega)^{\frac{1}{q}} \sum_{j\leqslant J} \mu^*(B_j)^{-\frac{1}{q}} + S(x)\sum_{j>J}\mu^*(B_j)^{\frac{1}{p}-\frac{1}{q}}\Big]$$
$$\leqslant C\big[C_0\mu(\Omega)^{\frac{1}{q}}\mu^*(B_J)^{-\frac{1}{q}} + S(x)\mu^*(B_J)^{\frac{1}{p}-\frac{1}{q}}\big], \tag{3.3}$$

where we used (1.5) to obtain the last estimate and C depends on p, q, p_0, C_σ and the constants $A_1, A_2, \theta_1, \theta_2$ in (1.5).

Fix $t > 0$ and let $x \in E = \{x \in \Omega\colon |f(x) - f_{B'}| > t\}$. Suppose first that $S(x)\mu^*(B_j)^{1/p} \leqslant C_0\mu(\Omega)^{1/q}$ for all j. Then also

$$S(x)\mu^*(B_j)^{\frac{1}{p}-\frac{1}{q}} \leqslant C_0\mu(\Omega)^{\frac{1}{q}}\mu^*(B_j)^{-\frac{1}{q}} \quad \text{for all } j,$$

and consequently by (3.3) with $J = 1$,

$$t < \big|f(x) - f_{B'}\big| \leqslant 2CC_0\mu(\Omega)^{\frac{1}{q}}\mu^*(B_1)^{-\frac{1}{q}}.$$

It follows that $\mu^*(B_1)/\mu(\Omega) \leqslant (2CC_0/t)^q$, and hence by (1.5) that

$$1/\wp \leqslant \frac{\mu^*(B_1)}{\mu(\Omega)} \leqslant \left(\frac{2CC_0}{t}\right)^q.$$

Then

$$\mu\big\{\Omega\colon |f - f_{B'}| > t\big\} \leqslant \mu(\Omega) \leqslant \mu(\Omega)\wp\left(\frac{2CC_0}{t}\right)^q,$$

which proves Theorem 1.2 in this case.

Next consider the case when $S(x)\mu^*(B_j)^{1/p} > C_0\mu(\Omega)^{1/q}$ for some $j \in \mathbb{N}$. We will first show that if $S(x)$ is finite, then there exists $J \in \mathbb{N}$ such that

$$\mu^*(B_J)^{\frac{1}{p}} \sim \frac{C_0\mu(\Omega)^{1/q}}{S(x)}, \tag{3.4}$$

with constants of equivalence depending only on the constants $A_1, A_2, \theta_1, \theta_2$ in (1.5). In fact, since $\mu^*(B_j)$ tends to 0 as $j \to \infty$ by (1.5), there exists $J \in \mathbb{N}$ for which $S(x)\mu^*(B_{J+1})^{1/p} \leqslant C_0\mu(\Omega)^{1/q}$ and $S(x)\mu^*(B_J)^{1/p} > C_0\mu(\Omega)^{1/q}$, and then (3.4) follows easily from (1.5).

Using (3.3) and (3.4), we obtain

$$t < \left|f(x) - f_{B'}\right| \leqslant C\left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^{1-\frac{p}{q}} S(x)^{\frac{p}{q}},$$

where C depends on $p, q, p_0, C_\sigma, A_1, A_2, \theta_1$ and θ_2. By definition of $S(x)$, there exists j such that

$$t \leqslant C\left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^{1-\frac{p}{q}} \left(A(B_j)\mu^*(B_j)^{\frac{1}{q}-\frac{1}{p}}\right)^{\frac{p}{q}}.$$

In deriving this estimate we assumed that $S(x) < \infty$, but the estimate obviously also holds for some B_j if $S(x) = \infty$. Hence, setting $B = B_j$ for j as above, we obtain a set B in $\mathfrak{F}$ such that

$$t\mu^*(B)^{\frac{1}{q}} \leqslant C\left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^{1-\frac{p}{q}} \left(A(B)\mu^*(B)^{\frac{1}{q}}\right)^{\frac{p}{q}}. \tag{3.5}$$

As x varies over $E = \{x \in \Omega\colon |f(x) - f_{B'}| > t\}$, let us denote the collection of such B by $\mathcal{B}_E$. Then using the assumption that $\mathfrak{F}$ is a cover of Vitali type with respect to (μ, μ^*), we can find a countable disjoint subcollection $\mathcal{B}'_E$ of $\mathcal{B}_E$ such that $\mu(E) \leqslant V_\mu \sum_{B_i \in \mathcal{B}'_E} \mu^*(B_i)$. Hence

$$\begin{aligned}
\mu\left\{x \in \Omega\colon \left|f(x) - f_{B'}\right| > t\right\} &= \mu(E) \\
&\leqslant V_\mu \sum_{B_i \in \mathcal{B}'_E} \mu^*(B_i) \\
&\leqslant C V_\mu \left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^{q-p} \sum_{B_i \in \mathcal{B}'_E} A(B_i)^p \mu^*(B_i)^{\frac{p}{q}} t^{-q} \quad \text{by (3.5)} \\
&\leqslant C V_\mu \left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^{q-p} \left(C_0\mu(\Omega)^{\frac{1}{q}}\right)^p t^{-q} \quad \text{by (1.6)} \\
&= C V_\mu C_0^q \mu(\Omega) t^{-q},
\end{aligned}$$

where C depends on $p, q, p_0, C_\sigma, A_1, A_2, \theta_1$ and θ_2. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.8. First, we will show that conditions (1)–(4) in Theorem 1.2 hold with $B' = B(x', \delta d(x'))$. Given $x \in \Omega$, let $\{\mathcal{Q}_i\}_{i=0}^L$ be as in Proposition 2.6. Define $\{Q_i^x\}_{i=1}^\infty$ by $Q_1^x = \mathcal{Q}_L = B'$, $Q_2^x = \mathcal{Q}_{L-1}, \ldots, Q_{L+1}^x = \mathcal{Q}_0 = B(x, \delta d(x))$, and $Q_{L+j}^x = 2^{1-j} Q_{L+1}^x = B(x, 2^{1-j}\delta d(x))$ if $j \geqslant 1$. Then (1.3) follows from Proposition 2.6(c) since the balls $\mathcal{Q}'_i$ in Proposition 2.6 are δ-balls and σ is δ-doubling by hypothesis. Also, by (1.13), condition (1.4) holds with the same functional $a(\cdot)$ as in (1.13), so condition (1) of Theorem 1.2 is valid for $\{Q_i^x\}$ with this choice of $a(\cdot)$.

We define $\{B_j^x\}_{j=1}^\infty$ by $B_1^x = B(x, 2^{N_x}\tau\delta d(x))$ and $B_j^x = 2^{j-1}B_1^x = B(x, 2^{N_x+1-j}\tau\delta d(x))$ if $j \geqslant 1$.

Clearly μ^* satisfies the ratio estimate in (1.5) for $\{B_j^x\}$ with $\theta_1 = (1/2)^{D_1}$ and $\theta_2 = (1/2)^{D_2}$. Moreover, since $B' = Q_1^x \subset \bigcup_i Q_i^x \subset B_1^x$, we have $\mu(B') \leqslant \mu(B_1^x) \leqslant \mu^*(B_1^x)$ and

$$\mu(\Omega) = \frac{\mu(\Omega)}{\mu(B_1^x)}\mu(B_1^x) \leqslant \frac{\mu(\Omega)}{\mu(B')}\mu^*(B_1^x).$$

Hence the first estimate in (1.5) holds for the pair (μ, μ^*) with $\wp = \mu(\Omega)/\mu(B')$, and we have verified condition (2) of Theorem 1.2.

Next, condition (3) of Theorem 1.2 with $\mathcal{C}(B)$ as in (1.12) has already been assumed in (1.15).

Hence (1.7) of Theorem 1.2 implies that (1.16) holds provided $\{B(x, 2^j\tau\delta d(x))\colon x \in \Omega, j \leqslant N_x\}$ is a Vitali-type cover with respect to (μ, μ^*). Since we have not assumed the existence of a doubling measure on all of H, we now provide some details about this covering. Let E be a measurable subset of Ω and let $\mathcal{F} = \{B(x, 2^{j_x}\tau\delta d(x))\colon x \in E\}$. To simplify notation, we momentarily denote the balls in $\mathcal{F}$ by $\{B_\alpha\}$. First set $R = \sup\{r(B_\alpha)\colon B_\alpha \in \mathcal{F}\}$. Then $R \leqslant 2\operatorname{diam}(\Omega) < \infty$. Now let $\mathcal{G}_j = \{B_\alpha \in \mathcal{F}\colon R/2^j < r(B_\alpha) \leqslant R/2^{j-1}\}$ for $j = 1, 2, \ldots$, and choose a maximal collection $\mathcal{F}_1$ of pairwise disjoint balls of $\mathcal{G}_1$. Since d is a quasimetric, it follows that there is a constant $C(\kappa)$ for which $\bigcup_{B_\alpha \in \mathcal{G}_1} B_\alpha \subset \bigcup_{B_\alpha \in \mathcal{F}_1} C(\kappa)B_\alpha$. Let $E_1 = \bigcup_{B_\alpha \in \mathcal{F}_1} B_\alpha$. Next, choose a maximal collection $\mathcal{F}_2$ of pairwise disjoint balls of $\{B_\alpha \in \mathcal{G}_2\colon B_\alpha \cap E_1 = \emptyset\}$, and note that

$$\bigcup_{B_\alpha \in \mathcal{G}_2} B_\alpha \subset \bigcup_{B_\alpha \in \mathcal{F}_2 \cup \mathcal{F}_1} C(\kappa)B_\alpha.$$

Let $E_2 = \bigcup_{B_\alpha \in \mathcal{F}_2 \cup \mathcal{F}_1} B_\alpha$ and choose a maximal collection $\mathcal{F}_3$ of pairwise disjoint balls of $\{B_\alpha \in \mathcal{G}_3\colon B_\alpha \cap E_2 = \emptyset\}$. Continuing the process, we obtain a collection $I' = \bigcup_{j \in \mathbb{N}} \mathcal{F}_j$ of pairwise disjoint balls such that

$$E \subset \bigcup_{B_\alpha \in \mathcal{F}} B_\alpha \subset \bigcup_{B_\alpha \in I'} C(\kappa)B_\alpha.$$

The collection is countable since $\sigma(\Omega) < \infty$ and $\sigma(B_\alpha) > 0$ for all $B_\alpha \in \mathcal{F}$. Then

$$\mu(E) \leqslant \sum_{B_\alpha \in I'} \mu\big(C(\kappa)B_\alpha\big) \leqslant \sum_{B_\alpha \in I'} \mu^*\big(C(\kappa)B_\alpha\big) \leqslant C \sum_{B_\alpha \in I'} \mu^*(B_\alpha)$$

since μ^* satisfies (1.14).

We will now prove part (b). Let Ω be a 1-John domain and fix τ, δ with $\tau \geqslant 1$ and $0 < \tau\delta < 1/(2\kappa^2)$. As noted in the remark following Theorem 2.9, Proposition 2.6 provides a collection $W = \{B\}$ of δ-balls for which the Boman chain conditions in the hypothesis of Theorem 2.9 hold. Moreover, by part (b) of Proposition 2.6, not only does W satisfy the bounded overlap condition (i) in Theorem 2.9, but also the enlarged balls $\{\tau B\}_{B \in W}$ have bounded intercepts. Consequently, W can be decomposed into K subfamilies $\{W_i\}_{i=1}^K$ such that the balls $\{\tau B\}_{B \in W_i}$ in each subfamily are disjoint; here K depends only on the bounded intercept constant (see the proof of Lemma 2.5). Using the assumptions in Theorem 1.8 that (1.13) and (1.17) hold (with $\sigma = \mu$ in (1.13)), we conclude from Theorem 2.9 that

$$\|f - f_{B'}\|^q_{L^q_\mu(\Omega)} \leqslant C \sum_{B \in W} \mu(B)a(B)^q \leqslant C\big(C_0'\big)^q \mu(\Omega)$$

with C depending on $q, \kappa, \tau, \delta, D_\mu$ and the 1-John constant. This proves Theorem 1.8(b). □

Proof of Theorem 1.10. Fix f and let $\tilde{f} = |f - f_{B',\sigma}|$. For $\omega > 0$, recall that

$$\tau_\omega \tilde{f}(x) = \min\{\tilde{f}(x), 2\omega\} - \min\{\tilde{f}(x), \omega\} = \begin{cases} \lambda & \text{if } \tilde{f}(x) \geqslant 2\omega, \\ \tilde{f}(x) - \omega & \text{if } \omega \leqslant \tilde{f}(x) < 2\omega, \\ 0 & \text{if } \tilde{f}(x) < \omega. \end{cases}$$

By hypothesis (see (1.20)), there are q, θ with $q > 0$ and $0 < \theta < 1$ such that

$$\sum_{B_\alpha \in I} \left(\sum_{Q \in \mathcal{C}(B_\alpha)} b(Q, \tau_\omega \tilde{f}) \right)^{q\theta} \mu^*(B_\alpha)^\theta = \sum A(B, \tau_\omega \tilde{f})^{q\theta} \mu^*(B)^\theta \leqslant h(\Omega, \tau_\omega \tilde{f})^{q\theta} \mu(\Omega)^\theta.$$

Moreover, (1.21) and (1.19) hold by hypothesis.

Applying Theorem 1.2 to the function $\tau_\omega \tilde{f}$, we have

$$\mu\{x \in \Omega\colon \left|\tau_\omega \tilde{f}(x) - (\tau_\omega \tilde{f})_{B',\sigma}\right| > t\} \leqslant Ch(\Omega, \tau_\omega \tilde{f})^q \wp V_\mu \mu(\Omega)/t^q. \tag{3.6}$$

Following the argument in [12, pp. 131, 132], let $\omega_k = 2^k \omega$ and define

$$\Omega_k = \{x \in \Omega\colon \omega_k < \tilde{f}(x) \leqslant \omega_{k+1}\},$$

for $k = 0, 1, 2, \ldots$. Let $\varepsilon > 0$. Observe that for $x \in \Omega_{k+1}$,

$$\begin{aligned} \omega_k = \tau_{\omega_k} \tilde{f}(x) &\leqslant \left|\tau_{\omega_k} \tilde{f}(x) - (\tau_{\omega_k} \tilde{f})_{B',\sigma}\right| + (\tau_{\omega_k} \tilde{f})_{B',\sigma} \\ &\leqslant \left|\tau_{\omega_k} \tilde{f}(x) - (\tau_{\omega_k} \tilde{f})_{B',\sigma}\right| + \tilde{f}_{B',\sigma} \\ &< \left|\tau_{\omega_k} \tilde{f}(x) - (\tau_{\omega_k} \tilde{f})_{B',\sigma}\right| + \omega/2 \end{aligned}$$

if we choose $\omega = 2\tilde{f}_{B',\sigma} + \varepsilon$. Note that this choice is positive even if $2\tilde{f}_{B',\sigma} = 0$. Hence, $\omega_{k-1} < |\tau_{\omega_k} \tilde{f}(x) - (\tau_{\omega_k} \tilde{f})_{B',\sigma}|$ for all $x \in \Omega_{k+1}$. We can now use (3.6) for each $\tau_{\omega_k} \tilde{f}$:

$$\begin{aligned} &\|f - f_{B',\sigma}\|^q_{L^q_\mu(\Omega)} = \|\tilde{f}\|^q_{L^q_\mu(\Omega)} \\ &\quad = \|\tilde{f}\|^q_{L^q_\mu(\{\Omega\colon \tilde{f} \leqslant 4\omega\})} + \sum_{k=1}^\infty \|\tilde{f}\|^q_{L^q_\mu(\Omega_{k+1})} \\ &\quad \leqslant (4\omega)^q \mu(\Omega) + \sum_{k=1}^\infty \omega_{k+2}^q \mu(\Omega_{k+1}) \\ &\quad \leqslant (4\omega)^q \mu(\Omega) + \sum_{k=1}^\infty \omega_{k+2}^q \mu\{x \in \Omega\colon \omega_{k-1} < \left|\tau_{\omega_k} \tilde{f}(x) - (\tau_{\omega_k} \tilde{f})_{B',\sigma}\right|\} \\ &\quad \leqslant (4\omega)^q \mu(\Omega) + \sum_{k=1}^\infty Ch(\Omega, \tau_{\omega_k} g)^q \wp V_\mu \mu(\Omega) \quad \text{(by (3.6))} \\ &\quad \leqslant (4\omega)^q \mu(\Omega) + Ch^*(\Omega, f)^q \wp V_\mu \mu(\Omega) \end{aligned}$$

by (1.21). Moreover, note that

$$\omega - \varepsilon = 2\tilde{f}_{B',\sigma} = \frac{2}{\sigma(B')}\int\limits_{B'} \tilde{f}\,d\sigma = \frac{2}{\sigma(B')}\int\limits_{B'} |f - f_{B',\sigma}|\,d\sigma.$$

By letting $\varepsilon \to 0$, it follows that (1.22) holds, which completes the proof. □

We would like to take this opportunity to point out a small gap in [12, p. 132]. The value $\lambda = 2|f|_{\Omega,\mu}$ on line 10 of that page can be estimated just as above.

4. Applications

We now mention without proofs a few more applications of our results.

Theorem 4.1. *Suppose Ω is a* 1*-John domain in $\mathbb{R}^n$ with the Euclidean metric and $0 < p_0 < \infty$. Let $0 < \delta < 1/2$, $1 \leqslant \tau < 1/(2\delta)$, $1 \leqslant p < q < \infty$, and σ, μ, w be weights with σ and μ δ-doubling on Ω. Suppose for each $f \in \mathrm{Lip}(\Omega)$ and each δ-ball B, there exists f_B so that*

$$\sigma(B)^{-\frac{1}{p_0}}\|f - f_B\|_{L^{p_0}_{\sigma}(B)} \leqslant C\frac{r(B)}{w(B)^{1/p}}\|\nabla f\|_{L^p_w(\tau B)},$$

where $f_{B(x,r)} \to f(x)$ as $r \to 0$ for μ-a.e. $x \in \Omega$. Then for all $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$,

$$\mu(\Omega)^{-1/q}\|f - f_{\Omega,\mu}\|_{L^q_\mu(\Omega)} \leqslant C\|\nabla f\|_{L^p_w(\Omega)} \tag{4.1}$$

$$\text{if } r(B)\mu(B)^{1/q} \leqslant Cw(B)^{1/p} \quad \text{for all } \delta\text{-balls } B. \tag{4.2}$$

The constant in (4.1) *depends on the one in* (4.2) *and on D_μ, D_σ, p, q and p_0.*

Theorem 4.2. *Let $1 \leqslant p < q < \infty$. Let μ, w be nonnegative locally integrable functions on $\mathbb{R}^n$ and $\mu(x)\,dx$, $w(x)\,dx$ be the corresponding absolutely continuous measures. Suppose μ is doubling with respect to the usual Euclidean metric and*

$$\left(\frac{l(Q)}{l(\tilde{Q})}\right)^{1-n}\left(\frac{\mu(Q)}{\mu(\tilde{Q})}\right)^{\frac{1}{q}}\left(\frac{(w^{-1/(p-1)})(Q)}{(w^{-1/(p-1)})(\tilde{Q})}\right)^{\frac{1}{p'}} \leqslant C \quad \text{for all cubes } Q, \tilde{Q} \text{ in } \mathbb{R}^n \text{ with } Q \subset \tilde{Q}, \tag{4.3}$$

where $l(Q)$ denotes the edgelength of Q and $(w^{-1/(p-1)})(Q)^{1/p'}$ denotes $\operatorname{esssup}_Q w^{-1}$ *if $p = 1$ and $(\int_Q w^{-1/(p-1)}\,dx)^{1/p'}$ if $p > 1$. Then for all cubes Q and all locally Lipschitz f,*

$$\frac{1}{\mu(Q)^{1/q}}\|f - f_Q\|_{L^q_\mu(Q)} \leqslant Cl(Q)^{1-n}\big(w^{-1/(p-1)}\big)(Q)^{1/p'}\|\nabla u\|_{L^p_w(Q)}. \tag{4.4}$$

Remark. In [23, Theorem 1B], (4.4) is shown to hold under the additional assumption that $w^{-1/(p-1)}$ satisfies reverse doubling. Theorem 4.2 also sharpens [6, Theorem 4.5].

Theorem 4.3. *Let $0 < \delta < 1/2$ and $1 \leqslant \tau < 1/(2\delta)$. Suppose Ω is a ϕ-John domain in $\mathbb{R}^n$ with respect to the usual Euclidean metric and $0 < p_0, \varepsilon < \infty$. Let σ, μ be Borel measures with σ δ-doubling and μ doubling on Ω. Let f be a function on Ω and v be a nonnegative measure so that for each δ-ball B in Ω, there exists f_B with*

$$\sigma(B)^{-1/p_0} \|f - f_B\|_{L^{p_0}_\sigma(B)} \leqslant v(B)^\varepsilon,$$

where $f_{B(x,r)} \to f(x)$ as $r \to 0$ for μ-a.e. $x \in \Omega$. Then if either $\varepsilon \geqslant 1$ or $\phi = \mathcal{J}_M$,

$$\frac{1}{\mu(\Omega)^{1/p}} \|f - f_{B'}\|_{L^p_\mu(\Omega)} \leqslant C v(\Omega)^\varepsilon, \quad 0 < p < \infty. \tag{4.5}$$

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