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Affine structures and a tableau model for E_6 crystals ☆

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ABSTRACT

We provide the unique affine crystal structure for type $E_6^{(1)}$ Kirillov–Reshetikhin crystals corresponding to the multiples of fundamental weights $s\Lambda_1$, $s\Lambda_2$, and $s\Lambda_6$ for all $s \geqslant 1$ (in Bourbaki's labeling of the Dynkin nodes, where 2 is the adjoint node). Our methods introduce a generalized tableaux model for classical highest weight crystals of type E and use the order three automorphism of the affine $E_6^{(1)}$ Dynkin diagram. In addition, we provide a conjecture for the affine crystal structure of type $E_7^{(1)}$ Kirillov–Reshetikhin crystals corresponding to the adjoint node.

1. Introduction

A uniform description of perfect crystals of level 1 corresponding to the highest root θ was given in [BFKL06]. A generalization to higher level s for certain nonexceptional types was studied in [Kod08]. These crystals B of level s have the following decomposition when removing the zero arrows [Cha01]:

$$B \cong \bigoplus_{k=0}^{s} B(k\theta), \tag{1.1}$$

where $B(\lambda)$ denotes the highest weight crystal with highest weight λ.

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In this paper, we provide the unique affine crystal structure for the Kirillov–Reshetikhin crystals $B^{r,s}$ of type $E_6^{(1)}$ for the Dynkin nodes $r = 1, 2$, and 6 in the Bourbaki labeling, where node 2 corresponds to the adjoint node (see Fig. 1). In addition, we provide a conjecture for the affine crystal structure for type $E_7^{(1)}$ Kirillov–Reshetikhin crystals of level s corresponding to the adjoint node.

Our construction of the affine crystals uses the classical decomposition (1.1) together with a promotion operator which yields the affine crystal operators. Combinatorial models of all Kirillov–Reshetikhin crystals of nonexceptional types were constructed using promotion and similarity methods in [Sch08,OS08,FOS09]. Perfectness was proved in [FOS10]. Affine crystals of type $E_6^{(1)}$ and $E_7^{(1)}$ of level 1 corresponding to minuscule coweights ($r = 1, 6$) were studied by Magyar [Mag06] using the Littelmann path model. Hernandez and Nakajima [HN06] gave a construction of the Kirillov–Reshetikhin crystals $B^{r,1}$ for all r for type $E_6^{(1)}$ and most nodes r in type $E_7^{(1)}$.

For nonexceptional types, the classical crystals appearing in the decomposition (1.1) can be described using Kashiwara–Nakashima tableaux [KN94]. We provide a similar construction for general types (see Theorem 2.6). This involves the explicit construction of the highest weight crystals $B(\Lambda_i)$ corresponding to fundamental weights Λ_i using the Lenart–Postnikov [LP08] model and the notion of pairwise weakly increasing columns (see Definition 2.1).

The promotion operator for the Kirillov–Reshetikhin crystal $B^{r,s}$ of type $E_6^{(1)}$ for $r = 1, 6$ is given in Theorem 3.13 and for $r = 2$ in Theorem 3.22. Our construction and proofs exploit the notion of composition graphs (Definition 3.10) and the fact that the promotion operator we choose has order three. As shown in Theorem 3.9, a promotion operator of order three yields a regular crystal. In Conjecture 3.26 we also provide a promotion operator of order two for the crystals $B^{1,s}$ of type $E_7^{(1)}$. However, for order two promotion operators the analogue of Theorem 3.9 is missing.

This paper is structured as follows. In Section 2, the fundamental crystals $B(\Lambda_1)$ and $B(\Lambda_6)$ are constructed explicitly for type E_6 and it is shown that all other highest weight crystals $B(\lambda)$ of type E_6 can be constructed from these. Similarly, $B(\Lambda_7)$ yields all highest weight crystals $B(\lambda)$ for type E_7. In Section 2.4, a generalized tableaux model is given for $B(\lambda)$ for general types. In particular, we introduce the notion of weak increase. The results are used to construct the affine crystals in Section 3. In Section 4, we give some details about the `Sage` implementation of the E_6, E_7, and $E_6^{(1)}$ crystals constructed in this paper. Some outlook and open problems are discussed in Section 5. Appendices A and B contain details about the proofs for the construction of the affine crystals, in particular the usage of oriented matroid theory.

2. A tableau model for finite-dimensional highest weight crystals

In this section, we describe a model for the classical highest weight crystals in type E. In Section 2.1, we introduce our notation and give the axiomatic definition of a crystal. The tensor product rule for crystals is reviewed in Section 2.2. In Section 2.3, we give an explicit construction of the highest weight crystals associated to the fundamental weights in types E_6 and E_7. In Section 2.4, we give a generalized tableaux model to realize all of the highest weight crystals in these types. The generalized tableaux are type-independent, and can be viewed as an extension of the Kashiwara–Nakashima tableaux [KN94] to type E. For a general introduction to crystals we refer to [HK02].

2.1. Axiomatic definition of crystals

Denote by $\mathfrak{g}$ a symmetrizable Kac–Moody algebra, P the weight lattice, I the index set for the vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{\alpha_i^\vee \in P^* \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. A $U_q(\mathfrak{g})$-*crystal* [Kas95] is a nonempty set B equipped with maps $\mathrm{wt}: B \to P$ and $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for all $i \in I$, satisfying

$$f_i(b) = b' \quad \Leftrightarrow \quad e_i(b') = b \quad \text{if } b, b' \in B,$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \quad \text{if } f_i(b) \in B,$$

$$\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b).$$

Here, we have

$$\varepsilon_i(b) = \max\{n \geqslant 0 \mid e_i^n(b) \neq \mathbf{0}\},$$
$$\varphi_i(b) = \max\{n \geqslant 0 \mid f_i^n(b) \neq \mathbf{0}\}$$

for $b \in B$, and we denote $\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$ by $\mathrm{wt}_i(b)$. A $U_q(\mathfrak{g})$-crystal B can be viewed as a directed edge-colored graph called the *crystal graph* whose vertices are the elements of B, with a directed edge from b to b' labeled $i \in I$, if and only if $f_i(b) = b'$. Given $i \in I$ and $b \in B$, the *i-string through b* consists of the nodes $\{f_i^m(b)\colon 0 \leqslant m \leqslant \varphi_i(b)\} \cup \{e_i^m(b)\colon 0 < m \leqslant \varepsilon_i(b)\}$.

Let $\{\Lambda_i \mid i \in I\}$ be the fundamental weights of $\mathfrak{g}$. For every $b \in B$ define $\varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i$ and $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i$. An element $b \in B$ is called *highest weight* if $e_i(b) = \mathbf{0}$ for all $i \in I$. We say that B is a *highest weight crystal* of highest weight λ if it has a unique highest weight element of weight λ. For a dominant weight λ, we let $B(\lambda)$ denote the unique highest-weight crystal with highest weight λ.

An *isomorphism* of crystals is a bijection $\Psi : B \cup \{\mathbf{0}\} \to B' \cup \{\mathbf{0}\}$ such that $\Psi(\mathbf{0}) = \mathbf{0}$, $\varepsilon(\Psi(b)) = \varepsilon(b)$, $\varphi(\Psi(b)) = \varphi(b)$, $f_i\Psi(b) = \Psi(f_i(b))$, and $\Psi(e_i(c)) = e_i\Psi(c)$ for all $b, c \in B$, $\Psi(b), \Psi(c) \in B'$ where $f_i(b) = c$.

When $\widetilde{\lambda}$ is a weight in an affine type, we call

$$\langle \widetilde{\lambda}, c\rangle = \sum_{i \in I} a_i^\vee \langle \widetilde{\lambda}, \alpha_i^\vee\rangle \tag{2.1}$$

the *level* of $\widetilde{\lambda}$, where c is the canonical central element and $\widetilde{\lambda} = \sum_{i\in I} \lambda_i\Lambda_i$ is the affine weight. In our work, we will often compute the 0-weight $\lambda_0\Lambda_0$ at level 0 for a node b in a classical crystal from the classical weight $\lambda = \sum_{i \in I\setminus\{0\}} \lambda_i\Lambda_i = \mathrm{wt}(b)$ by setting $\langle \lambda_0\Lambda_0 + \lambda, c\rangle = 0$ and solving for λ_0.

When $\mathfrak{g}$ is a finite-dimensional Lie algebra, every integrable $U_q(\mathfrak{g})$-module decomposes as a direct sum of highest weight modules. On the level of crystals, this implies that every crystal graph B corresponding to an integrable module is a union of connected components, and each connected component is the crystal graph of a highest weight module. We denote this by $B = \bigoplus B(\lambda)$ for some set of dominant weights λ, and we call these $B(\lambda)$ the *components* of the crystal.

Suppose that $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra and let $U_q'(\mathfrak{g})$ be the corresponding quantum algebra without derivation. The goal of this work is to study crystals $B^{r,s}$ that correspond to certain finite-dimensional $U_q'(\mathfrak{g})$-modules known as Kirillov–Reshetikhin modules. Here, r is a node of the Dynkin diagram and s is a nonnegative integer. The existence of the crystals $B^{r,s}$ that we study follows from results in [KKM+92], while the classical decomposition of these crystals is given in [Cha01].

2.2. Tensor products of crystals

Let $B_1, B_2, \ldots, B_L$ be $U_q(\mathfrak{g})$-crystals. The Cartesian product $B_1 \times B_2 \times \cdots \times B_L$ has the structure of a $U_q(\mathfrak{g})$-crystal using the so-called signature rule. The resulting crystal is denoted by $B = B_1 \otimes B_2 \otimes \cdots \otimes B_L$ and its elements $(b_1, \ldots, b_L)$ are written as $b_1 \otimes \cdots \otimes b_L$ where $b_j \in B_j$. The reader is warned that our convention is opposite to that of Kashiwara [Kas95]. Fix $i \in I$ and $b = b_1 \otimes \cdots \otimes b_L \in B$. The *$i$-signature* of b is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{- \cdots -}_{\varphi_i(b_1)\text{ times}} \quad \underbrace{+ \cdots +}_{\varepsilon_i(b_1)\text{ times}} \cdots \underbrace{- \cdots -}_{\varphi_i(b_L)\text{ times}} \quad \underbrace{+ \cdots +}_{\varepsilon_i(b_L)\text{ times}}.$$

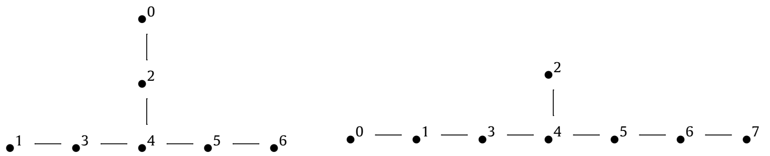


Fig. 1. Affine $E_6^{(1)}$ and $E_7^{(1)}$ Dynkin diagrams.

The *reduced i-signature* of b is the subword of the i-signature of b, given by the repeated removal of adjacent symbols $+-$ (in that order); it has the form

$$\underbrace{-\cdots-}_{\varphi_i \text{ times}} \quad \underbrace{+\cdots+}_{\varepsilon_i \text{ times}}.$$

If $\varphi_i = 0$ then $f_i(b) = \mathbf{0}$; otherwise

$$f_i(b_1 \otimes \cdots \otimes b_L) = b_1 \otimes \cdots \otimes b_{j-1} \otimes f_i(b_j) \otimes \cdots \otimes b_L$$

where the rightmost symbol $-$ in the reduced i-signature of b comes from b_j. Similarly, if $\varepsilon_i = 0$ then $e_i(b) = \mathbf{0}$; otherwise

$$e_i(b_1 \otimes \cdots \otimes b_L) = b_1 \otimes \cdots \otimes b_{j-1} \otimes e_i(b_j) \otimes \cdots \otimes b_L$$

where the leftmost symbol $+$ in the reduced i-signature of b comes from b_j. It is not hard to verify that this defines the structure of a $U_q(\mathfrak{g})$-crystal with $\varphi_i(b) = \varphi_i$ and $\varepsilon_i(b) = \varepsilon_i$ in the above notation, and weight function

$$\mathrm{wt}(b_1 \otimes \cdots \otimes b_L) = \sum_{j=1}^{L} \mathrm{wt}(b_j).$$

2.3. Fundamental crystals for type E_6 and E_7

Let $I = \{1, 2, 3, 4, 5, 6\}$ denote the classical index set for E_6. We number the nodes of the affine Dynkin diagram as in Fig. 1.

Classical highest-weight crystals $B(\lambda)$ for E_6 can be realized by the Lenart–Postnikov alcove path model described in [LP08]. We implemented this model in `Sage` and have recorded the crystal $B(\Lambda_1)$ in Fig. 2. This crystal has 27 nodes.

To describe our labeling of the nodes, observe that all of the i-strings in $B(\Lambda_1)$ have length 1 for each $i \in I$. Therefore, the crystal admits a transitive action of the Weyl group. Also, it is straightforward to verify that all of the nodes in $B(\Lambda_1)$ are determined by weight. For our work in Section 3, we also compute the 0-weight at level 0 of a node b in any classical crystal from the classical weight as described in Remark 3.4.

Thus, we label the nodes of $B(\Lambda_1)$ by weight, which is equivalent to recording which i-arrows come in and out of b. The i-arrows into b are recorded with an overline to indicate that they contribute negative weight, while the i-arrows out of b contribute positive weight.

By the symmetry of the Dynkin diagram, we have that $B(\Lambda_6)$ also has 27 nodes and is dual to $B(\Lambda_1)$ in the sense that its crystal graph is obtained from $B(\Lambda_1)$ by reversing all of the arrows. Reversing the arrows requires us to label the nodes of $B(\Lambda_6)$ by the weight that is the negative of the weight of the corresponding node in $B(\Lambda_1)$. Moreover, observe that $B(\Lambda_1)$ contains no pair of nodes

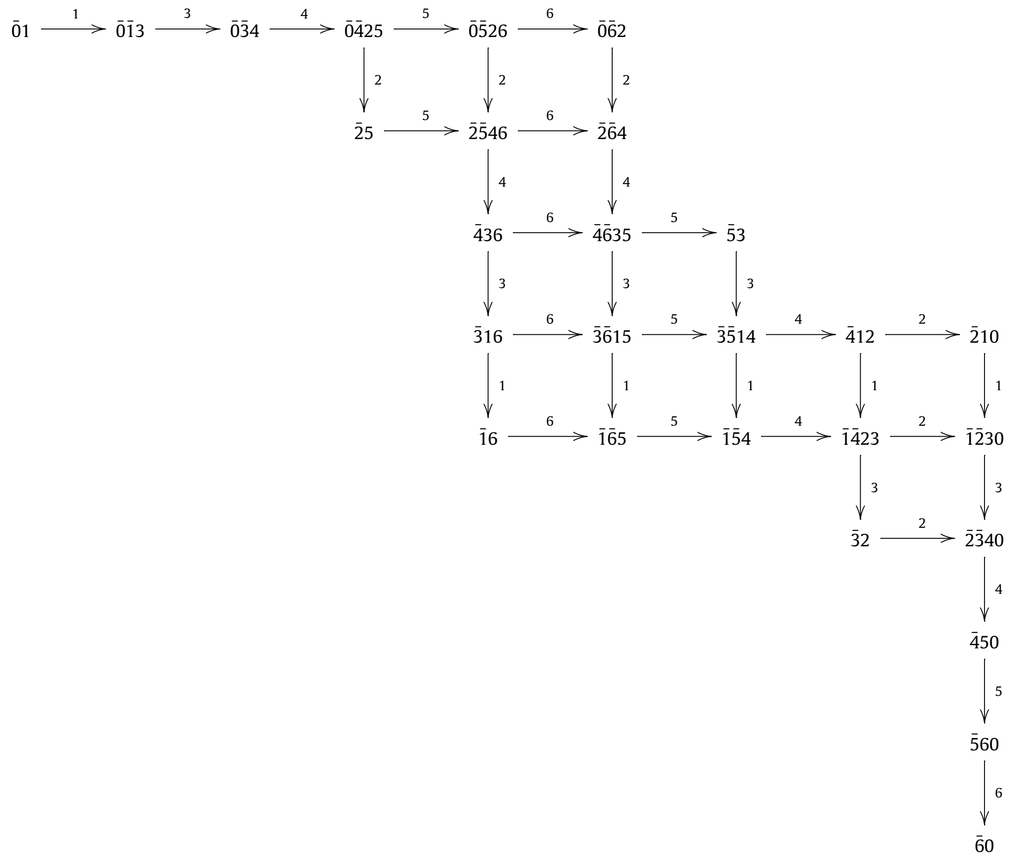


Fig. 2. Crystal graph for $B(\Lambda_1)$ of type E_6.

Table 1
Fundamental realizations for E_6.

	Generator	in	Dimension
$B(\Lambda_2)$	$2\bar{1}\bar{0}\otimes\bar{0}1$	$B(\Lambda_6)\otimes B(\Lambda_1)$	78
$B(\Lambda_3)$	$\bar{0}\bar{1}3\otimes\bar{0}1$	$B(\Lambda_1)^{\otimes 2}$	351
$B(\Lambda_4)$	$\bar{0}\bar{3}4\otimes\bar{0}\bar{1}3\otimes\bar{0}1$	$B(\Lambda_1)^{\otimes 3}$	2925
$B(\Lambda_5)$	$5\bar{6}\bar{0}\otimes 6\bar{0}$	$B(\Lambda_6)^{\otimes 2}$	351

with weights μ, $-\mu$, respectively. Hence, we can unambiguously label any node of $B(\Lambda_1)\cup B(\Lambda_6)$ by weight.

It is straightforward to show using characters that every classical highest-weight representation $B(\Lambda_i)$ for $i\in I$ can be realized as a component of some tensor product of $B(\Lambda_1)$ and $B(\Lambda_6)$ factors. On the level of crystals, the tensor products $B(\Lambda_1)^{\otimes k}$, $B(\Lambda_6)^{\otimes k}$ and $B(\Lambda_6)\otimes B(\Lambda_1)$ are defined for all k by the tensor product rule of Section 2.2. Therefore, we can realize the other classical fundamental crystals $B(\Lambda_i)$ as shown in Table 1. There are additional realizations for these crystals obtained by dualizing.

There is a similar construction for the fundamental crystals of type E_7. The highest weight crystal $B(\Lambda_7)$ has 56 nodes and these nodes all have distinct weights (see Fig. 3). Also, $\varphi_i(b)\leqslant 1$ and $\varepsilon_i(b)\leqslant 1$ for all $i\in\{1,2,\dots,7\}$ and $b\in B(\Lambda_7)$. Using character calculations, we can show that every

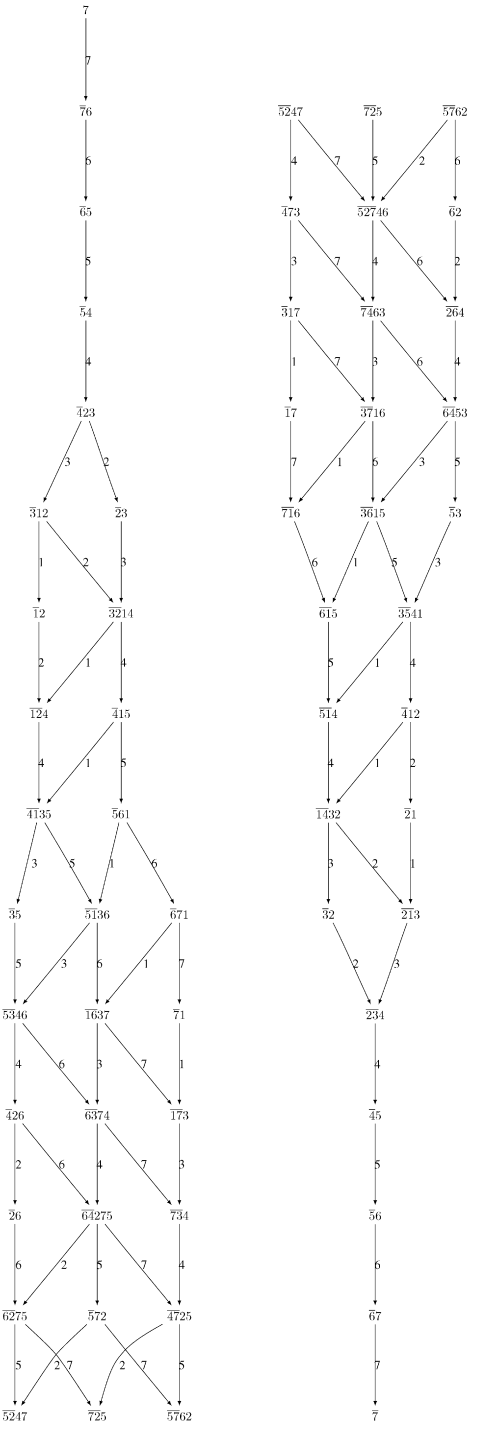


Fig. 3. $B(\Lambda_7)$ of type E_7.

Table 2
Fundamental realizations for E_7.

	Generator	in	Dimension
$B(\Lambda_1)$	$\bar{0}\bar{7}1 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 2}$	133
$B(\Lambda_2)$	$\bar{1}2 \otimes \bar{0}\bar{7}1 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 3}$	912
$B(\Lambda_3)$	$\bar{0}\bar{2}3 \otimes \bar{1}2 \otimes \bar{0}\bar{7}1 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 4}$	8645
$B(\Lambda_4)$	$\bar{0}\bar{5}4 \otimes \bar{0}\bar{6}5 \otimes \bar{0}\bar{7}6 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 4}$	365750
$B(\Lambda_5)$	$\bar{0}\bar{6}5 \otimes \bar{0}\bar{7}6 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 3}$	27664
$B(\Lambda_6)$	$\bar{0}\bar{7}6 \otimes \bar{0}7$	$B(\Lambda_7)^{\otimes 2}$	1539
$B(\Lambda_7)$	$\bar{0}7$	$B(\Lambda_7)$	56

classical highest-weight representation $B(\Lambda_i)$ appears in some tensor product of $B(\Lambda_7)$ factors. In Table 2, we display realizations for all of the classical fundamental crystals $B(\Lambda_i)$ in type E_7.

Green [Gre07,Gre08] has another construction of the 27-dimensional crystals $B(\Lambda_1)$ and $B(\Lambda_6)$ of type E_6, and the 56-dimensional crystal $B(\Lambda_7)$ of type E_7 in terms of full heaps, and also gives the connection of the fundamental E_6 crystals with the 27 lines on a cubic surface. A Littlewood–Richardson rule for type E_6 was given in [Hos07] using polyhedral realizations of crystal bases.

2.4. *Generalized tableaux*

In this section, we describe how to realize the crystal $B(\Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k})$ inside the tensor product $B(\Lambda_{i_1}) \otimes B(\Lambda_{i_2}) \otimes \cdots \otimes B(\Lambda_{i_k})$, where the Λ_i are all fundamental, or more generally dominant weights. Our arguments use only abstract crystal properties, so the results in this section apply to any finite type.

If b is the unique highest weight node in $B(\lambda)$ and c is the unique highest weight node in $B(\mu)$, then $B(\lambda + \mu)$ is generated by $b \otimes c \in B(\lambda) \otimes B(\mu)$. Iterating this procedure provides a recursive description of any highest-weight crystal embedded in a tensor product of crystals. Our goal is to give a nonrecursive description of the nodes of $B(\Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_k})$ for any collection of fundamental weights Λ_i.

For an ordered set of dominant weights $(\mu_1, \mu_2, \ldots, \mu_k)$ and for each permutation w in the symmetric group S_k, define

$$B_w(\mu_1, \ldots, \mu_k) = B(\mu_{w(1)}) \otimes B(\mu_{w(2)}) \otimes \cdots \otimes B(\mu_{w(k)})$$

so $B_e(\mu_1, \ldots, \mu_k)$ is $B(\mu_1) \otimes \cdots \otimes B(\mu_k)$ where $e \in S_k$ is the identity.

Definition 2.1. Let $(\mu_1, \mu_2, \ldots, \mu_k)$ be dominant weights. Then, we say that

$$b_1 \otimes b_2 \otimes \cdots \otimes b_k \in B(\mu_1) \otimes B(\mu_2) \otimes \cdots \otimes B(\mu_k)$$

is *pairwise weakly increasing* if

$$b_j \otimes b_{j+1} \in B(\mu_j + \mu_{j+1}) \subset B(\mu_j) \otimes B(\mu_{j+1})$$

for each $1 \leqslant j < k$.

Next, we fix an isomorphism of crystals

$$\Phi_w^{(\mu_1, \ldots, \mu_k)} : B_w(\mu_1, \ldots, \mu_k) \to B_e(\mu_1, \ldots, \mu_k)$$

for each $w \in S_k$. Observe that each choice of $\Phi_w^{(\mu_1, \ldots, \mu_k)}$ corresponds to a choice for the image of each of the highest-weight nodes in $B_w(\mu_1, \ldots, \mu_k)$.

Let b_j^* denote the unique highest weight node of the jth factor $B(\mu_j)$. Since we are fixing the dominant weights $(\mu_1, \ldots, \mu_k)$, we will sometimes drop the notation $(\mu_1, \ldots, \mu_k)$ from B_w and Φ_w in the proofs below.

Definition 2.2. Let w be a permutation and choose j to be the maximal integer such that w that fixes $\{1, 2, \ldots, j\}$. We say that $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ is a *lazy isomorphism* if the image of every highest weight node of the form

$$b_1 \otimes b_2 \otimes \cdots \otimes b_j \otimes b_{j+1}^* \otimes \cdots \otimes b_k^*$$

under $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ is equal to

$$b_1 \otimes b_2 \otimes \cdots \otimes b_j \otimes b_{w^{-1}(j+1)}^* \otimes \cdots \otimes b_{w^{-1}(k)}^*.$$

We want to choose our isomorphisms $\Phi_w^{(\mu_1,\ldots,\mu_k)}$ to be lazy, but we will see in the course of the proofs that our results do not otherwise depend upon the choice of $\Phi_w^{(\mu_1,\ldots,\mu_k)}$.

Definition 2.3. Let T be any subset of S_k, and $\{\Phi_w^{(\mu_1,\ldots,\mu_k)}\}_{w\in T}$ be a collection of lazy isomorphisms. We define $I^{(\mu_1,\ldots,\mu_k)}(T)$ to be

$$\bigcap_{w\in T} \Phi_w^{(\mu_1,\ldots,\mu_k)}\big(\{\text{pairwise weakly increasing nodes of } B_w(\mu_1, \ldots, \mu_k)\}\big) \subset B_e(\mu_1, \ldots, \mu_k).$$

Proposition 2.4. *Let T be any subset of S_k. Then, whenever $b \in I^{(\mu_1,\ldots,\mu_k)}(T)$ we have $e_i(b), f_i(b) \in I^{(\mu_1,\ldots,\mu_k)}(T)$.*

Proof. We first claim that the crystal operators e_i and f_i preserve the pairwise weakly increasing condition in any tensor product of highest weight crystals. Let

$$b = b_1 \otimes b_2 \otimes \cdots \otimes b_k$$

be a pairwise weakly increasing node in $B = B(\mu_1) \otimes \cdots \otimes B(\mu_k)$.

We need to show that $e_i(b)$ is pairwise weakly increasing. Suppose that e_i acts on the jth tensor factor in b, that is, $e_i(b) = b_1 \otimes \cdots \otimes e_i(b_j) \otimes \cdots \otimes b_k$. Hence it suffices to show that $b_{j-1} \otimes e_i(b_j) \in B(\mu_{j-1} + \mu_j)$ and $e_i(b_j) \otimes b_{j+1} \in B(\mu_j + \mu_{j+1})$. Since e_i acts on b_j in b, in the tensor product rule the leftmost unbracketed $+$ is associated to b_j. This means that any $+$ from b_{j-1} must be bracketed with a $-$ from b_j. But then $e_i(b_{j-1} \otimes b_j) = b_{j-1} \otimes e_i(b_j) \in B(\mu_{j-1} + \mu_j)$. Similarly, since e_i acts on b_j, not all $+$ in b_j are bracketed with $-$ in $b_{j+1} \otimes \cdots \otimes b_k$. But therefore, also not all $+$ in b_j are bracketed with $-$ in b_{j+1} and hence $e_i(b_j \otimes b_{j+1}) = e_i(b_j) \otimes b_{j+1} \in B(\mu_j + \mu_{j+1})$. The arguments for f_i are analogous.

Next, suppose that $b \in I^{(\mu_1,\ldots,\mu_k)}(T) \subset B_e$. Then, for all $w \in S_k$ we have $\Phi_w^{-1}(b)$ is pairwise weakly increasing in B_w. By the argument above, we then have that $e_i(\Phi_w^{-1}(b))$ is pairwise weakly increasing in B_w. Since Φ_w is an isomorphism, it commutes with e_i, so $\Phi_w^{-1}(e_i(b))$ is pairwise weakly increasing in B_w for all $w \in S_k$. Hence, $e_i(b) \in I^{(\mu_1,\ldots,\mu_k)}(T)$. The arguments for f_i are analogous. □

Corollary 2.5. *For any subset T of S_k, we have that $I^{(\mu_1,\ldots,\mu_k)}(T)$ is a direct sum of highest weight crystals $\bigoplus_\lambda B(\lambda)$ for some collection of weights λ.*

Proof. Proposition 2.4 implies that whenever $b \in I^{(\mu_1,\ldots,\mu_k)}(T)$, the entire connected component of the crystal graph containing b is in $I^{(\mu_1,\ldots,\mu_k)}(T)$. □

Theorem 2.6. *Fix a sequence* $(\mu_1, \ldots, \mu_k)$ *of dominant weights. Then,*

$$I^{(\mu_1,\ldots,\mu_k)}(S_k) \cong B(\mu_1 + \mu_2 + \cdots + \mu_k).$$

Proof. Let b_j^* be the unique highest weight node of B_j with highest weight μ_j for each $j = 1, \ldots, k$. Then $b^* = b_1^* \otimes b_2^* \otimes \cdots \otimes b_k^*$ generates $B(\mu_1 + \cdots + \mu_k)$ and this node lies in $I^{(\mu_1,\ldots,\mu_k)}(S_k)$.

Suppose there exists another highest weight node in $I^{(\mu_1,\ldots,\mu_k)}(S_k)$. Then, at least one of the factors b_j must have $\varepsilon_i(b_j) > 0$ for some i. Choose j to be the rightmost factor having $\varepsilon_i(b_j) > 0$ for some $i \in I$. Then fix some choice of i such that $\varepsilon_i(b_j) > 0$. Our highest weight node has the form

$$b = b_1 \otimes \cdots \otimes b_j \otimes b_{j+1}^* \otimes \cdots \otimes b_k^*.$$

In particular, $j < k$ since any rightmost factor of a highest weight tensor product must be highest weight.

Since b is highest weight, we have that all $+$ entries for factor b_j are canceled by $-$ entries lying to the right in the i-signature for the tensor product rule. Suppose that $b_{j'}^*$ is the leftmost factor for which a $-$ cancels a $+$ from b_j in the i-signature. Let w be the permutation that interchanges factors $j+1$ and j'. Then, by our choice of Φ_w we have that $\Phi_w^{-1}(b)$ is obtained from b just by interchanging the factors b_{j+1}^* and $b_{j'}^*$.

Hence, we have that $\Phi_w^{-1}(b)$ in B_w has an adjacent $+/-$ pair on factors $j, j+1$. Since this pair is part of a pairwise weakly increasing element, there must exist a sequence of $e_{i'}$ operations that brings $b_j \otimes b_{j'}^*$ to $b_j^* \otimes b_{j'}^*$. However, $e_{i'}$ can only operate on the first tensor factor in this pair because $b_{j'}^*$ is already highest weight. Moreover, we have that ε_i of the first factor and φ_i of the second factor are both positive. This remains true regardless of how we apply $e_{i'}$ operations where $i \neq i'$ by [Ste03, Axiom (P4)], [Ste07,DKK09]. We can potentially apply the e_i operation $\max\{\varepsilon_i(b_j) - \varphi_i(b_{j'}^*), 0\}$ times, but since $\varphi_i(b_{j'}^*) > 0$, we have that ε_i of the first factor will always remain positive. Hence, we can never reach $b_j^* \otimes b_{j'}^*$, a contradiction.

Thus, b^* is the unique highest weight node of $I^{(\mu_1,\ldots,\mu_k)}(S_k)$. □

Remark 2.7. The condition that there is a unique highest weight element that we used in the proof of Theorem 2.6 is equivalent to the hypothesis of [KN94, Proposition 2.2.1] from which the desired conclusion also follows.

Remark 2.8. Because we only require a constant amount of data to check the pairwise weakly increasing condition for each pair of tensor factors, Theorem 2.6 and its refinements will allow us to formulate arguments that apply to all highest-weight crystals simultaneously, regardless of the number of tensor factors.

When we are considering a specific highest-weight crystal, it may be computationally easier to generate $B(\mu_1 + \cdots + \mu_k)$ by simply applying f_i operations to the highest-weight node in all possible ways.

We will say that any node of $I^{(\mu_1,\ldots,\mu_k)}(S_k)$ is *weakly increasing*. It turns out that we can often take T to be much smaller than S_k by starting with $T = \{e\}$ and adding permutations to T until $I^{(\mu_1,\ldots,\mu_k)}(T)$ contains a unique highest weight node. In particular, the next result shows that we can take $T = \{e\}$ when we are considering a linear combination of two distinct fundamental weights.

Lemma 2.9. *Let* Λ_{i_1} *and* Λ_{i_2} *be distinct fundamental weights, and* $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$ *with* $k = k_1 + k_2$. *Then, the nodes of*

$$B(k_1\Lambda_{i_1} + k_2\Lambda_{i_2}) \subset B(\Lambda_{i_1})^{\otimes k_1} \otimes B(\Lambda_{i_2})^{\otimes k_2}$$

are precisely the pairwise weakly increasing tensor products $b_1 \otimes b_2 \otimes \cdots \otimes b_k$ *of* $B(\Lambda_{i_1})^{\otimes k_1} \otimes B(\Lambda_{i_2})^{\otimes k_2}$.

Proof. We order the fundamental weights as $(\Lambda_{i_1},\dots,\Lambda_{i_1},\Lambda_{i_2},\dots,\Lambda_{i_2})$ and apply the same argument as in the proof of Theorem 2.6 to see that any highest weight node in $I^{(\Lambda_{i_1},\dots,\Lambda_{i_1},\Lambda_{i_2},\dots,\Lambda_{i_2})}(\{e\})$ must be of the form

$$b_1 \otimes \cdots \otimes b_{k_1-1} \otimes b^*_{k_1} \otimes b^*_{k_1+1} \otimes \cdots \otimes b^*_k.$$

In this case, it is never necessary to apply Φ_w to reorder the factors because all of the factors to the right of factor k_1 must be the same.

Next, we let $j = k_1 - 1$. We have that $b_{j+1} = b^*_{j+1}$ and we work by downward induction to argue that b_j must be b^*_j. This follows because due to the pairwise weak increasing condition there exists a sequence of e_i that takes $b_j \otimes b^*_{j+1}$ to $b^*_j \otimes b^*_{j+1}$. The highest weight node of the fundamental crystal $B(\Lambda_{i_1})$ has a unique i_1-arrow. If $b_j \neq b^*_j$ then we could never traverse this edge because in the i_1-signature any $+$ would be canceled by a $-$ from b^*_{j+1}. Hence, $b_j = b^*_j$, and the induction continues.

Thus, there is a unique highest-weight node in $I^{(\Lambda_{i_1},\dots,\Lambda_{i_1},\Lambda_{i_2},\dots,\Lambda_{i_2})}(\{e\})$. □

All of the crystals in our work have classical decompositions that have been given by Chari [Cha01]. These crystals satisfy the requirement of Lemma 2.9 that at most two fundamental weights appear. On the other hand, Example 2.10 shows that no ordering of the factors in $B(\Lambda_2)\otimes B(\Lambda_1)\otimes B(\Lambda_6)$ in type E_6 admits an analogous weakly increasing condition that is defined using only pairwise comparisons.

Example 2.10. Observe that each of the following nodes in type E_6 is a counterexample to the condition required in [KN94, Proposition 2.2.1]. Each of the given nodes is highest weight, and pairwise weakly increasing, but none of the nodes correspond to the highest weight node of $B(\Lambda_1 + \Lambda_6 + \Lambda_2)$.

$$(3\bar{1}\bar{6} \otimes 1) \otimes u_1 \otimes u_6 \in B(\Lambda_2) \otimes B(\Lambda_1) \otimes B(\Lambda_6),$$
$$(5\bar{3} \otimes \bar{1}3) \otimes u_6 \otimes u_1 \in B(\Lambda_2) \otimes B(\Lambda_6) \otimes B(\Lambda_1),$$
$$\bar{2}5 \otimes u_6 \otimes u_2 \in B(\Lambda_1) \otimes B(\Lambda_6) \otimes B(\Lambda_2),$$
$$\bar{6}2 \otimes u_2 \otimes u_6 \in B(\Lambda_1) \otimes B(\Lambda_2) \otimes B(\Lambda_6),$$
$$\bar{2}3 \otimes u_1 \otimes u_2 \in B(\Lambda_6) \otimes B(\Lambda_1) \otimes B(\Lambda_2),$$
$$2\bar{1} \otimes u_2 \otimes u_1 \in B(\Lambda_6) \otimes B(\Lambda_2) \otimes B(\Lambda_1).$$

Here, u_i is the highest weight node of $B(\Lambda_i)$. Hence, it is not possible to obtain a pairwise weakly increasing condition that characterizes the nodes of $B(\Lambda_1 + \Lambda_6 + \Lambda_2)$.

Remark 2.11. In standard monomial theory [LS86], the condition that a tensor product of basis elements lies in $B(\lambda + \mu)$ can also be formulated as a comparison of the lift of these elements in Bruhat order [Lit96]. For several tensor factors, one needs to compare simultaneous lifts.

We now restrict to type E_6. Lemma 2.9 implies that we have a nonrecursive description of all $B(k\Lambda_i)$ determined by the finite information in $B(2\Lambda_i)$. In the case of particular fundamental representations, we can be more specific about how to test for the weakly increasing condition.

Proposition 2.12. *We have that $b_1 \otimes b_2 \in B(2\Lambda_1) \subset B(\Lambda_1)^{\otimes 2}$ if and only if b_2 can be reached from b_1 by a sequence of f_i operations in $B(\Lambda_1)$.*

Proof. This is a finite computation on $B(2\Lambda_1)$. □

The crystal graph for $B(\Lambda_1)$ of Fig. 2 can be viewed as a poset. Then Proposition 2.12 implies in particular that incomparable pairs in $B(\Lambda_1)$ are not weakly increasing.

There are 78 nodes in $B(\Lambda_2)$. We construct $B(\Lambda_2)$ as the highest weight crystal graph generated by $2\bar{1}\bar{0}\otimes\bar{0}1$ inside $B(\Lambda_6)\otimes B(\Lambda_1)$. Note that we only need to use the nodes in the "top half" of Fig. 2 and their duals. There are 2430 nodes in $B(2\Lambda_2)$.

Proposition 2.13. *We have that*

$$(b_1 \otimes c_1) \otimes (b_2 \otimes c_2) \in B(2\Lambda_2) \subset \big(B(\Lambda_6) \otimes B(\Lambda_1)\big)^{\otimes 2}$$

if and only if

(1) *b_2 can be reached from b_1 by f_i operations in $B(\Lambda_6)$, and c_2 can be reached from c_1 by f_i operations in $B(\Lambda_1)$, and*
(2) *whenever c_1 is dual to b_2, we have that there is a path of f_i operations from $(b_1 \otimes c_1)$ to $(b_2 \otimes c_2)$ of length at least 1 (so in particular, the elements are not equal) in $B(\Lambda_2)$.*

Proof. This is a finite computation on $B(2\Lambda_2)$. $\square$

3. Affine structure

In this section, we study the affine crystals of type $E_6^{(1)}$. We introduce the method of promotion to obtain a combinatorial affine crystal structure in Section 3.1 and the notion of composition graphs in Section 3.2. It is shown in Theorem 3.9 that order three twisted isomorphisms yield regular affine crystals. This is used to construct $B^{r,s}$ of type $E_6^{(1)}$ for the minuscule nodes $r = 1, 6$ in Section 3.3 and the adjoint node $r = 2$ in Section 3.4. In Section 3.5 we present conjectures for $B^{1,s}$ of type $E_7^{(1)}$.

3.1. Combinatorial affine crystals and twisted isomorphisms

The following concept is fundamental to this work.

Definition 3.1. Let $\widetilde{C}$ be an affine Dynkin diagram and C the associated finite Dynkin diagram (obtained by removing node 0) with index set I. Let $\dot{p}$ be an automorphism of $\widetilde{C}$, and B be a classical crystal of type C. We say that $\dot{p}$ *induces a twisted isomorphism of crystals* if there exists a bijection of crystals $p : B \cup \{\mathbf{0}\} \to B' \cup \{\mathbf{0}\}$ satisfying

$$p(b) = \mathbf{0} \quad \text{if and only if} \quad b = \mathbf{0}, \quad \text{and} \tag{3.1}$$

$$p \circ f_i(b) = f_{\dot{p}(i)} \circ p(b) \quad \text{and} \quad p \circ e_i(b) = e_{\dot{p}(i)} \circ p(b) \tag{3.2}$$

for all $i \in I \setminus \{\dot{p}^{-1}(0)\}$ and all $b \in B$.

We frequently abuse notation and denote B' by $p(B)$ even though the isomorphism $p : B \to p(B)$ may not be unique.

If we are given two classical crystals B and B', and there exists a Dynkin diagram automorphism $\dot{p}$ that induces a twisted isomorphism between B and B', then we say that B and B' are *twisted-isomorphic*.

Definition 3.2. Let B be a directed graph with edges labeled by I. Then B is called *regular* if for any 2-subset $J \subset I$, we have that the restriction of B to its J-arrows is a classical rank two crystal.

Definition 3.3. Let B be a classical crystal with index set I. Suppose $\widetilde{B}$ is a labeled directed graph on the same nodes as B and with the same I-arrows, but with an additional set of 0-arrows. If $\widetilde{B}$ is regular, then we say that $\widetilde{B}$ is a *combinatorial affine structure* for B.

Remark 3.4. Although we do not assume that $\widetilde{B}$ is a crystal graph for a $U_q'(\mathfrak{g})$-module, Kashiwara [Kas02,Kas05] has shown that the crystals of such modules must be regular and have weights at level 0. Therefore, we will compute the 0-weight $\lambda_0 \Lambda_0$ of the nodes b in a classical crystal from the classical weight $\lambda = \sum_{i \in I} \lambda_i \Lambda_i = \mathrm{wt}(b)$ using the formula given in Eq. (2.1) (recall that I in this section is the index set of the Dynkin diagram without 0).

Remark 3.5. Here are some consequences of Definitions 3.1 and 3.3.

(1) Any crystal $p(B)$ induced by $\dot{p}$ is just a classical crystal that is isomorphic to B up to relabeling. In particular, any graph automorphism $\dot{p}$ induces at least one twisted isomorphism p: If we view B as an edge-labeled directed graph, the image of p is given on the same nodes as B by relabeling all of the arrows according to $\dot{p}$. On the other hand, it is important to emphasize that there is no canonical labeling for the nodes of $p(B)$. Also, some crystal graphs may have additional symmetry which lead to multiple twisted isomorphisms of crystals associated with a single graph automorphism $\dot{p}$.

(2) For $b \in B$, we have $\varphi(p(b)) = \sum_{i \in I} \varphi_{\dot{p}^{-1}(i)}(b) \Lambda_i$ and $\varepsilon(p(b)) = \sum_{i \in I} \varepsilon_{\dot{p}^{-1}(i)}(b) \Lambda_i$. In addition, we can compute the 0-weight of any node in B by Remark 3.4. Therefore, $\dot{p}$ permutes all of the affine weights, in the sense that

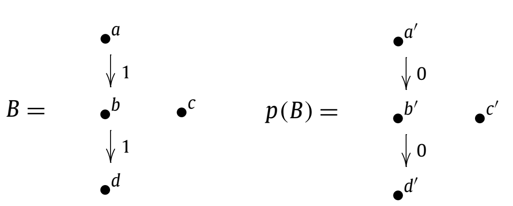
$$\mathrm{wt}_i(b) = \mathrm{wt}_{\dot{p}(i)}\big(p(b)\big) \quad \text{for all } b \in B \text{ and } i \in I \cup \{0\} .$$

(3) Since the node $\dot{p}(0)$ becomes the affine node in $p(B)$, it is sometimes possible to define a combinatorial affine structure for B "by promotion." Namely, we define f_0 on B to be $p^{-1} \circ f_{\dot{p}(0)} \circ p$. Note that in order for this to succeed, we must take the additional step of identifying the image $p(B)$ with a canonically labeled classical crystal so that we can infer the $f_{\dot{p}(0)}$ edges.

Example 3.6. The E_6 Dynkin diagram automorphism of order two that interchanges nodes 1 and 6 induces the dual map between $B(\Lambda_1)$ and $B(\Lambda_6)$.

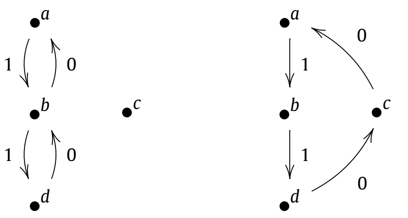
Example 3.7. Let $\dot{p}$ be the unique $E_6^{(1)}$ Dynkin diagram automorphism of order three sending node 0 to 1. There is no twisted isomorphism of $B(\Lambda_2)$ to itself that is induced by $\dot{p}$. To see this, consider the six nodes of weight 0 inside $B(\Lambda_2)$. Observe that there is precisely one node of weight 0 lying in the center of an i-string for each $i \in \{1, 2, \ldots, 6\}$. The twisted isomorphism p must send the node lying in the middle of a 6-string to one which lies only in the middle of a 0-string and is not connected to any other classical edges by Eq. (3.2). But no such node exists in $B(\Lambda_2)$. This is in agreement with (1.1) that an affine structure for the adjoint node exists on $B(\Lambda_2) \oplus B(0)$.

Example 3.8. Consider the crystal B of type A_1 shown below.



The only nontrivial graph automorphism $\dot{p}$ of the affine Dynkin diagram of type $A_1^{(1)}$ interchanges 0 and 1, which induces $p(B)$ as shown. However, constructing an affine structure on B by promotion requires choosing another map from $p(B)$ back to B.

By considering the level-0 weight, we must identify a' with d as well as d' with a. Since there is no restriction on $\varphi_0(b)$ nor $\varepsilon_0(b)$ for $b \in \{b, c\}$ from the given data, the other two nodes are undetermined. Hence, there are two identifications which give rise to distinct 0-arrows for B.



This example shows how twisted isomorphisms of order two can give rise to multiple affine structures.

The Dynkin diagram of $E_6^{(1)}$ has an automorphism of order three that we can use to construct combinatorial affine structures by promotion.

Theorem 3.9. *Let B be a classical E_6 crystal. Suppose there exists a bijection $p : B \to B$ that is a twisted isomorphism satisfying $p \circ f_1 = f_6 \circ p$, and suppose that p has order three. Then, there exists a combinatorial affine structure on B. This structure is given by defining f_0 to be $p^2 \circ f_1 \circ p$.*

Proof. If we apply p on the left and right of $pf_1 = f_6 p$, we obtain $ppf_1 p = pf_6 pp$. Since p has order three, this is

$$p^{-1} f_1 p = p f_6 p^{-1}. \tag{3.3}$$

Because p is a bijection on B, we may define 0-arrows on B by the map $p^{-1} f_1 p$. By the hypotheses, p must be induced by the unique Dynkin diagram automorphism $\dot{p}$ of order three that sends node 0 to 1.

To verify that this affine structure satisfies Definition 3.3, we need to check that restricting B to $\{0, i\}$-arrows is a crystal for all $i \in I$. Each of these restrictions corresponds to a rank 2 classical crystal, and Stembridge has given local rules in [Ste03] that characterize such classical crystals in simply laced types. Moreover, these rules depend only on calculations involving $\varphi_i(b)$ and $\varepsilon_i(b)$ at each node $b \in B$. Therefore, to check the restrictions for $i = 1, 2, 3, 4, 5$, it suffices by Eq. (3.2) to apply p and note that Stembridge's rules are satisfied for the restriction of B to $\{1, \dot{p}(i)\}$-arrows, since B is a classical crystal. Here, $\dot{p}(i) = 6, 3, 5, 4, 2$, respectively. To check the restriction for $i = 6$, we use Eq. (3.3) obtaining

$$ppf_6 = ppf_6 p^{-1} p = pp^{-1} f_1 pp = f_1 pp$$

and

$$ppf_0 = pppf_6 p^{-1} = f_6 pp.$$

These imply that we can apply $p^2 = p^{-1}$ and note that Stembridge's rules are satisfied for the restriction of B to $\{6, 1\}$-arrows, since B is a classical crystal.

Hence, we obtain a combinatorial affine structure for B. $\square$

From now on, we use the notation p to denote a twisted isomorphism induced by $\dot{p}$ sending

$$0 \mapsto 1 \mapsto 6 \mapsto 0, \qquad 2 \mapsto 3 \mapsto 5 \mapsto 2, \qquad 4 \mapsto 4.$$

Also, we let $\dot{p}$ act on the affine weight lattice as in Remark 3.5(2).

3.2. Composition graphs

Let $I = \{1, 2, \dots, 6\}$ be the index set for the Dynkin diagram of E_6, and $\widetilde{I} = I \cup \{0\}$ be the index set of $E_6^{(1)}$. Suppose $J \subset I$. Consider a classical crystal B of the form $\bigoplus B(k\Lambda)$ where Λ is a fundamental weight and we sum over some collection of nonnegative integers k. Let $H^J(B)$ denote the $(I \setminus J)$-highest weight nodes of B. We will study affine crystals with B as underlying classical crystal. For a given such affine crystal, let $H^{J;0}(B)$ be the $(\widetilde{I} \setminus J)$-highest weight nodes. Using the level 0 hypothesis of Remark 3.4, we can prove properties of $H^{J;0}(B)$ for any given affine crystal with B as underlying classical crystal.

Definition 3.10. Fix $J \subset I$ and form directed graphs G_J and $G_{J;0}$ as follows.

We construct the vertices of G_J and $G_{J;0}$ iteratively, beginning with all of the $(I \setminus J)$-highest weight nodes of $B(\Lambda)$. Then, we add all of the vertices $b \in B(\Lambda)$ such that

$$\begin{aligned}&\{i \in I\colon \varepsilon_i(b) > 0\}\\&\quad \subset J \cup \{i \in I\colon \text{there exists } b' \in G_J \text{ with } b \otimes b' \text{ pairwise weakly increasing and } \varphi_i(b') > 0\}\end{aligned}$$

to G_J. Moreover, if b also satisfies the property that there exists $b' \in G_{J;0}$ with $b \otimes b'$ pairwise weakly increasing and $\mathrm{wt}_0(b') > 0$ whenever $\mathrm{wt}_0(b) < 0$, then we add b to $G_{J;0}$. We repeat this construction until no new vertices are added. This process eventually terminates since $B(\Lambda)$ is finite.

The edges of G_J and $G_{J;0}$ are determined by the pairwise weakly increasing condition described in Definition 2.1. Note that some nodes may have loops. We call G_J and $G_{J;0}$ the *complete composition graph* for J and $J;0$, respectively.

Lemma 3.11. *Every element of $H^J(B)$ and $H^{J;0}(B)$ is a pairwise weakly increasing tensor product of vertices that form a directed path in G_J, respectively $G_{J;0}$, where the element in $B(0) \subset H^J(B)$ is identified with the empty tensor product.*

Proof. We perform induction on the number of tensor factors k to show that the algorithm in Definition 3.10 produces all of the elements of $H^J(B)$ and $H^{J;0}(B)$ from component $B(k\Lambda)$. The base case of $k = 1$ is satisfied because we initially add all of the $(I \setminus J)$-highest weight nodes of $B(\Lambda)$ to the complete composition graph, and $H^{J;0}(B) \subset H^J(B)$.

For the induction step, observe that we branch on the left by the tensor product rule. That is, when $b \otimes b'$ is highest weight, we must have that b' is highest weight. If there exists $b \in G_J$ with $\varepsilon_i(b) > 0$ where

$$i \notin J \cup \{i \in I\colon \varphi_i(b') > 0 \text{ for some } b' \in G_J \text{ such that } b \otimes b' \text{ is pairwise weakly increasing}\}$$

then no tensor product of nodes that includes b can ever cancel the $+$ from $\varepsilon_i(b) > 0$ in the tensor product rule. Therefore, no tensor product of nodes that includes b can be $(I \setminus J)$-highest weight.

Similarly for the case of $J;0$, if there exists $b \in G_{J;0}$ with $\mathrm{wt}_0(b) < 0$ and there is no $b' \in G_{J;0}$ with $b \otimes b'$ pairwise weakly increasing and $\mathrm{wt}_0(b') > 0$, then we can conclude that wt_0 of any tensor product of nodes that starts with b is negative. Since every rightmost factor of a highest weight tensor product must be highest weight, this would imply that no tensor product of nodes that includes b can be $(\widetilde{I} \setminus J)$-highest weight.

Hence, every $(I \setminus J)$-highest weight node is given by a pairwise weakly increasing tensor product of vertices from G_J, and every $(\widetilde{I} \setminus J)$-highest weight node is given by a pairwise weakly increasing tensor product of vertices from $G_{J;0}$. □

We say that the vertices of G_J are *transitively closed* if $b \otimes c$ is pairwise weakly increasing whenever $b \otimes b'$ and $b' \otimes c$ are pairwise weakly increasing for all $b, b', c \in G_J$. Although it is not obvious from

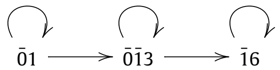


Fig. 4. Composition graph for $I \setminus \{1\}$-highest weight nodes in $B(\Lambda_1)$.

Definition 2.1 whether the pairwise weakly increasing condition is generally transitive, it is always a finite computation to verify that the vertices of G_J are transitively closed when J is fixed. Moreover, it is straightforward to verify that all of the vertex sets that explicitly appear in this work are transitively closed.

Therefore, we will typically draw only those edges of the complete composition graph G_J that cannot be inferred by transitivity, and we refer to this as the (*reduced*) *composition graph*. We will also abuse notation and refer to this reduced composition graph as G_J. We say that a *chain* is any collection of vertices that form a subgraph of a directed path in a reduced composition graph. Lemma 3.11 shows that we may identify nodes of $H^J(B)$ from component $B(k\Lambda)$ of B with chains in the reduced composition graph G_J having exactly k vertices. Analogues of all the definitions and statements given in the previous two paragraphs hold for $G_{J;0}$ and $H^{J;0}(B)$ as well.

We will see several examples of composition graphs in the following sections.

3.3. *Affine structures associated to Λ_1 and Λ_6*

Let $r \in \{1, 6\}$. By [KKM+92, Proposition 3.4.4], a crystal basis for the Kirillov–Reshetikhin module associated to $s\Lambda_r$ exists. We denote this crystal by $B^{r,s}$. It follows from [Cha01] that $B^{r,s} \cong B(s\Lambda_r)$ as classical crystals. In this section, we construct a combinatorial model for $B^{r,s}$ in the sense of Definition 3.3 using the order three Dynkin diagram automorphism of $E_6^{(1)}$.

Let $I = \{1, 2, 3, 4, 5, 6\}$ be the index set of the E_6 Dynkin diagram, $J = \{0, 2, 3, \ldots, 6\}$, and $K = I \setminus \{1\} = \{2, 3, 4, 5, 6\}$. In this section, we use the weakly increasing characterization given in Proposition 2.12. This characterization implies that the pairwise weakly increasing condition is transitive, so we draw reduced composition graphs.

Lemma 3.12. *For $r \in \{1, 6\}$, the K-highest weight nodes in $B(s\Lambda_r)$ are distinguished by their K-weights.*

Proof. The composition graph for the K-highest weight nodes for $B(\Lambda_1)$ is shown in Fig. 4. Therefore, by Lemma 3.11 all of the K-highest weight nodes in $B(s\Lambda_1)$ are of the form

$$\bar{0}1^{\otimes a} \otimes \bar{0}\bar{1}3^{\otimes b} \otimes \bar{1}6^{\otimes c}$$

and these nodes are all distinguished by their $\{3, 6\}$-weight together with $s = a + b + c$.

Similarly, the K-highest weight nodes for $B(s\Lambda_6)$ are of the form

$$\bar{0}6^{a} \otimes \bar{0}\bar{1}2^{b} \otimes \bar{1}0^{c}$$

which are also distinguished by their K-weight for fixed $s = a + b + c$. $\square$

Theorem 3.13. *Let $r \in \{1, 6\}$ and $s \geqslant 1$. There exists a unique twisted isomorphism $p : B(s\Lambda_r) \to B(s\Lambda_r)$ of order three, such that node $b \in B(s\Lambda_r)$ is mapped to node $p(b)$ with affine level-0 weight $\dot{p}(\mathrm{wt}(b))$.*

Proof. We state the proof for $r = 1$. The proof for $r = 6$ is analogous.

By constructing the composition graph shown in Fig. 5 and applying Lemma 3.11, the $I \setminus \{6\}$-highest weight nodes of $B(s\Lambda_1)$ all have the form

$$\bar{0}1^{\otimes a} \otimes \bar{0}\bar{6}2^{\otimes b} \otimes \bar{6}0^{\otimes c}.$$

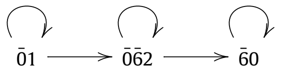


Fig. 5. Composition graph for $I \setminus \{6\}$-highest weight nodes in $B(\Lambda_1)$.

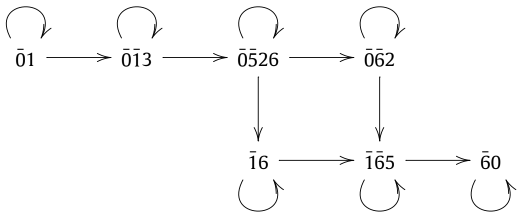


Fig. 6. Composition graph for $\widetilde{I} \setminus \{6, 1\}$-highest weight nodes in $B(\Lambda_1)$.

All of these nodes are uniquely determined by their affine level-0 weight

$$(c-b-a)\Lambda_0 + a\Lambda_1 + b\Lambda_2 - (b+c)\Lambda_6.$$

Any twisted isomorphism p induced by $\dot{p}$ must send such a node to one which is $I \setminus \{1\}$-highest weight, with affine level-0 weight

$$(c-b-a)\Lambda_1 + a\Lambda_6 + b\Lambda_3 - (b+c)\Lambda_0.$$

As we have seen in the proof of Lemma 3.12, the $I \setminus \{1\}$-highest weight nodes all have the form

$$\bar{0}1^{\otimes a'} \otimes \bar{0}\bar{1}3^{\otimes b'} \otimes \bar{1}6^{\otimes c'}$$

and are all uniquely determined by their affine level-0 weight

$$-\big(a'+b'\big)\Lambda_0 + \big(a'-b'-c'\big)\Lambda_1 + b'\Lambda_3 + c'\Lambda_6.$$

This system has the unique solution

$$a' = c, \qquad b' = b, \qquad c' = a,$$

and we can extend by Eq. (3.2) to define p on all of $B(s\Lambda_1)$.

If we apply p again, we send the $I \setminus \{1\}$-highest weight nodes to $\widetilde{I} \setminus \{6, 1\}$-highest weight nodes with affine level-0 weight $-(a'+b')\Lambda_1 + (a'-b'-c')\Lambda_6 + b'\Lambda_5 + c'\Lambda_0$. The composition graph $G_{6,1;0}$ is shown in Fig. 6 and by Lemma 3.11 every $\widetilde{I} \setminus \{6, 1\}$-highest weight node can be represented as a tensor product of nodes that form a path in this graph. The image of an $I \setminus \{1\}$-highest weight node h under p must have $\mathrm{wt}_2(p(h)) = 0$ and $\mathrm{wt}_3(p(h)) = 0$ so no $\bar{0}\bar{1}3$, $\bar{0}\bar{5}26$ nor $\bar{0}\bar{6}2$ nodes appear in the tensor product. Moreover, the multiplicity of $\bar{1}\bar{6}5$ must be equal to $\mathrm{wt}_3(h)$ and the multiplicity of $\bar{1}6$ must be equal to $\varphi_1(h)$. Finally, we must have $\mathrm{wt}_1(p(h)) = \mathrm{wt}_0(h)$ from which it follows that no $\bar{0}1$ nodes appear in the tensor product.

Hence, we have that p sends $\bar{0}1^{\otimes a'} \otimes \bar{0}\bar{1}3^{\otimes b'} \otimes \bar{1}6^{\otimes c'}$ to $\bar{1}6^{\otimes a'} \otimes \bar{1}\bar{6}5^{\otimes b'} \otimes \bar{6}0^{\otimes c'}$. Finally, observe that p sends these $\widetilde{I} \setminus \{6, 1\}$-highest weight nodes to $I \setminus \{6\}$-highest weight nodes with weight $-(a'+b')\Lambda_6 + (a'-b'-c')\Lambda_0 + b'\Lambda_2 + c'\Lambda_1$. Therefore, the twisted isomorphism p has order three. □

Corollary 3.14. *Let $s \geqslant 1$. The twisted isomorphism p of Theorem 3.13 defines a combinatorial affine crystal structure $\widetilde{B(s\Lambda_1)}$ on $B(s\Lambda_1)$. Moreover, if we restrict the arrows in $\widetilde{B(s\Lambda_1)}$ to J, which we denote by $\widetilde{B(s\Lambda_1)}|_J$, then*

$$\widetilde{B(s\Lambda_1)}|_J \cong B(s\Lambda_6). \tag{3.4}$$

The analogue of Corollary 3.14 for $B(s\Lambda_6)$ also exists.

Proof of Corollary 3.14. Since p of Theorem 3.13 has order three, it defines a combinatorial affine structure on $B(s\Lambda_1)$ by Theorem 3.9.

Any J-highest weight node b must also be an $I \setminus \{1\}$-highest weight node, and these all have the form

$$\bar{0}1^{\otimes a'} \otimes \bar{0}\bar{1}3^{\otimes b'} \otimes \bar{1}6^{\otimes c'}$$

with affine level-0 weight

$$-\bigl(a' + b'\bigr)\Lambda_0 + \bigl(a' - b' - c'\bigr)\Lambda_1 + b'\Lambda_3 + c'\Lambda_6.$$

If we further require that $\mathrm{wt}_0(b) \geqslant 0$, then we see that a' and b' must be 0. Hence, $b = \bar{1}6^{\otimes s}$ with J-weight $s\Lambda_6$. □

Theorem 3.15. *Let $\widetilde{B}$, $\widetilde{B}'$ be two affine type $E_6^{(1)}$ crystals. Suppose there exists a $\{1,2,3,4,5,6\}$-isomorphism Ψ_0 and a $\{0,2,3,4,5,6\}$-isomorphism Ψ_1 where*

$$\begin{aligned} \widetilde{B}|_{\{1,2,3,4,5,6\}} &\xrightarrow{\Psi_0} \widetilde{B}'\big|_{\{1,2,3,4,5,6\}} \cong B(s\Lambda_1), \\ \widetilde{B}|_{\{0,2,3,4,5,6\}} &\xrightarrow{\Psi_1} \widetilde{B}'\big|_{\{0,2,3,4,5,6\}} \cong B(s\Lambda_6). \end{aligned} \tag{3.5}$$

Then $\Psi_0(b) = \Psi_1(b)$ for all $b \in \widetilde{B}$ and so there exists a $\{0,1,2,3,4,5,6\}$-isomorphism $\Psi : \widetilde{B} \cong \widetilde{B}'$.

Proof. Set $K = \{2,3,4,5,6\}$. Note that if $\Psi_0(b) = \Psi_1(b)$ for a b in a given K-component $\mathcal{C}$, then $\Psi_0(b') = \Psi_1(b')$ for all $b' \in \mathcal{C}$ since $e_i\Psi_0(b') = \Psi_0(e_ib')$ and $e_i\Psi_1(b') = \Psi_1(e_ib')$ for $i \in K$. Furthermore, observe that Ψ_0 and Ψ_1 preserve weights by Remark 3.4. That is, $\mathrm{wt}(b) = \mathrm{wt}(\Psi_0(b)) = \mathrm{wt}(\Psi_1(b))$ for all $b \in \widetilde{B}$.

Since e_i commutes with Ψ_0 and Ψ_1 for $i \in K$, it follows that K-components in $\widetilde{B}$ must map to K-components in $\widetilde{B}'$. Restricted to I or J, the images of the K-components in $\widetilde{B}$ are also isomorphic to K-components in $B(s\Lambda_1)$ under Ψ_0 and to K-components in $B(s\Lambda_6)$ under Ψ_1. However, the K-highest weight elements in $B(s\Lambda_1)$ and $B(s\Lambda_6)$ are determined by their weights by Lemma 3.12. Hence we must have $\Psi_0(b) = \Psi_1(b)$ for all $b \in \widetilde{B}$. □

Corollary 3.16. *For $r \in \{1,6\}$ and $s \geqslant 1$, the combinatorial affine structure $\widetilde{B(s\Lambda_r)}$ of Corollary 3.14 is isomorphic to the Kirillov–Reshetikhin crystal $B^{r,s}$.*

Proof. By [Cha01], $B^{r,s} \cong B(s\Lambda_r)$ for $r = 1,6$ as a classical crystal. By [KKM+92, Proposition 3.4.4], $B^{r,s}$ for $r = 1,6$ exists since it is irreducible as a classical crystal.

Let us now restrict to $r = 1$ as the case $r = 6$ is analogous. To show that $B^{1,s} \cong B(s\Lambda_6)$ as a J-crystal, it suffices to show that there exists a corresponding highest weight vector since the crystal is irreducible. However, the element of level-0 weight $s(\Lambda_6 - \Lambda_1)$ is precisely this element.

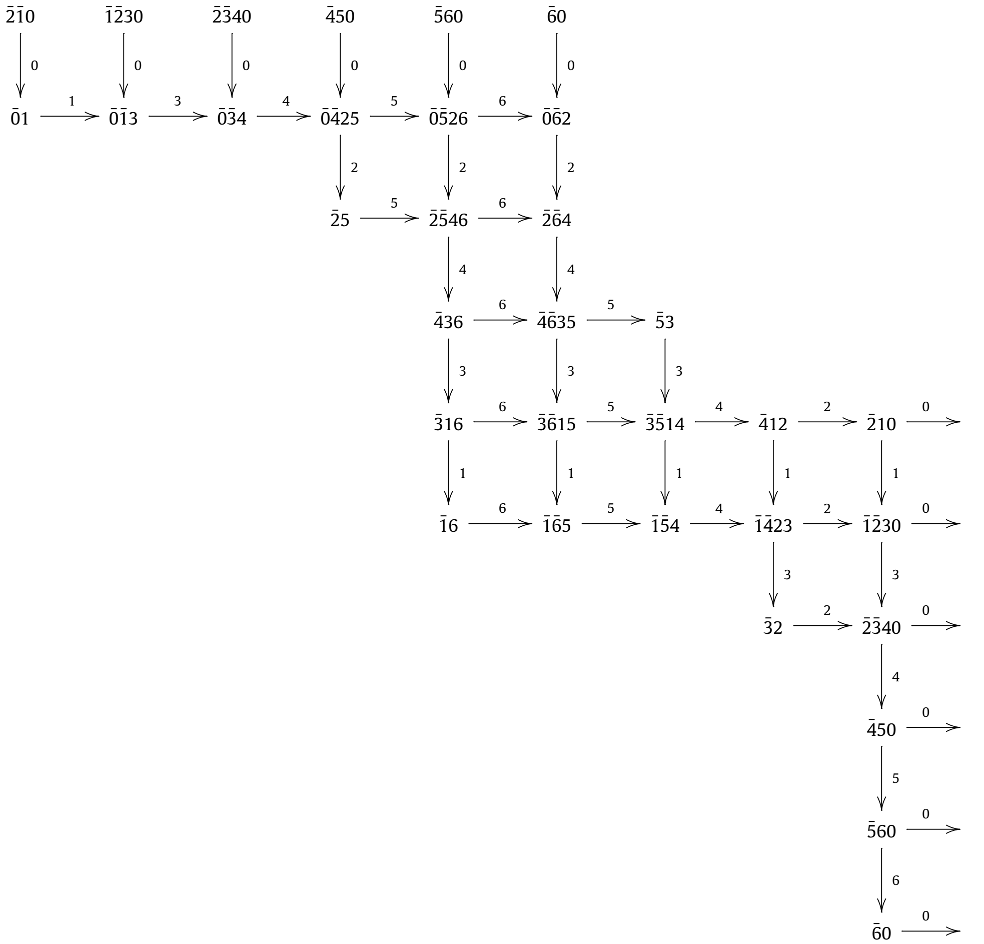


Fig. 7. Crystal graph for $B^{1,1}$ of type $E_6^{(1)}$.

Since $B^{1,r}|_I \cong \widetilde{B(s\Lambda_1)}|_I \cong B(s\Lambda_1)$ and $B^{1,r}|_J \cong \widetilde{B(s\Lambda_1)}|_J \cong B(s\Lambda_6)$ by the above arguments and (3.4), by Theorem 3.15 we must have $B^{1,s} \cong \widetilde{B(s\Lambda_1)}$ as affine crystals. □

The resulting affine crystal $B^{1,1}$ is shown in Fig. 7.

3.4. *Affine structures associated to* Λ_2

By [KKM+92, Proposition 3.4.5], a crystal basis $B^{2,s}$ for the Kirillov–Reshetikhin module associated to $s\Lambda_2$ exists. It follows from [Cha01] that $B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\Lambda_2)$ as classical crystals. We will refer to $B(k\Lambda_2)$ as the *kth component* of $\bigoplus_{k=0}^{s} B(k\Lambda_2)$. In this section, we will show how to construct a combinatorial affine structure for $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ using Theorem 3.9.

We use the weakly increasing characterization given in Proposition 2.13 for our work in this section. Let H_s^J denote the $(I \setminus J)$-highest weight nodes of $\bigoplus_{k=0}^{s} B(k\Lambda_2)$. The composition graphs for $J = \{6\}$ and $J = \{1\}$ are shown in Figs. 8 and 9, respectively. Observe that the nodes a and c were added to $H_1^{\{6\}}$ in the course of the algorithm described in Definition 3.10 to obtain G_6. The nodes of weight 0 do not have loops by Proposition 2.13. A finite computation shows that the vertex sets of

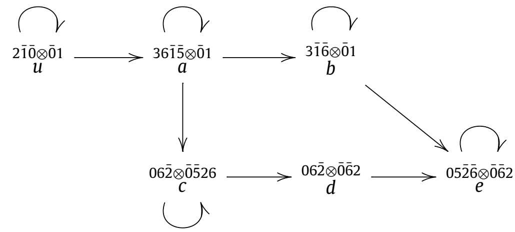


Fig. 8. Composition graph G_6 for $I \setminus \{6\}$-highest weight nodes.

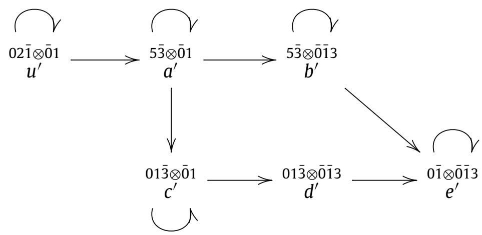


Fig. 9. Composition graph G_1 for $I \setminus \{1\}$-highest weight nodes.

these composition graphs are transitively closed, so Lemma 3.11 models the nodes of $H_s^{\{6\}}$ and $H_s^{\{1\}}$ as chains in G_6 and G_1, respectively.

Example 3.17. We see from the composition graph that

$$(2\bar{1}\bar{0} \otimes \bar{0}1) \otimes (2\bar{1}\bar{0} \otimes \bar{0}1) \otimes (06\bar{2} \otimes \bar{0}\bar{5}26) \otimes (05\bar{2}\bar{6} \otimes \bar{0}\bar{6}2)$$

is a typical node in $H_4^{\{6\}}$.

Definition 3.18. Let $\mathcal{C}(m)$ denote the set

$$\{(L_2, L_3, L_5) \in \mathbb{Z}_{\geqslant 0}\colon\ L_2 + L_3 + L_5 = m\}$$

of *weak compositions* of m into 3 parts.

Proposition 3.19. *There is a bijection from the $I \setminus \{6\}$-highest weight nodes of $B(k\Lambda_2)$ to $\bigcup_{m=0}^{k} \mathcal{C}(m)$ such that a node corresponding to the weak composition $L_2 + L_3 + L_5 = m$ has $I \setminus \{6\}$-weight $L_2\Lambda_2 + L_3\Lambda_3 + L_5\Lambda_5$.*

In particular, the $I \setminus \{6\}$-highest weight nodes of $B(k\Lambda_2)$ are determined by their $\{2, 3, 5\}$-weight, and for any such node b, we have

$$k = \varphi_6(b) + \mathrm{wt}_2(b) + \mathrm{wt}_3(b) + \mathrm{wt}_5(b).$$

Proof. By Lemma 3.11, the $I \setminus \{6\}$-highest weight nodes of $B(k\Lambda_2)$ correspond to chains of length k in G_6. Moreover, we claim that for each value of k and weak composition $L_2 + L_3 + L_5 = m$ with $0 \leqslant m \leqslant k$, there exists a unique chain of length k in G_6 having $I \setminus \{6\}$-weight $L_2\Lambda_2 + L_3\Lambda_3 + L_5\Lambda_5$.

Denote the multiplicities of the vertices by u, a, b, c, d, e corresponding to the labeling in Fig. 8. All of these multiplicities must be nonnegative, and we also have $d \in \{0, 1\}$ by Proposition 2.13. There are two maximal chains in G_6 and we will write a system of linear equations for each of them.

The equations among the multiplicities that are induced by the upper maximal chain of the graph are

$$L_2 = u, \qquad L_5 = e - a,$$
$$L_3 = a + b, \qquad k = u + a + b + e$$

and we can solve these to obtain

$$a = k - (L_2 + L_3 + L_5),$$
$$e = k - (L_2 + L_3),$$
$$b = 2L_3 + L_5 + L_2 - k.$$

Note that $a, e \geqslant 0$, but b may be < 0.

The equations induced by the lower maximal chain are

$$L_2 = u, \qquad L_5 = e - c - a,$$
$$L_3 = a, \qquad k = u + a + c + d + e$$

and we can solve these to obtain

$$2e + d = k + L_5 - L_2$$

which has a unique solution in nonnegative integers with $d \in \{1, 0\}$, and

$$c = e - (L_5 + L_3).$$

Now, $a, d, e \geqslant 0$. But $c \geqslant 0$ if and only if $2c \geqslant 0$ if and only if

$$k + L_5 - L_2 - d - 2(L_5 + L_3) = k - L_2 - L_5 - 2L_3 - d \geqslant 0.$$

This occurs when $d = 0$ and $b \leqslant 0$ or when $d = 1$ and $b < 0$. Moreover, the solutions for the two chains in the graph agree when $b = 2L_3 + L_5 + L_2 - k = 0$. Hence, we obtain a unique solution in all cases of the parameters k, L_2, L_3, L_5.

In addition, we have that φ_6 and ε_6 are uniquely determined by L_2, L_3, L_5 and k. The upper path equations give

$$\varphi_6 = a = k - L_2 - L_3 - L_5 \quad \text{and} \quad \varepsilon_6 = b + 2e = k - L_2 + L_5.$$

The lower path equations give

$$\varphi_6 = a + 2c + d = L_3 + k + L_5 - L_2 - 2(L_3 + L_5) = k - L_2 - L_3 - L_5 \quad \text{and}$$
$$\varepsilon_6 = d + 2e = k - L_2 + L_5.$$

So φ_6 and ε_6 agree in both cases.

Finally, ε_i and φ_i for $i = 1, 4$ of any solution is zero. $\square$

Remark 3.20. Proposition 3.19 can also be interpreted as a branching rule from classical E_6 to D_5.

Corollary 3.21. *The $I \setminus \{6\}$-highest weight nodes of $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ are uniquely determined by their $\{2, 3, 5\}$-weight together with φ_6. The $I \setminus \{1\}$-highest weight nodes of $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ are uniquely determined by their $\{2, 3, 5\}$-weight together with φ_1.*

Proof. The first statement follows directly from Proposition 3.19, and the second statement has an analogous proof.

The composition graph for the $I \setminus \{1\}$-highest weight nodes is shown in Fig. 9. Note that $d' \otimes d'$ is not weakly increasing. When we set up the analogous set of equations to solve for the multiplicities u', a', b', c', d', e' in terms of the parameters k, L_2, L_3, L_5, we obtain equations derived from those in the proof of Proposition 3.19 by fixing Λ_2 and interchanging Λ_3 with Λ_5. $\square$

We are now in a position to state our main result.

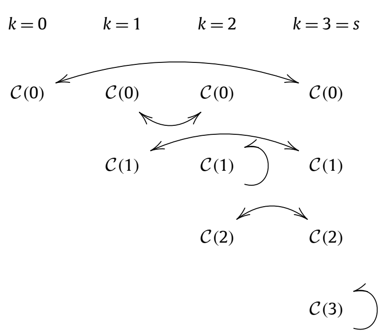
Theorem 3.22. *There exists a unique twisted isomorphism* $p : \bigoplus_{k=0}^{s} B(k\Lambda_2) \to \bigoplus_{k=0}^{s} B(k\Lambda_2)$ *of order three. This isomorphism sends an* $I \setminus \{6\}$*-highest weight node* b *from component* k *to the unique* $I \setminus \{1\}$*-highest weight node* b' *in component* $(s-k) + (\mathrm{wt}_2(b) + \mathrm{wt}_3(b) + \mathrm{wt}_5(b))$ *satisfying* $\mathrm{wt}_{\dot{p}(i)}(b') = \mathrm{wt}_i(b)$ *for each* $i \in \{2, 3, 5\}$.

The proof of this theorem is given at the end of this section. We first discuss some consequences, examples, and preliminary results.

Corollary 3.23. *The twisted isomorphism* p *of Theorem* 3.22 *defines a combinatorial affine crystal structure which is isomorphic to the Kirillov–Reshetikhin crystal* $B^{2,s}$.

Proof. By Theorem 3.9, p yields a combinatorial affine structure for $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ via Eq. (3.2). The results of Chari [Cha01] show that $B^{2,s}$ has the same classical decomposition. By [KMOY07, Theorem 6.1], we have that if a combinatorial affine structure for $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ exists, then it is isomorphic to the Kirillov–Reshetikhin crystal $B^{2,s}$. $\square$

Example 3.24. Suppose $s = 3$. Then, $H_s^{\{6\}}$ decomposes into $(s+1)$ components according to which summand $B(k\Lambda_2)$ the node lies in. Each of these components further decomposes as $\bigcup_{m=0}^{k} \mathcal{C}(m)$ by Proposition 3.19. Hence, we have the following schematic of $H_s^{\{6\}}$ in which the twisted isomorphism p reflects the $\mathcal{C}(m)$ components along rows. The twisted isomorphism p also twists the weights according to $\dot{p}$, which is not shown explicitly. The resulting node lies in $H_s^{\{1\}}$.



To compute $p(b)$ for

$$b = (2\bar{1}\bar{0} \otimes \bar{0}1) \otimes (06\bar{2} \otimes \bar{0}\bar{5}26) \otimes (05\bar{2}\bar{6} \otimes \bar{0}\bar{6}2)$$

we observe that $\mathrm{wt}_2(b) = 1$, $\mathrm{wt}_3(b) = 0$, $\mathrm{wt}_5(b) = 0$ so the composition associated to b is $(1, 0, 0)$. According to Theorem 3.22, p maps b to the unique chain of length 1 in G_1 corresponding to the composition $(0, 1, 0)$, namely $b' = 0\bar{1} \otimes \bar{0}\bar{1}3$. In general, we define $f_0(b)$ by $p^{-1} \circ f_1 \circ p(b)$. In this case, $f_0(b) = 0$.

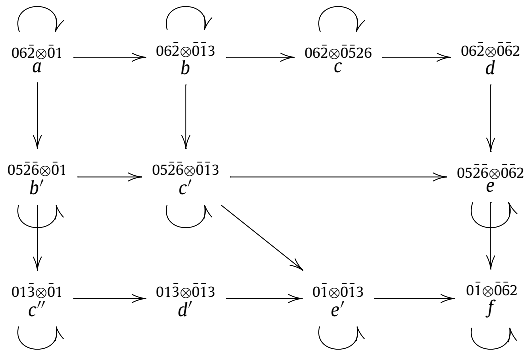


Fig. 10. Graph $G_{6,1;0}$ of weakly increasing $\widetilde{I}\setminus\{6,1\}$-highest weight nodes.

The composition graph for the $(\widetilde{I}\setminus\{6,1\})$-highest weight nodes is shown in Fig. 10. This graph was constructed using the algorithm described in Definition 3.10. It is more complicated than the composition graphs G_6 and G_1 because we are taking highest weight nodes with respect to the complement of two classical Dynkin diagram nodes. Also, we use the level 0 hypothesis to compute affine weights and our composition graph includes only those nodes that can contribute to chains having 0-highest weight. A finite computation shows that the vertex set of $G_{6,1;0}$ is transitively closed, so the $\widetilde{I}\setminus\{6,1\}$-highest weight nodes correspond to chains in $G_{6,1;0}$ by Lemma 3.11.

In order to prove Theorem 3.22, we study how p maps chains from G_1 to chains in $G_{6,1;0}$.

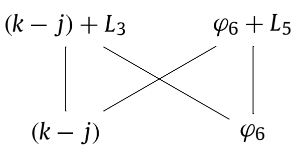
Lemma 3.25. *Let b be an $I\setminus\{1\}$-highest weight node of $B(j\Lambda_2)$ corresponding to the weak composition (L_2, L_3, L_5). Then, for every $j \leqslant k \leqslant s$, there exists a unique $\widetilde{I}\setminus\{6,1\}$-highest weight node b' in $B(k\Lambda_2)$ such that $\mathrm{wt}_i(b') = \mathrm{wt}_{\dot{p}^{-1}(i)}(b) = L_{\dot{p}^{-1}(i)}$ for $i \in \{2,3,5\}$. Moreover, $\varphi_1(b') = k - j$.*

Proof. In Appendix A, we solve the equations describing how to map an $I\setminus\{1\}$-highest weight node from component j to an $\widetilde{I}\setminus\{6,1\}$-highest weight node of component k, using the equation for φ_1 from Corollary 3.21 which must become φ_6 in the image.

As shown in Appendix A, there is one system of linear equations for each of the 6 maximal chains in $G_{6,1;0}$. The set of parameters for which each case is valid is shown below.

Case 1. $(k-j) + L_3 \leqslant \varphi_6$,
Case 2. $(k-j) \leqslant \varphi_6 \leqslant (k-j) + L_3 \leqslant \varphi_6 + L_5$,
Case 3. $\varphi_6 \leqslant (k-j) \leqslant (k-j) + L_3 \leqslant \varphi_6 + L_5$,
Case 4. $\varphi_6 \leqslant (k-j) \leqslant \varphi_6 + L_5 \leqslant (k-j) + L_3$,
Case 5. $(k-j) \leqslant \varphi_6 \leqslant \varphi_6 + L_5 \leqslant (k-j) + L_3$,
Case 6. $\varphi_6 + L_5 < (k-j)$.

Observe that these cover all possible values of the parameters, because if we are not in Case 1 nor Case 6, then we have the partial order of parameters shown below.



This partial order has exactly four linear extensions corresponding precisely to Cases 2–5.

We must also show that if a particular set of parameters (L_2, L_3, L_5, j, k) is satisfied by multiple cases, then the solutions obtained from each case all agree. This can be done by hand for the sys-

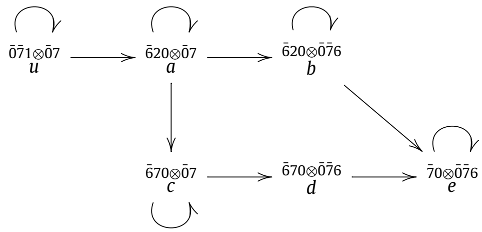


Fig. 11. Composition graph G_7 for $I \setminus \{7\}$-highest weight nodes in $B(\Lambda_1)$ for E_7.

tems described in Appendix A. In Appendix B we also describe an effective procedure that can be automated to establish this fact.

Observe that in every solution, $k - j$ must be nonnegative. Moreover, in every case, φ_1 of the solution is $k - j$. $\square$

Proof of Theorem 3.22. Fix a weight $L_2\Lambda_2 + L_3\Lambda_3 + L_5\Lambda_5$ and a component $j \leqslant s$. There is a unique $I \setminus \{1\}$-highest weight node b corresponding to these parameters by Corollary 3.21. Any twisted isomorphism p induced from the Dynkin diagram automorphism $\dot{p}$ sends b to an $\widetilde{I} \setminus \{6,1\}$-highest weight node $p(b)$ in some component, say k, and $p(p(b))$ is an $I \setminus \{6\}$-highest weight node in some component, say j'.

By Lemma 3.25, we have that a solution $p(b)$ exists and that $\varphi_1(p(b))$ is $(k - j)$. Hence,

$$j' - (L_2 + L_3 + L_5) = (k - j) \geqslant 0$$

by Corollary 3.21.

We suppose that p has order three, and work by downward induction on j, starting from the fact that nodes of component $j = s$ must go to component $k = s$, which goes to component $j' = L_2 + L_3 + L_5$. As j decreases, if we ever have $k < s$, then φ_1 with respect to $\widetilde{I} \setminus \{6,1\}$ is less than $(s - j)$. This implies that $j' < (s - j) + (L_2 + L_3 + L_5)$, and so we would map $p(b)$ onto an $I \setminus \{6\}$-highest weight node that has already appeared in the image of p. Hence, we find that $k = s$ always. This specifies a unique solution of order three for p. $\square$

3.5. A conjecture for E_7

Recall the Dynkin diagram of type $E_7^{(1)}$ shown in Fig. 1. Let $\dot{p}$ denote the unique automorphism of this diagram, so $\dot{p}$ has order two and sends the affine node 0 to node 7.

The adjoint node in E_7 is node 1, and [Cha01] has given the decomposition $B^{1,s} = \bigoplus_{k=0}^{s} B(k\Lambda_1)$ of the corresponding Kirillov–Reshetikhin crystal into classical crystals. We can form the composition graph for $J = \{7\}$ and the result is shown in Fig. 11.

This graph is essentially the same as the composition graph G_1 that we obtained for $B(\Lambda_2)$ in E_6. In particular, the classical weights $\Lambda_1, \Lambda_2, \Lambda_6, \Lambda_7$ that appear in G_7 for type E_7 correspond to $\Lambda_2, \Lambda_5, \Lambda_3, \Lambda_1$ in G_1 of type E_6. Our solution to the equations associated with G_1 in E_6 shows that there exists a unique $I \setminus \{7\}$-highest weight node of $B(k\Lambda_1)$ in E_7 having weight $L_1\Lambda_1 + L_2\Lambda_2 + L_6\Lambda_6$. That is, the $I \setminus \{7\}$-highest weight nodes of $B(k\Lambda_1)$ are in bijection with weak compositions with 3 parts. Moreover, we have that $k = \varphi_7(b) + \mathrm{wt}_1(b) + \mathrm{wt}_2(b) + \mathrm{wt}_6(b)$ for such nodes b.

Define $p : \bigoplus_{k=0}^{s} B(k\Lambda_1) \to \bigoplus_{k=0}^{s} B(k\Lambda_1)$ on the $I \setminus \{7\}$-highest weight nodes by sending $b \in B(k\Lambda_1)$ to the unique $I \setminus \{7\}$-highest weight node b' in component $(s-k) + (\mathrm{wt}_1(b) + \mathrm{wt}_2(b) + \mathrm{wt}_6(b))$ satisfying $\mathrm{wt}_{\dot{p}(i)}(b') = \mathrm{wt}_i(b)$ for each $i \in \{1,2,6\}$.

Since $\dot{p}$ does not have order three, Theorem 3.9 does not apply to prove that this construction gives a combinatorial affine structure. To get a sense of the ambiguity that can arise when working with twisted isomorphisms of order two, consider Example 3.8. It remains to show that if we define 0-arrows by $f_0 = p \circ f_7 \circ p$, then the restriction to $\{0, i\}$-arrows is a crystal for all $i \in I$. The argument given in the proof of Theorem 3.9 shows that this is true for all $i \neq 7$. Moreover, we conjecture that this is true for $i = 7$ as well.

Conjecture 3.26. *Define $p : \bigoplus_{k=0}^{s} B(k\Lambda_1) \to \bigoplus_{k=0}^{s} B(k\Lambda_1)$ as described above, and let $f_0 = p \circ f_7 \circ p$. Then f_0 commutes with f_7 so we obtain a combinatorial affine structure on $\bigoplus_{k=0}^{s} B(k\Lambda_1)$, which is isomorphic to $B^{1,s}$ of type $E_7^{(1)}$.*

We have verified this conjecture for $s \leqslant 2$.

4. Sage implementation

As illustrated in the following examples, we have implemented the crystals described in this paper in `Sage` [WSea09] and `Sage-Combinat` [SCc09]. For more information see the documentation of `Sage-Combinat` and `Sage`, in particular the crystal documentation.[1]

Sage Example 4.1. For type E_6, the building block $B(\Lambda_1)$ of Fig. 2 is accessible as follows:

```
sage: C = CrystalOfLetters(['E',6])
sage: C.list()
[[1], [-1, 3], [-3, 4], [-4, 2, 5], [-2, 5], [-5, 2, 6], [-2, -5, 4, 6],
[-4, 3, 6], [-3, 1, 6], [-1, 6], [-6, 2], [-2, -6, 4], [-4, -6, 3, 5],
[-3, -6, 1, 5], [-1, -6, 5], [-5, 3], [-3, -5, 1, 4], [-1, -5, 4], [-4, 1, 2],
[-1, -4, 2, 3], [-3, 2], [-2, -3, 4], [-4, 5], [-5, 6], [-6], [-2, 1], [-1, -2, 3]]
```

The crystal can be plotted as

```
sage: G = C.digraph()
sage: G.show(edge_labels=true, figsize=12, vertex_size=1)
```

or

```
sage: view(C, viewer = 'pdf', tightpage = True)
```

The dual crystal $B(\Lambda_6)$ can be constructed as

```
sage: C = CrystalOfLetters(['E',6], dual = True)
```

The crystal $B(\Lambda_7)$ of type E_7 can be accessed in a similar fashion. Fig. 3 was constructed as follows:

```
sage: C = CrystalOfLetters(['E',7])
sage: C.latex_file(filename.tex)
```

Sage Example 4.2. The classical crystals for type E_6 (and similarly for E_7) corresponding to arbitrary dominant weights can be constructed as follows:

```
sage: C = CartanType(['E',6])
sage: Lambda = C.root_system().weight_lattice().fundamental_weights()
sage: T = HighestWeightCrystal(C, dominant_weight=Lambda[1]+Lambda[6]+Lambda[2])
sage: T.highest_weight_vector()
[[1], [[2, -1], [1]], [6]]
sage: T.cardinality()
34749
```

[1] http://www.sagemath.org/doc/reference/combinat/crystals.html.

Sage Example 4.3. The Kirillov–Reshetikhin crystals $B^{r,s}$ for $r = 1, 6, 2$ for type E_6 are also implemented:

```
sage: K = KirillovReshetikhinCrystal(['E',6,1], 1,1)
sage: K.cardinality()
27
sage: K = KirillovReshetikhinCrystal(['E',6,1], 6,1)

sage: K = KirillovReshetikhinCrystal(['E',6,1], 2,1)
sage: K.classical_decomposition()
Finite-dimensional highest weight crystal of type ['E', 6] and
dominant weight(s) [0, Lambda[2]]
sage: b = K.module_generator(); b
[[[2, -1], [1]]]
sage: b.e(0)
[]
sage: b.e(0).e(0)
[[[-1], [-2, 1]]]
```

5. Outlook

In the case of $r = 3$, by [Cha01] the classical decomposition is $B^{r,s} \cong \bigoplus_{\substack{j+k=s\\ j,k\geqslant 0}} B(j\Lambda_3 + k\Lambda_6)$. It is possible to form a composition graph that includes nodes from both $B(\Lambda_3)$ and $B(\Lambda_6)$ so that weakly increasing chains of vertices correspond to $(I \setminus J)$-highest weight nodes. However, it is straightforward to verify that even for $s = 1$, the $I \setminus \{1\}$-highest weight nodes are not uniquely determined by the statistics $(\varepsilon_1, \ldots, \varepsilon_6, \varphi_1, \ldots, \varphi_6)$, in contrast to the cases $r = 1, 2, 6$ that we have considered in this work. Hence one would first have to find vertices within each component which can be distinguished using a suitable statistics, and then construct the corresponding composition graph. The case $r = 5$ is essentially the same as the $r = 3$ case.

The ε and φ statistics are the most obvious quantities preserved by twisted isomorphism, and the fact that we were able to identify highest weight nodes by their statistics allowed us to solve the equations that proved our twisted isomorphism in fact had order three.

The classical decomposition of $B^{4,s}$ of type $E_6^{(1)}$ was conjectured in [HKO+99] and proved by Nakajima [Nak03]. As it involves more than two distinct fundamental weights, our tableau model and composition graphs would likely be substantially more complicated than those we have used for the cases $r = 1, 2, 6$.

As already mentioned in Section 3.5, the method of composition graphs for the adjoint Kirillov–Reshetikhin crystal $B^{1,s}$ of type $E_7^{(1)}$ is applicable and analogous to type $E_6^{(1)}$. However, to prove that the result is indeed an affine combinatorial crystal requires the analogue of Theorem 3.9 for twisted isomorphisms of order two. The Dynkin diagram $E_8^{(1)}$ does not have nontrivial automorphisms. Hence a new strategy is required.

It was conjectured in [HKO+02, Conjecture 2.1] that the crystals $B^{r,s}$ of type $E_6^{(1)}$ are perfect. The proof for the crystals considered in this paper is still outstanding.

All Kirillov–Reshetikhin crystals can in principle be constructed from those of simply-laced type using virtual crystals. In particular, the KR crystals for type $F_4^{(1)}$ and $E_6^{(2)}$ can be constructed from those of type $E_6^{(1)}$ (see [OSS03, Example 3.1]). Hence the construction of all type E KR crystals is an important undertaking.

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For our computer explorations we used and implemented new features in the open-source mathematical software `Sage` [WSea09] and its algebraic combinatorics features developed by the `Sage-Combinat` community [SCc09]; we are grateful to Nicolas M. Thiéry for all his support. Fig. 3 was produced using `graphviz`, `dot2tex`, and `pgf/tikz`.

Appendix A

Here, we set up and solve the linear equations describing how to map an $I \setminus \{1\}$-highest weight node from component j to an $\widetilde{I} \setminus \{6,1\}$-highest weight node of component k, using the equation for $\varphi_1 = j - (\mathrm{wt}_2 + \mathrm{wt}_3 + \mathrm{wt}_5)$ which must become φ_6 in the image. The cases correspond to the 6 maximal chains in the directed graph $G_{6,1;0}$.

Case (1).

$$\begin{aligned}
a+b+c+d+e+f &= k,\\
\varphi_6 = a+b+2c+d &= j - L_2 - L_3 - L_5,\\
L_2 &= -a-b+f,\\
L_3 &= b,\\
L_5 &= -c+e
\end{aligned}$$

with solution

$$\begin{aligned}
f &= (k-j) + L_2 + L_3,\\
a &= (k-j),\\
2c+d &= 2j-k-2L_3-L_2-L_5 = \varphi_6 - (k-j) - L_3,\\
e &= c + L_5
\end{aligned}$$

valid if $(k-j)+L_3 \leqslant \varphi_6$.

Case (2).

$$\begin{aligned}
a+b+c'+e+f &= k,\\
\varphi_6 = a+b &= j - L_2 - L_3 - L_5,\\
L_2 &= -a-b-c'+f,\\
L_3 &= b+c',\\
L_5 &= c'+e
\end{aligned}$$

with solution

$$\begin{aligned}
f &= (k-j) + L_2 + L_3,\\
a &= (k-j),\\
b &= 2j-k-L_2-L_3-L_5 = \varphi_6 - (k-j),
\end{aligned}$$

$$c' = -2j + k + 2L_3 + L_2 + L_5 = L_3 + (k-j) - \varphi_6,$$
$$e = 2j - k - 2L_3 - L_2 = L_5 - L_3 + \varphi_6 - (k-j)$$

valid if $(k-j) \leqslant \varphi_6 \leqslant (k-j) + L_3 \leqslant \varphi_6 + L_5$.

Case (3).

$$a + b' + c' + e + f = k,$$
$$\varphi_6 = a = j - L_2 - L_3 - L_5,$$
$$L_2 = -a - b' - c' + f,$$
$$L_3 = c',$$
$$L_5 = b' + c' + e$$

with solution

$$f = (k-j) + L_2 + L_3,$$
$$b' = -2j + k + L_2 + L_3 + L_5 = (k-j) - \varphi_6,$$
$$e = 2j - k - L_2 - 2L_3 = \varphi_6 - (k-j) - L_3 + L_5$$

valid if $\varphi_6 \leqslant (k-j) \leqslant (k-j) + L_3 \leqslant \varphi_6 + L_5$.

Case (4).

$$a + b' + c' + e' + f = k,$$
$$\varphi_6 = a = j - L_2 - L_3 - L_5,$$
$$L_2 = -a - b' - c' + f,$$
$$L_3 = c' + e',$$
$$L_5 = b' + c'$$

with solution

$$f = \varphi_6 + L_2 + L_5 = j - L_3,$$
$$e' = k - 2j + L_2 + 2L_3 = (k-j) - \varphi_6 + L_3 - L_5,$$
$$c' = L_3 - e' = \varphi_6 - (k-j) + L_5,$$
$$b' = L_5 - c' = (k-j) - \varphi_6$$

valid if $\varphi_6 \leqslant (k-j) \leqslant \varphi_6 + L_5 \leqslant (k-j) + L_3$.

Case (5).

$$\begin{aligned} a+b+c'+e'+f&=k,\\ \varphi_6=a+b&=j-L_2-L_3-L_5,\\ L_2&=-a-b-c'+f,\\ L_3&=b+c'+e',\\ L_5&=c' \end{aligned}$$

with solution

$$\begin{aligned} f&=L_2+L_5+\varphi_6=j-L_3,\\ e'&=k-2j+L_2+2L_3=(k-j)-\varphi_6-L_5+L_3,\\ b&=L_3-L_5-e'=2j-k-L_2-L_3-L_5=\varphi_6-(k-j),\\ a&=\varphi_6-b=(k-j) \end{aligned}$$

valid if $0 \leqslant (k-j) \leqslant \varphi_6 \leqslant \varphi_6+L_5 \leqslant (k-j)+L_3$.

Case (6).

$$\begin{aligned} a+b'+c''+d'+e'+f&=k,\\ \varphi_6=a&=j-L_2-L_3-L_5,\\ L_2&=-a-b'+f,\\ L_3&=-c''+e',\\ L_5&=b' \end{aligned}$$

with solution

$$\begin{aligned} f&=\varphi_6+L_2+L_5=j-L_3,\\ d'+2e'&=(k-j)-\varphi_6+2L_3-L_5,\\ c''&=e'-L_3 \end{aligned}$$

valid if $\varphi_6+L_5<(k-j)$ because

$$0 \leqslant e'-L_3 \quad\Leftrightarrow\quad 0 \leqslant 2e'-2L_3 \quad\Leftrightarrow\quad 0<2e'-2L_3+d'$$

and $c'' \geqslant 0$ implies $e' \geqslant 0$.

Appendix B

Here, we prove that, whenever a set of parameters (L_2, L_3, L_5, j, k) is satisfied by two distinct cases from the systems described in Appendix A, then the solutions we obtain in each case agree.

Since $\varphi_1 = k - j$ in every solution by Lemma 3.25 and $\varphi_6 = j - (\mathrm{wt}_2 + \mathrm{wt}_3 + \mathrm{wt}_5)$ encodes j, we have that any solution $b \in H_s^{\{6,1\};0}$ for the parameters (L_2, L_3, L_5, j, k) must have prescribed values for $(\mathrm{wt}_2(b), \mathrm{wt}_3(b), \mathrm{wt}_5(b), \varphi_1(b), \varphi_6(b))$. Hence, to prove the uniqueness of the solution, it suffices to show that the nodes of $H_s^{\{6,1\};0}$ are uniquely determined by $(\mathrm{wt}_2, \mathrm{wt}_3, \mathrm{wt}_5, \varphi_1, \varphi_6)$.

Proposition B.1. *Let $b \in B(k\Lambda_2)$ and $b' \in B(k'\Lambda_2)$ be $\widetilde{I} \setminus \{6,1\}$-highest weight nodes. If $\mathrm{wt}_i(b) = \mathrm{wt}_i(b')$ for $i = 2, 3, 5$ and $\varphi_j(b) = \varphi_j(b')$ for $j = 1, 6$, then $b = b'$.*

Proof. Let $A = (a_{i,j})$ be the matrix where $a_{i,j}$ is the ith entry of $(\mathrm{wt}_2, \mathrm{wt}_3, \mathrm{wt}_5, \varphi_1, \varphi_6)$ applied to the jth entry of $(a, b, b', c, c', c'', d, d', e, e', f)$ from $G_{6,1;0}$. Then,

$$A = \begin{bmatrix} -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If there were two solutions for a given set of parameters $(\mathrm{wt}_2, \mathrm{wt}_3, \mathrm{wt}_5, \varphi_1, \varphi_6)$ then we could subtract them to obtain a vector in the nullspace of A. Moreover, the positive coordinates of this vector would correspond to nodes in $G_{6,1;0}$ that all lie on a maximal chain, and similarly for the negative coordinates of the vector.

The nullspace of A is spanned by the rows of the following matrix.

$$\begin{bmatrix} a & b & b' & c & c' & c'' & d & d' & e & e' & f \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \end{bmatrix}.$$

Although the nullspace is nontrivial, observe that no basis vector actually corresponds to a valid relation because in every case we have that either the positive entries or the negative entries in the basis vector violate the constraint that the multiplicities lie on a maximal chain in $G_{6,1;0}$, so as to form a weakly increasing tensor product.

Next, we show that these chain constraints are actually violated for every vector in the nullspace of A. To see this, consider that every minimal linear dependence among the columns $\{u_1, \dots, u_{11}\}$ of A has the form $\sum_{i=1}^{11} c_i u_i = 0$. Define $\mathrm{sgn}(x)$ to be 0, -1, or 1, if x is 0, < 0 or > 0, respectively. The collection of all sign vectors $(\mathrm{sgn}(c_1), \dots, \mathrm{sgn}(c_{11}))$ obtained from minimal linear dependencies among the columns of A forms what are known as the *circuits* of an *oriented matroid*. Moreover, there is a formula to find these circuits that is given in terms of certain minors of A.

To be precise, let $\{v_1, \dots, v_{11}\}$ denote the columns of A. Then, we define $\chi_A : \{1, \dots, 11\}^5 \to \{-1, 0, 1\}$ by $\chi_A(i_1, \dots, i_5) = \mathrm{sgn}\det(v_{i_1}, \dots, v_{i_5})$. Consider

$$C : \{1, \dots, 11\}^6 \to \{-1, 0, 1\}^{11}$$

where $C(i_1, \dots, i_6)$ is defined by

$$\big(\chi_A\big((i_1,\dots,i_6)\setminus 1\big)(-1)^{j(1)+1},\ \chi_A\big((i_1,\dots,i_6)\setminus 2\big)(-1)^{j(2)+1},\ \dots,\ \chi_A\big((i_1,\dots,i_6)\setminus 11\big)(-1)^{j(11)+1}\big).$$

Here, $j(m)$ denotes the index j such that $i_j = m$, and we interpret $\chi_A((i_1,\dots,i_6)\setminus m)$ as 0 if $m \notin (i_1,\dots,i_6)$. It then follows from [BLVS⁺99, Section 1.5] that the circuits are precisely the set

$$\big\{C(i_1,\dots,i_6)\colon\ (i_1,\dots,i_6)\in\{1,\dots,11\}^6\big\}\setminus(0,0,\dots,0).$$

Using this formula, we have computed that A has 81 circuits and determined that each of them violates the chain constraints from $G_{6,1;0}$. Therefore, we have that there is a unique solution for any given set of parameters $(\mathrm{wt}_2,\mathrm{wt}_3,\mathrm{wt}_5,\varphi_1,\varphi_6)$. □

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