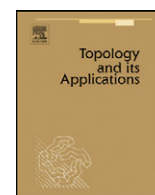Contents lists available at ScienceDirect

Topology and its Applications

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On finite groups acting on homology 4-spheres and finite subgroups of SO(5)

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A R T I C L E   I N F O

A B S T R A C T

We show that a finite group which admits a faithful, smooth, orientation-preserving action on a homology 4-sphere, and in particular on the 4-sphere, is isomorphic to a subgroup of the orthogonal group SO(5), by explicitly determining the various groups which can occur (up to an indeterminacy of index two in the case of solvable groups). As a consequence we obtain also a characterization of the finite groups which are isomorphic to subgroups of the orthogonal groups SO(5) and O(5).

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1. Introduction

We are interested in the class of finite groups which admit an orientation-preserving action on a homology 4-sphere, and in particular on the 4-sphere S^4, and to compare it with the class of finite subgroups of the orthogonal group SO(5) acting linearly on S^4. As a general hypothesis, all actions in the present paper will be faithful, orientation-preserving and smooth (or locally linear).

By the recent geometrization of finite group actions on 3-manifolds, every finite group action on the 3-sphere is conjugate to an orthogonal action; in particular, the finite groups which occur are exactly the well-known finite subgroups of the orthogonal groups SO(4) or O(4). Finite groups acting on arbitrary homology 3-spheres are considered in [9,14]; here some other finite groups occur and the situation is still not completely understood (if we dropped our standing hypothesis of local linearity, then these groups might arise in dimension four).

It is no longer true that finite group actions on the 4-sphere are conjugate to orthogonal actions; for example, it is well known that the Smith conjecture does not hold in dimension four (that is, the fixed point set of a periodic diffeomorphism of the 4-sphere may be a knotted 2-sphere). However, the class of finite groups admitting an action on the 4-sphere, and more generally also on a homology 4-sphere, should still coincide with the class of finite subgroups of the orthogonal group SO(4). Up to an indeterminacy of index two in the case of solvable groups, this is a consequence of the following main result of the present paper which explicitly determines the various groups which can occur.

Theorem. *A finite group G which admits an orientation-preserving action on a homology 4-sphere is isomorphic to one of the following groups:*

(i) *a subgroup of the Weyl group $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$;*

(ii) $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$ *or* $\mathbb{S}_6$;

(iii) *an orientation-preserving subgroup of $O(3) \times O(2)$ (and hence of SO(5));*

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(iv) *a subgroup of* SO(4), *or a 2-fold extension of such a group.*

If G *is nonsolvable* (*but most likely also in general*), (iv) *can be replaced by the stronger*:

(iv′) *a subgroup of* O(4) $\subset$ SO(5).

Note that the different cases of the theorem are not mutually exclusive. Since all groups in the theorem (except maybe the 2-fold extensions in (iv)) are subgroups of SO(5) we have the following (cf. [5, Question 6 and Problem 9]):

Corollary 1. *A finite group* G *which admits an orientation-preserving action on a homology 4-sphere is isomorphic to a subgroup of* SO(5), *or possibly, if* G *is solvable, to a 2-fold extension of a subgroup of* SO(4).

The group W in the theorem can be described as follows. Consider the semidirect product $\tilde{W} = (\mathbb{Z}_2)^5 \rtimes \mathbb{S}_5$ where the symmetric group $\mathbb{S}_5$ acts on the normal subgroup $(\mathbb{Z}_2)^5$ by permuting the components (Weyl group or wreath product $\mathbb{Z}_2 \wr \mathbb{S}_5$). This semidirect product acts orthogonally on euclidean 5-space by inversion and permutation of coordinates, and its subgroup of index two of orientation-preserving elements is the semidirect product $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$ in the theorem (composing the orientation-reversing elements in the original action of $\mathbb{S}_5$ on $\mathbb{R}^5$ with $-\mathrm{id}_{\mathbb{R}^5}$); note that $\tilde{W} = W \oplus \mathbb{Z}_2$, with $\mathbb{Z}_2 = \langle -\mathrm{id}_{\mathbb{R}^5} \rangle$.

The symmetric group $\mathbb{S}_6$ acts on the 5-simplex by permuting its vertices, and on its boundary which is the 4-sphere; composing the orientation-reversing elements with $-\mathrm{id}_{\mathbb{S}^4}$ one obtains an orientation-preserving orthogonal action of $\mathbb{S}_6$ on S^4.

The finite subgroups of SO(4) and O(4) are well known, see e.g. [4]. The group SO(4) is isomorphic to the central product $S^3 \times_{\mathbb{Z}_2} S^3$ of two unit quaternion groups S^3. There is a 2-fold covering of Lie groups $S^3 \to$ SO(3) whose kernel is the central involution of S^3. The finite subgroups of SO(3) are the polyhedral groups, that is cyclic $\mathbb{Z}_n$, dihedral $\mathbb{D}_{2n}$, tetrahedral $\mathbb{A}_4$, octahedral $\mathbb{S}_4$ or dodecahedral $\mathbb{A}_5$; the finite subgroups of S^3 are their preimages in S^3 which are cyclic, binary dihedral (or generalized quaternion) $\mathbb{D}_{4n}^*$, binary tetrahedral $\mathbb{A}_4^*$, binary octahedral $\mathbb{S}_4^*$ or binary dodecahedral $\mathbb{A}_5^*$.

A consequence of the theorem and its proof is also the following characterization of the finite subgroups of the orthogonal groups SO(5).

Corollary 2. *Let* G *be a finite subgroup of the orthogonal group* SO(5). *Then one of the following cases occurs*:

(i) G *is conjugate to a subgroup of* $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$;
(ii) G *is isomorphic to* $\mathbb{A}_5, \mathbb{S}_5, \mathbb{A}_6$ *or* $\mathbb{S}_6$;
(iii) G *is conjugate to an orientation-preserving subgroup of* O(4) $\times$ O(1) *or* O(3) $\times$ O(2).

Here O(4) $\times$ O(1) and O(3) $\times$ O(2) are considered as subgroups of the orthogonal group O(5) in the natural way. We note that, since $\mathbb{Z}_2 = \langle -\mathrm{id}_{\mathbb{R}^5} \rangle$ is orientation-reversing and central in O(5), Corollary 2 can easily be extended to the case of O(5), by just adding all groups $G \times \mathbb{Z}_2$ in the cases (i) and (ii) (note that $W \times \mathbb{Z}_2 \cong \tilde{W} \cong (\mathbb{Z}_2)^5 \rtimes \mathbb{S}_5$), and by admitting arbitrary finite subgroups in the case (iii).

Since $\mathbb{A}_5$ occurs as an irreducible subgroup of all three orthogonal groups SO(3), SO(4) and SO(5), the cases of Corollary 2 are again not mutually exclusive; see the character tables in [3] for the irreducible representations of the groups in (ii).

It should be noted that the proof of Corollary 2 is considerably easier than the proof of the theorem. For both the theorem and Corollary 2 one has to determine the finite simple groups which act on a homology 4-sphere, resp. admit an orthogonal action on the 4-sphere. In the case of the theorem this is based on [8, Theorem 1] and employs the Gorenstein–Harada classification of the finite simple groups of sectional 2-rank at most four; for Corollary 2, this can be replaced by much shorter arguments from the representation theory of finite groups (see the remark at the end of Section 3).

Finite groups acting on other types of 4-manifolds (including $S^2 \times S^2$ and $\mathbb{R}^4$) are discussed in [10,11]; see also the survey [5].

Finally, we summarize the missing cases of the theorem (orientation-preserving as well as orientation-reversing) in the following conjectures whose solution would complete the classification of the finite groups acting on a homology 4-sphere.

Conjecture 1. *Suppose that a finite solvable group* G *acts locally linearly, orientation-preservingly, and without a global fixed point on a homology 4-sphere, with a subgroup* G_0 *of index two fixing exactly two points* (*in particular,* G_0 *is isomorphic to a subgroup of* SO(4)). *Then* G *is isomorphic to a subgroup of* O(4) $\subset$ SO(5).

Conjecture 2. *Suppose that* G *is a finite group acting on a homology 4-sphere, with some elements which do not preserve orientation. Suppose that the index-two orientation-preserving subgroup* $G_0 \subset G$ *is isomorphic to a subgroup of* SO(5). *Then* G *is isomorphic to a subgroup of* O(5).

2. Preliminaries

We introduce some notation and give some comments on the general structure of the proof of the theorem. Let G be a finite group acting on a homology 4-sphere as in the theorem. We shall consider the *maximal semisimple normal subgroup* E of G. Recall that a *semisimple group* is a central product of quasisimple groups, and a *quasisimple group* is a perfect central extension of a simple group (see [13, Chapter 6.6] or [7, Chapter 6.5]). Crucial for the proof of the theorem is [8, Lemma 4.3] which states that the maximal semisimple normal group E of G is either trivial or isomorphic to one of the following groups:

$$\mathbb{A}_5, \quad \mathbb{A}_6, \quad \mathbb{A}_5^*, \quad \mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$$

($\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ denotes the central product of two binary dodecahedral groups $\mathbb{A}_5^*$); we note that this uses the Gorenstein–Harada classification of the finite simple groups of sectional 2-rank at most four. In the next section, we shall consider these different cases separately. Note that, if E is nontrivial, we have to prove the stronger version (iv′) of the theorem.

If E is nontrivial we shall consider the centralizer $C = C_G(E)$ of E in G and the subgroup $\tilde{E}$ of G generated by E and C; note that $G/\tilde{E}$ is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(E) = \mathrm{Aut}(E)/\mathrm{Inn}(E)$ of E which is small for the possible groups E.

If E is trivial, the group G may be solvable or nonsolvable. In this case, we shall consider the *Fitting subgroup* F of G instead, that is the maximal nilpotent normal subgroup of G. For an arbitrary finite group, the subgroup F^* generated by the Fitting subgroup F and the maximal semisimple normal subgroup E is called the *generalized Fitting subgroup* of G. As its main property, the generalized Fitting subgroup F^* contains its centralizer in G, hence G/F^* is isomorphic to a subgroup of the outer automorphism group of F^*. If E is trivial, the generalized Fitting subgroup F^* coincides with the Fitting subgroup F; in particular, if G is nontrivial, then also $F^* = F$ is nontrivial. Also, since F is nilpotent it is the direct product of its Sylow p-subgroups, for different primes p, and each of these is normal in G.

The following three lemmas will be frequently used in the proof of the theorem, see [8, Lemma 4.1] for a proof of Lemma 1 (note that, in the statement of case (a) of Lemma 4.1 in [8], it is written erroneously 0-dimensional instead of 2-dimensional). We note that, by general Smith fixed point theory, the fixed point set of an orientation-preserving periodic diffeomorphism of prime power order of a homology 4-sphere is either a 0-sphere or a 2-sphere (that is, a homology sphere of even codimension).

Lemma 1. *For a prime p, let A be an elementary abelian p-group acting orientation-preservingly on a homology 4-sphere.*

(a) *If A has rank two and p is odd, then A contains exactly two cyclic subgroups whose fixed point set is a 2-sphere, and the fixed point set of A is a 0-sphere.*
(b) *If A has rank two and $p = 2$, then A contains at least one involution whose fixed point set is a 2-sphere; in the case of three involutions with a 2-sphere as fixed point set the group A has a 1-sphere as fixed point set.*
(c) *If A has rank three and $p = 2$, then A can contain either one or three involutions whose fixed point set is a 0-sphere.*
(d) *If A has rank four and $p = 2$, then A contains exactly five involutions whose fixed point set is a 0-sphere.*

Lemma 2. *Let G be a finite group acting orientation-preservingly on a homology 4-sphere. Suppose that G contains a cyclic normal group H of prime order p such that the fixed point set of H is a 0-sphere S^0; then G contains, of index at most two, a subgroup isomorphic to a subgroup of $\mathrm{SO}(4)$. Moreover, if G acts orthogonally on S^4 then G is conjugate to a subgroup of $\mathrm{O}(4) \times \mathrm{O}(1)$, and hence of $\mathrm{SO}(5)$.*

Proof. The group G leaves invariant the fixed point set S^0 of H which consists of two points. A subgroup of index at most two fixes both points and acts orthogonally and orientation-preservingly on a 3-sphere (the boundary of a regular invariant neighborhood of one of the two fixed points).

If the action of G is an orthogonal action on the 4-sphere then G acts orthogonally on the equatorial 3-sphere of the 0-sphere S^0 and hence is a subgroup of $\mathrm{O}(4) \times \mathrm{O}(1)$, up to conjugation. $\square$

Lemma 3. *Let G be a finite group acting orientation-preservingly on a homology 4-sphere. Suppose that G contains a cyclic normal group H of prime order p such that the fixed point set of H is a 2-sphere S^2; then G is isomorphic to a subgroup of $\mathrm{O}(3) \times \mathrm{O}(2)$. Moreover, if G acts orthogonally on S^4 then G is conjugate to a subgroup of $\mathrm{O}(3) \times \mathrm{O}(2)$, and hence of $\mathrm{SO}(5)$.*

Proof. The group G leaves invariant the fixed point set S^2 of H. A G-invariant regular neighbourhood of S^2 is diffeomorphic to the product of S^2 with a 2-disk, so G acts on its boundary $S^2 \times S^1$ (preserving its Seifert fibration by circles). Now it is well known that every finite group action on $S^2 \times S^1$ is standard and conjugate to a subgroup of its isometry group $\mathrm{O}(3) \times \mathrm{O}(2)$ (see [12, Theorem 8.4]).

If G acts orthogonally on S^4 then the group G leaves invariant S^2, the corresponding 3-dimensional subspace in $\mathbb{R}^5$ as well as its orthogonal complement, so up to conjugation we are in $\mathrm{O}(3) \times \mathrm{O}(2)$. $\square$

3. Proof of the theorem (and of Corollary 2)

We consider the various possibilities for the maximal normal semisimple subgroup E of G as listed in Section 2.

3.1. Suppose that E is isomorphic to $\mathbb{A}_5$.

Let C denote the centralizer of E in G and $\tilde{E}$ the subgroup of G generated by E and C. Then $G/\tilde{E}$ is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(E) = \mathrm{Aut}(E)/\mathrm{Inn}(E)$ of $E \cong \mathbb{A}_5$ which has order two (see [3]). If the centralizer C of E in G is trivial then $E = \tilde{E}$ and G is isomorphic to either $\mathbb{A}_5$ or $\mathbb{S}_5$ (case (ii) of the theorem).

Suppose that C is nontrivial. We prove first that C is cyclic or dihedral. Let S be a Sylow 2-subgroup of E. The group S is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and its three involutions t_1, t_2 and t_3 are conjugate. By Lemma 1, all three involutions have a 2-sphere as fixed point set and the whole group S has a 1-sphere as fixed point set. Let S^2 denote the fixed point set of t_1; note that t_2 and t_3 act as reflections in the same 1-sphere of S^2.

Suppose that a nontrivial element f in C acts trivially on S^2. Then S and f generate an abelian group and have a common fixed point on S^2; the isotropy group of such a fixed point has a faithful orthogonal representation on euclidean space $\mathbb{R}^4$ (the tangent space of the fixed point). Since the action of G is orientation-preserving, the group generated by S and f acts faithfully on the linear subspace of dimension two orthogonal to the tangent space of S^2; this gives a contradiction since only cyclic and dihedral groups act orthogonally and faithfully on $\mathbb{R}^2$.

So it follows that the action of C on S^2 is faithful. We can also suppose that the action of C on S^2 is orientation-preserving, otherwise we compose the orientation-reversing elements of C with t_2 obtaining a new group isomorphic to C and with an orientation-preserving action. We consider the finite groups which admit a faithful, orientation-preserving action on the 2-sphere; C cannot be isomorphic to $\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$ since the reflections t_2 and t_3 of S^2 in a circle do not commute with such a group. So we conclude that C is cyclic or dihedral.

Suppose that C contains a cyclic subgroup C' of prime order p which is normal in G; this is always the case if C is either cyclic, or dihedral of order strictly greater than four. By Smith fixed point theory, the fixed point set of C' is either a 0-sphere or a 2-sphere.

If the fixed point set of C' is a 2-sphere S^2, we shall prove that we are in case (iii) of the theorem. We consider the subgroup C'' consisting of elements of G which act trivially on S^2; it is normal and cyclic (it acts as a group of rotations around S^2). The group $E \cong \mathbb{A}_5$ acts by conjugation on C''; since E is simple and the automorphism group of C'' is abelian, E acts trivially on C''. If there is an element g in G not contained in $\tilde{E}$, then g acts nontrivially on E and in particular g is not contained in C''; this implies that the factor group G/C'', which acts faithfully on S^2, contains a subgroup isomorphic to the symmetric group $\mathbb{S}_5$. However $\mathbb{S}_5$ does not act faithfully on the 2-sphere, so it follows that G and $\tilde{E}$ coincide, G is isomorphic to $E \times C''$ and we are in case (iii) of the theorem. (Alternatively, one can use Lemma 3.)

On the other hand, if the fixed point set of C' is a 0-sphere S^0 that is consists of two points, we shall prove that we are in case (iv) of the theorem. The group G contains G_0, a subgroup of index at most two that fixes both points; this subgroup acts faithfully and orientation-preservingly on a 3-sphere which is the boundary of a regular invariant neighborhood of one of the two fixed points. By the classification of finite subgroups of SO(4) in [4], it follows that G_0 is isomorphic to $\mathbb{A}_5 \times \mathbb{Z}_2$; the fixed point set of the involution in the center of G_0 is S^0. If $G_0 = G$ we are done. Suppose that G_0 has index two, let f be an element in G but not in G_0. Since the automorphism group of $\mathbb{A}_5$ is isomorphic to $\mathbb{S}_5$ (see [3]), we can suppose that f^2 acts trivially on $E \cong \mathbb{A}_5$ and f^2 is in the center of G_0. By the Lefschetz fixed point theorem f has non-empty fixed point set; if f^2 is the involution in the center of G_0, the fixed point set of f is S^0 and this is impossible. We can suppose that f is an involution. If f does not act trivially on E, we obtain that $G \cong \mathbb{S}_5 \times \mathbb{Z}_2$ that is a subgroup of O(4) (case 51 in [4]). If f acts trivially on E we have $G \cong \mathbb{A}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and this is again a subgroup of O(4) (case 49 in [4]).

Finally we can suppose that $C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is dihedral of order four. By Lemma 1 we have three possibilities for the fixed point sets of the involutions in C: the fixed point set of each involution is a 2-spheres, or the fixed point set of exactly one involution is a 0-sphere, or the fixed point set of exactly one involution is a 2-sphere. In the last two possibilities an involution in C is central in G (it is not conjugate to any other element in C) and we are in the previous case. If the fixed point set of each involution is a 2-sphere, by Lemma 1 the global fixed point set of C is a 1-sphere S^1. The subgroup consisting of elements of G acting trivially on S^1 is normal in G. Since $\mathbb{A}_5$ does not act faithfully on a 1-sphere, the action of E and hence also of $\tilde{E}$ on S^1 has to be trivial. Then $\tilde{E} = E \times C$ is contained in the isotropy group of any point of S^1 and acts faithfully and orthogonally on $\mathbb{R}^4$ (the tangent space of the point). The group $E \times C$ fixes pointwise the tangent space of S^1 and hence acts faithfully and orthogonally also on the orthogonal subspace of dimension 3. But $E \times C \cong \mathbb{A}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ does not act faithfully and orthogonally on $\mathbb{R}^3$ (or S^2). This contradiction completes the proof in case 3.1.

3.2. Suppose next that E is isomorphic to either $\mathbb{A}_5^*$ or $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$.

By Smith fixed point theory, the fixed point set of the central involution t of E is either a 0-sphere or a 2-sphere.

Suppose that the fixed point set of t is a 2-sphere S^2; we shall show that this case really does not occur. Note that E cannot be isomorphic to $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ since no quotient group of $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ by a cyclic subgroup (the group fixing point-wise S^2) admits an action on a 2-sphere. So there remains the case $E \cong \mathbb{A}_5^*$. Then E admits a faithful action on a regular

neighborhood of S^2 which is diffeomorphic to the product of S^2 with a 2-disk (since the self-intersection number of S^2 in a homology 4-sphere has to be zero), and on its boundary $S^2 \times S^1$ (preserving its Seifert fibration by circles). However by [12, Theorem 8.4] every action of a finite group on $S^2 \times S^1$ is standard (preserves the product structure, up to conjugation), and in particular $\mathbb{A}_5^*$ does not admit a faithful action on $S^2 \times S^1$. So also this case does not occur.

On the other hand, if the fixed point set of t is a 0-sphere S^0 we shall prove that we are in case (iv) of the theorem. A subgroup G_0 of index at most two of G fixes both points of S^0 and is isomorphic to a subgroup of SO(4). If $G = G_0$ we are done, so we can suppose that G_0 has index two in G. Let f be an element in G but not in G_0.

Suppose first that f normalizes G_1, a subgroup of G_0 isomorphic to $\mathbb{A}_5^*$ containing t in its center; this is always the case if $E \cong \mathbb{A}_5^*$. The group G_1 fixes both points and G_1 acts orthogonally and orientation-preservingly on a 3-sphere which is the boundary of a regular invariant neighborhood of one of the two fixed points set. The action of $\mathbb{A}_5^*$ on S^3 has to be free (by the classification of the finite subgroups of SO(5), see [4], or by [9, Theorem 2 and Lemma 6]), so the fixed point set of any element in G_1 is S^0. Up to composition by an element of G_1, we can suppose either that f centralizes G_1 or that f is represented in $\mathrm{Aut}(G_1) \cong \mathbb{S}_5$ by a transposition; in any case f centralizes h, an element of order three in G_1. We can suppose that the order of f is a power 2^m of two. Since $f \notin G_0$, f exchanges the two points of S^0, the fixed point set of h. By the Lefschetz fixed point theorem fh has a fixed point not in S^0; then the same holds for $(fh)^{2^m} = f^{2^m} h^{2^m} = h^{\pm 1}$ which is a contradiction, so this case cannot occur.

We can suppose that E is isomorphic to $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ and f exchanges by conjugation the two quasisimple components. Since $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ is maximal among the groups acting orientation-preservingly on a 3-sphere, the subgroup G_0 coincides with E. We describe $E \cong \mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ as the factor group of $\mathbb{A}_5^* \times \mathbb{A}_5^* = \{(x, y) \mid x, y \in \mathbb{A}_5^*\}$ by the normal subgroup $\{(c, c), (1, 1)\}$ where c is the central involution in $\mathbb{A}_5^*$. By [13, Theorem 6.11] the automorphism group of E is isomorphic to a semidirect product of the normal subgroup $\mathrm{Aut}(\mathbb{A}_5) \times \mathrm{Aut}(\mathbb{A}_5) \cong \mathbb{S}_5 \times \mathbb{S}_5$ with a group of order two generated by the automorphism $\phi(x, y) = (y, x)$. Since $f^2 \in E$ and f exchanges the two components, up to composition of f by an element of E, we can suppose that the automorphism induced by f is either $\phi(x, y) = (y, x)$ or $\phi'(x, y) = (\sigma(y), \sigma(x))$ where σ is a non-inner automorphism of order two of $\mathbb{A}_5^*$. In any case we can suppose that f^2 acts trivially by conjugation on E, that is f^2 is contained in the center of E. By the Lefschetz fixed point theorem f has non-empty fixed point set. If f^2 is the nontrivial element in the center of E, the fixed point set of f coincides with S^0, the fixed point set of the involution in the center of E, and this is impossible. We obtain that f is an involution and that G is a split extension of E by the subgroup of order two generated by f. The two possible automorphisms induced by f, that are ϕ and ϕ', are conjugate in the automorphism group of E, hence the possible extension of E by f is unique up to isomorphism. In this case G has to be isomorphic to the unique extension of $\mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$ that appears in the list of finite subgroups of O(4) (the case 50 in [4]).

3.3. The case $E \cong \mathbb{A}_6$ is considered in [8, Theorem 2.1], and we will not repeat the arguments here; one shows that G is isomorphic to either $\mathbb{A}_6$ or $\mathbb{S}_6$, so we are in case (ii) of the theorem.

Finally we come to the last case:

3.4. Suppose that E is trivial.

Now G may be solvable or nonsolvable; if G is nonsolvable we have to prove the stronger version (iv') of the theorem. In the following, we start by proving version (iv) of the theorem, and comment on the nonsolvable case in the appropriate instance.

Let F denote the Fitting subgroup of G; since E is trivial, F coincides with the generalized Fitting subgroup and contains its centralizer in G (see Section 2). Also, the nilpotent group F is the direct product of its Sylow p-subgroups, for different primes p, and each of these is normal in G.

Suppose that the subgroup F of G has a nontrivial Sylow p-subgroup S_p with p odd. We consider the maximal elementary abelian subgroup Z in the center of S_p; the subgroup Z is normal in G. By Smith fixed point theory (see [8, Lemma 2.3]), the rank of Z is either one or two. If the rank is one, either Lemma 2 or Lemma 3 applies and we are done. If the rank of Z is two, by Lemma 1 the fixed point set of Z is a 0-sphere S^0. The group G leaves invariant the 0-sphere and we can conclude as in Lemma 2.

We can therefore suppose that the Fitting subgroup F is a 2-group. Let Z denote again the maximal elementary abelian subgroup contained in the center of F; the subgroup Z is normal in G. By Smith theory (see [8, Lemma 2.3]), the rank of Z is at most four, and accordingly we consider the following four cases.

If Z has rank one we are done by Lemma 2 or 3.

Suppose that Z has rank two. The group Z contains three involutions. If exactly one involution in Z has a 2-sphere or a 0-sphere as fixed point set, G has again a cyclic normal subgroup and Lemma 2 or Lemma 3 applies. By Lemma 1 the unique other possibility is that all three involutions have a 2-sphere as fixed point set; in this case the fixed point set of Z is a 1-sphere S^1, invariant under the actions of G. The boundary of a G-invariant regular neighbourhood of S^1 is diffeomorphic to $S^2 \times S^1$. Since finite group actions on $S^2 \times S^1$ are standard (see [12, Theorem 8.4]), G is isomorphic to a subgroup of its isometry group O(3) × O(2). Moreover, if G acts orthogonally on S^4 then it leaves invariant S^1, the corresponding subspace of dimension two in $\mathbb{R}^5$ and its orthogonal complement, hence is conjugate to a finite subgroup of O(3) × O(2).

If Z has rank three then by Lemma 1 it has either one or three involutions with a 0-sphere as fixed point set. In the first case we have a cyclic normal subgroup and Lemma 2 applies. In the second case the three involutions with a 0-sphere as fixed point set generate Z since, by Lemma 1, a subgroup of rank two cannot contain three involutions with a 0-sphere as fixed point set. The group G permutes by conjugation these three involutions, and their product is an involution which is central in G and has a 2-sphere as fixed point set; in this case we are done by Lemma 3.

Finally we suppose that Z has rank four. We note that, by [8, Lemma 4.2], we are exactly in this situation if G is nonsolvable (that is, in all cases of 3.4 considered so far the group G is solvable and version (iv) of the theorem applies).

By Lemma 1 the group Z contains at least one involution t which has a 0-sphere S^0 as fixed point set. The group F leaves invariant S^0 and a subgroup F_0 of index at most two of F fixes both points of S^0. The group F_0 acts faithfully on a 3-sphere that is the boundary of a regular invariant neighborhood of one of the two fixed points. Moreover F_0 contains in its center an elementary abelian 2-subgroup of rank three (the group $F_0 \cap Z$); since the only 2-group acting on the 3-sphere with this property is $(\mathbb{Z}_2)^3$ (see [9, Propositions 2 and 3]), it follows that the Fitting subgroup F coincides with Z and is an elementary abelian 2-group of rank four.

Recall that F contains its centralizer in G. By Lemma 1, F contains exactly five involutions with 0-dimensional fixed point set; they generate the group F because by Lemma 1 the subgroups of index two contain at most three such involutions. We conclude that G/F acts faithfully on the set of the five involutions of F with 0-dimensional fixed point set, and hence G/F is isomorphic to a subgroup of $\mathbb{S}_5$.

Suppose first that G is solvable. Considering the solvable subgroups of $\mathbb{S}_5$, one can distinguish two cases then. If there is a subgroup $\mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$ of $Z \cong (\mathbb{Z}_2)^4$, invariant under the action of G/F and hence normal in G, then one concludes as above that cases (iii) or (iv) of the theorem apply. The only other possibility is that G/F is cyclic of order five or dihedral of order ten, and then it is easy to see that G is the semidirect product of Z with G/F and hence isomorphic to a subgroup of $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$.

On the other hand, if G is nonsolvable, then G/F is isomorphic to $\mathbb{S}_5$ or $\mathbb{A}_5$. Moreover, the action of $\mathbb{S}_5$ or $\mathbb{A}_5$ on the five involutions with 0-dimensional fixed point set is the standard permutation action, and the action of $\mathbb{S}_5$ or $\mathbb{A}_5$ on $F \cong (\mathbb{Z}_2)^4$ coincides with the action in the semidirect products $W = (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$ or its subgroup $W_0 = (\mathbb{Z}_2)^4 \rtimes \mathbb{A}_5$ of index two. So it remains to show that G is in fact such a semidirect product (that is, a split extension).

Suppose first that G/F is isomorphic to $\mathbb{A}_5$. Then G is a perfect group of order 960. The perfect groups of small order have been determined in [6] (implemented also in the group theory software GAP), and there are exactly two perfect groups of order 960 [6, cases 4.1 and 4.2, p. 118]. Both are semidirect products $(\mathbb{Z}_2)^4 \rtimes \mathbb{A}_5$, one is isomorphic to the subgroup W_0 of index two of W, so G is isomorphic to W_0 in this case. (We note that there are exactly two subgroups $\mathbb{A}_5$ of the automorphism group $\mathrm{GL}(4, 2) \cong \mathbb{A}_8$ of $(\mathbb{Z}_2)^4$, up to conjugation, corresponding to the two actions of $\mathbb{A}_5$ on $(\mathbb{Z}_2)^4$, see also [6, case 0.1, p. 116]. One may then consider the second cohomology $H^2(\mathbb{A}_5; (\mathbb{Z}_2)^4)$ which is trivial for both actions [6, p. 118]; since $H^2(\mathbb{A}_5; (\mathbb{Z}_2)^4)$ classifies extension of $(\mathbb{Z}_2)^4$ with factor group $\mathbb{A}_5$ [2, Theorem IV.3.12], the only possible extensions are the two semidirect products.)

Now suppose that G/F is isomorphic to $\mathbb{S}_5$. By the first case, G contains $W_0 = (\mathbb{Z}_2)^4 \rtimes \mathbb{A}_5$ as a subgroup of index two. Now there is exactly one embedding of $\mathbb{A}_5$ into W_0, up to conjugation: this can be checked directly (e.g. by GAP), or it follows from the fact that $H^1(\mathbb{A}_5; (\mathbb{Z}_2)^4)$ is trivial for the action corresponding to W_0 [6, p. 118] (the first cohomology $H^1(\mathbb{A}_5; (\mathbb{Z}_2)^4)$ classifies injections of $\mathbb{A}_5$ into the semidirect product up to conjugation, see [2, Proposition IV.2.3]). Let x be a coset representative of W_0 in G, representing an element of order two in the factor group $\mathbb{S}_5$ of G, so $x^2 \in (\mathbb{Z}_2)^4$. By the preceding, we can assume that $x \mathbb{A}_5 x^{-1} = \mathbb{A}_5$. Then necessarily $x^2 = 1$ (since otherwise $x^2 \mathbb{A}_5 x^{-2} \neq \mathbb{A}_5$), and hence $\mathbb{A}_5$ and x generate a subgroup $\mathbb{S}_5$ of G and G splits as a semidirect product $(\mathbb{Z}_2)^4 \rtimes \mathbb{S}_5$ isomorphic to W.

This completes the proof of the theorem.

Concerning the proof of Corollary 2 in the last case $F \cong (\mathbb{Z}_2)^4$, suppose that G acts orthogonally on S^4 and consider the corresponding orthogonal action of G on $\mathbb{R}^5$. In this case the five 1-dimensional subspaces of $\mathbb{R}^5$ which are the fixed point sets of the five involutions in F are pairwise orthogonal and, up to conjugation, we can suppose that F is the orientation-preserving subgroup of the group generated by the inversion of the coordinates in $\mathbb{R}^5$ (every elementary abelian 2-subgroup of an orthogonal group is diagonalizable). Any element in the normalizer of F in $\mathrm{SO}(5)$ leaves invariant the set of five 1-dimensional subspaces which are the fixed point sets of the five involutions in F, and the elements leaving invariant each of these 1-dimensional subspaces are contained in F. The normalizer of F in $\mathrm{SO}(5)$ is generated by F and by the group of permutations of the coordinates, and in particular G is conjugated to a subgroup of the group W.

As noted in the introduction, for the proof of Corollary 2 the Gorenstein–Harada classification of the finite simple groups of sectional 2-rank at most four can be replaced by arguments from the representation theory of finite groups. In fact, the finite groups which admit a faithful, irreducible representation in degree five (or equivalently, are irreducible subgroups of the linear group $\mathrm{SL}(5, \mathbb{C})$) have been determined in [1], and the simple groups occurring are the alternating groups $\mathbb{A}_5$ and $\mathbb{A}_6$, the linear fractional group $\mathrm{PSL}(2, 11)$ and the unitary or symplectic group $\mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$ (or, in another notation, the group $\mathrm{U}_4(2) \cong \mathrm{S}_4(3)$); in dimensions less than five there occurs in addition the linear fractional group $\mathrm{PSL}(2, 7)$, with an irreducible representation in dimension three. Since none of these groups except $\mathbb{A}_5$ and $\mathbb{A}_6$ admits a faithful *real* representation in dimension five (see [3]), we are left with the groups $\mathbb{A}_5$ and $\mathbb{A}_6$.

Acknowledgement

We thank the referee for his detailed and useful comments improving the presentation of the paper.

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