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Folded Tilting Complexes for Brauer Tree Algebras

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We give dual one-sided tilting complexes producing inverse equivalences of the derived category of a Brauer star algebra and a Brauer tree algebra of the same type, folded according to an additional combinatorial structure on the Brauer tree. We relate this to the two-sided two-term tilting complex of Rouquier in the case of a group block, showing that it induces the "completely folded" case for each one-sided complex.

1. INTRODUCTION

Let G be a tree with e edges. If one of the $e+1$ vertices is designated as "exceptional" and assigned a multiplicity m, $m \geqslant 1$, then f is called a *Brauer tree* of type (e, m). Brauer trees arise naturally in the modular theory of group representations in order to describe blocks of group algebras with cyclic defect group.

In [R1], the first author gave a tilting complex which would tilt an arbitrary block with cyclic defect group of a group algebra to a *Brauer star* of type (e, m), in which the exceptional vertex in the center and all other vertices are terminal. Unlike the various recursively defined tiltings studied afterwards in [M, Z], the original complex went from the given block to the algebra corresponding to the star in a single step. We will refer to this as the tree-to-star complex.

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In [SZ1]–[SZ3] the second author and her doctoral student considered the opposite problem: finding a combinatorial construction of a one-step star-to-tree tilting complex going from the Brauer star to an arbitrary algebra determined by a Brauer tree, both of type (e, m). The tilting complexes considered, designated two-restricted, were direct sums of indecomposables each involving no more than two projectives, and a complete classification for all such complexes was given. It turned out that the number of such complexes for a given tree is e times the product of the valencies of the non-exceptional vertices, and they can be classified by imposing an additional structure called a "pointing" on the Brauer tree.

In this paper, we will show that if the tree-to-star complex and the star-to-tree complex are "folded" according to the same pointing on the Brauer tree, then we have one-sided complexes T and $\hat{T}$ which induce inverse equivalences between the derived categories. This implies the existence of a two-sided complex C such that C is chain homotopy equivalent to T and the dual C^* is chain homotopy equivalent to $\hat{T}$, when considered as complexes of left modules. We then show that in the case of a group block, with T and $\hat{T}$ "completely folded," the two-term sequence of Rouquier [Rou] will give the desired C.

We will give the precise definitions of the various ideas used here in Section 2. In Section 3, we will define the folding for the tree-to-star complex $\hat{T}$ determined by a pointing and prove the duality of T and $\hat{T}$. In Section 4, we will consider the construction of the two-sided complex in certain special cases.

2. DEFINITIONS AND PREVIOUS RESULTS

Each Brauer tree of type (e, m) determines a Brauer tree algebra, a finite-dimensional algebra with e simple modules, in which each indecomposable module is uniserial or biserial. A complete description of this algebra in terms of composition series is given in [A]; a description in terms of quivers and relations is provided in [S]. Each simple module Σ_i corresponds to an edge, and the two series of composition factors in the indecomposable projective P_i are obtained by a counterclockwise circuit of the edges at the two end vertices. This circuit is made once unless the vertex is exceptional, in which case the circuit is made m times, returning to Σ_i in the end. If the vertex is terminal and not exceptional, then the circuit provides only two copies of Σ_i, and thus P_i is uniserial. If the vertex is terminal and exceptional, then we get $m + 1$ copies of Σ_i as one of the two series of composition factors. We will not need the full details in this paper, but it will be critical that there are no maps between indecomposable projectives which are not adjacent to a common vertex.

We recall the following definition from [R1].

DEFINITION. Let R be a commutative Noetherian ring. A *tilting complex* for a finite-dimensional R-algebra A is a complex T of finitely generated projective A-modules such that

(1) $\mathrm{Hom}_{D^b(A)}(T[n], T]) = 0$ for $n \neq 0$.

(2) The indecomposable summands of T generate the subcategory $K^b(P_A)$ of the derived category $D^b(A)$ consisting of bounded complexes of finitely generated projectives as a triangulated category.

If T satisfies only condition (1), it will be called a *partial* tilting complex.

Remark. The definition of the standard triangles of $D^b(A)$ in terms of mapping cones implies that if $u: X \to Y$ is a chain map between two elements of a triangulated subcategory $\mathscr{D}$ of $D^b(A)$, then the mapping cone $\mathrm{Cone}(X \to Y)$ also lies in $\mathscr{D}$. Since all the complexes in $K^b(P_A)$ can be built up recursively from the projective indecomposables, it suffices to show that we can obtain each projective indecomposable in order to establish (2).

In [R1] the first author gave a combinatorial construction of a tilting complex T which will tilt a given Brauer tree algebra to a Brauer star algebra.

(1) In degree zero, for each edge i adjacent to the exceptional vertex, take a stalk complex P_i.

(2) For each edge not adjacent to the exceptional vertex, if $i_1, i_2, \ldots, i_s = j$ is a path connecting the edge j to the exceptional vertex, add the indecomposable complex

$$P_{i_1} \to P_{i_2} \to \cdots \to P_{i_j},$$

where each map is a uniquely determined non-zero homomorphism determined by the structure of the Brauer tree algebra, where the term P_{i_1} is in degree zero.

For this paper, we will refer to $\hat{T}$ as a *tree-to-star complex*.

For a completely different purpose, connected with deformations, Zakay-Illouz studied certain combinatorial tilting complexes going in the opposite directions, from the Brauer star of type (e, m) to an arbitrary Brauer tree of type (e, m) [Za].

DEFINITION. A partial tilting complex for the Brauer star algebra will be called *two-restricted* if it is a direct sum of shifts of the following *elementary complexes*, where the first non-zero term is in degree 0.

(1) $S_i : 0 \to Q_i \to 0,$

(2) $T_{jk} : 0 \to Q_j \xrightarrow{h_{jk}} Q_k \to 0,$

where the maps h_{jk} have maximal rank among homomorphisms from Q_j to Q_k.

The first section of the thesis determined necessary and sufficient conditions for a two-restricted complex to be a partial tilting complex. We will not need the full classification here. We will use only:

(a) Each indecomposable projective Q_i occurs in the same degree in each elementary complex.

(b) After an appropriate cyclic renumbering of the vertices of the Brauer tree, each complex T_{jk} has $j < k$.

There are, in addition, a third set of conditions which we may summarize as *non-overlap conditions*. These are much easier to describe in terms of the full tilting complex, so we will postpone the discussion till we have given the connection between the Brauer tree and the tilting complex.

DEFINITION. Let G be a Brauer tree, identified with a planar diagram in which the edges are represented by line segments and the ordering of the edges at each vertex corresponds to a counterclockwise circuit of the vertex. A *pointing* of the tree consists of the choice of one sector at each non-exceptional vertex, indicated by placing a point in that sector. The resulting tree with this additional structure is called a *pointed Brauer tree*.

The main theorem of [Za], written up in [SZ1], states that the two-restricted tilting complexes of the Brauer star of type (e, m), modulo cyclic permutation of the star, are in one-to-one correspondence with the different pointed Brauer trees of type (e, m).

We give here the algorithm for computing a complex from the corresponding Brauer tree.

(1) Pick an arbitrary branch at the exceptional vertex as a starting point, and let the exceptional vertex be numbered as 0.

(2) Number all non-exceptional vertices by taking a Green's walk [G] around the tree in a counterclockwise direction, assigning a number to each vertex whenever the corresponding point is reached.

(3) Give each edge the same number as the vertex farthest from the exceptional vertex.

(4) Define a two-restricted complex $T = \oplus_{i=1}^{e} R_i$ by recursion on the distance from the exceptional vertex.

(a) If edge i is adjacent to the exceptional vertex, then R_i is the stalk complex S_i in degree zero.

(b) If $i_i, i_2, \ldots, i_j = i$ are the numbers assigned to the edges in a minimal path from the exceptional vertex to edge i, then we distinguish two cases:

(b.1) $i_j < i_{j-1}$: In this case, where edge i is between the entering edge at vertex i_{j-1} and the point in the counterclockwise ordering, we set

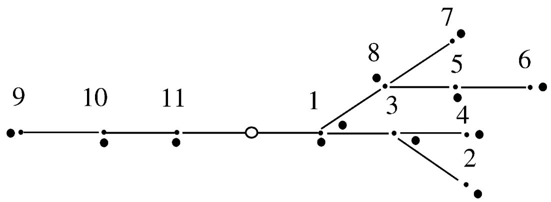
$$R_{i_j} = T_{i_j i_{j-1}}[n_j],$$

where n_j is the shift required to ensure that P_{j-1} is in the same degree in $R_{i_{j-1}}$ and R_{i_j}.

(b.2) $i_{j-1} < i_j$: In this case, when the edge i is between the point and the entering edge in the counterclockwise ordering, then we set $R_{i_j} = T_{i_{j-1}i_j}[n_{j-1}]$, where again n_{j-1} is a shift which will line up the copies of $P_{i_{j-1}}$ in R_{i_j} and $R_{i_{j-1}}$.

Remark. In all cases, we use only T_{jk} with $j < k$.

EXAMPLE 1.



$$0 \to P_1 \to 0$$

$$0 \to P_1 \to P_3 \to 0$$

$$0 \to P_2 \to P_3 \to 0$$

$$0 \to P_3 \to P_4 \to 0$$

$$0 \to P_1 \to P_8 \to 0$$

$$0 \to P_5 \to P_8 \to 0$$

$$0 \to P_5 \to P_6 \to 0$$

$$0 \to P_7 \to P_8 \to 0$$

$$0 \to P_{11} \to 0$$

$$0 \to P_{10} \to P_{11} \to 0$$

$$0 \to P_9 \to P_{10} \to 0$$

There are two particular pointings worthy of special mention. In the first, the point is always immediately after the entering edge, and the resulting numbering is that given by the standard Green's walk. We will call the resulting complex "completely unfolded." In the second, the point alternates, first immediately after the entering vertex and then immediately before it. This gives a "completely folded" two-term tilting complex. Each of these special pointings has a dual version with the location of the point reversed.

3. FOLDED TREE-TO-STAR COMPLEXES

Given a pointed Brauer tree, we get a folded star-to-tree tilting complex T which is unique up to cyclic permutation of the Brauer star. In this section, we want to build a folded tree-to-star complex such that T and $\hat{T}$ will induce inverse equivalences.

We define the numbering of the vertices and edges of the Brauer star as (1)–(4) of the algorithm in Section 2.

($\hat{5}$) We build the complex $\hat{T} = \oplus \hat{R}_i$ by recursion on the distance from the exceptional vertex.

($\hat{a}$) If edge i is adjacent to the exceptional vertex, then $\hat{R}_i$ is the stalk complex with P_i in degree zero.

($\hat{b}$) If $i_1, \ldots, i_j = i$ are the numbers assigned to the edges in a minimal path from the exceptional vertex, assume we know by recursion that $R_{i_{j-1}}$ contains one copy of $P_{i_{j-1}}$, then we distinguish two cases:

($\hat{b}.1$) $i_{j-1} < i_j$: We set

$$\hat{R}_{i_j} : P_{i_j} \to R_{i_{j-1}},$$

where the map is induced by a non-zero map from $P_{i_j} \to P_{i_{j-1}}$, which is unique up to isomorphism because P_{i_j} is not adjacent to the exceptional vertex.

($\hat{b}.2$) $i_j < i_{j-1}$: We set

$$\hat{R}_{i_j} : \hat{R}_{i_{j-1}} \to P_{i_j},$$

where the map is induced by a non-zero map from $P_{i_{j-1}} \to P_{i_j}$, again unique up to isomorphism.

Remark. We have to note the duality: In T the smaller of the two indices adjacent in the tree is in lower degree, whereas in $\hat{T}$ it is in higher degree. For example, the complex $\hat{T}$ for the previous example is

$$\begin{array}{ccccccccc}
 & & & & 0 & \to & P_1 & \to & 0 \\
 & & 0 & \to & P_3 & \to & P_1 & \to & 0 \\
 & & 0 & \to & P_3 & \to & P_1 \oplus P_2 & \to & 0 \\
0 & \to & P_4 & \to & P_3 & \to & P_1 & \to & 0 \\
 & & 0 & \to & P_8 & \to & P_1 & \to & 0 \\
 & & 0 & \to & P_8 & \to & P_1 \oplus P_5 & \to & 0 \\
 & & 0 & \to & P_8 \oplus P_6 & \to & P_1 \oplus P_5 & \to & 0
\end{array}$$

$$
\begin{array}{ccccccccccc}
0 & \to & P_8 & \to & P_1 \oplus P_7 & \to & 0 & & & & \\
 & & 0 & \to & P_{11} & \to & 0 & & & & \\
 & & 0 & \to & P_{11} & \to & P_{10} & \to & 0 & & \\
 & & 0 & \to & P_{11} & \to & P_{10} & \to & P_9 & \to & 0
\end{array}
$$

In [SZ2], it was proven that the complex T is a tilting complex from the Brauer star to the Brauer tree algebra for the pointed Brauer tree with which we started.

PROPOSITION 1. *The endomorphism ring of the complex $\hat{T}$ associated with a pointed Brauer tree is the Brauer star algebra.*

Proof. The various branches at the exceptional vertex determine a partition of $1, \ldots, e$ into disjoint intervals. Let $i_1, \ldots, i_r$ be the distinct branches at the exceptional vertex, with R_{i_k} given by the stalk complex with P_{i_k} in degree zero. Let us define integers s_k and t_k such that the interval determined by the branch i_k is $I_k = [s_k, \ldots, t_k]$. By the definition of the complex $\hat{T}$, each $\hat{R}_j$ for $j \in I_k$ contains a single copy of P_{i_k}, always in degree zero. We claim that we can define homomorphisms $f_j : \hat{R}_j \to \hat{R}_{j+1}$ for j, $j+1 \in I_k$ which are the identity on P_{i_k}, and homomorphisms $f_{t_k} : \hat{R}_{t_k} \to \hat{R}_{s_{k+1}}$ which give the map with minimal quotient on the degree zero terms $P_{i_k} \to P_{i_{k+1}}$. (Here we understand $i_{r+1} = s_1$.)

$j \neq t_k$: We have the following cases to consider:

Case 1. $j = i_k$, or j lies on a path connecting $j+1$ to i_k. Since none of the intermediate edges has a number assigned between j and $j+1$, their points must be on the left as we go out, and thus we get

$$i_k.\ldots.j.j_1.j_2.\ldots.j_\ell.j+1.$$

We conclude that $j_1 > j_2 > \cdots > j_\ell > j+1$. Thus

$$
\begin{tikzcd}
 & \hat{R}_j & & \\
\hat{R}_{j+1} : & & & \\
 & P_{j_1} \arrow[uu] \arrow[r] & P_{j_2} \arrow[r] & \cdots \arrow[r] & P_{j+1}.
\end{tikzcd}
$$

We let f_j be the canonical injection of $\hat{R}_j$ into $\hat{R}_{j+1}$.

Case 2. $j+1 = i_k$ or $j+1$ lies on a path connecting j to i_k. By an argument dual to that in case 1, we have a path

$$i_k, \ldots, j+1, j_1, j_2, \ldots, j_\ell, j,$$

with $j_1 < j_2 < \cdots < j < j+1$. We then have

$$\begin{tikzcd}
 & & & \hat{R}_{j+1} \arrow[d] \\
\hat{R}_j : & & & \\
P_j \arrow[r] & P_{j_\ell} \arrow[r] & \ldots P_{j_2} \arrow[r] & P_{j_1}
\end{tikzcd}$$

and $f_j : R_j \to R_{j+1}$ is the surjection whose kernel is the subcomplex $P_j \to \cdots \to P_{j_1}$.

Case 3. The paths connecting j and $j+1$ to i_k diverge at $h \neq j, j+1$.

Case 3a. $h < j, j+1$. Let $j_1, j_2, \ldots, j_\ell, j$ and $j_1', \ldots, j_{\ell'}, j+1$ be the two extensions. Then by the construction of the numbering of the pointed graph, we must have

$$h < j_1 < j_2 < \cdots < j_\ell < j < j+1 < j'_{\ell'} < \cdots < j'_2 < j'_1.$$

We thus have

$$\begin{tikzcd}
\hat{R}_j : & P_j \arrow[r] & P_{j_\ell} \cdots \arrow[r] & P_{j_2} \arrow[r] & P_{j_1} \arrow[r] & \hat{R}_h \\
\hat{R}_{j+1} : & P_{j'_1} \arrow[r] \arrow[dr] & P_{j'_2} \arrow[r] & \cdots \arrow[r] & P_{j'_{\ell'}} \arrow[r] & P_{j+1} \\
 & & \hat{R}_h & & &
\end{tikzcd}$$

The homomorphism $f_j : \hat{R}_j \to \hat{R}_{j+1}$ can be defined by sending $P_{j_1} \to P_{j'_1}$ by the appropriate map at vertex h. Then f_j is well defined on P_{j_2} since the composition $P_{j_1} \to P_{j_2} \to P_h$ is zero because j_2 and h are not adjacent vertices.

Case 3b. $k > j, j+1$. This is dual to Case 3a.

$j = t_k$: Map the copy of P_{i_k} in $\hat{R}_{t_k}$ to the copy of $P_{i_{k+1}}$ in $\hat{R}_{j+1}$ by a homomorphism with minimal quotient. The composition of this map with any of the other maps in $\hat{R}_{t_k}$ or $\hat{R}_{s_{k+1}}$ is zero, so f_{t_k} is well defined.

It is clear that none of these maps is homotopic to zero, and that the result of one circuit starting at P_{i_k} is to map P_{i_k} to the maximal proper submodule with the same top. Thus the result of m circuits will be to map P_{i_k} to its socle. Now take any $j \in I_k$, let ε_j^m be the result of m circuits. We claim that ε_j^m is not homotopic to zero and that $f_j \circ \varepsilon_j^m$ *is* homotopic to zero.

A full proof of this claim involves treating all of the cases in the definition of f_j separately, so we will give only a sketch of the proof. Suppose that the common part of the paths from j and $j+1$ to i_k is $i_k = j_0, j_1, \ldots, j_h$. For any index i, let $\bar{\varepsilon}_i : P_i \to P_i$ be the non-identity homomorphism with quotient of minimal length. Let $s_i : P_i \to P_i$ be the map of the projective P_i onto its socle. Thus $s_{i_k} = \bar{\varepsilon}_i^m$, and $s_{i_\ell} = \bar{\varepsilon}_{i_\ell}$, for $\ell \neq 0$.

One can show that the map from $\hat{R}_{j_h}$ to $\hat{R}_{j_h}$ which is $\bar{\varepsilon}_{i_k}^m$ on P_{i_k}, $(-1)^h s_{j_h}^m$ on P_{j_h} and zero elsewhere is homotopic to zero by a homotopy $\tilde{T}$. The homotopy $\tilde{T}$ is a direct sum of homomorphisms $h_{j_i j_{i+1}}$ for $j_i > j_{i+1}$ and $h_{j_{i+1} j_i}$ for $j_{i+1} > j_i$. This demonstrates that ε_j^m is not homotopic to zero, since it is homotopic to the well-defined map $\hat{s}_{j_n} : \hat{R}_j \to \hat{R}_j$ which is equal to $(-1)^{h+1} s_{j_n}$ on P_{j_n} and zero elsewhere. If j_1' is the first edge in the direction of j, and j_1'' is the first edge in the direction of $j+1$, then the map from $\hat{R}_{j_1'}$ to $\hat{R}_{j_1''}$ which is $\bar{\varepsilon}_{i_k}^m$ on P_{i_k} and zero elsewhere is homotopic to zero by an extension $\tilde{T}'$ of $\tilde{T}$ which produces a zero map at P_{j_h}. We describe the map T_0 giving the extension from $\tilde{T}$ to $\tilde{T}'$.

$j_h = j$

$$\begin{tikzcd} & P_{j_h} \arrow[dl, "T_0=(-1)^{h+1}h_{j_h j_1''}"'] \arrow[d, "(-1)^{h+1}s_h"] \\ P_{j_1''} \arrow[r] & P_{j_h} \end{tikzcd}$$

$j_h = j+1$ is dual.

$j_h < j_1' < j < j+1 < j_1''$

$$\begin{tikzcd} P_{j_1''} \arrow[r] \arrow[d, "0"'] & P_{j_h} \arrow[dl, "T_0"'] \arrow[d, "(-1)^{h+1}s_h"] \\ P_{j_1''} \arrow[r] & P_{j_h} \end{tikzcd}$$

$j_1' < j < j+1 < j_1'' < j_h$ is dual. The same homotopy $\tilde{T}'$ shows that $\hat{R}_j \xrightarrow{f_j \circ \varepsilon_j^m} \hat{R}_{j+1}$ is homotopic to zero.

We have actually shown that $\mathrm{End}_B(\hat{T}^\bullet)$ is isomorphic to the opposite algebra A^0 of A but since $A \xrightarrow{\sim} A^0$, we have the desired result. ∎

PROPOSITION 2. *The complexes $T^\bullet$ and $\hat{T}^\bullet$ determined by a given pointed Brauer tree determine inverse equivalences of categories.*

Proof. In the previous proposition, we established a correspondence between the projectives $Q_1, \ldots, Q_e$ of b and the components $\hat{R}_1, \ldots, \hat{R}_e$ of the tilting complex $\hat{T}$. Knowing, from theoretical considerations, that an inverse equivalence exists, we will determine the inverses to the indecomposable projectives $P_1, \ldots, P_e$ of B using mapping cones.

Each branch at the exceptional vertex is sufficient to generate all the projectives appearing in that branch, so we will assume as in the proof of Proposition 1 that we are concerned with the branch at i_k, represented by an interval $[s_k, \ldots, t_k]$. The individual projectives are generated recursively according to their distance from the exceptional vertex with Q_{i_k} corresponding to P_{i_k}. If $i_k = j_0, j_1, \ldots, j_h$ is a minimal path, then we consider two cases.

$j_{h-1} < j_h : \hat{R}_{j_h} : P_{j_h} \rightarrow \hat{R}_{j_{h-1}}$. Suppose that in $\hat{R}_{j_{h-1}}$, the projective $P_{j_{h-1}}$ occurs in degree $-n_{h-1}$. Then $\hat{R}_{j_h} = \mathrm{Cone}(P_{j_{h-1}}[n_{h-1}] \rightarrow \hat{R}_{j_{h-1}})$.

Applying the inverse functor $F^{-1} : (D^b(B) \rightarrow D^b(A))$ and using the fact that $F^{-1}(\hat{R}_j) = Q_j$, we get

$$Q_{j_h} \xrightarrow{\sim} \mathrm{Cone}(F^{-1}(P_{j_{h-1}})[n_{h-1}] \xrightarrow{u} Q_{j_{h-1}}).$$

We conclude that $F^{-1}[P_{j_{h-1}}[n_{h-1}]] = T_{j_{h-1}j_h}$ and the map u is the identity on $Q_{j_{h-1}}$. After adjusting the shift, which commutes with the functor F^{-1}, we conclude that

$$F^{-1}(P_{j_{h-1}}) \xrightarrow{\sim} T_{j_{h-1}j_h}[-n_{h-1}].$$

Thus in the tilting complex T, $Q_{j_{h-1}}$ occurs in degree n_{h-1}.

The case $j_h < j_{h-1}$ is precisely dual. ∎

4. TWO-TERM TILTING COMPLEXES

Let us now suppose that we have a two-term two-sided tilting complex C, as in [Rou], with the following properties. Let A be a basic Brauer star algebra, and let B be a Brauer tree algebra. Let M be a $B - A$ bimodule, which is projective both as a left B-module and as a right A-module, such that

$$(*) \qquad \begin{aligned} M \otimes_A M^* &\xrightarrow{\sim} B \oplus \text{ projective } B\text{-bimodules}, \\ M \otimes_B M^* &\xrightarrow{\sim} A \oplus \text{ projective } A\text{-bimodules}. \end{aligned}$$

Suppose that $C : 0 \rightarrow N \rightarrow M \rightarrow 0$ is a complex of $B - A$ bimodules with N a projective $B - A$ bimodule and satisfying $(*)$ in degree zero. Suppose further that $C \otimes_A C^*$ is homotopy equivalent to B (as a stalk complex) and $C^* \otimes_B C$ is homotopy equivalent to A. Then C as a complex of projective B-modules is a one-sided tilting complex from B to A and C^* as a complex of B modules is a one-sided tilting complex from A to B, giving inverse equivalences.

PROPOSITION 3. *If C is a two-term, two-sided tilting complex of $B - A$ bimodules, then, regarded as complexes of left modules, C is chain homotopy equivalent to a tree-to-star, completely folded tilting complex $\hat{T}$, and C^* is chain homotopy equivalent to a two-restricted completely folded star-to-tree complex T obtained from the same pointing of the Brauer tree.*

Proof. Since, by Proposition 2, the complexes T and $\hat{T}$ obtained from a given Brauer tree give inverse equivalences, it suffices to show that, considered as a complex of A-modules, C^* is chain homotopy equivalent to a two-restricted one-sided tilting complex. For this, the only information we need about our complex is that it is a two-term complex of projectives, so we will consider the more general case

$$D: 0 \to V \xrightarrow{u} W \to 0$$

with

$$V = \bigoplus_{j=1}^{s} V_j, \qquad V_j \xrightarrow{\sim} P_{i_j}, \qquad V_j = \langle v_j \rangle,$$
$$W = \bigoplus_{\ell=1}^{t} W_\ell, \qquad W_\ell \xrightarrow{\sim} P_{k_\ell}, \qquad W_\ell = \langle w_\ell \rangle.$$

Choose an element x of the Brauer star algebra A which generates the radical, with $x^{\hat{n}} \neq 0$, $x^{\hat{n}+1} = 0$, for $\hat{n} = em + 1$.

If $u = 0$, then D decomposes into a direct sum of stalk complexes, which is surely two-restricted, so we may assume that $u \neq 0$. We first want to prove, by induction on s, that D is a direct sum of stalk complexes and complexes of the form $P_i \to P_k$. For $s = 0$ we have direct sum of stalk complexes, so we assume $s > 0$ and that every D with smaller s decomposes.

We have

$$u(V) \not\subseteq x^{\hat{n}+1} W = \{0\},$$
$$u(V) \subseteq x^0 W = W.$$

Therefore, there is a unique integer g, $\hat{0} \leqslant g \leqslant \hat{n}$, such that

$$u(V) \not\subseteq x^{g+1} W, \qquad u(V) \subseteq x^g W.$$

There is some j such that $u(V_j) \subseteq x^{g+1} W$, and some ℓ such that

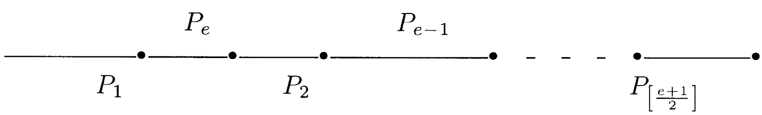
$$\pi_\ell(u(v_j)) = c x^g w_\ell, \qquad c \in k,$$

where π_ℓ is the projection of W on W_ℓ. Since $u(v_j) \in x^g W$, there is an element $w \in W$ such that $u(v_j) = x^g(w)$, and we can make a change of coordinates substituting w for w_ℓ so that $u(v_j) = x^g w_\ell$, since $u(v_j) = v(w)$ implies that $\langle w \rangle \xrightarrow{\sim} P_{k_\ell} \xrightarrow{\sim} \langle w_\ell \rangle$ and $\pi_\ell(w) = cw_\ell$ with $c \neq 0$.

We now make a change of coordinates in V such that $\pi_\ell(v_i) = 0$ for every $i \neq j$, by adding appropriate linear combinations of $x^t v_j$ to v_i. It is then possible to split off $V_j \to W_\ell$, and get a complex D' with smaller s.

It remains to show that the summands $P_i \xrightarrow{h} P_k$ are of type T_{ik}. If $i = k$, with h an isomorphism, it is homotopic to zero. If u factors through ε_i, then it was shown in [SZ1] that the complex cannot occur in a tilting complex. Thus we must have $P_i \xrightarrow{h_{ik}} P_k$ as desired. D is two-restricted and thus, by the main theorem in [SZ2], it corresponds to a pointed Brauer tree. Since it is a two-term, it is completely folded. Proposition 2 then finishes the theorem. ∎

EXAMPLE. $G = PSL(2,p)$ as in [Rou]. Take the numbering starting at the non-exceptional vertex



Our numbering is one greater than that in [Rou], so that each P_i has a simple top of dimension $2i-1$ (instead of $2i+1$). The module M in this case is $OG\hat{e}$, where $\hat{e}$ is the idempotent of the principal block and contains $2i-1$ copies of each P_i. The module M, as a B module, contains p copies of each P_i, since dim $Q_i^* = p$. The B-module $\hat{T}$ is $\oplus_{i=1}^{e} \hat{R}_j$. Since each P_i occurs at most once in $\hat{R}_j$, and the number of $\hat{R}_j$ in which P_i occurs in the distance of P_i from the non-exceptional vertex, we get

$$\hat{T}\colon \bigoplus_{j=\lceil\frac{e+1}{2}\rceil+1}^{e} P_j^{2(e-j+1)} \to \bigoplus_{j=1}^{\lceil\frac{e+1}{2}\rceil} P_i^{2j-1},$$

$$C\colon \bigoplus_{j=\lceil\frac{e+1}{2}\rceil+1}^{e} P_j^{p} \to \bigoplus_{j=1}^{e} P_i^{2j-1}.$$

Since we know that $\hat{T}$ is homotopic to C, we see that, for each $j = [\frac{e+1}{2}] + 1, \ldots, e$, and, we must split off from C $2j-1$ copies of $P_j \xrightarrow{\text{id}} P_j$.

$$\begin{aligned} p - (2j-1) &= (2e+1) - (2j+1) \\ &= 2(e-j+1). \end{aligned}$$

We get just the right number of copies of P_i left for $\hat{T}$.

5. APPLICATIONS

Although two-sided tilting complexes exist whenever there is a one-sided tilting complex, in fact there are few known examples. We hope that having the matched pairs T and $\hat{T}$ will help us work out a recursive procedure for constructing the two-sided complexes. We would be particularly interested in finding a two-sided complex with only one term which is not projective as a bimodule.

Should we be able to work out a procedure to construct the two-sided complexes $C - C^*$ for the $T - \hat{T}$ pairs, then we would get a large number of elements in the derived Picard group of the Brauer star, by considering two different points for a fixed Brauer tree. If we have $C - C^*$ for one pointing and $C' - C'^*$ for another pointing, then $C \otimes_B C'$ will be a self-equivalence of A. It is possible that these complexes generate the entire derived Picard group of A. If this is true, then any two-sided tilting complex between A and B is a composition of two-sided tilting complexes whose corresponding one-sided tilting complexes are the tree-to-star and star-to-tree complexes associated with pointed Brauer trees.

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