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THE STRUCTURE OF MEDIAN GRAPHS

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A median graph is a connected graph, such that for any three vertices u, v and w there is exactly one vertex x that lies simultaneously on a shortest (u, v)-path, a shortest (v, w)-path and a shortest (w, u)-path. It is proved that a median graph can be obtained from a one-vertex graph by an expansion procedure. From this characterization some nice properties are derived.

0. Introduction

In [2] the concept of median graph was introduced. It was shown that there is a close relation between median graphs and some at first sight fairly distinct mathematical structures. One of these is a special class of Helly hypergraphs. This class consists of the hypergraphs with vertex-set V and edge-set $E \subset P(V)$, such that

$$A \in E \Leftrightarrow V \setminus A \in E$$

and

$$E' \subset E, \bigcap E' = \emptyset \Rightarrow \exists A, B \in E' : A \cap B = \emptyset.$$

In the sequel a structural characterization of median graphs is given and some nice properties are derived.

With some minor adaptations the terminology of Bondy and Murty [1] is adopted.

1. Definitions and preliminaries

Let $G = (V, E)$ be a simple loopless graph with vertex-set V, edge-set E and distance function d. The graph G is a *median graph* if G is connected, and for any three vertices u, v and w of G there is exactly one vertex x, called the *median* of u, v and w and denoted by (u, v, w), such that

$$d(u, x) + d(x, v) = d(u, v)$$
$$d(v, x) + d(x, w) = d(v, w)$$
$$d(w, x) + d(x, u) = d(w, u).$$

The notion of median graph was introduced in [2]. All trees and the n-cubes are median graphs. It is easily seen that a median graph is bipartite (as will be proved later).

A *cutset* in a connected graph is a minimal disconnecting edge-set (a bond in [1]).

A *cutset colouring* of a connected graph is a proper edge colouring of the graph (adjacent edges have different colours), such that for any colour i, the set of edges assigned colour i is a cutset (or the empty set). Necessary conditions for the existence of a cutset colouring of a connected graph G are:

- G is loopless (because of the minimality, a cutset contains no loops);
- G is simple (two edges joining the same pair of vertices have to be coloured differently, but they must belong to the same cutset);
- G is bipartite (a cutset contains an even number of edges of each circuit (in [1] called cycle); so a circuit contains an even number of edges of each colour in the cutset colouring, and thus has even length);
- G does not contain $K_{2,3}$ as a subgraph (if a cutset contains an edge of $K_{2,3}$, then it contains at least two adjacent edges of $K_{2,3}$).

A connected graph is said to be *uniquely cutset colourable*, if the graph admits, up to the labelling of the colours, exactly one cutset colouring.

Note that, if we want to establish a cutset colouring of a connected graph, we are forced to assign the same colour to non-adjacent edges in any circuit of length four. Hence the n-cube is uniquely cutset colourable with n colours.

A *convex subgraph* of a graph G is an induced subgraph $G[W]$ of G such that with any two vertices $u, v \in W$ all shortest (u, v)-paths in G lie entirely in $G[W]$. Clearly a convex subgraph of a median graph is itself a median graph.

For subsets S, T of the vertex-set of a graph, $[S, T]$ denotes the set of edges with one end in S and the other in T.

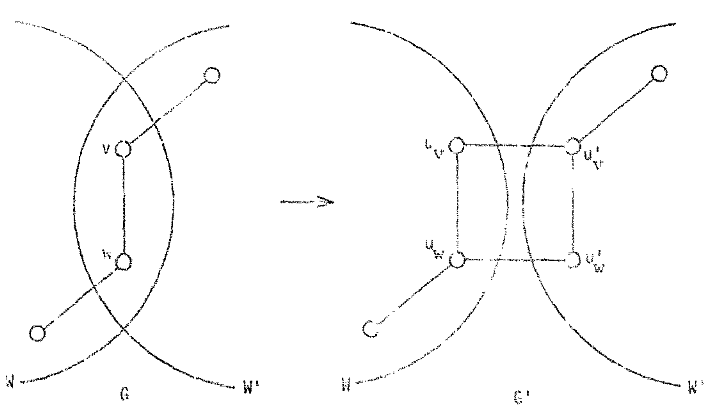

Fig. 1.

Let G be a simple loopless graph. Let $W, W' \subset V$ be such that $W \cup W' = V$, $W \cap W' \neq \emptyset$ and $[W \setminus W', W' \setminus W] = \emptyset$.

The *expansion* of G with respect to W and W' is the graph G' constructed as follows:

(i) replace each vertex $v \in W \cap W'$ by two vertices u_v, u'_v, which are joined by an edge;

(ii) join u_v to the neighbours of v in $W \setminus W'$ and u'_v to those in $W' \setminus W$;

(iii) if $v, w \in W \cap W'$ and $vw \in E$, then join u_v to u_w and u'_v to u'_w.

In Fig. 1 the construction of an expansion of a graph is illustrated by an example.

If $G[W]$ and $G[W']$ are convex subgraphs of G, then G' will be called a *convex expansion* of G.

2. The structure of median graphs

Theorem. *A graph G is a median graph iff G can be obtained from a one-vertex graph by a sequence of convex expansions.*

The proof of the theorem will be given in the next section. In the course of the proof several properties of median graphs will be inferred, which are interesting in their own right and which elucidate the structure of median graphs. Therefore they are stated here as corollaries to the theorem.

Corollary 1. *A median graph is uniquely cutset colourable.*

Corollary 2. *Let G be a median graph and $F = [W, V \setminus W]$ a colour of the cutset colouring of G. Set $U = \{u \in W \mid u$ end of an edge of $F\}$ and $U' = \{u \in V \setminus W \mid u$ end of an edge of $F\}$. Then*

(i) *$G[U]$, $G[U']$, $G[W]$ and $G[V \setminus W]$ are convex subgraphs of G, and thus are median graphs;*

(ii) *the mapping $f: U \to U'$ defined by*

$$f(u) = u' \quad \text{iff} \quad uu' \in F$$

induces a colour preserving isomorphism between $G[U]$ and $G[U']$.

Note that not all uniquely cutset colourable graphs are median graphs. Deleting a k-cube from an n-cube, with $n - 3 \geq k \geq 0$, produces a uniquely cutset colourable graph, which is not a median graph.

From the theorem, the following characterization (cf. [2]) of median graphs is easily deduced by induction on the number of vertices, or on the number of colours in the cutset colouring.

Corollary 3. *A graph G is a median graph iff G is a connected induced subgraph of an n-cube such that with any three vertices of G their median in the n-cube is also a vertex of G.*

Remark. Other characterizations of median graphs obtained in [2] can also be derived from the theorem.

3. Proof of the theorem

First the *if part* of the proof.

Let G be a median graph. Let G' be a convex expansion of G, say, with respect to W and W'. Let

$$U = \{u_v \mid v \in W \cap W'\}$$

and

$$U' = \{u'_v \mid v \in W \cap W'\},$$

where u_v and u'_v are as in Section 1.

Set $Z = (W \setminus W') \cup U$ and $Z' = (W' \setminus W) \cup U'$. Then $Z \cup Z'$ is the vertex-set of G'.

From the definition of expansion, it is clear that $G'[Z] \cong G[W]$, $G'[Z'] \cong G[W']$ and $G'[U] \cong G'[U']$. Furthermore, it follows that $G'[Z]$ and $G'[Z']$ are convex subgraphs of G'. And thus, since $G[W]$ and $G[W']$ are median graphs, any three vertices in Z (or in Z') have a unique median in G'.

Take three vertices of G', not all in Z or in Z', say $a, b \in Z$ and $c \in Z'$. Let x be the median of the vertices in G corresponding to a, b and c. Now a shortest path in G' from a vertex in Z to a vertex in Z' can be obtained from a shortest path in G between the corresponding vertices, by "adding" an edge between U and U' to the path. Moreover, all vertices on a shortest (a, b)-path in G' lie in Z. Hence the vertex of G' in Z corresponding to x is the uniquely determined median of a, b and c in G'.

Although it goes against the advice of [3], the proof of the *only if part* of the theorem is broken into a succession of steps. This is done, because the proof is rather lengthy, and also because several properties of median graphs can thereby be stated separately.

Let $G = (V, E)$ be a median graph. Fix an edge $e = ab$ of G and define

$$W_a := \{w \in V \mid d(a, w) + 1 = d(b, w)\},$$
$$W_b := \{w \in V \mid d(a, w) = d(b, w) + 1\},$$
$$F := [W_a, W_b],$$
$$U_a := \{u \in W_a \mid u \text{ end of an edge in } F\},$$
$$U_b := \{u \in W_b \mid u \text{ end of an edge in } F\}$$

Note that W_a is the set of all vertices that are nearer to a than to b, and W_b of all vertices that are nearer to b than to a. Clearly $a \in W_a$ and $b \in W_b$.

(0) If $u, v \in V$ are joined by an edge, then $(u, v, w) = u$ or v, but clearly not both, for all $w \in V$.

Proof. Apply the definition of median.

(1) $W_a = \{w \in V \mid (a, w, b) = a\}$, and $W_b = \{w \in V \mid (a, w, b) = b\}$

Proof. Follows directly from (0).

(2) $W_b = V \setminus W_a$.

Proof. Use (0) and (1).

(3) G is bipartite.

Proof. Since ab is an arbitrary edge of G, assertion (2) implies that for any two adjacent vertices u and v there exists no vertex w in G, such that $d(u, w) = d(v, w)$. So G contains on odd circuits.

(4) For $v \in W_a$ each shortest (a, v)-path lies entirely in $G[W_a]$, for $v \in W_b$ each shortest (b, v)-path lies entirely in $G[W_b]$.

Proof. Use the definition of W_a and W_b.

(5) F is a cutset.

Proof. Assertion (2) implies that F is a disconnecting edge-set. And (4) implies that $G[W_a]$ and $G[W_b]$ are connected, so F is minimal.

(6) If $u \in U_a$, $v \in U_b$ such that $uv \in F$, then $d(u, a) = d(v, b)$.

Proof. Since $uv \in F$, we have

$$d(v, b) + 1 = d(v, a) \leq d(u, a) + 1 = d(u, b) \leq d(v, b) + 1.$$

(7) F is a matching.

Proof. Assume on the contrary that there exists a vertex u, say $u \in U_a$, such that u has two distinct neighbours $v, v' \in U_b$.

According to (3) the graph G contains no triangles, so $d(v, v') = 2$. Furthermore, (6) implies that $d(v, a) = d(v', a) = d(u, a) + 1$, so we have $(v, v', a) = u$.

Since $v, v' \in W_b$, it follows from the definition of W_b, that b lies on a shortest (a, v)-path, as well as on a shortest (a, v')-path. Hence $(v, v', b) = (v, v', a)$.

But (4) implies that $(v, v', b) \subseteq W_b$. So $u \in W_b \cap U_a \subset W_b \cap W_a$, thus establishing a contradiction with $W_a \cap W_b = \emptyset$.

(8) For $u \in U_a$ each shortest (a, u)-path lies entirely in $G[U_a]$, for $u \in U_b$ each shortest (b, u)-path lies entirely in $G[U_b]$.

Proof. We only prove the first assertion, using induction on $d(a, u)$.

Let v be the neighbour of u in U_b and let $w \in W_a$ be a neighbour of u such that $d(a, w) = d(a, u) - 1$.

Then (6) implies $d(v, b) = d(a, u) = d(a, w) + 1 = d(b, w)$. Moreover, $d(v, w) = 2$. Hence (v, w, b) is a common neighbour of v and w. According to (4), we have $(v, w, b) \in W_b$ and thus $w \in U_a$.

(9) For any edge uv of G such that $u \in U_a$ and $v \in U_b$, we have $W_a = \{w \in V \mid (u, w, v) = u\}$ and $W_b = \{w \in V \mid (u, w, v) = v\}$.

Proof. First let u be a neighbour of a, and thus v a neighbour of b.

We shall prove that $W_a \subseteq \{w \in V \mid (u, w, v) = u\}$. Then similarly $W_b \subseteq \{w \in V \mid (u, w, v) = v\}$ and thus, using (1), both equalities hold.

Take a vertex $w \in W_a$. Set $d(a, w) = k$. Then $d(b, w) = k + 1$.

If $(a, u, w) = u$, then $d(u, w) = d(a, w) - 1 = k - 1$. Now

$$d(u, w) + 1 \geqslant d(v, w) \geqslant d(b, w) - 1 = k + 1 - 1 > k - 1 = d(u, w).$$

And hence $d(v, w) = d(u, w) + 1$.

If $(a, u, w) = a$, then $d(u, w) = d(a, w) + 1 = k + 1 = d(b, w)$. So $(u, w, b) = a$ and $d(v, w) = k$ or $k + 2$, since G is bipartite. But $d(v, w) = k$ implies $a = (u, w, b) = v$, which is a contradiction. So again $d(v, w) = d(u, w) + 1$.

The general case now follows by induction on $d(a, u)$, using (8).

(10) $G[U_a]$ and $G[U_b]$ are convex subgraphs of G, and thus they are median graphs.

Proof. Take $u, u' \in U_a$. Let v' be the neighbour of u' in U_b. According to (9) we also could have used the edge $u'v'$ instead of the edge ab for the definition of W_a and W_b, and thus of F, U_a and U_b. This implies that assertion (8) still holds, when a is replaced by u'. So we can conclude that each shortest (u', u)-path lies entirely in $G[U_a]$. Hence $G[U_a]$ is a convex subgraph of G.

Similarly $G[U_b]$ is convex.

(11) $G[W_a]$ and $G[W_b]$ are convex subgraphs of G, and thus they are median graphs.

Proof. Use (10).

(12) The mapping $f: U_a \to U_b$, defined by $f(u) = v$ iff $uv \in F$, induces an isomorphism between $G[U_a]$ and $G[U_b]$.

Proof. Since F is a matching between U_a and U_b, the mapping f is bijective.

Take $u, u' \in U_a$. According to (9) assertion (6) still holds when we replace a by u' and b by $f(u')$. That is, $d(u, u') = d(f(u), f(u'))$. So $uu' \in E$ iff $f(u)f(u') \in E$.

(13) G is uniquely cutset colourable.

Proof. Take an edge uv of G, with $u \in U_a$ and $v \in U_b$. Set $W_u = \{w \in V \mid (u, w, v) = u\}$ and $W_v = \{w \in V \mid (u, w, v) = v\}$. Assertion (9) implies that $F = [W_u, W_v]$. That is, the edge uv defines the same cutset-matching F as the edge ab. Since ab is an arbitrary edge of G, it follows that G is cutset colourable.

Now let $a = u_0, u_1, \ldots, u_p = u$ be a path from a to u in $G[U_a]$. Then $b = f(u_0), f(u_1), \ldots, f(u_p) = v$ is a path from b to v in $G[U_b]$. As observed in Section 1, non-adjacent edges in a circuit of length four in G are to be assigned the same colour in any cutset colouring of G. So the edges $ab = u_0f(u_0)$, $u_1f(u_1), \ldots, u_pf(u_p) = uv$ are to be assigned the same colour in any cutset colouring of G. And thus it follows that the cutset colouring of G, constructed above, is unique.

(14) The isomorphism f, defined in (12), is colour preserving.

Proof. Let uu' be an edge in $G[U_a]$. Then $u, f(u), f(u'), u', u$ is a circuit of length four in G.

(15) G can be obtained as a convex expansion from a graph with less vertices.

Proof. Let G' be the graph constructed from $G[V \setminus U_b]$, by joining each vertex $u \in U_a$ to all neighbours of $f(u)$ in $W_b \setminus U_b$. In other words, G' is obtained from G by "contracting" F.

Set $W_b' = (W_b \setminus U_b) \cup U_a$.

Now $G'[W_a] = G[W_a]$, $G'[W_b'] \cong G[W_b]$ and $G'[U_a] = G[U_a] \cong G[U_b]$. Clearly G is the expansion of G' with respect to W_a and W_b'.

Take $u, v \in U_a$. No shortest (u, v)-path in G passes through a vertex of $V \setminus U_a$, and no shortest $(f(u), f(v))$-path in G passes through a vertex of $V \setminus U_b$. So all shortest (u, v)-paths in G' lie entirely in $G'[U_a]$. That is, $G'[U_a]$ is a convex subgraph of G'. Since there are no edges in G' between $W_a \setminus W_b'$ and $W_b' \setminus W_a$, it follows from the convexity of $G'[U_a]$, that $G'[W_a]$ and $G'[W_b']$ also are convex subgraphs of G'.

Assertion (11) now implies that any three vertices in W_a have a unique median in G'. The same holds for three vertices in W_b'. Let $u, v \in W_a$ and $w \in W_b' \setminus W_a$. Then u, v and w have a unique median, say, x in G. Note that $x \in W_a$. Let P be a shortest path in G from w to u. Then P passes through exactly one edge, say $f(u_p)u_p$, from F. Let $f(u_1)$ be the first vertex of P that lies in U_b. Since $G[U_b]$ is convex, P is of the form $P_1, f(u_1), f(u_2), \ldots, f(u_p), u_p, P_2$, where $P_1, f(u_1)$ is the subpath of P from w to $f(u_1)$ and u_p, P_2 is the subpath of P from u_p to u. Now the path $Q = P_1, f(u_1), u_1, u_2, \ldots, u_p, P_2$ also is a shortest path from w to u in G. And Q contains exactly one vertex of U_b. It is clear that, in determining the median of u, v and w in G, we can confine ourselves to paths of the same form as Q. When we "contract" the edge $f(u_1)u_1 \in F$ in Q, then we obtain a shortest path Q' from w to u in G'. Any shortest (w, u)-path in G' can be obtained in this way from a path of "type Q". From these observations it is clear, that x is the unique median of u, v and w in G'.

Similarly, $w \in W_a \setminus W_b'$ and $u, v \in W_b'$ have a unique median in G'. Thus G' is a median graph.

This finishes the proof of the theorem.

Note added in proof

It was brought to my attention by L. Nebeský that some of the above results are contained in his paper "Median graphs", Comment. Math. Univ. Carolinae 12 (1971) 317–325.

References

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