Note

A new formula for an evaluation of the Tutte polynomial of a matroid

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Abstract

Given a matroid \( M \) and its Tutte polynomial \( T_M(x, y) \), \( T_M(0, 1) \) is an invariant of \( M \) with various interesting combinatorial and topological interpretations. Being a Tutte–Grothendieck invariant, \( T_M(0, 1) \) may be computed via deletion–contraction recursions. In this note we derive a new recursion formula for this invariant that involves contractions of \( M \) through the circuits containing a fixed element of \( M \).

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1. Introduction

Given a matroid \( M \), let \( T_M(x, y) \) denote its Tutte polynomial. We define the \( \alpha \)-invariant of \( M \) to be the evaluation of \( T_M(x, y) \) at \( (x, y) = (0, 1) \), i.e.,

\[ \alpha(M) := T_M(0, 1). \]

The main purpose of this note is to present a new recursion formula for computing \( \alpha(M) \). First we discuss some combinatorial and topological interpretations of \( \alpha(M) \) that we will use in the following section. Most definitions and results from matroid theory and matroid complexes that we use in this section are standard and we refer the reader to [1,5] for further discussions.

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The Möbius invariant of \( M \) is defined to be \( \tilde{\mu}(M) := |\mu_L(0, \hat{1})| = T_M(1, 0) \), where \( \mu_L \) is the Möbius function on the lattice \( L \) of flats of \( M \). Since \( T_M(x, y) = T_{M^*}(y, x) \), where \( M^* \) denotes the dual matroid of \( M \), it follows that \( \alpha(M) = \tilde{\mu}(M^*) \). For this reason \( \alpha(M) \) is also called the Möbius coinvariant of \( M \). We note that \( \tilde{\mu}(M^*) \) counts the number of internal activity zero bases in \( M \) with respect to any ordering \( \omega \) of the ground set elements of \( M \) [1].

Also recall that the independent sets of a matroid \( M \) form a simplicial complex \( \text{IN}(M) \), called the independence complex of \( M \). The \( f \)-vector of \( \text{IN}(M) \) is the sequence \( (f_0, f_1, \ldots, f_r) \), where \( f_i = \) the number of cardinality \( i \) independent sets in \( M \). The \textit{unsigned reduced Euler number} of \( \text{IN}(M) \) is given by \( \tilde{\alpha}(\text{IN}(M)) = \sum_{i=0}^r (-1)^i f_{r-i} \), which is also well-known to equal \( T_M(0, 1) \). Moreover, \( \text{IN}(M) \) is shellable, hence it has the homotopy type of a wedge of \( (r-1) \)-dimensional spheres. It follows that \( \alpha(M) = |\tilde{\alpha}(\text{IN}(M))| = \text{rk} \tilde{H}_{r-1}(\text{IN}(M)) \) [1], where \( \tilde{H}_{r-1}(\text{IN}(M)) \) is the \( (r-1) \)th reduced homology group of \( \text{IN}(M) \) which is free abelian.

Since \( \alpha(M) \) is a Tutte–Grothendieck invariant [2], i.e., an evaluation of the Tutte polynomial \( T_M(x, y) \), it satisfies the deletion–contraction recursions: if \( e \in M \) is not an isthmus nor a loop, then \( \alpha(M) = \alpha(M-e) + \alpha(M/e) \). If \( e \) is a loop, then \( \alpha(M) = \alpha(M-e) \). It follows that \( \alpha(M) = 1 \) when \( M \) consists of loops only for \( \alpha(M) = 1 \) when \( M \) is empty. Moreover, if \( M \) has an isthmus \( e \), then \( \alpha(M) = 0 \) for \( T_M(x, y) = x T_{M/e}(x, y) = 0 \) when \( x = 0 \). In this note we present a new recursion formula for \( \alpha(M) \) involving contractions of \( M \) through circuits.

2. A new formula for \( \alpha(M) \)

The new formula will be a consequence of Crapo’s complementation theorem for the Möbius function on finite lattices [3]. Let us recall Crapo’s theorem. For a finite lattice \( L \) let \( e \) be an atom in \( L \) and \( e^\perp = \{ e' \in L | e \wedge e' = 0, e \vee e' = \hat{1} \} \). For \( x, y \in L \) let \( \zeta(x, y) = 1 \) if \( x \leq y \) and \( 0 \) otherwise. Then Crapo’s theorem states that

\[
\mu(\hat{0}, \hat{1}) = \sum_{x, y \in e^\perp} \mu(\hat{0}, x) \zeta(x, y) \mu(y, \hat{1}).
\]

**Theorem 1.** Given a matroid \( M \), let \( e \in M \) be an element in the ground set that is neither an isthmus nor a loop. Let \( C_e \) be the set of all circuits in \( M \) containing \( e \). Then we have

\[
\alpha(M) = \sum_{C \in C_e} \alpha(M/C),
\]

where \( M/C \) is the contraction of \( M \) through \( C \).

**Proof.** We will assume that the contraction \( M/C \) is obtained first by deleting the element \( e \), and then contracting every remaining element of \( C \) in order to avoid contracting a loop. First we prove the “dual” statement of the theorem as follows. Let \( L(M) \) be the geometric lattice of flats of \( M \). Given an atom \( e \in L(M), e^\perp \) is the set of all hyperplanes not containing \( e \).
By Crapo’s theorem we have
\[ \bar{\mu}(M) = \sum_{H \in \mathcal{E}_e} \bar{\mu}(H), \]
where \( \bar{\mu}(M) = |\mu_L(M)(\hat{0}, \hat{1})| \) if \( M \) is loopless and \( \bar{\mu}(M) = 0 \) otherwise.

Now the theorem follows by applying this equation to the dual \( M^* \) of \( M \). More precisely, recall that \( \bar{\mu}(M) = \bar{\mu}(M^*) \) for any matroid \( M \). A hyperplane \( H^* \) in \( M^* \) not containing \( e \) is precisely the complement of a circuit in \( M \) containing \( e \), i.e., \( H^* = M^* - C \) for some \( C \in \mathcal{C}_e \). Therefore we have
\[ \bar{\mu}(H^*) = \bar{\mu}(M^*|_{M^* - C}) = \bar{\mu}((M/C)^*) = \chi(M/C) \]
Now the result follows. \( \square \)

**Remark.** There is also a direct proof of this theorem applying Crapo’s theorem to the lattice \( P \) (with a formal \( \hat{1} \)) of simplices in \( \text{IN}(M) \) ordered by inclusion. Recall that the order complex \( \mathcal{P} \) of \( P \) is a simplicial complex whose simplices are chains in \( P \). Since \( \mathcal{P} \) is the first barycentric subdivision of \( \text{IN}(M) \) \([4]\), \( \mathcal{P} \) and \( \text{IN}(M) \) have the same homotopy type. Therefore,
\[ |\mu_P(\hat{0}, \hat{1})| = |\mathcal{P}(\mathcal{P})| = |\mathcal{P}(\text{IN}(M))| = \chi(M). \]

Now given an atom \( e \in P \), one can show that \( e^\perp \) is partitioned as
\[ e^\perp = \bigcup_{C \in \mathcal{C}_e} [C - e, \hat{1}], \]
and that for each \( C \in \mathcal{C}_e \), the half open interval \( [C - e, \hat{1}] \) is isomorphic to the poset of independent sets in \( M/C \), where the isomorphism is given by \( x \mapsto x - (C - e) \). Then the theorem follows from an application of Crapo’s theorem to \( \mu_P(\hat{0}, \hat{1}) \) together with this isomorphism.

As an application of this theorem we compute the \( \chi \)-invariant of a class of paving matroids. Recall that a matroid \( M \) of rank \( r \geq 2 \) is a paving matroid if and only if \( M \) has no circuits of cardinality \( \leq r - 2 \), i.e., every circuit in \( M \) has corank 0 or 1. We define a paving matroid to be symmetric if every hyperplane of \( M \) has the same cardinality. In particular, if \( M \) is a symmetric paving matroid with \( n \) elements in each hyperplane, and if \( C \) is a circuit of corank 1 with closure \( \bar{C} \), then \( |\bar{C}| = n \).

**Corollary 2.** Let \( M \) be a symmetric paving matroid of rank \( r \geq 2 \) with \( m \) elements in the ground set and \( n \) elements in each hyperplane. Fix a ground set element \( e \in M \) and let \( c_0 \) and \( c_1 \) be the number of circuits in \( \mathcal{C}_e \) with coranks 0 and 1, respectively. Then \( \chi(M) = c_0 + (m - n - 1)c_1 \).

**Proof.** If \( C \in \mathcal{C}_e \) has corank 0, then \( \chi(M/C) = 1 \) because, \( M/C \) is empty or has loops only. If \( C \in \mathcal{C}_e \) has corank 1, then \( M/C \) is a rank 1 matroid with \( m - n \) parallel elements (and loops). Then we have \( \chi(M/C) = m - n - 1 \) either from a straight forward application
of Theorem 1 or from the Tutte polynomial $T_{M/C}(x, y) = x + y + y^2 + \cdots + y^{m-n-1}$. Now the result follows. □

**Examples.** (1) In the uniform matroid $U_{k,n}$ every circuit has corank 0. Moreover, for any $e \in U_{k,n}$, a circuit containing $e$ is a subset of cardinality $k + 1$ containing $e$. It follows that $\alpha(U_{k,n}) = c_0 = \binom{n-1}{k}$.

(2) For the Fano matroid $F$ and any $e \in F$, we have $c_0 = 4$ and $c_1 = 3$. Every circuit $C$ of corank 1 is a hyperplane in $F$, and has cardinality 3. Since the cardinality of $F$ is 7, we have $\alpha(F) = c_0 + 3c_1 = 13$.

(3) Now we discuss an example that is not a paving matroid. Let $M$ be the cycle matroid of the complete graph $K_5$ with the ground set $E(K_5)$. Given any $e \in E(K_5)$, $\mathcal{C}_e$ consists of three circuits $C$ of length 3, six circuits $C'$ of length 4, and six circuits $C''$ of length 5. By Theorem 1, we have $\alpha(M) = 3 \alpha(M/C) + 6 \alpha(M/C') + 6 \alpha(M/C'')$. It is easily seen that $\alpha(M/C'') = 1$ and $\alpha(M/C') = 3$. Now $M' = M/C$ has rank 2 with two sets of three parallel edges and one simple edge. Here, we apply Theorem 1 again by letting $e$ to be the simple edge in $M'$. Since there are nine circuits of length 3 in $M'$ containing $e$, we have $\alpha(M') = 9$. Therefore, $\alpha(M) = 27 + 18 + 6 = 51$. Note that the $f$-vector of $\text{IN}(M)$ is readily seen to be $(1, 10, 45, 110, 125)$. Hence we also confirm that $\alpha(M) = |\tilde{\chi}(\text{IN}(M))| = |-1 + 10 - 45 + 110 - 125| = 51$.

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**References**


