Stabilization of switched linear systems
with time-delay in detection of switching signal

Guangming Xie *, Long Wang

Center for Systems and Control, LTCS and Department of Mechanics and Engineering Science,
Peking University, Beijing, 100871, PR China

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Abstract

A feedback stabilization problem for switched linear systems with time-delay in detection of
switching signal is formulated. First, online state feedback controller design method for asymptotic
stability and exponential stability is given. Then, offline state feedback controller design method for
asymptotic stability and exponential stability is given as well.

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1. Introduction

Switched linear systems are an important class of hybrid dynamic systems which consist of
a family of linear time-invariant systems and a switching law specifying the switching
between them. In recent years, there has been increasing interest in the control problems
of switched systems due to their significance both in theory and applications.

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* Corresponding author.
E-mail address: xiegming@mech.pku.edu.cn (G. Xie).
There are some papers on the definition and determination of controllability and reachability of switched systems [1–6]. Ezzine and Haddad gave a necessary and sufficient condition for single-periodic controllability of linear switched systems under the assumption that the mode switching sequence is periodic [1]. Xie and Zheng extended the results to the multiple-periodic controllability case [3]. Sun and Zheng gave a sufficient condition and a necessary condition for controllability of general switched linear systems [2]. Then Xie and Wang established a sufficient and necessary condition for controllability and extended the results to linear switched systems with time-delays [4–6].

Meanwhile, there have been a lot of studies on stability analysis and design of switched systems [8–20,25–27]. Liberzon and Morse summarize three basic problems regarding stability and design of switched systems [7]. They are: (i) stability for arbitrary switching sequences; (ii) stability for certain useful classes of switching sequences; (iii) construction of stabilizing switching sequences. For problem (i), finding conditions under which there exists a common Lyapunov function for the system is a typical approach [8–10]. For problem (ii), multiple Lyapunov functions method, an extension of classical Lyapunov theory, is the main tool [11–13]. For problem (iii), there are many results available [14–20]. Pettersson and Lennartson show that the search for Lyapunov functions can be formulated as a linear matrix inequality (LMI) problem [14]. Xu and Antsaklis give a necessary and sufficient condition for the asymptotic stabilizability of switched systems consisting of several second-order subsystems with unstable foci [15]. If the condition holds, an asymptotically stabilizing switching law can be obtained. Hu, Xu, Antsaklis and Michel discuss the robustness of this kind of stabilizing control laws [16]. Hespanha and Morse prove that exponential stability is achieved when the number of switches in any finite interval grows linearly with the length of the interval, and the growth rate is sufficiently small [17]. Wicks, Peleties and DeCarlo show that there exists a switching law for the stabilization of systems with $N = 2$ if there is a stable convex combination of $A_1$ and $A_2$ [18]. Li, Wen and Soh generalize this result to arbitrary $N$ with two assumptions: (i) a basis of $\mathbb{R}^n$ can be selected from $\bigcup_{i=1}^{N} \widehat{X}_i$, where $\widehat{X}_i$ is the set of stable eigenvectors of $A_i$; (ii) denote such a basis as $\{X_{1,1}, \ldots, X_{1,r_1}, \ldots, X_{N,1}, \ldots, X_{N,r_N}\}$, then $\text{Span}\{X_{i,1}, \ldots, X_{i,r_i}\}$ are invariant under $A_i$, $i = 1, \ldots, N$ [19]. Schinkel, Wang and Hunt discuss the methods for stable and robust controller design in switched linear systems [25]. Under the assumption that all subsystems are completely controllable and are in controller canonical form, a stable state feedback design method and a robust state feedback design method were presented such that the system is stable or robustly stable for arbitrary switching signal [25].

A common assumption in the above results is that the detection of the switching signal is instant. However, in many real switched systems, the switching signal is created by some unknown or nondeterministic function, for example, unknown abrupt phenomena such as component and interconnection failures. One can not detect the changing of the switching signal instantly, but only after a time period. All the above results become ineffective in such a case. To our knowledge, there is little work concerning this phenomenon. In this paper, we formulate the state feedback stabilization problem for switched linear systems with time-delay in the detection of the switching signal. Then, online state feedback design and offline state feedback design is investigated, respectively.

This paper is organized as follows. Section 2 formulates the problem. Section 3 is the main result of this paper. Finally, we provide the conclusion in Section 4.
Notations. We use standard notations throughout this note. \( \mathbb{R} \) is the set of real number, \( \mathbb{R}^+ \) is the set of nonnegative real number. Given a vector \( v \in \mathbb{C}^n \), \( \bar{v} \) is the conjugate of \( v \). Given a positive scalar \( m \), \( [m] \) is the maximum integer smaller than \( m \). The norm \( \| \cdot \| \) is the \( \infty \)-norm.

2. Problem formulation and preliminaries

Consider a switched linear system given by
\[
\begin{align*}
\dot{x}(t) &= A_{\delta(t)}x(t) + B_{\delta(t)}u(t), \\
\gamma(t) &= \delta(t - \tau),
\end{align*}
\]
(1)
where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R} \) is the single input, the right continuous function \( \delta(t) : \mathbb{R}^+ \to \{1, 2, \ldots, N\} \) is the switching signal created by some unknown or nondeterministic function (e.g., unexpected fault, change of working points, etc.). Moreover, \( \delta(t) = i \) implies that the subsystem \( (A_i, B_i) \) is activated, \( i = 1, \ldots, N \). \( \gamma(t) \) is the detection function of \( \delta(t) \), the time-delay \( \tau > 0 \) means that one cannot detect which subsystem \( (A_i, B_i) \) is being activated instantly, but after a time period \( \tau \).

Assumption 1. \( (A_i, B_i) \) is controllable, for any \( i = 1, \ldots, N \).

Assumption 2. If subsystem \( (A_i, B_i) \) is activated, then it will hold at least for a period of \( h_i > \tau \), for any \( i = 1, \ldots, N \).

Remark 1. Assumption 1 is reasonable for real systems, since controllability is generic. Assumption 2 is also reasonable since we require the real systems be “finite time finite switching.”

In this paper, we try to use piecewise constant state feedback to stabilize the system (1), i.e., once we detect which subsystem is being activated, we can find an appropriate constant state feedback for this subsystem. For system (1), we introduce the piecewise constant state feedback as follows:
\[
\begin{align*}
u(t) &= K_{\gamma(t)}x(t),
\end{align*}
\]
(2)
where \( K_i \) is to be designed, \( i = 1, \ldots, N \), then we get the closed-loop system
\[
\begin{align*}
\dot{x}(t) &= A_{\delta(t)}x(t) + B_{\delta(t)}K_{\gamma(t)}x(t).
\end{align*}
\]
(3)

Definition 1 (Asymptotic stabilizability). System (1) is said to be asymptotically stabilizable via state feedback, if for any switching signal \( \delta(t) \), the closed-loop system (3) satisfies
\[
\lim_{t \to \infty} x(t) = 0.
\]

Definition 2 (Exponential stabilizability). System (1) is said to be exponentially stabilizable via state feedback, if for any switching signal \( \delta(t) \), there exist two constants \( C > 0 \) and \( \beta > 0 \) such that the closed-loop system (3) satisfies \( \|x(t)\| \leq C \|x(0)\| \exp(-\beta t) \).
Since the detection of the switching signal is with a time delay, the switching of the state feedback is with a time-delay too. Given a switching signal

$$\delta(t) = \delta_m, \quad \text{if } t \in [t_m, t_m), \quad m = 1, 2, \ldots,$$

where $t_0 = 0$, and $t_m - t_{m-1} > \tau$, $m = 1, 2, \ldots$, the evolution of the closed-loop system can be described as in Fig. 1.

As is well known, for an LTI system, if it is controllable, then we can find a constant state feedback such that the closed-loop system is asymptotically stable or exponentially stable [21–23]. By Fig. 1, even if the time-delay is zero, the stability of all $A_i + B_i K_i$, $i = 1, \ldots, N$, does not guarantee the stability of the whole system under arbitrary switching, since there may not exist a common Lyapunov function for all the subsystems [12].

Before giving the main results, a basic lemma is introduced as follows.

**Lemma 1** [24]. Consider the matrix

$$A = \begin{bmatrix}
0 & 1 \\
& \ddots & \ddots \\
& & 0 & 1 \\
-\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1}
\end{bmatrix}$$

and suppose its distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ are given. Then we have

$$\|\exp(Ah)\| \leq \exp\left(\max_{j=1, \ldots, n} \text{Re}(\lambda_j)h\right) \times \max_{k=0, \ldots, n-1} \left(\sum_{j=1}^{n} |\lambda_j|^k\right)^{\frac{1}{n}} \frac{\left(1 + \max_{i=1, \ldots, n} |\lambda_i|\right)^{n-1}}{\left(\min_{1\leq i \leq j \leq n} |\lambda_i - \lambda_j|\right)^{n-1}}.$$  \hspace{1cm} (4)

Here we extend Lemma 1 to a more general case.
Lemma 2. Given two LTI systems \((A, B), (\tilde{A}, \tilde{B})\) and two scalars \(h, \tau > 0\), suppose \((A, B)\) is controllable and \(\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0\), denote \(\tilde{\alpha} = [\alpha_0, \ldots, \alpha_{n-1}]\), and

\[
F = \begin{bmatrix} A^{n-1}B, \ldots, AB, B \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \cdots & \cdots & 1 \end{bmatrix}.
\]

Let \(u = Kx\) be a constant state feedback such that the closed-loop system \(\dot{x} = (A + BK)x\) is stable, denote the distinct eigenvalues of the matrix \(A + BK\) as \(\lambda_1, \ldots, \lambda_n\), then we have

\[
\ln\left(\frac{\|\exp[(\tilde{A} + \tilde{B}K)\tau]\exp[(A + BK)h]\|}{\|F\| + \ln n + \max_{i=1, \ldots, n} \Re(\lambda_i)h + (n - 1) \ln \left(\frac{1 + \max_{i=1, \ldots, n} |\lambda_i|^2}{\min_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|}\right)}\right) \leq \left(\|A\| + \|B\|\|F\| + 1 + \sum_{i=1}^{n} (1 + |\lambda_i|)\right) \tau
\]

\[
+ \ln \|F^{-1}\| \|F\| + \ln n + \max_{i=1, \ldots, n} \Re(\lambda_i)h + (n - 1) \ln \left(\frac{1 + \max_{i=1, \ldots, n} |\lambda_i|^2}{\min_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|}\right).
\]

Proof. See Appendix A. \(\square\)

In the sequel, we first consider single input systems, i.e., suppose that \(B_i \in \mathbb{R}^n\), \(i = 1, \ldots, N\). For system (1), consider each subsystem \((A_i, B_i), i = 1, \ldots, N\), suppose \(\det(sI - A_i) = s^n + \alpha_{i,n-1}s^{n-1} + \cdots + \alpha_{i,1}s + \alpha_{i,0}\), denote \(\tilde{\alpha}_i = [\alpha_{i,0}, \ldots, \alpha_{i,n-1}]\), and

\[
F_i = \begin{bmatrix} A_i^{n-1}B_i, \ldots, A_iB_i, B_i \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{i,n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i,1} & \cdots & \cdots & 1 \end{bmatrix}.
\]

3. Controller design

In this section, two kinds of feedback stabilization mechanisms will be investigated.

The first mechanism is online-type, i.e., there is no feedback in the beginning, once we acquire the value of the switching signal, we add an appropriate constant feedback; afterwards, once we detect that the switching signal is changed and the system is switched from \((A_i, B_i)\) to \((A_j, B_j)\), then we add an appropriate state feedback \(K_j\) according to the information of \((A_i, B_i)\) and its state feedback \(K_i\) for \((A_j, B_j)\).
The second mechanism is offline-type, i.e., for each subsystem $(A_i, B_i)$, a state feedback $K_i$ has been designed beforehand, once we detect which subsystem is being activated then the corresponding state feedback is selected to be added.

The main difference between the two mechanisms is that in the online-type one, for the same subsystem, we may design many different controllers since the design is real-time and depends on the information of the former subsystem and its controller; while in the offline-type one, for the same subsystem, we may use the same controller which has been designed beforehand.

The controller design is based on moving all poles of all subsystems to appropriate positions in the left-hand side of the s-plane.

3.1. Online state feedback controller design

In this subsection, we present online state feedback controller design.

**Theorem 1.** For system (1), suppose Assumptions 1 and 2 are satisfied, then the system is asymptotically stabilizable by online state feedback mechanism.

**Proof.** For system (1), given any switching signal

$$\delta(t) = t_m, \quad \text{if } t \in [t_m-1, t_m), \ m = 1, 2, \ldots,$$

where $t_0 = 0$, and $t_m - t_{m-1} > h_i$, $m = 1, 2, \ldots$, we will design appropriate state feedback $u(t) = K_i x(t)$, $t \in [t_m-1 + \tau, t_m + \tau)$, such that $\lim_{t \to \infty} \|x(t)\| = 0$.

Denote $A_{i0} = A_i$, $B_{i0} = B_i$ and $K_{i0} = 0$. It is easy to see that for $m = 1, 2, \ldots$

$$x(t_m) = \exp[(A_{i_m} + B_{i_m} K_{i_m})(tm - t_{m-1} - \tau)] \exp[(A_{i_m} + B_{i_m} K_{i_m})\tau] x(t_{m-1}).$$

Thus, if given a scalar $0 < \rho < 1$, we can find $K_{im}$, $m = 1, 2, \ldots$, such that

$$\| \exp[(A_{i_m} + B_{i_m} K_{i_m})(tm - t_{m-1} - \tau)] \exp[(A_{i_m} + B_{i_m} K_{i_m})\tau] \| \leq \rho,$$

then, we have $\lim_{t \to \infty} \|x(t_m)\| = 0$. This implies that system (1) is asymptotically stable.

Now we will show that such a $K_{im}$ can be found. Denote the $n$ eigenvalues of $A_{i_m} + B_{i_m} K_{i_m}$ as

$$\lambda_j = -z - \frac{j - 1}{n - 1} (1 + z), \ j = 1, \ldots, n,$$

where $z > 0$. Then, we have

$$\max_{i = 1, \ldots, n} \text{Re}(\lambda_i)(tm - t_{m-1} - \tau) + (n - 1) \ln \left( \frac{1 + \max_{i = 1, \ldots, n} |\lambda_i|}{\min_{1 \leq i \leq j \leq n} |\lambda_i - \lambda_j|} \right)$$

$$= -z(tm - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z).$$

Denote

$$f(z) = -z(h_i - \tau) + (n - 1) \ln 4(n - 1)(1 + z).$$
It is easy to verify that \( \lim_{z \to +\infty} f(z) = -\infty \). Thus, there must exist \( z^* > 0 \) such that
\[
f(z^*) < -\left( \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| + \ln \| F_{m-1}^{-1} \| F_m \| + \ln n \right) + \ln \rho.
\]
This implies that for any \( t_m \) with \( t_m - t_{m-1} > h_m \),
\[
-z(t_m - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z)
\leq -\left( \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| + \ln \| F_{m-1}^{-1} \| F_m \| + \ln n \right) + \ln \rho.
\]
Thus, we can find an appropriate \( K_m \) such that the eigenvalues of \( A_{m-1} + B_{m-1} K_{m-1} \) are
\[
\lambda_j = -z^* - \frac{j - 1}{n} (1 + z^*), \quad j = 1, \ldots, n.
\]
Then, we have
\[
\ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})(t_m - t_{m-1} - \tau)] \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| 
\leq \ln \left[ \exp[(A_{m-1} + B_{m-1} K_{m-1})(t_m - t_{m-1} - \tau)] \| + \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| 
\right.
\leq \ln \| F_{m-1}^{-1} \| F_m \| + \ln n + \max_{i=1, \ldots, n} \text{Re} (\lambda_i)(t_m - t_{m-1} - \tau)
\]
\[
(1 + \max_{i=1, \ldots, n} |\lambda_i|)^2
\]
\[
+ (n - 1) \ln \left( \min_{1 \leq i \leq j \leq n} |\lambda_i - \lambda_j| \right) + \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \|
\]
\[
= \ln \| F_{m-1}^{-1} \| F_m \| + \ln n - z^* (t_m - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z^*)
\]
\[
+ \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| 
\leq \ln \| F_{m-1}^{-1} \| F_m \| + \ln n - (\ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| 
\]
\[
+ \ln \| F_{m-1}^{-1} \| F_m \| + \ln n) + \ln \rho + \ln \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \|
\]
\[
= \ln \rho.
\]
Hence, we have \( \| \exp[(A_{m-1} + B_{m-1} K_{m-1})(t_m - t_{m-1} - \tau)] \| \exp[(A_{m-1} + B_{m-1} K_{m-1})\tau] \| \leq \rho \). □

We can extend Theorem 1 to the exponential stabilizability case.

**Theorem 2.** For system (1), suppose Assumptions 1 and 2 are satisfied, then the system is exponentially stabilizable by online state feedback mechanism.

**Proof.** The proof is similar to that of Theorem 1. For system (1), given any switching signal
\[
\delta(t) = i_m, \quad \text{if } t \in [t_{m-1}, t_m), \quad m = 1, 2, \ldots,
\]
where \( t_0 = 0 \), and \( t_m - t_{m-1} > h_m, \quad m = 1, 2, \ldots, \) we will find appropriate state feedback \( u(t) = K_m x(t), \quad t \in [t_{m-1} + \tau, t_m + \tau), \) such that system (1) is exponentially stable.

Denote \( A_{i_0} = A_{i_0}, B_{i_0} = B_{i_0}, K_{i_0} = 0 \). It is easy to see that for \( m = 1, 2, \ldots, \)
\[
x(t_m) = \exp[(A_{i_m} + B_{i_m} K_{i_m})(t_m - t_{m-1} - \tau)] \exp[(A_{i_m} + B_{i_m} K_{i_m})\tau] x(t_{m-1}).
\]
Thus, if given \( \beta > 0 \), we can find \( K_m, m = 1, 2, \ldots, \) such that
Thus, we can find an appropriate $K_{im}$ such that the eigenvalues of $A_{im} + B_{im} K_{im}$ are

$$\lambda_j = -z - \frac{j - 1}{n - 1} (1 + z), \quad j = 1, \ldots, n,$$

where $z > 0$. Then, we have

$$\max_{i=1, \ldots, n} \Re(\lambda_i) (t_m - t_{m-1} - \tau) + (n - 1) \ln \left( \frac{1 + \max_{i=1, \ldots, n} |\lambda_i|}{\min_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|} \right)$$

$$= -z (t_m - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z).$$

Denote

$$f(z) = -z(h_{im} - \tau) + (n - 1) \ln 4(n - 1)(1 + z).$$

It is easy to verify that $\lim_{z \to +\infty} f(z) = -\infty$. Thus, there must exist $z^* > \beta$ such that $f(z^*) < -\ln \left( \| \exp[(A_{im} + B_{im} K_{im-1})\tau] \| + \ln \| F_{im}^{-1} \| F_{im} \| + \ln n \right) - \beta h_{im}.$

This implies that for any $t_m$ with $t_m - t_{m-1} > h_{im},$

$$-z(t_m - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z)$$

$$\leq -\ln \left( \| \exp[(A_{im} + B_{im} K_{im-1})\tau] \| + \ln \| F_{im}^{-1} \| F_{im} \| + \ln n \right) - \beta (t_m - t_{m-1}).$$

Thus, we can find an appropriate $K_{im}$ such that the eigenvalues of $A_{im} + B_{im} K_{im}$ are

$$\lambda_j = -z^* - \frac{j - 1}{n - 1} (1 + z^*), \quad j = 1, \ldots, n.$$

Then, we have

$$\ln \left\| \exp[(A_{im} + B_{im} K_{im})(t_m - t_{m-1} - \tau)] \exp[(A_{im} + B_{im} K_{im-1})\tau] \right\|$$

$$\leq \ln \left\| \exp[(A_{im} + B_{im} K_{im})(t_m - t_{m-1} - \tau)] \right\| + \ln \left\| \exp[(A_{im} + B_{im} K_{im-1})\tau] \right\|$$

$$\leq \ln \| F_{im}^{-1} \| F_{im} \| + \ln n + \max_{i=1, \ldots, n} \Re(\lambda_i) (t_m - t_{m-1} - \tau)$$

$$+ (n - 1) \ln \left( \frac{1 + \max_{i=1, \ldots, n} |\lambda_i|}{\min_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|} \right) + \ln \left\| \exp[(A_{im} + B_{im} K_{im-1})\tau] \right\|$$

$$= \ln \| F_{im}^{-1} \| F_{im} \| + \ln n - z^*(t_m - t_{m-1} - \tau) + (n - 1) \ln 4(n - 1)(1 + z^*)$$

$$+ \ln \left\| \exp[(A_{im} + B_{im} K_{im-1})\tau] \right\|.$$
\[
\ln \|F_{im}^{-1}\| \|F_{im}\| + \ln n - (\ln \|\exp[(A_{im} + B_{im}K_{im-1})\tau]\| \\
+ \ln \|F_{in}^{-1}\| \|F_{in}\| + \ln n) \\
- \beta(t_m - t_{m-1}) + \ln \|\exp[(A_{im} + B_{im}K_{im-1})\tau]\| \\
= -\beta(t_m - t_{m-1}).
\]

Hence, we have \[\|\exp[(A_{im} + B_{im}K_{im})(t_m - t_{m-1} - \tau)]\exp[(A_{im} + B_{im}K_{im-1})\tau]\| \leq \exp[-\beta(t_m - t_{m-1})]. \]

3.2. Offline state feedback controller design

In this subsection, we present offline state feedback controller design.

**Theorem 3.** For system (1), suppose Assumptions 1 and 2 are satisfied, if there exist \(z^*_i > 0\), \(i = 1, \ldots, N\), such that
\[
\max_{j=1, \ldots, N} \|A_j\| + \max_{j=1, \ldots, N} \|B_j\| \|F_i\| \left(\|\tilde{\alpha}_i\| - 1 + \frac{3^\eta(1 + z^*_i)^\eta}{2^n}\right)\tau \\
+ \ln \|F^{-1}_i\| \|F_i\| + \ln n - z^*_i h_i + (n - 1) \ln 4(n - 1)(1 + z^*_i) < 0,
\]

then, we can find appropriate state feedback \(K_i\), \(i = 1, \ldots, N\), such that the system is asymptotically stabilizable by offline state feedback mechanism.

**Proof.** Since (8) holds, there exists \(\rho < 0\) such that
\[
\max_{j=1, \ldots, N} \|A_j\| + \max_{j=1, \ldots, N} \|B_j\| \|F_i\| \left(\|\tilde{\alpha}_i\| - 1 + \frac{3^\eta(1 + z^*_i)^\eta}{2^n}\right)\tau \\
+ \ln \|F^{-1}_i\| \|F_i\| + \ln n - z^*_i h_i + (n - 1) \ln 4(n - 1)(1 + z^*_i) < \rho.
\]

Thus, we can find an appropriate \(K_i\) such that the eigenvalues of \(A_i + B_iK_i\) are
\[
\lambda_{i,j} = -z^*_i - \frac{j - 1}{n - 1}(1 + z^*_i), \quad j = 1, \ldots, n.
\]

By Lemma 2, for any \(i \neq j, t > h_i\), we have
\[
\ln \|\exp[(A_j + B_jK_j)\tau]\exp[(A_i + B_iK_i)(t - \tau)]\| \\
\leq \left[\|A_j\| + \|B_j\| \|F\| \left(\|\tilde{\alpha}_i\| - 1 + \prod_{m=1}^{n} (1 + |\lambda_{i,m}|)\right)\tau + \ln \|F^{-1}_i\| \|F_i\| + \ln n \\
+ \max_{m=1, \ldots, n} \Re(\lambda_{i,m})(t - \tau) + (n - 1) \ln \left(\frac{1 + \max_{m=1, \ldots, n} |\lambda_{i,m}|}{\min_{1 \leq j_1 < j_2 \leq n} |\lambda_{i,j_1} - \lambda_{i,j_2}|}\right).
\]

Since

\[
\|F_{im}^{-1}\| \|F_{im}\| + \ln n - \ln \|\exp[(A_{im} + B_{im}K_{im})\tau]\| \\
+ \ln \|F_{in}^{-1}\| \|F_{in}\| + \ln n \\
- \beta(t_m - t_{m-1}) + \ln \|\exp[(A_{im} + B_{im}K_{im-1})\tau]\| \\
= -\beta(t_m - t_{m-1}).
\]
\[
\prod_{m=1}^{n} (1 + |\lambda_i,m|) = (1 + z_i^*) \left(1 + z_i^* + \frac{1}{n-1} (1 + z_i^*) \right) \cdots (2 + 2z_i^*) \\
\leq \left(1 + z_i^* + \frac{z_i^*}{2} \right) \leq 3^n (1 + z_i^*)^n
\]

we have

\[
\ln \| \exp[(A_j + B_j K_i)\tau] \exp[(A_i + B_i K_i)(t - \tau)] \| \\
\leq \left[ \max_{j=1,\ldots,N} \| A_j \| + \max_{j=1,\ldots,N} \| B_j \| \| F_i \| \left( \| \bar{G}_i \| - 1 + \frac{3^n (1 + z_i^*)^n}{2^n} \right) \right] \tau \\
+ \ln \| F_i^{-1} \| \| F_i \| + \ln n - z_i^* h_i + (n-1) \ln 4(n-1) (1 + z_i^*) < \rho. \tag{10}
\]

For system (1), given any switching signal \( \delta(t) = i_m \), if \( t \in [t_m - 1, t_m) \), \( m = 1, 2, \ldots \), where \( t_0 = 0 \), and \( t_m - t_{m-1} > h_{i_m}, \ m = 1, 2, \ldots \). Denote \( A_{i_0} = A_{i_1}, B_{i_0} = B_{i_1} \) and \( K_{i_0} = 0 \). It is easy to see that for \( m = 1, 2, \ldots \),

\[
x(t_m + \tau) = \exp[(A_{i_m+1} + B_{i_{m+1}} K_{i_m})\tau] \\
\times \exp[(A_{i_m} + B_{i_m} K_{i_m})(t_m - t_{m-1} - \tau)] x(t_{m-1} + \tau).
\]

By (10), we have

\[
\exp[(A_{i_m+1} + B_{i_{m+1}} K_{i_m})\tau] \exp[(A_{i_m} + B_{i_m} K_{i_m})(t_m - t_{m-1} - \tau)] \leq \exp(\rho),
\]

then, we have \( \lim_{m \to \infty} \| x(t_m + \tau) \| = 0 \). This implies that the closed-loop system is asymptotically stable. \( \Box \)

**Remark 2.** Given an integer \( i \in \{1, \ldots, N\} \), consider the scalar functions

\[
g(z) = C_1 + C_2 (1 + z)^n + C_3 \ln(1 + z), \quad f(z) = C_4 z, \quad z > 0, \tag{11}
\]

where

\[
C_1 = \left[ \max_{j=1,\ldots,N} \| A_j \| + \max_{j=1,\ldots,N} \| B_j \| \| F_i \| (\| \bar{G}_i \| - 1) \right] \tau \\
+ \ln \| F_i^{-1} \| \| F_i \| + \ln n + (n-1) \ln 4(n-1),
\]

\[
C_2 = \frac{3^n}{2^n} \max_{j=1,\ldots,N} \| B_j \| \| F_i \| \tau, \quad C_3 = n - 1 \quad \text{and} \quad C_4 = h_i.
\]

It is obvious that there exists \( z_i^* \) such that (8) holds if and only if the set

\[
\Omega = \{ z > 0 \ | \ g(z) < f(z) \}
\]

is not empty. If \( \tau < h_i \), then it is possible that the set \( \Omega \) is not empty (see Fig. 2).
Remark 3. From the proof of Theorem 3, it is easy to see that if we can select $K_1, \ldots, K_N$ such that
\[
\exp[(A_j + B_j K_i)\tau]\exp[(A_i + B_i K_i)h_i] < 1, \quad \forall i, j = 1, \ldots, N,
\] (12)
then the closed-loop system is asymptotically stabilizable. The inequalities (8) is just a sufficient condition for inequalities (12). Thus, even if (8) is not satisfied, the system may be stabilized by offline state feedback mechanism.

Moreover, for exponential stabilizability, we have the following theorem.

Theorem 4. For system (1), suppose Assumptions 1 and 2 are satisfied, if there exist $\beta > 0, z^*_i > \beta, i = 1, \ldots, N$, such that
\[
\left[ \max_{j=1,\ldots,N} \|A_j\| + \max_{j=1,\ldots,N} \|B_j F_i\| \left( \|\bar{a}_i\| - 1 + \frac{3^n(1 + z^*_i)^n}{2^n} \right) \right] \tau
+ \ln \left\| F_i^{-1} \right\| F_i \| + \ln n - z^*_i h_i + (n - 1) \ln 4(n - 1) \left( 1 + z^*_i \right) < -\beta h_i,
\] (13)
then, we can select appropriate state feedback $K_i, i = 1, \ldots, N$, such that the system is exponential stabilizable by offline state feedback mechanism.

Proof. The proof is similar to that of Theorem 3. □

Remark 4. Theorems 1–4 are applicable to single-input systems. As is well known, given an multiple-input LTI system $(A, B)$, where $B \in \mathbb{R}^{n \times p}$, if it is controllable, then we can find appropriate matrix $K \in \mathbb{R}^{p \times n}$ and vector $H \in \mathbb{R}^p$ such that the single input LTI system $(A + BK, BH)$ is still controllable (see [22]). Based on this fact, we can apply these theorems to multiple-input systems by using a set of single-input subsystems to replace the original multiple-input subsystems.
4. Example

Here, a numerical example is presented to illustrate our results.

**Example 1.** Consider the 2-dimensional switched linear system, with

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

the time-delay \( \tau = 0.1 \) and \( h_1 = h_2 = 1 \).

It is easy to see that the system in Example 1 satisfies Assumptions 1 and 2, hence it is asymptotically stabilizable. However, there do not exist \( z_1^*, z_2^* \) satisfying (8), hence we do not know whether the system can be asymptotically stabilized by offline state feedback mechanism. Moreover, if we take \( K_1 = [-100 - 20] \) and \( K_2 = [-20 - 100] \), it is easy to verify that (12) holds. Thus, the system is asymptotically stabilizable by offline state feedback mechanism as well.

If the time constants change to \( h_1 = h_2 = 4 \gg \tau = 0.1 \), then we can obtain by direct computation that there exist \( z_1^* = z_2^* = 10 \) satisfying (8). Furthermore, we can find the state feedback \( K_1 = [-210 - 31], K_2 = [-31 - 210] \) such that the closed-loop system is asymptotically stable.

5. Conclusion

A stabilization problem for switched linear systems with time-delay in detection of switching signal has been formulated. First, online state feedback controller design method for asymptotic stability and exponential stability has been given. Then, offline state feedback controller design method for asymptotic stability and exponential stability has been given as well.

Appendix A

**Proof of Lemma 2.** First, since

\[
\max_{k=0,\ldots,n-1} \left( \sum_{j=1}^{n} |\lambda_j|^k \right) \leq n \max_{j=1,\ldots,n} \left( \max_{j=1,\ldots,n} |\lambda_j| \right)^{n-1} \leq n \left( 1 + \max_{j=1,\ldots,n} |\lambda_j| \right)^{n-1},
\]

we can loosen the inequality (4) to

\[
\|\exp(Ah)\| \leq n \exp \left( \max_{j=1,\ldots,n} \text{Re}(\lambda_j)h \right) \left( 1 + \max_{i=1,\ldots,n} |\lambda_i| \right)^{2n-2} \left( \min_{1 \leq i \leq j \leq n} |\lambda_i - \lambda_j| \right)^{n-1}.
\]

Let \( A_c = F^{-1}AF, B_c = F^{-1}B \) and \( K_c = KF \). Thus, we know that \( (A_c, B_c) \) is in controller canonical form [21,22]. Then, we have
\[
\ln \left( \left\| \exp \left( \tilde{A} + \tilde{B}K \right) \right\| \exp \left( \left( A + BK \right) h \right) \right) \\
\leq \ln \left( \left\| \exp \left( \tilde{A} + \tilde{B}K \right) \right\| \left\| \exp \left( A + BK \right) h \right\| \right) \\
\leq \ln \left\| \exp \left( \tilde{A} + \tilde{B}K \right) \right\| + \ln \left\| \exp \left( A + BK \right) h \right\| \\
\leq \ln \left( \left( \tilde{A} + \tilde{B}K \right) \tau \right) + \ln \left\| \exp \left( A + BK \right) h \right\| \\
\leq \left( \tilde{A} + \tilde{B}KcF^{-1} \right) \tau \right) + \left\| \tilde{B} \right\| \left\| F \right\| \left\| F^{-1} \right\| \tau \\
+ \ln \left\| F \right\| \left\| F^{-1} \right\| + \ln \left[ \left\| \exp \left( A + BKc \right) h \right\| \right].
\]

Since \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A + BK \), they are the eigenvalues of \( A_c + BK_c \) as well. Denote \( K_c = [K_{c,1}, \ldots, K_{c,n}] \). Then, we know that \( \lambda_1, \ldots, \lambda_n \) are the \( n \) roots of the equation
\[
s^n + (\alpha_{n-1} - K_{c,n})s^{n-1} + \cdots + (\alpha_2 - K_{c,2})s + (\alpha_1 - K_{c,1}) = 0.
\]

Thus, we have
\[
\alpha_{i-1} - K_{c,i} = (-1)^i \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^i \lambda_{j_k}, \quad \text{for } i = 1, \ldots, n.
\]

It follows that
\[
\| K_c - \tilde{\alpha} \| = \sum_{i=1}^n |\alpha_{i-1} - K_{c,i}| \leq \sum_{i=1}^n \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^i |\lambda_{j_k}|
\]
\[
= \prod_{i=1}^n \left( 1 + |\lambda_i| \right) - 1. \quad \text{(16)}
\]

By (15), we have
\[
\ln \left\| \exp \left( (A_c + BK_c)h \right) \right\| \leq \ln n + \max_{i=1,\ldots,n} \text{Re}(\lambda_i)h
\]
\[
+ (n - 1) \ln \left( \frac{1 + \max_{i=1,\ldots,n} |\lambda_i|}{\min_{1 \leq i \neq j \leq n} |\lambda_i - \lambda_j|} \right). \quad \text{(17)}
\]

Hence, by (16) and (17), we know that (5) holds. \( \square \)

References