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Some characterizations of strongly preinvex functions

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Abstract

In this paper, a new class of generalized convex function is introduced, which is called the strongly α-preinvex function. We study some properties of strongly α-preinvex function. In particular, we establish the equivalence among the strongly α-preinvex functions, strongly α-invex functions and strongly $\alpha\eta$-monotonicity under some suitable conditions. As special cases, one can obtain several new and previously known results for α-preinvex (invex) functions.

Keywords: Preinvex functions; η-Monotone operators; Invex functions; Pseudomonotone invex functions

1. Introduction

In recent years, the concept of convexity has been generalized and extended in several directions using novel and innovative techniques. An important and significant generalization of convex functions is the introduction of invex function, which was introduced by Hanson [1]. This concept is particularly interesting from optimization view point, since it provides a broader setting to study the optimization and mathematical programming problems. Ben-Israel and Mond [2] introduced a class of convex functions, which is called the

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preinvex function. It is known that the differentiable preinvex functions are invex functions, but the converse is not true. However, Mohan and Neogy [3] have shown that the preinvex functions and invex functions are equivalent under certain conditions. Weir and Mond [4] and Noor [5] have shown that the preinvex functions preserve some nice properties that convex functions have. Jeyakumar and Mond [6] introduced and studied another class of generalized convex functions, which is known as strongly α-invex function. It has been shown [6,7] that α-preinvex (α-invex) have useful and important applications in generalized convex programming and multiobjective optimization. We note that the concept of the strongly (α)-invex function defined in [6,7] is misleading. Compare Definitions 2.9–2.11 with those of Jeyakumar [7] and Jeyakumar and Mond [6].

Motivated and inspired by the research going on in this fascinating field, we introduce a new class of generalized functions, which is called strongly α-preinvex functions. We also introduce several new concepts of strongly $\alpha\eta$-monotonicities. We establish the relationship among the strongly α-preinvex, strongly α-invex and $\alpha\eta$-monotonicities under some suitable and appropriate conditions. We also give a necessary condition for strongly $\alpha\eta$-pseudomonotone invex functions. As special cases, one can obtain several new and correct versions of the previously known results for various classes of preinvex and invex functions.

2. Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle .,. \rangle$ and $\|.\|$ the inner product and norm respectively. Let $F : K \to H$ and $\eta(.,.) : K \times K \to R$ be continuous functions. Let $\alpha : K \times K \to R \setminus \{0\}$ be a bifunction. First of all, we recall the following well-known results and concepts.

Definition 2.1. Let $u \in K$. Then the set K is said to be α-invex at u with respect to $\eta(.,.)$ and $\alpha(.,.)$, if, for all $u, v \in K$, $t \in [0, 1]$,

$$u + t\alpha(v, u)\eta(v, u) \in K.$$

K is said to be an α-invex set with respect to η and α, if K is α-invex at each $u \in K$. The α-invex set K is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(v, u) = 1$ and $\eta(v, u) = v - u$ is an invex set, but the converse is not true.

Remark 2.1.

(i) If $\alpha(v, u) = 1$, then the set K is called the invex (η-connected) set; see [4,6,7].
(ii) If $\eta(v, u) = v - u$ and $0 < \alpha(v, u) < 1$, then the set K is called the star-shaped.
(iii) If $\alpha(v, u) = 1$ and $\eta(v, u) = v - u$, then the set K is called the convex set.

From now onward K is a nonempty closed α-invex set in H with respect to $\alpha(.,.)$ and $\eta(.,.)$, unless otherwise specified.

Definition 2.2. The function F on the α-invex set K is said to be α-preinvex with respect to α and η, if

$$F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant (1-t)F(u) + tF(v), \quad \forall u, v \in K,\ t \in [0,1].$$

The function F is said to be α-preconcave if and only if $-F$ is α-preinvex. Note that every convex function is a preinvex function, but the converse is not true. For example, the function $F(u) = -|u|$ is not a convex function, but it is a preinvex function with respect to η and $\alpha(v,u) = 1$, where

$$\eta(v,u) = \begin{cases} v - u, & \text{if } v \leqslant 0,\ u \leqslant 0 \text{ and } v \geqslant 0,\ u \geqslant 0, \\ u - v, & \text{otherwise.} \end{cases}$$

For the applications in generalized convex programming and multiobjective optimization see [6,7].

Definition 2.3. The function F on the α-invex set K is called quasi α-preinvex with respect to α and η, if

$$F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant \max\big\{F(u), F(v)\big\}, \quad \forall u, v \in K,\ t \in [0,1].$$

Definition 2.4. The function F on the α-invex set K is said to be logarithmic α-preinvex with respect to α and η, if

$$F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant \big(F(u)\big)^{1-t}\big(F(v)\big)^{t}, \quad u, v \in K,\ t \in [0,1],$$

where $F(.) > 0$.

From the above definitions, we have

$$\begin{aligned} F\big(u + t\alpha(v,u)\eta(v,u)\big) &\leqslant \big(F(u)\big)^{1-t}\big(F(v)\big)^{t} \leqslant (1-t)F(u) + tF(v) \\ &\leqslant \max\big\{F(u), F(v)\big\} < \max\big\{F(u), F(v)\big\}. \end{aligned}$$

For $t = 1$, Definitions 2.2 and 2.4 reduce to:

Condition A.

$$F\big(u + \alpha(v,u)\eta(v,u)\big) \leqslant F(v), \quad \forall u, v \in K,$$

which plays an important part in studying the properties of the α-preinvex (α-invex) functions. Some properties of the α-preinvex functions have been studied in [9,12]. For $\alpha(v,u) = 1$, Condition A reduces to the following for preinvex functions.

Condition B.

$$F\big(u + \eta(v,u)\big) \leqslant F(v), \quad \forall u, v \in K.$$

For the applications of Condition B see [9,12].

Definition 2.5. The function F on the α-invex set K is said to be pseudo α-preinvex with respect to α and η, if there exists a strictly positive function $b(.,.)$ such that

$$F(v) \leqslant F(u) \quad \Rightarrow \quad F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant F(u) + t(t-1)b(u,v),$$
$$u, v \in K,\ t \in [0,1].$$

Lemma 2.1. *If the function F is α-preinvex function on K with respect to α and η, then F is pseudo α-preinvex function with respect to α and η.*

Proof. Without loss of generality, we assume that $F(v) < F(u)$, $\forall u, v \in K$. For every $t \in [0,1]$, we have

$$\begin{aligned} F\big(u + t\alpha(v,u)\eta(v,u)\big) &\leqslant (1-t)F(u) + tF(v) < F(u) + t(t-1)\big\{F(u) - F(v)\big\} \\ &= F(u) + t(t-1)b(v,u), \end{aligned}$$

where $b(v,u) = F(v) - F(u) > 0$. Thus it follows that the function F is pseudo α-preinvex function with respect to α and η, the required result. □

Definition 2.6. The differentiable function F on K is said to be an α-invex function with respect to α and η, if

$$F(v) - F(u) \geqslant \big\langle \alpha(v,u)F'(u), \eta(v,u)\big\rangle, \quad \forall u, v \in K,$$

where $F'(u)$ is the differential of F at $u \in K$. The concepts of the α-invex and α-preinvex functions have played very important role in the development of convex programming; see [6,7]. Note that for $\alpha(v,u) = 1$, Definition 2.6 is mainly due to Hanson [1].

Definition 2.7. An operator $T : K \to H$ is said to be:

(i) strongly $\alpha\eta$-monotone, iff, there exists a constant $\alpha_1 > 0$ such that

$$\big\langle \alpha(v,u)Tu, \eta(v,u)\big\rangle + \big\langle \alpha(u,v)Tv, \eta(u,v)\big\rangle \leqslant -\alpha_1\big\{\big\|\eta(v,u)\big\|^2 + \big\|\eta(u,v)\big\|^2\big\},$$
$$\forall u, v \in K;$$

(ii) $\alpha\eta$-monotone, iff,

$$\big\langle \alpha(v,u)Tu, \eta(v,u)\big\rangle + \big\langle \alpha(u,v)Tv, \eta(u,v)\big\rangle \leqslant 0, \quad \forall u, v \in K;$$

(iii) strictly $\alpha\eta$-monotone, iff,

$$\big\langle \alpha(v,u)Tu, \eta(v,u)\big\rangle + \big\langle \alpha(u,v)Tv, \eta(u,v)\big\rangle < 0, \quad \forall u, v \in K;$$

(iv) strongly $\alpha\eta$-pseudomonotone, iff, there exists a constant $\nu > 0$ such that

$$\big\langle \alpha(v,u)Tu, \eta(v,u)\big\rangle + \nu\big\|\eta(v,u)\big\|^2 \geqslant 0 \quad \Rightarrow \quad -\big\langle \alpha(u,v)Tv, \eta(u,v)\big\rangle \geqslant 0,$$
$$\forall u, v \in K;$$

(v) $\alpha\eta$-pseudomonotone, iff,

$$\big\langle \alpha(v,u)Tu, \eta(v,u)\big\rangle \geqslant 0 \quad \Rightarrow \quad \big\langle \alpha(u,v)Tv, \eta(u,v)\big\rangle \leqslant 0, \quad \forall u, v \in K;$$

(vi) quasi $\alpha\eta$-monotone, iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle > 0 \quad \Rightarrow \quad \langle \alpha(u,v)Tv, \eta(u,v)\rangle \leqslant 0, \quad \forall u, v \in K;$$

(vii) strictly $\alpha\eta$-pseudomonotone, iff,

$$\langle \alpha(v,u)Tu, \eta(v,u)\rangle \geqslant 0 \quad \Rightarrow \quad \langle \alpha(u,v)Tv, \eta(u,v)\rangle < 0, \quad \forall u, v \in K.$$

Note that for $\alpha(v,u) = 1$, $\forall u, v \in K$, the α-invex set K becomes an invex set. In this case, Definition 2.7 is exactly the same as in [9,12]. In addition, if $\alpha(v,u) = 1$ and $\eta(v,u) = v - u$, then the α-invex set K is the convex set K and consequently Definition 2.7 reduces to the one in [14,15] for the convex set K.

We now define the concept of strongly α-preinvex and α-invex functions on the α-invex set K.

Definition 2.8. A function F on the set K is said to be strongly α-preinvex, if there exists a constant $\mu > 0$ such that

$$F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant (1-t)F(u) + tF(v) - t(1-t)\mu\|\eta(v,u)\|^2, \quad \forall u, v \in K.$$

Note that for $\mu = 0$, strongly α-preinvex functions reduces to α-preinvex functions as defined in Definition 2.2.

Definition 2.9. A differentiable function F on the set K is said to strongly α-invex function if there exists a constant $\mu > 0$ such that

$$F(v) - F(u) \geqslant \langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K,$$

where $F'(u)$ is the differential of a function F at $u \in K$. Clearly Definition 2.8 includes Definition 2.3 as a special case.

From Definition 2.9, we have the following concepts.

Definition 2.10. A differentiable function F on K is said to be strongly pseudo $\alpha\eta$-invex function, iff, there exists a constant $\mu > 0$ such that

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \mu\|\eta(u,v)\|^2 \geqslant 0 \quad \Rightarrow \quad F(v) - F(u) \geqslant 0, \quad \forall u, v \in K.$$

Definition 2.11. A differentiable function F on K is said to be strongly quasi α-invex, if there exists a constant $\mu > 0$ such that

$$F(v) \leqslant F(u) \quad \Rightarrow \quad \langle \alpha(v,u)F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \leqslant 0, \quad \forall u, v \in K.$$

Definition 2.12. The function F on the α-invex set K is said to be pseudo $\alpha\eta$-invex, if

$$\langle \alpha(v,u)F'(u), \eta(v,u)\rangle \geqslant 0 \quad \Rightarrow \quad F(v) \geqslant F(u), \quad \forall u, v \in K.$$

Note that, if $\alpha(v,u) = 1$, then the α-invex set K is exactly the invex set K and consequently Definitions 2.10–2.12 are exactly the same as in [9]. In particular, if $\eta(v,u) =$

$-\eta(v,u)$, $\forall u, v \in K$, that is, the function $\eta(.,.)$ is skew-symmetric and $\alpha(v,u)=1$, then Definitions 2.8–2.12 reduces to the ones in [10,12]. This shows that the concepts introduced in this paper represent an improvement of the previously known ones. All the concepts defined above play important and fundamental part in the mathematical programming and optimization problems; see [6,7].

We also need the following assumption regarding the functions $\eta(.,.)$ and $\alpha(.,.)$.

Condition C. Let $\eta(.,.): K \times K \to H$ and $\alpha(.,.): K \times K \to R \setminus \{0\}$ satisfy the assumptions

$$\eta\big(u, u + t\alpha(v,u)\eta(v,u)\big) = -t\eta(v,u),$$
$$\eta\big(v, u + t\alpha(v,u)\eta(v,u)\big) = (1-t)\eta(v,u), \quad \forall u, v \in K,\ t \in [0,1].$$

Clearly for $t = 0$, we have $\eta(u,u) = 0$, $\forall u \in K$. One can easily show [12] that

$$\eta\big(u + t\alpha(v,u)\eta(v,u), u\big) = t\eta(v,u), \quad \forall u, v \in K.$$

Note that for $\alpha(v,u) = 1$, Condition C collapses to the following condition, which is due to Mohan and Neogy [3].

Condition D. Let $\eta(.,.): K \times K \to H$ satisfy the assumptions

$$\eta\big(u, u + t\eta(v,u)\big) = -t\eta(v,u),$$
$$\eta\big(v, u + t\eta(v,u)\big) = (1-t)\eta(v,u), \quad \forall u, v \in K,\ t \in [0,1].$$

For the applications of Condition D see [7,12,13].

3. Main results

In this section, we consider some basic properties of strongly α-preinvex functions on the α-invex set K.

Theorem 3.1. *Let F be a differentiable function on the α-invex set K. Let Condition* C *hold and $\alpha(u,z) = \alpha(v,z)$, $\forall u, v, z \in K$. Then the function F is a strongly α-preinvex function if and only if F is a strongly α-invex function.*

Proof. Let F be a strongly α-preinvex function on the α-invex set K. Then, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t\alpha(v,u)\eta(v,u) \in K$ and

$$F\big(u + t\alpha(v,u)\eta(v,u)\big) \leqslant (1-t)F(u) + tF(v) - t(1-t)\mu\big\|\eta(v,u)\big\|^2, \quad \forall u, v \in K,$$

which can be written as

$$F(v) - F(u) \geqslant \frac{F(u + t\alpha(v,u)\eta(v,u)) - F(u)}{t} + (1-t)\mu\big\|\eta(v,u)\big\|^2.$$

Letting $t \to 0$ in the above inequality, we have

$$F(v) - F(u) \geqslant \big\langle \alpha(v,u)F'(u), \eta(v,u)\big\rangle + \mu\big\|\eta(v,u)\big\|^2,$$

which implies that F is a strongly α-invex functions.

Conversely, let F be a strongly α-invex function on the α-invex function K. Then, using Condition C, we have

$$\begin{aligned} F(v) - F(v_t) &\geqslant \bigl\langle \alpha(v, v_t)F'(v_t), \eta(v, v_t)\bigr\rangle + \mu\bigl\|\eta(v, v_t)\bigr\|^2 \\ &= (1-t)\alpha(v, v_t)\bigl\langle F'(v_t), \eta(v, u)\bigr\rangle + \mu(1-t)^2\bigl\|\eta(v, u)\bigr\|^2. \end{aligned} \tag{3.1}$$

In a similar way, we have

$$\begin{aligned} F(u) - F(v_t) &\geqslant \bigl\langle \alpha(u, v_t)F'(v_t), \eta(u, v_t)\bigr\rangle + \mu\bigl\|\eta(u, v_t)\bigr\|^2 \\ &= -t\alpha(u, v_t)\bigl\langle F'(v_t), \eta(v, u)\bigr\rangle + t^2\mu\bigl\|\eta(v, u)\bigr\|^2. \end{aligned} \tag{3.2}$$

Multiplying (3.1) by t and (3.2) by $(1-t)$ and adding the resultant, we have

$$F\bigl(u + t\alpha(v, u)\eta(v, u)\bigr) \leqslant (1-t)F(u) + tF(v) - \mu t(1-t)\bigl\|\eta(v, u)\bigr\|^2,$$

showing that F is a strongly α-preinvex function. □

Theorem 3.2. *Let F be differentiable function on the α-invex set K. If the function F is a strongly α-invex function, then its differential $F'(u)$ is strongly $\alpha\eta$-monotone. Conversely, if the function $\alpha(v, u)$ is a symmetric function, that is, $\alpha(v, u) = \alpha(u, v)$, $\forall u, v \in K$, then F is a strongly α-invex function provided Conditions* A *and* C *hold.*

Proof. Let F be a strongly α-invex function on the α-invex set K. Then

$$F(v) - F(u) \geqslant \bigl\langle \alpha(v, u)F'(u), \eta(v, u)\bigr\rangle + \mu\bigl\|\eta(v, u)\bigr\|^2, \quad \forall u, v \in K. \tag{3.3}$$

Changing the role of u and v in (3.3), we have

$$F(u) - F(v) \geqslant \bigl\langle \alpha(u, v)F'(v), \eta(u, v)\bigr\rangle + \mu\bigl\|\eta(u, v)\bigr\|^2, \quad \forall u, v \in K. \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\begin{aligned} &\bigl\langle \alpha(v, u)F'(u), \eta(v, u)\bigr\rangle + \bigl\langle \alpha(u, v)F'(v), \eta(u, v)\bigr\rangle \\ &\quad \leqslant -\mu\bigl\{\bigl\|\eta(v, u)\bigr\|^2 + \bigl\|\eta(u, v)\bigr\|^2\bigr\}, \end{aligned} \tag{3.5}$$

which shows that F' is strongly $\alpha\eta$-monotone.

Conversely, let the differential $F'(u)$ be strongly $\alpha\eta$-monotone. Then

$$\bigl\langle \alpha(v, u)F'(v), \eta(u, v)\bigr\rangle \leqslant -\bigl\langle \alpha(u, v)F'(u), \eta(v, u)\bigr\rangle - \mu\bigl\{\bigl\|\eta(v, u)\bigr\|^2 + \bigl\|\eta(u, v)\bigr\|^2\bigr\},$$

which can be written as

$$\bigl\langle F'(v), \eta(u, v)\bigr\rangle \leqslant -\bigl\langle F'(u), \eta(v, u)\bigr\rangle - \bar{\mu}\bigl\{\bigl\|\eta(v, u)\bigr\|^2 + \bigl\|\eta(u, v)\bigr\|^2\bigr\}, \tag{3.6}$$

since $\alpha(v, u)$ is a symmetric function and $\bar{\mu} = \mu/\alpha(v, u)$.

Since K is an α-invex set, $\forall u, v \in K$, $t \in [0, 1]$, $v_t = u + t\alpha(v, u)\eta(v, u) \in K$. Taking $v = v_t$ in (3.6) and using Condition C, we have

$$\begin{aligned}\big\langle F'(v_t), \eta(u, v_t)\big\rangle &\leqslant -\big\langle F'(u), \eta(v_t, u)\big\rangle - \bar{\mu}\big\{\big\|\eta(v_t, u)\big\|^2 + \big\|\eta(u, v_t)\big\|^2\big\} \\ &= -t\big\langle F'(u), \eta(v, u)\big\rangle - 2t^2\bar{\mu}\big\|\eta(v, u)\big\|^2,\end{aligned}$$

which implies that

$$\big\langle F'(v_t), \eta(v, u)\big\rangle \geqslant \big\langle F'(u), \eta(v, u)\big\rangle + 2\bar{\mu}t\big\|\eta(v, u)\big\|^2. \tag{3.7}$$

Let

$$g(t) = F\big(u + t\alpha(v, u)\eta(v, u)\big), \quad \forall u, v \in K,\ t \in [0, 1].$$

Then from (3.7), we have

$$\begin{aligned}g'(t) &= \big\langle \alpha(v, u)F'\big(u + t\eta(v, u)\big), \eta(v, u)\big\rangle \\ &\geqslant \big\langle \alpha(v, u)F'(u), \eta(v, u)\big\rangle + 2\bar{\mu}\alpha(v, u)t\big\|\eta(v, u)\big\|^2 \\ &= \big\langle \alpha(v, u)F'(u), \eta(v, u)\big\rangle + 2\mu t\big\|\eta(v, u)\big\|^2.\end{aligned} \tag{3.8}$$

Integrating (3.8) between 0 and 1, we have

$$g(1) - g(0) \geqslant \big\langle \alpha(v, u)F'(u), \eta(v, u)\big\rangle + \mu\big\|\eta(v, u)\big\|^2,$$

that is,

$$F\big(u + \alpha(v, u)\eta(v, u)\big) - F(u) \geqslant \big\langle \alpha(v, u)F'(u), \eta(v, u)\big\rangle + \mu\big\|\eta(v, u)\big\|^2.$$

By using Condition A, we have

$$F(v) - F(u) \geqslant \big\langle \alpha(v, u)F'(u), \eta(v, u)\big\rangle + \mu\big\|\eta(v, u)\big\|^2,$$

which shows that the function F is a strongly invex function on the invex set K. □

Note that if $\alpha(v, u) = 1$, then Theorem 3.2 collapses to the following result for strongly invex (preinvex) functions.

Theorem 3.3. *Let Conditions* B *and* D *hold. The differentiable function F on the invex set K is invex* (*preinvex*) *function if and only if its differential F' is η-monotone.*

For $\mu = 0$, Theorem 3.2 reduces to the following result for the α-invex (α-preinvex) functions.

Theorem 3.4. *Let F be a differentiable function and let Conditions* C *and* A *hold. If $\alpha(v, u)$ is asymmetric function, then the function F is α-invex* (α*-preinvex*) *function if and only if its differential F' is $\alpha\eta$-monotone.*

We now give a necessary condition for strongly pseudo $\alpha\eta$-invex function, which is also a generalization and refinement of results proved in [10,12].

Theorem 3.5. *Let the differential $F'(u)$ of a function $F(u)$ be strongly $\alpha\eta$-pseudomonotone. If Conditions* A *and* C *hold, then the function F is strongly pseudo $\alpha\eta$-invex function.*

Proof. Let $F'(u)$ be strongly $\alpha\eta$-pseudomonotone. Then, $\forall u, v \in K$,

$$\langle \alpha(v,u)F'(u), \eta(v,u) + \mu \|\eta(v,u)\|^2 \rangle \geqslant 0,$$

implies that

$$-\langle \alpha(u,v)F'(v), \eta(u,v) \rangle \geqslant 0. \tag{3.9}$$

Since K is an α-invex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t\alpha\eta(v,u) \in K$. Taking $v = v_t$ in (3.9) and using Condition C, we have

$$\langle F'(v_t), \eta(v,u) \rangle \geqslant 0. \tag{3.10}$$

Let

$$g(t) = F(v_t) \equiv F\big(u + t\alpha(v,u)\eta(v,u)\big), \quad \forall u, v \in K,\ t \in [0,1].$$

Then, using (3.10), we have

$$g'(t) = \langle \alpha(v,u)F'(v_t), \eta(v,u) \rangle \geqslant 0.$$

Integrating the above relation between 0 and 1, we have

$$g(1) - g(0) \geqslant 0,$$

that is,

$$F(v_t) - F(u) \geqslant 0,$$

which implies, using Condition A,

$$F(v) - F(u) \geqslant 0,$$

showing that F is strongly pseudo $\alpha\eta$-invex function. □

As special cases of Theorem 3.5, we have the following:

Corollary 3.1. *Let the differential $F'(u)$ of a function $F(u)$ on the α-invex set K be $\alpha\eta$-pseudomonotone. If Conditions* A *and* C *hold, then F is pseudo $\alpha\eta$-invex function.*

Corollary 3.2. *Let the differential $F'(u)$ of a function $F(u)$ on the α-invex set K be strongly η-pseudomonotone. If Conditions* A *and* C *hold, then F is strongly pseudo η-invex function.*

Corollary 3.3. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K be strongly η-pseudomonotone. If Conditions* B *and* D *hold, then F is strongly pseudo η-invex function.*

Corollary 3.4. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K be η-pseudomonotone. If Conditions* B *and* D *hold, then F is pseudo invex function.*

4. Conclusions

In this paper, we have defined some new concepts of strongly α-preinvex (α-invex) functions and strongly $\alpha\eta$-monotone operators. We have established some new relationships among various concepts of preinvex (invex) functions. As special cases, one can obtain several refined and correct versions of the previously known results [6,7,10,12,13]. For the applications in variational inequalities and equilibrium problems, see [8,10–13] and the references therein.

References

[1] M.A. Hanson, On sufficiency of the Kuhn–Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545–550.
[2] A. Ben-Israel, B. Mond, What is invexity? J. Aust. Math. Soc. Ser. B 28 (1986) 1–9.
[3] S.R. Mohan, S.K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995) 901–908.
[4] T. Weir, B. Mond, Preinvex functions in multiobjective optimization, J. Math. Anal. Appl. 136 (1988) 29–38.
[5] M.A. Noor, Generalized convex functions, Panamer. Math. J. 4 (1994) 73–89.
[6] V. Jeyakumar, B. Mond, On generalized convex mathematical programming, J. Aust. Math. Soc. Ser. B 34 (1992) 43–53.
[7] V. Jeyakumar, Strong and weak invexity in mathematical programming, Methods Oper. Res. 55 (1985) 109–125.
[8] M.A. Noor, Invex equilibrium problems, J. Math. Anal. Appl. 302 (2005) 463–475.
[9] M.A. Noor, Properties of preinvex functions, Preprint, Etisalat College of Engineering, Sharjah, UAE, 2004.
[10] G. Ruiz-Gaezion, R. Osuna-Gomez, A. Rufian-Lizan, Generalized invex monotonicity, European J. Oper. Res. 144 (2003) 501–512.
[11] X.Q. Yang, Generalized convex functions and vector variational inequalities, J. Optim. Theory Appl. 79 (1993) 563–580.
[12] X.M. Yang, X.Q. Yang, K.L. Teo, Criteria for generalized invex monotonicities, European J. Oper. Res., in press.
[13] X.M. Yang, X.Q. Yang, K.L. Teo, Generalized invexity and generalized invariant monotonicity, J. Optim. Theory Appl. 117 (2003) 607–625.
[14] S. Schaible, Generalized monotonicity: Concepts and uses, in: F. Giannessi, A. Maugeri (Eds.), Variational Inequalities and Network Equilibrium Problems, Plenum Press, New York, 1995, pp. 289–299.
[15] D.L. Zhu, P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, SIAM J. Optim. 6 (1996) 714–726.