# A fixed point formula for the index of multi-centered $\mathcal{N} = 2$ black holes

**Jan Manschot,**[a] **Boris Pioline**[b] **and Ashoke Sen**[c]

[a]*Institut de Physique Théorique, CEA Saclay, CNRS-URA 2306,*
*91191 Gif sur Yvette, France*

[b]*Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589,*
*Université Pierre et Marie Curie,*
*4 place Jussieu, 75252 Paris cedex 05, France*

[c]*Harish-Chandra Research Institute,*
*Chhatnag Road, Jhusi, Allahabad 211019, India*

*E-mail:* jan.manschot@cea.fr, pioline@lpthe.jussieu.fr, sen@hri.res.in

ABSTRACT: We propose a formula for computing the (moduli-dependent) contribution of multi-centered solutions to the total BPS index in terms of the (moduli-independent) indices associated to single-centered solutions. The main tool in our analysis is the computation of the refined index $\mathrm{Tr}\,(-y)^{2J_3}$ of configurational degrees of freedom of multi-centered BPS black hole solutions in $\mathcal{N} = 2$ supergravity by localization methods. When the charges carried by the centers do not allow for scaling solutions (i.e. solutions where a subset of the centers can come arbitrarily close to each other), the phase space of classical BPS solutions is compact and the refined index localizes to a finite set of isolated fixed points under rotations, corresponding to collinear solutions. When the charges allow for scaling solutions, the phase space is non-compact but appears to admit a compactification with finite volume and additional non-isolated fixed points. We give a prescription for determining the contributions of these fixed submanifolds by means of a 'minimal modification hypothesis', which we prove in the special case of dipole halo configurations.

# Contents

## 1   Introduction and summary

In $\mathcal{N} = 2$ supersymmetric string vacua, BPS states in suitable large charge limits can be represented as multi-centered black hole solutions of $\mathcal{N} = 2, D = 4$ supergravity [1–3]. To compute the moduli-dependent index $\Omega(\gamma; t)$ associated with such configurations, one needs to combine two independent sets of data.

The first part of the data are the indices $\Omega^S(\gamma)$ associated with single centered BPS black holes carrying electromagnetic charges $\gamma$. In the supergravity approximation, the index is given by the exponential of the Bekenstein-Hawking entropy [4, 5], and is independent of the asymptotic values of the scalar fields $t$ (within a given basin of attraction) by virtue of the attractor phenomenon [6–8]. Effects of classical higher derivative corrections to the low energy effective action can be incorporated by using Wald's modification of the Bekenstein-Hawking formula [9–12], while quantum corrections to the index can in principle be computed using the quantum entropy function formalism [13–15].

The second part of the data is the index of the supersymmetric quantum mechanics describing multi-centered black hole configurations. In this description, the centers are treated as pointlike, entropy-less particles carrying (in general, mutually non-local) electromagnetic charges $\alpha_1, \ldots, \alpha_n$, and kept in equilibrium by balance of forces [1]. The space of solutions of this mechanical problem is a $2(n-1)$-dimensional symplectic space $\mathcal{M}_n$ with an hamiltonian action of the rotation group SO(3), which can be quantized by the standard procedure of geometric quantization [2, 16]. Unlike the first part of the data, the index of these configurational degrees of freedom, which we denote by $g(\{\alpha_i\}; t)$, depends sensitively on the asymptotic values of the scalar fields $t$. In our previous work [17], reviewed in [18], we showed how to compute the jump of this index across a wall of marginal stability by localization with respect to a U(1) subgroup of SO(3) corresponding to rotations along the $z$-axis. In this work, we extend the techniques of [17] to compute the configurational index away from the walls of marginal stability, and propose a formula to combine this result with the indices associated to single-centered black holes in order to compute the total BPS index.

The approach of [17] was based on several simplifying facts. First it was shown that since identical centers do not interact, one can replace the Bose-Fermi statistics carried by the centers by Boltzmann statistics, provided the index $\Omega(\alpha_i)$ carried by the centers — or more generally the 'refined index'[1] $\Omega_{\text{ref}}(\alpha_i, y) = \text{Tr}'(-y)^{2J_3}$, where $J_3$ denotes the generator of rotations along the $z$ axis and $y$ is a real parameter — is replaced by an effective

---

[1]Here $\text{Tr}'$ denotes the trace over states carrying a fixed set of charges after removing the contribution from the fermion zero modes associated with broken supersymmetry generators. While $\text{Tr}'(-y)^{2J_3}$ is in general not a protected index away from $y = 1$ (except in rigid $\mathcal{N} = 2$ field theories), it is essential to

rational index $\bar{\Omega}_{\text{ref}}(\alpha_i, y)$ [17, 19–21]. Thus, even when some of the $\alpha_i$'s are identical, one may still treat the centers as distinguishable. Second, it was assumed that the 'refined index' of this quantum mechanical problem is related to the equivariant volume (i.e. the integral of $y^{2J_3}$ over $\mathcal{M}_n$) by a simple overall multiplicative renormalization required for consistency with angular momentum quantization. Third, it was important that, in the case of loosely bound constituents relevant for wall-crossing, the phase space $\mathcal{M}_n$ was compact and the action of $J_3$ had only isolated fixed points, corresponding to collinear multi-centered solutions along the $z$-axis. Under these circumstances, the localization theorem of Duistermaat and Heckman [22] can be used to express the configurational index $g(\{\alpha_i\}, t)$, or rather its refined generalization $g(\{\alpha_i\}; y, t)$, as a finite sum over the fixed points of $J_3$. This result was found to match the prediction of the known wall crossing formulae from supergravity [23, 24] and mathematics [19, 20] in all cases where it was tested. As we shall show in this work, provided $\mathcal{M}_n$ is compact and the fixed points are isolated, the multiplicative renormalisation postulated in [17] can in fact be derived from the Atiyah-Bott Lefschetz fixed point theorem [25, 26, 34, 35], a quantum-mechanical version of the Duistermaat-Heckman formula.

In this paper, we show that the same approach can be used to find the spectrum of multi-centered black hole solutions at a generic point in the moduli space, given the indices associated to single-centered black holes. There are however some important differences:

1. The set of collinear multi-centered solutions must not only satisfy the BPS equilibrium conditions of [1], but also lead to a regular metric. This condition was automatically satisfied for loosely bound states near a wall of marginal stability, but needs to be checked when computing the index at a generic point in moduli space and for generic charges. This condition is expected to rule out all but finitely many decompositions of the total charge $\gamma$ into a sum $\sum_{i=1}^{n} \alpha_i$ [23]. It can also rule out certain connected components of $\mathcal{M}_n$ even when the decomposition $\gamma = \sum_{i=1}^{n} \alpha_i$ is allowed, see section 3.3.1 for an example. A necessary (but not sufficient) condition is that the collinear configuration be regular along the $z$-axis.

2. For some range of charges carried by the centers, the phase space includes 'scaling solutions', i.e. regions where the relative distances between a subset (or all) the centers can become arbitrarily small [2, 23, 27–29]. As a result, the space $\mathcal{M}_n$ is non-compact, and the sum over regular collinear configurations fails to produce a sensible answer. In particular, it does not have a finite limit as $y \to 1$, and cannot be interpreted as a sum of characters of SO(3) (nor of its double cover SU(2)). However, despite being non-compact, $\mathcal{M}_n$ turns out to have a finite symplectic volume, suggesting that it may admit a compactification. In the case of 'dipole halo' configurations, introduced in [16, 29], it is straightforward to construct the compactification explicitly. The resulting space $\hat{\mathcal{M}}_n$ still admits an hamiltonian action of SO(3), but the fixed points of $J_3$ are no longer isolated, in particular there is a codimension 4 submanifold of

---

allow for $y \neq 1$, at least as the intermediate steps, in order for localization methods to apply. It would be interesting to understand the dependence of $\text{Tr}'(-y)^{2J_3}$ on the string coupling and other hypermultiplet fields.

fixed points where the total angular momentum vanishes, and which parametrizes scaling solutions. We shall assume that $\mathcal{M}_n$ always admits a compactification $\hat{\mathcal{M}}_n$, although we shall not require the details of its construction.

3. Due to the fact that the action of $J_3$ on $\hat{\mathcal{M}}_n$ has non-isolated fixed points, the equivariant volume and equivariant index are no longer related by a simple multiplicative renormalization. While the 'refined index' could still be in principle computed by localization using the Atiyah-Bott theorem, this would require a complete understanding of the compactification $\hat{\mathcal{M}}_n$ which we have not achieved so far. Instead, we propose a 'minimal modification hypothesis' which determines the contribution of these scaling regions from that of the regular, well-separated collinear fixed points. Our prescription amounts to requiring that scaling solutions contribute with the smallest possible angular momentum compatible with the final result being a character of SU(2). This prescription is motivated by the fact that classically (ignoring angular momentum quantization), scaling solutions carry zero total angular momentum. While we do not have a proof of this hypothesis, we shall demonstrate that it is consistent with wall-crossing and with the split attractor flow conjecture [23]. For a special class of 'dipole halo' configurations, where the moduli space of multi-centered solutions is fully understood, we shall verify that our prescription agrees with the geometric quantization of $\mathcal{M}_n$ performed in [16, 29], for an arbitrary number of centers.

We shall now summarize our proposal. We denote by $\Omega_{\mathrm{ref}}^{S}(\alpha, y)$ the index $\mathrm{Tr}'(-y)^{2J_3}$ carried by a single centered black hole with charge $\alpha$ (here, $\mathrm{Tr}'$ denotes the trace after factoring out the bosonic and fermionic zero modes). Since single centered black holes carry zero angular momentum [30, 31], we expect that $\Omega_{\mathrm{ref}}^{S}(\alpha, y)$ is independent of $y$. However we shall not make use of this information in our analysis, and proceed with general $\Omega_{\mathrm{ref}}^{S}(\alpha, y)$. Let us denote by $\Omega_{\mathrm{ref}}(\gamma, y) \equiv \mathrm{Tr}'(-y)^{2J_3}$ the total contribution to the index from single and multi-centered black hole solutions carrying total charge $\gamma$, and by

$$\bar{\Omega}_{\mathrm{ref}}(\gamma, y) = \sum_{m|\gamma} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} \, \Omega_{\mathrm{ref}}(\gamma/m, y^m) \,. \tag{1.1}$$

First consider the case when there are no scaling solutions. Then our proposal for $\bar{\Omega}_{\mathrm{ref}}(\gamma, y)$ is:

$$\bar{\Omega}_{\mathrm{ref}}(\gamma, y) = \sum_{\substack{\{\alpha_i \in \Gamma\} \\ \sum_i \alpha_i = \gamma}} \frac{1}{\mathrm{Aut}(\{\alpha_i\})} \, g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n; y) \, \bar{\Omega}_{\mathrm{ref}}^{S}(\alpha_1; y) \cdots \bar{\Omega}_{\mathrm{ref}}^{S}(\alpha_n; y) \,. \tag{1.2}$$

Here $\Gamma$ is the charge lattice, $\mathrm{Aut}(\{\alpha_i\})$ is the symmetry factor appropriate for Maxwell-Boltzmann statistics,[2] and $\bar{\Omega}_{\mathrm{ref}}^{S}(\alpha, y)$ is the 'rational refined index', related to the refined index by

$$\bar{\Omega}_{\mathrm{ref}}^{S}(\alpha, y) = \sum_{m|\alpha} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} \Omega_{\mathrm{ref}}^{S}(\alpha/m, y^m) \,. \tag{1.3}$$

---

[2]$\mathrm{Aut}(\{\alpha_i\})$ is defined as the order of the subgroup of the permutation group of $n$ elements which preserves the ordered set $(\alpha_1, \ldots, \alpha_n)$, for a fixed (arbitrary) choice of ordering. Thus if the set $\{\alpha_i\}$ consists of $r_1$ copies of $\beta_1$, $r_2$ copies of $\beta_2$ etc. then $|\mathrm{Aut}(\{\alpha_i\})| = \prod_k r_k!$.

The coefficient $g_{\text{ref}}$ is the refined index of the configurational degrees of freedom of $n$-centered BPS black hole solutions. By localization, it evaluates to

$$g_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y) = (-1)^{\sum_{i<j} \alpha_{ij} + n - 1} \left[ (y - y^{-1})^{1-n} \sum_p s(p)\, y^{\sum_{i<j} \alpha_{ij} \, \text{sign}[z_j - z_i]} \right], \quad (1.4)$$

where $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$ is the (Dirac-Schwinger-Zwanziger) symplectic inner product between $\alpha_i$ and $\alpha_j$, the sum over $p$ represents sum over all collinear solutions to the BPS equilibrium conditions (2.24), (2.10), and $s(p)$ takes value $\pm 1$ as determined from eq. (2.29). As mentioned above, the set of possible decompositions of $\gamma$ that contributes to the sum (1.2) is expected to be finite [23]. We can recover the integer invariant $\Omega_{\text{ref}}$ from the rational invariant $\bar{\Omega}_{\text{ref}}$ using the inverse formula [17]

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{m | \gamma} \mu(m)\, m^{-1} (y - y^{-1})(y^m - y^{-m})^{-1} \bar{\Omega}_{\text{ref}}(\gamma/m, y^m) \quad (1.5)$$

where $\mu(m)$ is the Möbius function. Expressing this in terms of $\Omega_{\text{ref}}^S$ using (1.2), (1.3) we can arrive at an expression of the form

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} G(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}), \quad (1.6)$$

for some function $G$. This expresses the total index in terms of the indices of single centered solutions. Of course, the sum in (1.6) also includes the contribution from single centered black holes given by $\Omega_{\text{ref}}^S(\gamma, y)$.

In the presence of scaling solutions, (1.6) cannot be the full answer for the following reason. If we follow the procedure outlined above ignoring the presence of scaling solutions, and denote the corresponding functions $G$ by $G_{\text{coll}}$, we shall find that the functions $G_{\text{coll}}(\{\beta_i\}, \{m_i\}; y)$ are not Laurent polynomials in $y$, i.e. finite linear combinations of $y^{\pm m}$ with integer $m$. Hence the corresponding expression (1.6) cannot be interpreted as the generating function of the spectrum of a quantum mechanical system with quantized angular momentum. Our prescription for taking into account the effect of scaling solutions is to modify (1.6) to

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \, \sum_i m_i \beta_i = \gamma}} G_{\text{coll}}(\{\beta_i\}, \{m_i\}; y) \prod_i \left( \Omega_{\text{ref}}^S(\beta_i, y^{m_i}) + \Omega_{\text{scaling}}(\beta_i, y^{m_i}) \right), \quad (1.7)$$

where $\Omega_{\text{scaling}}(\alpha, y)$ is given by

$$\Omega_{\text{scaling}}(\alpha, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}), \quad (1.8)$$

for some function $H(\{\beta_i\}, \{m_i\}; y)$. To determine $H$ we substitute (1.8) into (1.7) to express the latter equation as

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} G(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}), \quad (1.9)$$

for some functions $G$. We fix $H$ by requiring that $G(\{\beta_i\}, \{m_i\}; y)$ be a Laurent polynomial in $y$. The ambiguity of adding to $H$ a Laurent polynomial is resolved by using the minimal modification hypothesis, which requires that $H$ must be symmetric under $y \to y^{-1}$ and vanish as $y \to \infty$. An iterative procedure for determining the functions $H$ and hence $G$ has been described in section 5.

One advantage of our construction compared to the split attractor flow conjecture of [23] is that it allows us to compute the contributions from scaling solutions. Since the latter are usually stable across walls of marginal stability (as illustrated for three centers in section 2.3 and figure 1 below), their contributions cannot be obtained from attractor flow trees. This advantage is however mitigated by the fact that we do not know how to determine *a priori* which decompositions $\gamma = \sum \alpha_i$ lead to regular multi-centered solutions, except by checking (2.10) numerically. It should also be emphasized that in cases where the BPS spectrum is described by some quiver quantum mechanics with non-zero superpotential, the configurational degrees of freedom carried by scaling solutions do not exhaust, by far [29], the exponentially large number of states present on the Higgs branch [23]. These Higgs branch states have macroscopic entropy and should be included as part of the single-centered configurations counted by $\Omega_{\text{ref}}^S(\gamma)$.

The rest of the paper is organized as follows. In section 2, we review the structure of the phase space of multi-centered BPS solutions in $\mathcal{N} = 2$ supergravity, establish the finiteness of its symplectic volume, and compute the classical and quantum refined index (also known as equivariant volume and equivariant index) by localization. In section 3, we investigate several examples of three-centered solutions in a simple one-modulus supergravity, paying attention to the regularity condition (2.10) and to the contributions of scaling solutions. In section 4 we describe our proposal for the index of multi-centered black hole configuration when there are no scaling solutions, and show the consistency of this formula with wall crossing and with the split attractor flow conjecture. In section 5 we describe our proposal for modifying the result of section 4 in the presence of scaling solutions. In section 6 we prove the validity of our prescription in the solvable case of dipole halo configurations. Further technical details are relegated to appendices: in appendix A we provide a detailed analysis of certain sign rules which govern the contributions of collinear fixed points. In appendix B we prove that in the absence of scaling solutions the right hand side of (1.4) can be expressed as a Laurent polynomial in $y$. Explicit computations of the equivariant volume and index for dipole halos with 4 and 5 centers can be found in appendix C.

## 2 The phase space of multi-centered configurations

We begin by reviewing some relevant properties of supersymmetric multi-centered black hole solutions in $\mathcal{N} = 2$ supergravity. Such solutions fall into the stationary metric ansatz

$$\mathrm{d}s^2 = -e^{2U} \left(\mathrm{d}t + \mathcal{A}\right)^2 + e^{-2U} \mathrm{d}\vec{r}^2 \tag{2.1}$$

where the scale function $U$, the Kaluza-Klein one-form $\mathcal{A}$ and the vector multiplet scalars $t^a, a = 1, \ldots, n_v$ depend on the coordinate $\vec{r}$ on $\mathbb{R}^3$.

### 2.1 Equilibrium and regularity conditions

Let $\Gamma$ denote the charge lattice. Locally in the moduli space of the theory a charge vector $\alpha \in \Gamma$ may be split into its electric and magnetic components $(p^\Lambda, q_\Lambda)$. Given two such vectors $\alpha$ and $\alpha'$ we can define a symplectic product

$$\langle \alpha, \alpha' \rangle = q_\Lambda p'^\Lambda - q'_\Lambda p^\Lambda . \tag{2.2}$$

For $n$ centers located at $\vec{r}_1, \ldots, \vec{r}_n$, carrying electromagnetic charges $\alpha_1, \ldots, \alpha_n$ in the charge lattice $\Gamma$, the values of the vector multiplet scalars and of the scale factor $U$ are obtained by solving the "attractor equations" [1].

$$- 2 \, e^{-U(\vec{r})} \, \mathrm{Im} \left[ e^{-\mathrm{i}\phi} Y \left( (t^a(\vec{r})) \right) \right] = \beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|}, \qquad \phi = \arg Z_\gamma, \tag{2.3}$$

where $Y(t^a) = -e^{\mathcal{K}/2}(X^\Lambda(t), F_\Lambda(t))$ is the symplectic section afforded by the special geometry of the vector multiplet moduli space, $\mathcal{K} = -\ln i(F_\Lambda \bar{X}^\Lambda - \bar{F}_\Lambda X^\Lambda)$, and $Z_\gamma$ is the central charge

$$Z_\gamma = \langle \gamma, Y(t_\infty) \rangle \tag{2.4}$$

associated to the total charge $\gamma = \alpha_1 + \cdots + \alpha_n$. $Y(t_\infty)$ denotes the value of $Y$ at infinity. The constant vector $\beta$ on the right-hand side of (2.3) is determined in terms of the asymptotic values of the moduli at infinity $t^a_\infty$ by

$$\beta = -2 \, \mathrm{Im} \left[ e^{-\mathrm{i}\phi} Y(t_\infty) \right] . \tag{2.5}$$

The locations $\vec{r}_i$ are subject to the equilibrium conditions (also known as integrability equations) [1]

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{\alpha_{ij}}{r_{ij}} = c_i , \tag{2.6}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$, $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$, and the real constants

$$c_i \equiv 2 \, \mathrm{Im} \left( e^{-\mathrm{i}\phi} Z_{\alpha_i} \right) \tag{2.7}$$

depend on the the asymptotic values of the moduli. Since $\phi = \arg Z_\gamma$, these constants satisfy $\sum_{i=1}^n c_i = 0$. Finally, $\mathcal{A}$ is given by

$$*_3 \, d\mathcal{A} = \left\langle d \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|}, \beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right\rangle , \tag{2.8}$$

where $*_3$ denotes Hodge dual in 3 flat dimensions. The conditions (2.6) guarantee the existence of a Kaluza-Klein connection $\mathcal{A}$ such that the above configuration is a supersymmetric solution of the equations of motion.

It follows from (2.3) that the scale factor $U$ is given by evaluating the Bekenstein-Hawking entropy function $S(\gamma)$ on the harmonic function appearing on the right-hand side of (2.3) [3],

$$e^{-2U(\vec{r})} = \frac{1}{\pi} S \left( \beta + \sum_{i=1}^n \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right) . \tag{2.9}$$

In order for the solution to be physical one must require that the scale factor be everywhere real and positive [23],[3]

$$S \left( \beta + \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right) > 0 \,, \qquad \forall \, \vec{r} \in \mathbb{R}^3 \,, \tag{2.10}$$

where $\vec{r}_i$ is the location of the $i$-th center. This is necessary to ensure that there exists a regular solution to the attractor equations (2.3) at all points in $\mathbb{R}^3$.

## 2.2 Symplectic structure and equivariant volume

Leaving aside the regularity condition (2.10) for now, let us denote by $\mathcal{M}_n(\{\alpha_{ij}\}; \{c_i\})$ the space of solutions $\{\vec{r}_1, \dots, \vec{r}_n\}$ to the equilibrium conditions (2.6), modulo overall translations of the centers. $\mathcal{M}_n$ is a (possibly disconnected) $2n - 2$-dimensional submanifold of $\mathbb{R}^{3n-3} \backslash \Delta$, where $\Delta$ is the locus in $\mathbb{R}^{3n-3}$ where two or more of the centers $\vec{r}_i$ coincide. $\mathbb{R}^{3n-3} \backslash \Delta$ is equipped with the closed two-form[4]

$$\omega = \frac{1}{4} \sum_{\substack{i,j \\ i<j}} \alpha_{ij} \frac{\epsilon^{abc} \, \mathrm{d} r_{ij}^a \wedge \mathrm{d} r_{ij}^b \, r_{ij}^c}{|r_{ij}|^3} = \frac{1}{2} \sum_{\substack{i,j \\ i<j}} \alpha_{ij} \, \sin \theta_{ij} \, \mathrm{d} \theta_{ij} \wedge \mathrm{d} \phi_{ij} \,, \tag{2.11}$$

where $\theta_{ij}, \phi_{ij}$ are the polar angles parametrizing the direction of the vector $\vec{r}_{ij}$ with respect to a fixed unit vector $\vec{u}$, for example $\vec{u} = (0, 0, 1)$. For generic values of $c_i$, the restriction of $\omega$ to $\mathcal{M}_n$ is non-degenerate,[5] and provides $\mathcal{M}_n$ with a symplectic structure [16].

By construction, the symplectic form $\omega$ is invariant under SO(3) rotations. The moment map associated to infinitesimal rotations is the angular momentum

$$\vec{J} = \frac{1}{2} \sum_{\substack{i,j \\ i<j}} \alpha_{ij} \, \frac{\vec{r}_{ij}}{|r_{ij}|} = \frac{1}{2} \sum_i c_i \, \vec{r}_i \,, \tag{2.12}$$

where the second equality follows by using (2.6). After some further algebraic manipulations, the norm of $\vec{J}$ can be written as [16]

$$j \equiv \sqrt{\vec{J}^2} = \sqrt{-\frac{1}{4} \sum_{\substack{i,j \\ i \neq j}} c_i \, c_j \, r_{ij}^2} \,. \tag{2.13}$$

---

[3]This condition is somewhat too strong, since it rules out the 'empty hole' configurations representing e.g. the BPS states which become massless at the conifold point [1]. A more accurate requirement would be that regions with $S \leq 0$ are shielded by a shell where the moduli lie at singular conifold-type points, such that the regions inside the shell can be replaced by flat regions with constant values of the scalars. To avoid this complication we shall restrict to charges $\alpha_i$ which satisfy $D(\alpha_i) \geq 0$, where $D$ is the quartic polynomial such that $S = \sqrt{D}/\pi$ in the large volume limit.

[4]Our normalisation differs by a factor of two from the one used in [16]. This ensures that $\omega/2\pi$ has integer periods, see section 2.5.

[5]A notable exception occurs when $c_i = 0$, where the space of solutions admits an exact dilation symmetry $r_{ij} \to \varepsilon \, r_{ij}$, along which the two-form $\omega$ becomes degenerate. This case corresponds to multi-centered solutions asymptotic to $AdS_2 \times S^2$, and will become relevant in the analysis of scaling solutions below.

For $n \geq 3$ centers, the orbits of the SO(3) action are generically 3-dimensional, except on the two-dimensional subspace $\mathcal{M}_{n;\text{coll}} \subset \mathcal{M}_n$ corresponding to collinear configurations. Removing this locus, the quotient of $\mathcal{M}_n \backslash \mathcal{M}_{n;\text{coll}}$ by the SO(3) action is a $(2n-5)$-dimensional Poisson manifold $\tilde{\mathcal{M}}_n$. Let $\mathcal{J} = j(\mathcal{M}_n) \subset \mathbb{R}^+$ be the range[6] spanned by the total angular momentum $j$ on the space of configurations $\mathcal{M}_n$. The symplectic leaves of $\tilde{\mathcal{M}}_n$ are the hypersurfaces $\tilde{\mathcal{M}}_n(j)$ with fixed total angular momentum $j \in \mathcal{J}$. If one so wishes, one may parametrize $\tilde{\mathcal{M}}_n$ by $2n-5$ relative distances $r_{ij}$ (suitably chosen among the $n(n-1)/2$ radii $r_{ij}$, and subject to triangle inequalities), and the leaves $\tilde{\mathcal{M}}_n(j)$ by their projectivization. Using Euler angle coordinates $\theta, \phi, \sigma$ on SO(3), the symplectic form on $\mathcal{M}_n$ may then be decomposed into

$$\omega = j \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi - \mathrm{d}j \wedge (\mathrm{d}\sigma + \cos\theta \mathrm{d}\phi) + \tilde{\omega} \, , \tag{2.14}$$

where $\tilde{\omega}$ is the symplectic form on the symplectic leaf $\tilde{\mathcal{M}}(j)$. Eq. (2.14) follows by requiring the invariance of $\omega$ under the vector fields $\partial_\sigma, \partial_\phi, (\partial_\sigma - \cos\theta \partial_\phi)/\sin\theta$, generalizing the argument in [16] to an arbitrary number of centers.

Our goal in this work will be to determine the 'refined index' $\mathrm{Tr}'(-y)^{2J_3}$ of the supersymmetric quantum mechanics of $n$-centered BPS configurations, where $J_3$ is the angular momentum operator along the $z$-axis.[7] We defer to section 2.5 a detailed discussion of this quantum mechanics, and focus for now on the classical version of the refined index, the phase space integral [17]

$$g_{\text{classical}}(\{\alpha_i\}; y) \equiv \frac{(-1)^{\sum_{i<j} \alpha_{ij} - n + 1}}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n} e^{2\nu J_3} \omega^{n-1} \, , \tag{2.15}$$

where throughout this paper, we denote

$$\nu \equiv \ln y \, . \tag{2.16}$$

Such integrals over a symplectic manifold of the exponential of the moment map of some Hamiltonian action are well studied in the mathematical literature under the name of 'equivariant volume' (see e.g. [32] for a survey). The convergence of the integral (2.15) will be addressed in section 2.3. The sign $(-1)^{\sum_{i<j} \alpha_{ij} - n + 1}$ in (2.15) is inserted for reasons which will become clear in section 2.5. Leaving aside this sign, the equivariant volume (2.15) is expected to be a good approximation to $\mathrm{Tr}'y^{2J_3}$ in the classical limit, where all symplectic products $\alpha_{ij}$ are scaled to infinity and $y \to 1$ (this last condition ensuring that the function $y^{2J_3}$ varies slowly on the phase space).

Using the description of $\mathcal{M}_n$ as SO(3) $\times \tilde{\mathcal{M}}_n$ we can rewrite the integral (2.15) as

$$g_{\text{classical}}(\{\alpha_i\}; y) \equiv \frac{(-1)^{\sum_{i<j} \alpha_{ij} - n + 1}}{(2\pi)^{n-1}(n-3)!} \int_{\mathcal{J}} j \, \mathrm{d}j \int_{\text{SO(3)}} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\sigma \, e^{2\nu j \cos\theta} \int_{\tilde{\mathcal{M}}_n(j)} \tilde{\omega}^{n-3} \, . \tag{2.17}$$

---

[6]In general, $\mathcal{J}$ is bounded, and consists of a set of intervals in $\mathbb{R}^+$. The regularity condition (2.10) may rule out certain intervals in $\mathcal{J}$.

[7]We shall use the symbol $J_3$ to denote both the quantum angular momentum operator as well as the classical angular momentum (2.12). The correct interpretation should be clear from the context.

Carrying out the angular integral, we arrive at

$$g_{\text{classical}}(\{\alpha_i\}; y) \equiv (-1)^{\sum_{i<j} \alpha_{ij} - n + 1} \int_{\mathcal{J}} \mathrm{d}j \, \frac{\sinh(2\nu j)}{\nu} \, \tilde{g}_{\text{classical}}(\{\alpha_i\}, j) \quad (2.18)$$

where

$$\tilde{g}_{\text{classical}}(\{\alpha_i\}, j) = \frac{1}{(2\pi)^{n-3}(n-3)!} \int_{\tilde{\mathcal{M}}_n(j)} \tilde{\omega}^{n-3} \quad (2.19)$$

is the symplectic volume of the leaf $\tilde{\mathcal{M}}_n(j)$.

## 2.3 Scaling solutions and finiteness of the equivariant volume

In order to assess the convergence of the equivariant volume (2.15), it is useful to start with the simplest three-body case, which was discussed in detail in [16]. Recall that the solutions to the equilibrium conditions

$$\frac{\alpha_{12}}{r_{12}} + \frac{\alpha_{13}}{r_{13}} = c_1 \,, \quad -\frac{\alpha_{12}}{r_{12}} + \frac{\alpha_{23}}{r_{23}} = c_2 \,, \quad -\frac{\alpha_{13}}{r_{13}} - \frac{\alpha_{23}}{r_{23}} = c_3 = -c_1 - c_2 \quad (2.20)$$

can be parametrized by [16]

$$r_{12} = \frac{\alpha_{12}}{\rho}, \quad r_{23} = \frac{\alpha_{23}}{\rho + c_2}, \quad r_{13} = \frac{\alpha_{31}}{\rho - c_1} \,, \quad (2.21)$$

where $\rho$ runs over the subset of $\mathbb{R}$ satisfying $r_{ij} > 0$ and the triangular inequalities $r_{12} \leq r_{13} + r_{23}$, $r_{23} \leq r_{12} + r_{13}$, $r_{13} \leq r_{12} + r_{23}$. In general, the allowed range of $\rho$ consists of at most two intervals, possibly reaching $\pm\infty$, whose finite endpoints correspond to collinear configurations (see figure 1 for a pictorial determination of the allowed range of $\rho$ as a function of the constants $c_i$). Trading $\rho$ for the total angular momentum $j$ using (2.13), one finds that the range $\mathcal{J}$ of the latter is always a bounded interval $[j_-, j_+]$.

When $\alpha_{12}$, $\alpha_{23}$ and $\alpha_{31}$ are positive and satisfy the same triangular inequalities, the region $\rho \to \infty$ is included in the allowed range, and corresponds to scaling solutions where the three centers come arbitrarily close to each other [16]. From the second equality in (2.12) it follows that such configurations have vanishing total angular momentum $j$ in the strict limit $\rho \to \infty$, hence $j_- = 0$.

Even though the space $\mathcal{M}_3$ is non-compact when such solutions are allowed, its equivariant volume $g_{\text{classical}}(\{\alpha_i\}; y)$ (in particular, its symplectic volume) is in fact finite. To see this, note that the reduced symplectic space $\tilde{\mathcal{M}}_3(j)$ consists of a single point when $j \in [j_-, j_+]$, or is empty otherwise. Thus,

$$g_{\text{classical}}(\alpha_1, \alpha_2, \alpha_3; y) = \frac{1}{2}(-1)^{\alpha_{12} + \alpha_{23} + \alpha_{13}} \frac{\cosh(2\nu j_+) - \cosh(2\nu j_-)}{\nu^2} \quad (2.22)$$

In the presence of scaling solutions, $j_- = 0$, but the integral is still convergent. In fact the space $\mathcal{M}_3$ may be compactified by adding one point , corresponding to scaling solutions in the strict $\rho = \infty$ limit. Since $j = 0$ in this limit, this point is fixed under the action of SO(3).
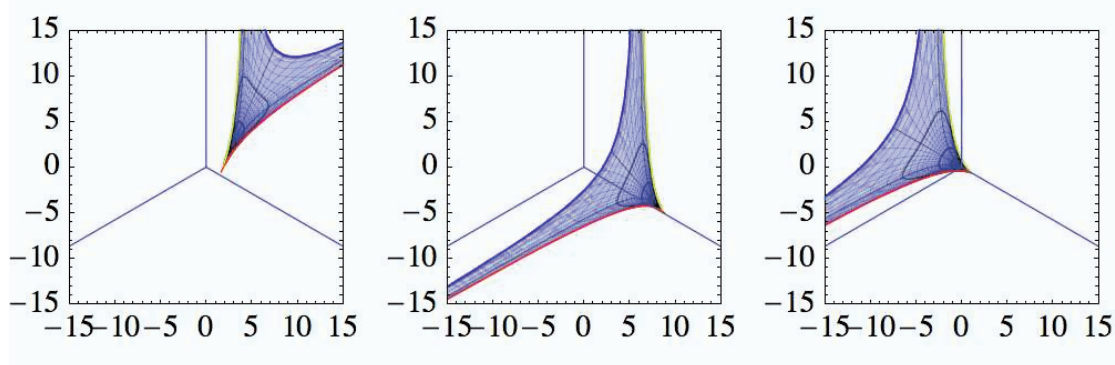
**Figure 1**. Phase diagram for 3-center configurations, in the plane $(x,y) = (-\frac{3}{2}(c_2+c_3), \frac{\sqrt{3}}{2}(c_2-c_3))$. The shaded area corresponds to the values of $c_i \, r_{12}$ reached by varying the position of the third center, keeping the position of the centers 1 and 2 and $\alpha_{ij}$ fixed. The range of $r_{12}$ (and therefore, of the parameter $\rho$ in (2.21)) at fixed values of $c_i$ can be determined by intersecting the shaded area with a radial line extending from the origin through the point $c_i$ (Thus if the radial line passing through 0 and $\vec{c}$ meets the boundary of the shaded region at $\vec{c}^{(1)}$ and $\vec{c}^{(2)}$, then the range of $r_{12}$ is from $|\vec{c}^{(1)}|/|\vec{c}|$ to $|\vec{c}^{(2)}|/|\vec{c}|$. On the other hand if the origin falls inside the shaded region then the range of $r_{12}$ varies from 0 to $|\vec{c}^{(1)}|/|\vec{c}|$). The blue, red and yellow lines correspond to collinear configurations in order 132, 213, 321 (or their reverse), respectively. The lines with $\arg(x+iy) = \pi/2, -5\pi/6, -\pi/6$ denote the locus where $c_1 = 0, c_2 = 0, c_3 = 0$, respectively. i) $\alpha_{12} > 0, \alpha_{23} > 0, \alpha_{13} > 0$: solutions exist in the sector $c_1 > 0, c_3 > 0$ ii) $\alpha_{12} > 0, \alpha_{23} > 0, \alpha_{13} < 0, \alpha_{12} > \alpha_{31} + \alpha_{23}$ (non-scaling regime): solutions exist in the sectors $c_1 > 0$ or $c_2 < 0$ iii) $\alpha_{12} > \alpha_{23} > 0, \alpha_{13} < 0, \alpha_{12} < \alpha_{31} + \alpha_{23}$ (scaling regime): solutions exist for all values of $c_i$. The figures were produced using $(\alpha_{12}, \alpha_{23}, \alpha_{13}) = (1,3,2), (6,3,-2), (1,3,-3)$ in cases i), ii), iii), respectively.

We now turn to solutions with arbitrary number of centers. In the special case where the centers can be ordered via a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $\alpha_{\sigma(i)\sigma(j)} \geq 0$ whenever $i > j$,[8] it is straightforward to see that the $r_{ij}$'s are bounded from below by a non-zero $r_{\min}$, therefore scaling solutions do not exist. Away from walls of marginal stability or threshold stability,[9] the $r_{ij}$'s are also bounded from above, and the phase space $\mathcal{M}_n$ is therefore compact. In such cases, the equivariant volume (2.15) is manifestly finite.

If on the other hand, the centers cannot be ordered such that $\alpha_{\sigma(i),\sigma(j)} \geq 0$ whenever $i > j$, then there may exist a subset $A$ of $(1, \ldots, n)$ for which there exist vectors $\vec{r}_i \in \mathbb{R}^3$ $(i \in A)$ satisfying

$$\sum_{\substack{j \in A \\ j \neq i}} \frac{\alpha_{ij}}{|\vec{r}_{ij}|} = 0, \qquad \forall \quad i \in A. \tag{2.23}$$

---

[8]This is equivalent to the condition that the associated quiver (with $n$ nodes and $|\alpha_{ij}|$ arrows from $i$ to $j$ if $\alpha_{ij} > 0$, or from $j$ to $i$ if $\alpha_{ij} < 0$) contains no oriented closed loop. This condition is sufficient for the absence of scaling solutions, but not necessary, as illustrated by the 3-center case. It is in particular obeyed when all charges $\alpha_i$ belong to a positive cone in a two-dimensional charge lattice, the case studied in [17].

[9]Walls of marginal or threshold stability correspond to loci where $\sum_{i \in A} c_i = 0$ for a proper subset $A$ of $\{1, \ldots, n\}$, such that the centers in the subset $A$ can move off to infinity. Threshold stability corresponds to the special case where the charge $\gamma_A = \sum_{i \in A} \alpha_i$ of the subset $A$ and the charge $\gamma - \gamma_A$ of its complement are mutually local [33].

In this case the centers in the subset $A$ may form a scaling solution, i.e. may come arbitrarily close to each other. In this case, the space $\mathcal{M}_n$ is non-compact. However, we shall now outline a proof that the equivariant volume (2.15) is in fact finite even in this case.

We start by considering potential divergences from 'maximally scaling solutions', where all centers come arbitrarily close to each other. Since from eq. (2.11) the symplectic form $\omega$ is invariant under $r_{ij} \to \varepsilon r_{ij}$, the volume element $\omega^{n-1}$ is superficially expected to behave as $d\varepsilon/\varepsilon$ in the region $\varepsilon \to 0$, suggesting a logarithmic divergence of the equivariant volume. However, notice that in the $\varepsilon \to 0$ limit, the symplectic form $\omega$ becomes degenerate in the direction corresponding to overall dilations of the configuration (as noted in footnote 5 above). This dilation is an exact symmetry of the equilibrium equations (2.6) at $c_i = 0$, but is broken for finite $\varepsilon$ at non-zero $c_i$. Thus, the naive estimate $\omega^{n-1} \sim d\varepsilon/\varepsilon$ must be modified by an integer power[10] $k \geq 1$ of the dimensionless parameter $c\varepsilon$ (where $c$ denotes a function of the $c_i$ homogeneous of degree one). In this case the volume element $\omega^{n-1} \sim (c\,\varepsilon)^k d\varepsilon/\varepsilon$ vanishes at $c_i = 0$. Thus, the contribution of the region $\varepsilon \to 0$ to the equivariant volume is finite, and there is no divergence coming from 'maximally scaling solutions'.

We should also consider situations where $m$ of the $n$ centers come arbitrarily close to each other. The phase space near this configuration is locally a product of the phase space of an $(n - m + 1)$ centered configuration, in which all the $m$ close-by centers are regarded as a single center, and that of a $m$-centered scaling solution. The symplectic volume of the former is manifestly finite, while the symplectic volume of the latter is finite by the argument given earlier. This can be easily generalized to the cases where there are several subsets, each containing a set of centers which come close to each other. Thus, this reasoning indicates that the equivariant volume (2.15) is finite even in the presence of scaling solutions. In section 6, we shall verify this claim in the case of dipole halo configurations, where the phase space admits a natural compactification into a toric symplectic manifold, whose volume can be evaluated explicitly. It is an important problem to construct a compactification of $\mathcal{M}_n$ in the general case involving scaling solutions.

## 2.4 A fixed point formula for the equivariant volume

Having established that the equivariant volume (2.15) is finite, we shall now evaluate it by localization. For this purpose, we need to determine the fixed points of the action of $J_3$ on $\mathcal{M}_n$. In general, the fixed points correspond either to collinear solutions along the $z$ axis, or to the coinciding limit of scaling solutions approaching a point on this axis. First we restrict to the case where the charges $\alpha_i$ do not allow for scaling solutions. In this case, $\mathcal{M}_n$ is compact and all fixed points are isolated.

Collinear solutions lying along the $z$ axis satisfy a one-dimensional version of the equi-

---

[10]In fact, using the decomposition (2.18) of the equivariant volume and the second relation (2.12), one finds that the volume element $j\mathrm{d}j$ along the angular momentum direction scales as $c^2\varepsilon\mathrm{d}\varepsilon$. If the symplectic volume of the reduced phase space $\tilde{\mathcal{M}}_n(j)$ has a finite non-zero value at $j = 0$, which is the case for general 3-centered configurations or for dipole halo configurations with an arbitrary number of centers, it follows that the volume element goes as $\varepsilon d\varepsilon$ near $\varepsilon = 0$.

librium conditions (2.6),

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i , \qquad \sum_i z_i = 0 , \qquad (2.24)$$

where the second equation fixes the translational zero-mode. These equations then select the critical points of the 'superpotential' [17]

$$\hat{W}(\lambda, \{z_i\}) = -\sum_{i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i] \ln|z_j - z_i| - \sum_i (c_i - \lambda/n) z_i \qquad (2.25)$$

as a function of $n+1$ variables $\lambda, z_1, \cdots z_n$. In the vicinity of one such fixed point $p$, the angular momentum $J_3$ and the symplectic form $\omega$ take the form

$$J_3 = \frac{1}{2} \sum_{i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i] - \frac{1}{4} M_{ij}(p) (x_i x_j + y_i y_j) + \cdots ,$$

$$\omega = \frac{1}{2} M_{ij}(p) \, \mathrm{d}x_i \wedge \mathrm{d}y_j + \cdots , \qquad (2.26)$$

where $M_{ij}(p)$ is the Hessian matrix of $\hat{W}(\lambda, \{z_i\})$ with respect to $z_1, \ldots, z_n$, and $(x_i, y_i)$ are coordinates in the plane transverse to the $z$-axis at the center $i$, subject to the condition $\sum_i x_i = \sum y_i = 0$. Except for the overall translational zero-mode, the matrix $M_{ij}$ is non-degenerate, and the critical points are isolated. Since $\mathcal{M}_n$ is compact, we have a finite set of collinear fixed points $p$. It will be useful to note that $\widehat{\det} M = -\det \hat{M}$, where $\widehat{\det} M$ denotes the determinant of the $(n-1) \times (n-1)$ matrix $M$ with the first column and first row removed (i.e. the Hessian of $\hat{W}$ as a function of $z_2, \ldots, z_n$), and $\hat{M}$ is the Hessian of $\hat{W}$ as a function of the $n+1$ variables $\lambda, z_1, \ldots, z_n$ [17].

Since $\mathcal{M}_n$ is compact and the fixed points are isolated, the Duistermaat-Heckman formula [22] allows to express the equivariant volume (2.15) as a sum over fixed points. The contribution of each fixed point $p$ is given by the formal Gaussian integral[11] obtained by replacing the moment map by its quadratic approximation around $p$, and the symplectic form by its value at the same point. Thus the net contribution is given by

$$g_{\text{classical}}(\{\alpha_i\}; y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} + n - 1}}{(2\pi)^{n-1}} \sum_p \int \prod_{i=2}^n \mathrm{d}x_i \, \mathrm{d}y_i \, \frac{(\widehat{\det} M(p))}{2^{n-1}}$$

$$\exp\left\{ \nu \sum_{\substack{i,j=1 \\ i<j}}^n \alpha_{ij} \operatorname{sign}[z_j - z_i] - \frac{\nu}{2} \sum_{\substack{i,j=2 \\ i<j}}^n M_{ij}(p) (x_i x_j + y_i y_j) \right\} . \qquad (2.27)$$

After carrying out the Gaussian integral over $x_i, y_i$, we arrive at [17]

$$g_{\text{classical}}(\{\alpha_i\}; y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} + n - 1}}{(2 \ln y)^{n-1}} \sum_p s(p) \, y^{\sum_{i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i]} , \qquad (2.28)$$

---

[11]This Gaussian integral is ill-defined since the quadratic form is in general not positive definite, but the Duistermaat-Heckman formula guarantees that formal Gaussian integration leads to the correct result.

where the coefficient $s(p)$ is given by

$$s(p) = \text{sign} \widehat{\det} M(p) = -\text{sign} \det \hat{M}(p) \,. \tag{2.29}$$

$-s(p)$ is recognized as Morse index of the critical point $p$ of $\hat{W}$. It is worthwhile noting that the prefactor $(2\nu)^{n-1}$ in the denominator of (2.28) originates from the determinant of the quadratic form in the $2n - 2$ transverse directions to the fixed point.

In the presence of scaling solutions, $\mathcal{M}_n$ is non-compact and the Duistermaat-Heckman formula does not apply directly. In the examples that we shall study in section 6, it appears that $\mathcal{M}_n$ admits a natural compactification $\hat{\mathcal{M}}_n$, which introduces additional, non-isolated fixed points. Their contribution can in principle be evaluated using a suitable generalization of the Duistermaat-Heckman formula [34]. For lack of a complete understanding of the compactification $\hat{\mathcal{M}}_n$, we shall not pursue this approach. Instead, one may give a 'minimal modification prescription' which determines the contribution of the non-isolated fixed points by requiring that the Fourier transform of the equivariant volume $\int_{\mathbb{R}} g_{\text{classical}}(\{\alpha_i\}, \nu) e^{2i\nu m} d\nu$ becomes a piecewise polynomial function of $m$ with compact support, as must be the case for any compact symplectic manifold with a Hamiltonian action [22]. Since our interest lies eventually in the equivariant index rather than the equivariant volume, we shall not explain this prescription here, referring to section 6 and section C for direct computations of the equivariant volume for dipole halo configurations, and identifications of the corresponding fixed points.

Finally, we return to the regularity condition (2.10). In the case considered in [17], namely multi-centered solutions whose charges all lie in a positive cone of a two-dimensional lattice, it appears that this condition is automatically satisfied near the wall. In general, this need not be so. On physical grounds, we do not expect that this condition should cut out regions of a given connected component of $\mathcal{M}_n$. On the other hand, it is quite possible that this condition could forbid certain connected components of $\mathcal{M}_n$ altogether. In this case, we simply need to omit the corresponding fixed points from the sum (2.28). We shall see an example of this phenomenon in section 3.2 below.

## 2.5 A fixed point formula for the refined index

We now turn to the problem of quantizing the configurational degrees of freedom of $n$-centered black hole solutions and the computation of the refined index $\text{Tr}'(-y)^{2J_3}$, of which (2.28) is supposed to be the classical approximation.

We start by recalling and expanding upon some general aspects of the quantization of BPS multi-centered black hole solutions, first laid out in [2, 16]. As in these references, we assume that the Hilbert space of BPS states, in principle obtained by first quantizing the full configuration space of $\mathcal{N} = 2$ supergravity (or more generally, string theory) and then restricting to the subspace annihilated by 4 supercharges, can be alternatively obtained by first restricting to classical solutions annihilated by the same number of supercharges, and then quantizing.

Since each BPS solution of the type (2.3) preserves 4 supercharges, one expects that the locations $\vec{r}_i$ should be quantized into operators in a quantum mechanics with $\mathcal{N} = 4$

supersymmetries, such that the classical supersymmetric vacua are in one-to-one correspondence with solutions to the equilibrium conditions (2.6). The final result should not be sensitive to the details of the potential in this quantum mechanics as long as the space of classical supersymmetric vacua is given by (2.6). One way to obtain the Lagrangian is to consider the Coulomb branch of an $\mathcal{N} = 4$ supersymmetric quiver quantum mechanics with $n$ vector multiplets (each including 3 real scalars $\vec{r}_i$, a U(1) gauge field $A_i$ and four fermions $\psi_i$) and, for each pair of centers $i, j$, $|\alpha_{ij}|$ chiral multiplets with charge $\text{sign}(\alpha_{ij})$ under $A_i$ and $-\text{sign}(\alpha_{ij})$ under $A_j$. On the Coulomb branch,[12] the chiral multiplets are massive and can be integrated out at one-loop. The result is an $\mathcal{N} = 4$ supersymmetric Lagrangian for the positions $\vec{r}_i$ and their fermionic partners $\psi_i$, with a potential which vanishes when the $3n$ scalars satisfy the equilibrium conditions (2.6). In addition, there is a first order coupling $\int \lambda$ to the one-form [2]

$$\lambda = \frac{1}{2} \sum_{\substack{i,j \\ i<j}} \alpha_{ij} \frac{\epsilon^{abc} r_{ij}^a \, \mathrm{d} r_{ij}^b \, u^c}{|\vec{r}_{ij}|(|\vec{r}_{ij}| + \vec{r}_{ij} \cdot \vec{u})} = \frac{1}{2} \sum_{\substack{i,j \\ i<j}} \alpha_{ij} \left(1 - \cos\theta_{ij}\right) \mathrm{d}\phi_{ij}, \qquad (2.30)$$

where $\vec{u}$ is the same unit vector used in (2.11). The degrees of freedom associated to the center of mass motion are decoupled and can be removed by setting $\sum_i \vec{r}_i = \sum_i \psi_i = 0$, so that the bosonic configuration space consists of ($\mathbb{R}^{3n-3} \backslash \Delta$), where $\Delta$ is the coinciding locus. Quantum mechanically, the Hilbert space consists of square integrable sections of $\tilde{S} \otimes \tilde{\mathcal{L}}$ over $\mathbb{R}^{3n-3} \backslash \Delta$, where $\tilde{S}$ is the trivial bundle of rank $2^{2n-2}$ over $\mathbb{R}^{3n-3}$ obtaining by quantizing the $4(n-1)$ fermionic modes $\psi^i$, and $\tilde{\mathcal{L}}$ is a complex line bundle with first Chern class[13] $c_1(\tilde{\mathcal{L}}) = \omega = \mathrm{d}\lambda$. BPS states correspond to supersymmetric ground states of this $\mathcal{N} = 4$ quantum mechanics.

For the purposes of studying these ground states, one may return to the classical Coulomb branch Lagrangian, integrate out the fluctuations transverse to the zero-energy submanifold $\mathcal{M}_n \subset \mathbb{R}^{3n-3}$, and then quantize the system. While we have not studied this problem in detail, we assume, in the spirit of [16], that BPS states correspond to harmonic spinors on $\mathcal{M}_n$, i.e. sections of $S \otimes \mathcal{L}$ annihilated by $D$, where $S$ is the total spin bundle[14] of $\mathcal{M}_n$, $\mathcal{L}$ is the restriction of $\tilde{\mathcal{L}}$ to $\mathcal{M}_n$ and $D$ is the Dirac operator for the metric $g_{\mu\nu}$ induced from the flat metric on $\mathbb{R}^{3n-3} \backslash \Delta$, twisted by the line bundle $\mathcal{L}$. The Dirac operator $D$ decomposes as $D = D_+ + D_-$, where $D_\pm$ maps $S_\pm$ to $S_\mp$ where $S = S_+ \oplus S_-$ is the decomposition into spinors with positive and negative chirality. Moreover, the action of SO(3) on $\mathcal{M}_n$ lifts to an action of SU(2) on $S_\pm \otimes \mathcal{L}$. The refined index $\text{Tr}'(-y)^{2J_3}$ is then

---

[12]The quiver quantum mechanics also has a Higgs branch. In some cases, including the wall-crossing problem considered in [17], the BPS spectra on the Coulomb branch and Higgs branch are identical, but this is not so in general, and the Higgs branch states cannot always be viewed as multi-centered black holes.

[13]The normalisation chosen in (2.11) ensures that $\omega \in H^2(\mathbb{R}^{3n-3} \backslash \Delta, \mathbb{Z})$, moreover $\mathbb{R}^{3n-3} \backslash \Delta$ is simply connected so $\tilde{\mathcal{L}}$ is uniquely defined. The unit vector $\vec{u}$ in (2.30) correspond to a choice of Dirac strings for the line bundle $\tilde{\mathcal{L}}$.

[14]We assume that $\mathcal{M}_n$ admits a spin structure, unlike a generic symplectic manifold which only admits a spin$_c$ structure [35]. By integrating out the massive fermions around the supersymmetric vacua, it should be possible to construct the spin bundle $S$ as a rank $2^{n-1}$ subbundle of $\tilde{S}$, but we have not investigated this in detail.

given by

$$g_{\rm ref}(\{\alpha_i\}; y) = {\rm Tr}_{\,{\rm Ker}D_+}(-y)^{2J_3} + {\rm Tr}_{\,{\rm Ker}D_-}(-y)^{2J_3} \ . \tag{2.31}$$

We shall assume that ${\rm Ker}D_- = 0$, so that the refined index $g_{\rm ref}(\{\alpha_i\}; y)$ reduces to

$$g_{\rm ref}(\{\alpha_i\}; y) = {\rm Tr}_{\,{\rm Ker}D_+}(-y)^{2J_3} - {\rm Tr}_{\,{\rm Ker}D_-}(-y)^{2J_3} \ . \tag{2.32}$$

When $\mathcal{M}_n$ is Kähler, as is the case for $n = 2, 3$ or for the dipole halo configurations studied in section 6, and $\alpha_{ij}$ large enough, the fact that ${\rm Ker}D_- = 0$ can be proved as follows. In this case $S_+$ and $S_-$ are isomorphic to $\Lambda^{\rm even}(T^{(0,1)}_{\mathbb{C}}\mathcal{M}_n) \otimes K^{1/2}$ and $\Lambda^{\rm odd}(T^{(0,1)}_{\mathbb{C}}\mathcal{M}_n) \otimes K^{1/2}$, respectively, where $K$ is the canonical bundle of $\mathcal{M}_n$, i.e. the complex line bundle of $(n-1, 0)$ forms (the square root exists since $\mathcal{M}_n$ is assumed to be spin). In this case the Kodaira vanishing theorem states that the cohomology groups $H^{(0,q)}(\mathcal{M}_n, \mathcal{L} \otimes K^{1/2})$ all vanish except for $q = 0$. This shows that ${\rm Ker}D_- = 0$. In general, we do not how to prove this assumption, but it is supported by the fact that it leads to results in agreement with wall-crossing. Since the refined index ${\rm Tr}'(-y)^{2J_3}$ is not protected[15] in $\mathcal{N} = 2$ supergravity, for the purposes of computing the index ${\rm Tr}'(-1)^{2J_3}$ it suffices to make the weaker assumption that ${\rm Tr}_{\,{\rm Ker}D_-}(-1)^{2J_3} = 0$.

Under this assumption, $g_{\rm ref}(\{\alpha_i\}; -y)$ reduces to the equivariant index of the Dirac operator $D$. Using the Lefschetz fixed point formula established in [25, 26, 34, 38],[16] assuming that $\mathcal{M}_n$ is compact, the equivariant index can be written as the integral

$$g_{\rm ref}(\{\alpha_i\}; -y) = \int_{\mathcal{M}_n} {\rm Ch}(\mathcal{L}, \nu)\, \hat{A}(\mathcal{M}_n, \nu)|_{n-1} \tag{2.33}$$

where ${\rm Ch}(\mathcal{L}, \nu)$ is the equivariant Chern character of the line bundle $\mathcal{L}$, and $\hat{A}(\mathcal{M}_n, \nu)$ is the equivariant $\hat{A}$-genus of $\mathcal{M}_n$, defined by

$${\rm Ch}(\mathcal{L}, \nu) = \exp\left(2\nu J_3 + \frac{1}{2\pi}\omega\right) , \quad \hat{A}(\mathcal{M}_n, \nu) = \det\left(\frac{2\nu L + \frac{1}{2\pi}\mathcal{R}}{2 \sinh \frac{1}{2}\left(2\nu L + \frac{1}{2\pi}\mathcal{R}\right)}\right) . \tag{2.34}$$

Here, $J_3$ is the moment map of the action of rotations around the $z$ axis, $L$ is the endomorphism of the holomorphic tangent bundle $T^{(1,0)}\mathcal{M}_n$ induced by the same action, $\mathcal{R}$ is the curvature two-form on $T^{(1,0)}\mathcal{M}_n$, and $A|_p$ denotes the degree $2p$ part of a multi-form $A$. The integral (2.33) can be evaluated by localization [38], leading to

$$g_{\rm ref}(\{\alpha_i\}; -y) = \int_{\mathcal{M}_n^{\rm fixed)}} \frac{{\rm Ch}(\mathcal{L}, \nu)\, \hat{A}(\mathcal{M}_n, \nu)}{{\rm Eu}(N\mathcal{M}_n^{\rm fixed})}\bigg|_p \tag{2.35}$$

where $\mathcal{M}_n^{\rm fixed)} \subset \mathcal{M}_n$ denotes the fixed point locus, of dimension $2p$, and

$${\rm Eu}(N\mathcal{M}_n^{\rm fixed}) = \det\left(2\nu L + \frac{1}{2\pi}\mathcal{R}\right) \tag{2.36}$$

is the equivariant Euler character of the normal bundle of $\mathcal{M}_n^{\rm fixed}$.

---

[15]In $\mathcal{N} = 2$ gauge theories at low energies, it may be possible to compute the protected spin character (PSC) [36] by quantizing the space of classical multi-centered Abelian dyons along the lines of [37]. Since the PSC is protected, it is plausible that the configurational index will be directly equal to the equivariant index (2.32), without the need to assume that ${\rm Ker}D_-$ vanishes. The vanishing of ${\rm Ker}D_- = 0$ in this context is closely related to the weak positivity conjecture of [36].

[16]We are grateful to M. Vergne for guidance into the math literature.

In the absence of scaling solutions, $\mathcal{M}_n$ is compact and all fixed points are isolated, so $N\mathcal{M}_n^{\text{fixed}} = T\mathcal{M}_n$ and the Euler character cancels the numerator in the $\hat{A}$-genus. Moreover, $J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\text{sign}[z_j - z_i]$ and the operator $L$, representing the action of $J_3$ on $T^{(1,0)}(\mathcal{M}_n)$, has eigenvalues $\pm 1$, with $\det L = s(p)$. This leads to the following explicit formula for the quantum refined index:[17]

$$g_{\text{ref}}(\{\alpha_i\}; -y) = (y - 1/y)^{n-1} \sum_p s(p)\, y^{\sum_{i<j} \alpha_{ij}\text{sign}[z_j - z_i]}. \tag{2.37}$$

Hence, after changing[18] $y \to -y$,

$$g_{\text{ref}}(\{\alpha_i\}; y) = \frac{(-1)^{\sum_{i<j} \alpha_{ij} + n - 1}}{(y - 1/y)^{n-1}} \sum_p s(p)\, y^{\sum_{i<j} \alpha_{ij}\text{sign}[z_j - z_i]}. \tag{2.38}$$

In particular, the refined index (2.38) is related to the equivariant volume (2.28) by an overall rescaling by a factor of $(\nu/\sinh\nu)^{n-1}$. This multiplicative renormalization was postulated in [17] on the basis of angular momentum quantization. As the present derivation shows, this multiplicative renormalization in fact follows from the Atiyah-Bott Lefschetz fixed point formula for the equivariant index of the Dirac operator on $\mathcal{M}_n$.

Eq. (2.38) will be our main tool for computing the index of multi-centered black holes in the absence of scaling solutions. Although it was derived under various assumptions, we shall carry out various consistency checks which build our confidence in this formula. In the presence of scaling solutions, the space $\mathcal{M}_n$ is non-compact and its compactification includes non-isolated fixed points for the action of $J_3$. Our limited understanding of the geometry of the fixed submanifold $\mathcal{M}_n^{\text{fixed}}$ prevents us from computing the index using (2.35), nevertheless in section 5 we shall give a prescription for computing the contributions from these non-isolated fixed points, from the knowledge of the isolated ones.

## 3 Case studies in a one-modulus supergravity model

In this section, we analyze several examples of three-centered solutions in the context of a simple $\mathcal{N} = 2$ supergravity model with a single modulus. This model, introduced in section 3.1, arises by compactifying type IIA string theory on a Calabi-Yau 3-fold $\mathcal{X}$ with $b_2 = 1$ in the large volume limit. The first example in section 3.2 illustrates the importance of the regularity condition (2.10), while the second example in section 3.3.1 shows the necessity of including contributions from fixed points associated with scaling solutions. This example is a particular case of a general class of dipole halo configurations which will be discussed further in section 6. The example in section 3.3.2 illustrates the role of the regularity condition (2.10) in deciding whether a given scaling solution is physical or not.

---

[17]The same localization techniques show that the index of the untwisted Dirac operator $\int_{\mathcal{M}_n} \hat{A} = (y - 1/y)^{n-1} \sum_p s(p)$ vanishes, in particular is integer, consistently with the fact that $\mathcal{M}_n$ is spin.

[18]The fact that $y \to -y$ changes the refined index by an overall sign shows that for fixed $\alpha_{ij}$ and $n$, all states carry the same parity of $2J_3$.

### 3.1 One-modulus model

We consider a simple supergravity model with one vector multiplet, governed by the pre-potential

$$F(X^0, X^1) = -\kappa (X^1)^3/6X^0 + \frac{1}{2}\mathcal{A}(X^1)^2 + \frac{1}{24}\mathcal{B}X^0 X^1 \,. \tag{3.1}$$

This model arises in the large volume limit of type IIA string theory compactified on a Calabi-Yau 3-fold $\mathcal{X}$ with $b_2 = 1$. In this case $\kappa = \int_\mathcal{X} \omega^3$ is the cubic self-intersection of the unique generator $\omega$ of of $H^2(\mathcal{X}, \mathbb{Z})$, $\mathcal{B} = \int \omega \wedge c_2$ is an integer such that $\mathcal{B} = -2\kappa \mod 12$ and $\mathcal{A}$ is an half-integer constant such that $\mathcal{A} = \kappa/2 \mod 1$.[19] The ratio $X^1/X^0$ is the complexified Kähler modulus $t$:

$$t = B + \mathrm{i}J = \frac{X^1}{X^0} \,, \tag{3.2}$$

with $B$ the NS 2-form potential and $J$ the real Kähler modulus. The central charge of a BPS state with electromagnetic charges $(p^\Lambda, q_\Lambda)$ is given by

$$
\begin{aligned}
Z &= e^{\mathcal{K}/2}(p^\Lambda F_\Lambda - q_\Lambda X^\Lambda) \\
&= e^{\mathcal{K}/2} X^0 \left[ \frac{1}{6}\kappa\, t^3 p^0 - \frac{1}{2}\kappa\, t^2 p^1 - t\tilde{q}_1 - \tilde{q}_0 \right] \,,
\end{aligned} \tag{3.3}
$$

where $e^{-\mathcal{K}} = \mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) = \frac{4}{3}\kappa J^3 |X^0|^2$ is the Kähler potential, and

$$\tilde{q}_1 = q_1 - \frac{\mathcal{B}}{24}p^0 - \mathcal{A}p^1, \qquad \tilde{q}_0 = q_0 - \frac{\mathcal{B}}{24}p^1 \,. \tag{3.4}$$

With the above conditions on $\mathcal{A}, \mathcal{B}$, the charges $p^0$, $p^1$, $q_1$ and $q_0$ are quantized in integer units [39] and represent (up to a sign) $D6$, $D4$, $D2$ and $D0$-brane charges. We shall denote

$$\alpha = (p^0, p^1, q_1, q_0), \qquad \tilde{\alpha} = (p^0, p^1, \tilde{q}_1, \tilde{q}_0) \,. \tag{3.5}$$

Note that given a pair of vectors $\alpha$, $\alpha'$ we have $\langle \alpha, \alpha' \rangle = \langle \tilde{\alpha}, \tilde{\alpha}' \rangle$. Solving the attractor equations gives for the Bekenstein-Hawking entropy $\pi|Z_\alpha(X^I_{\mathrm{attr.}})|^2$ of a large black hole in this model [40]:

$$S = \pi \sqrt{D(p^0, p^1, \tilde{q}_1, \tilde{q}_0)}, \tag{3.6}$$

where $D$ is the quartic polynomial

$$D(p^0, p^1, \tilde{q}_1, \tilde{q}_0) = \frac{\kappa^2}{9} \left[ 3\frac{(\tilde{q}_1 p^1)^2}{\kappa^2} - 18\frac{\tilde{q}_0\, p^0\, \tilde{q}_1\, p^1}{\kappa^2} - 9\frac{\tilde{q}_0^2\,(p^0)^2}{\kappa^2} - 6\frac{(p^1)^3\,\tilde{q}_0}{\kappa} + 8\frac{p^0\,(\tilde{q}_1)^3}{\kappa^3} \right]. \tag{3.7}$$

For a multi-centered black hole solution with charges $\alpha_i$, the consistency condition (2.10) can now be expressed as

$$D\left( \tilde{\beta} + \sum_{i=1}^n \frac{\tilde{\alpha}_i}{|\vec{r} - \vec{r}_i|} \right) > 0 \,, \qquad \forall \quad \vec{r} \in \mathbb{R}^3 \,. \tag{3.8}$$

---

[19]These congruence conditions are special cases of the conditions $\frac{1}{2}\kappa_{abc}p^b p^c - \mathcal{A}_{ab}p^b = 0 \mod 1$, $\frac{1}{6}\kappa_{abc}p^a p^b p^c + \frac{\mathcal{B}_a}{12}p^a \mod 1$ for all integer vector $p^a$, see [39]. For the quintic, we have $\kappa = 5$, $\mathcal{A} = -11/2$, $\mathcal{B} = 50$.

For single centered supersymmetric black holes $D(p^0, p^1, \tilde{q}_1, \tilde{q}_0) > 0$ implies (3.8) for $J \gg 0$. For multi-center black holes (3.8) does not follow from regularity of the near-horizon regions but must be checked independently.

We shall conclude this subsection with some comments on the effect of $\alpha'$ corrections to the supergravity action. One source of corrections originates from world-sheet instantons which contribute to the prepotential $F$. It is straightforward in principle to incorporate such corrections in the analysis of multi-centered solutions. In addition, there are also higher derivative corrections to the four-dimensional effective action which greatly complicate the construction of exact multi-centered solutions [11, 41]. While these corrections are essential in computing the index $\Omega^S(\alpha)$ of certain single centered black holes [12], their effect on the functions $g_{\text{ref}}(\{\alpha_i\}; y)$ governing the index of multi-centered black holes is mild, since the detailed information about the action and the solution is needed only to determine whether a given collinear solution has a non-singular metric or not; but given such a solution the contribution to $g_{\text{ref}}(\{\alpha_i\}; y)$ given in (2.38), and its generalization to scaling solutions which we shall discuss later, is independent of the action. Thus in our analysis we shall ignore the effect of these higher derivative corrections. For explicit computation we shall also set $\mathcal{A}$ and $\mathcal{B}$ to zero for convenience.

## 3.2 Non-scaling solutions: $D6 - D6 - \overline{D6}$

We now consider a 3-centered configuration of $D6$-branes with fluxes, carrying charges $e^{U\omega}$, $e^{V\omega}$ and $-e^{(U+V)\omega}$ where $U, V$ are positive integers. This example was studied in detail in [23], section 5.2.2. To compute the corresponding charge vectors, recall that the central charge associated with a $D6$-brane carrying charge $e^{U\omega}$ is proportional to

$$- \int_{\mathcal{X}} e^{-t\omega} \wedge \left[ e^{U\omega} \left( 1 + \frac{c_2}{24} \right) \right] . \tag{3.9}$$

Comparing this with (3.3), and using the fact that $\int c_2 \wedge \omega = \mathcal{B}$, $\int \omega \wedge \omega \wedge \omega = \kappa$, we see that the first center carries charges

$$\tilde{\alpha}_1 = \left( 1, U, -\frac{1}{2}\kappa U^2 - \frac{1}{24}\mathcal{B}, \frac{1}{6}\kappa U^3 + \frac{1}{24}\mathcal{B}U \right) , \tag{3.10}$$

where we have determined the overall normalization of $\tilde{\alpha}_1$ by using the fact that for a single $D6$-brane $p^0 = 1$. Similarly the second and the third centers carry charges

$$
\begin{aligned}
\tilde{\alpha}_2 &= \left( 1, V, -\frac{1}{2}\kappa V^2 - \frac{1}{24}\mathcal{B}, \frac{1}{6}\kappa V^3 + \frac{1}{24}\mathcal{B}V \right) , \\
\tilde{\alpha}_3 &= \left( -1, -(U+V), \frac{1}{2}\kappa (U+V)^2 + \frac{1}{24}\mathcal{B}, -\frac{1}{6}\kappa (U+V)^3 - \frac{1}{24}\mathcal{B}(U+V) \right) ,
\end{aligned}
\tag{3.11}
$$

so that the total charge is given by

$$\tilde{\gamma} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 = \left( 1, 0, \kappa UV - \frac{1}{24}\mathcal{B}, -\frac{1}{2}\kappa UV(U+V) \right) . \tag{3.12}$$

It is easy to see that for integer $U$ and $V$, the charges $\tilde{q}_i$ are not integers in general, but $q_i$ computed via (3.4) are. Using $\alpha_{ij} = \tilde{\alpha}_{ij}$, the integer symplectic products are then given by

$$\alpha_{21} = \frac{1}{6}\kappa\,(V-U)^3 + \frac{1}{12}\mathcal{B}(V-U)\,, \quad \alpha_{23} = \frac{1}{6}\kappa\,U^3 + \frac{1}{12}\mathcal{B}U\,, \quad \alpha_{13} = \frac{1}{6}\kappa\,V^3 + \frac{1}{12}\mathcal{B}V\,. \quad (3.13)$$

Our goal is to find the index of supersymmetric bound states in the large volume limit. We choose as a concrete example

$$\kappa = 6\,, \quad \mathcal{B} = 0\,, \quad U = 1\,, \quad V = 2\,, \quad B + iJ = 3\,i\,. \quad (3.14)$$

This choice differ from that of [23] in the choice of $\kappa$, but this is a simple normalization factor and still allows us to compare our findings with the results of [23]. The value of $J$ is chosen to be large enough to lie in the large volume chamber. The precise value of $J$ affects the numerical values of the solutions to eq. (2.24), but is otherwise irrelevant. We use the projective symmetry of special geometry to fix $X^0 = 1$. For the values of charges and moduli corresponding to (3.14), this leads to

$$\alpha_{12} = -1\,, \qquad \alpha_{23} = 1\,, \qquad \alpha_{13} = 8\,,$$
$$c_1 = \frac{73}{477}\sqrt{318}\,, \quad c_2 = \frac{170}{477}\sqrt{318}\,, \quad \beta = \sqrt{\frac{53}{6}}\left(\frac{7}{159}, \frac{2}{53}, \frac{63}{53}, -\frac{18}{53}\right)\,. \quad (3.15)$$

Numerical analysis of (2.24) leads to the following collinear solutions:[20]

$$
\begin{array}{|c|r|r|r|r|c|c|}
\hline
\sigma & z_{\sigma(1)} & z_{\sigma(2)} & z_{\sigma(3)} & \rho & s(\sigma) & \text{reg.} \\
\hline
132 & 0 & 2.59 & 2.75 & -0.363 & -1 & \checkmark \\
231 & -2.75 & -2.59 & 0 & -0.363 & -1 & \checkmark \\
123 & 0 & 2.37 & 2.54 & -0.422 & 1 & \checkmark \\
321 & -2.54 & -2.37 & 0 & -0.422 & 1 & \checkmark \\
123 & 0 & 0.195 & 1.02 & -5.14 & -1 & \times \\
321 & -1.02 & -0.195 & 0 & -5.14 & -1 & \times \\
213 & -0.182 & 0 & 0.974 & -5.49 & 1 & \times \\
312 & -0.974 & 0 & 0.182 & -5.49 & 1 & \times \\
\hline
\end{array}
\qquad (3.16)
$$

In this table, we have displayed the permutation $\sigma$ of 123 specifying the order of the $z_i$'s for a given solution according to $z_{\sigma(1)} < z_{\sigma(2)} < z_{\sigma(3)}$, the location of the centers $z_{\sigma(i)}$, the value of the parameter $\rho$ in the parametrization given in (2.21), the value of the sign $s(p)$, and in the last column, we indicated by $\checkmark$ solutions obeying the regularity condition (2.10), and by $\times$ those which do not (it turns out that the region where the discriminant $D$ becomes negative intersects the $z$-axis, and therefore can be found by plotting $D$ as a function of $z$). Note that the parameter $\rho$ takes the same value for two configurations related by a reversal of the $z$-axis. More generally, solutions of (2.6) satisfying the triangular inequalities arise from the disconnected intervals $-.422 < \rho < -.363$ and $-5.49 < \rho < -5.14$ along the

---

[20]We have presented the solution in the $z_1 = 0$ gauge instead of $\sum_i z_i = 0$ gauge.

$\rho$ axis, and correspond to general non-collinear solutions with angular momentum in the intervals $0.444 < j^2 < 0.694$ and $0.25 < j^2 < 0.444$, respectively. Only the first interval leads to solutions which satisfy the regularity solution (2.10).

According to our prescription only the solutions marked by $\sqrt{}$ must be included in the sum in (2.38), and we get

$$g_{\rm ref}(\alpha_1, \alpha_2, \alpha_3; y) = (-1)^{\alpha_{21}+\alpha_{23}+\alpha_{13}} \frac{1}{\sinh^2 \nu} \sinh(\alpha_{23}\nu) \sinh((\alpha_{13} - \alpha_{21})\nu), \quad \nu \equiv \ln y,$$

$$(3.17)$$

in agreement with [23]. Had we ignored the constraint (3.8) and included contributions from all the solutions given in (3.16), we would have got

$$(-1)^{\alpha_{21}+\alpha_{23}+\alpha_{13}} \frac{1}{\sinh^2 \nu} \sinh(\alpha_{13}\nu) \sinh((\alpha_{23} + \alpha_{21})\nu). \tag{3.18}$$

This last expression is indeed the result produced by quiver quantum mechanics, but as emphasized in [23], quiver quantum mechanics in this case fails to give the correct result. We see that the difference between the correct result and quiver quantum mechanics is accounted for by the additional constraint (3.8) that must be imposed on the solutions besides (2.24).

## 3.3 Scaling solutions

Consider a three centered black hole solution with the centers carrying charges $\alpha_1$, $\alpha_2$ and $\alpha_3$. Suppose further that $\alpha_{12} = 3$, $\alpha_{23} = 4$ and $\alpha_{31} = 5$. In this case $\alpha_{12}$, $\alpha_{23}$ and $\alpha_{31}$ satisfy a triangle inequality, showing that the scaling solution can exist. Thus this provides us with a laboratory for studying the role of scaling solutions in the computation of the index of multi-centered black hole solutions.

For now we focus on the constraints imposed by eqs.(2.24) without worrying about (2.10); we shall return to this later. Since we have two moduli $J$ and $B$ at our disposal we can adjust them to set $c_1$ and $c_2$ as we like (within an appropriate range), $c_3$ is then fixed to be $-c_1 - c_2$. Let us use this freedom to choose a point in the moduli space where $c_i = -\Lambda \sum_j \alpha_{ij}$ for some positive constant $\Lambda$. We can now look for solutions to (2.24). Numerical analysis shows that there are only two collinear configurations which contribute, corresponding to the alignments 123 and 321. Furthermore both contribute with positive sign. As a result the net contribution is given by $(y-y^{-1})^{-2} (y^{\alpha_{12}+\alpha_{13}+\alpha_{23}} + y^{-\alpha_{12}-\alpha_{13}-\alpha_{23}})$.

This causes a puzzle since this does not have a finite $y \to 1$ limit. We must however recall that there are also scaling solutions to Denef's constraints which correspond to all three points approaching each other. Clearly this is also a fixed point of $J_3$ if we choose the point of approach to lie on the $z$ axis. These however will not show up in the numerical determination of the fixed points which assumes from the beginning that the centers have finite separation.

The fixed point associated with the scaling solution has $J_3 = 0$ and hence it contributes a constant to the index. There may be additional factors from the integration measure near the scaling solution. These can in principle be determined from a detailed analysis of the

scaling solution. However we can try to guess the contribution from the scaling solution by requiring that the total contribution has a finite $y \to 1$ limit. Our first guess would be

$$g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3; y) = (y - y^{-1})^{-2} \left( y^{\alpha_{12} + \alpha_{13} + \alpha_{23}} + y^{-\alpha_{12} - \alpha_{13} - \alpha_{23}} - 2 \right). \tag{3.19}$$

For $\alpha_{12} + \alpha_{13} + \alpha_{23}$ even, the numerator has a factor of $(y - y^{-1})^2$, and after canceling this against the denominator we are left with a Laurent polynomial in $y$. This is a sensible result, with the coefficient of $y^m$ counting $(-1)^m$ times the number of states with $J_3 = m/2$. But (3.19) does not lead to a sensible spectrum for odd values of $\alpha_{12} + \alpha_{13} + \alpha_{23}$, since the factor of $(y - y^{-1})^2$ in the denominator is not cancelled. Our proposal is to replace the subtraction constant 2 by $y + y^{-1}$ in this case so that $(y - y^{-1})^{-2}$ factor in the denominator is cancelled and we again get a Laurent polynomial in $y$.

The above analysis shows that we can predict the existence of and the contribution from a scaling solution from the results on the non-scaling solutions which are simpler to find. This is the general procedure we shall follow, i.e. assume that the role of the fixed points associated with the scaling solution is to essentially make the final result satisfy desirable properties. This by itself will not fix the contribution uniquely. For example in (3.19) we could have taken the subtraction term to be $y^2 + y^{-2}$ instead of 2. To fix this ambiguity we shall make a further assumption, that the extra contribution due to the scaling solutions vanish as $y \to \infty$. This fixes the correction terms uniquely. We shall call this the 'minimal modification hypothesis'. We shall describe the general rule for such replacements in section 5. While we have no *a priori* justification for this assumption, it seems to work in all known cases.

For specific choices of the prepotential we also need to verify that the solutions we consider satisfy the constraint (3.8). This can be easily implemented by testing (3.8) for each of the two collinear solutions described above and if we find that (3.8) fails for these solutions then we would conclude that the subtraction terms associated with the scaling solutions must also vanish. Thus there will be no contribution to the index from this configuration. For implementing this condition we need to work with specific charge vectors. We shall now discuss two examples, — one where this condition is satisfied, and another where it fails.

### 3.3.1 Example 1: $\overline{D6} - D6 - D0$

Let us take a $D6$-$\overline{D6}$-brane pair with flux as in the last subsection and a pure $D0$-charge. In particular we choose

$$\begin{aligned}
\tilde{\alpha}_1 &= (1, -U, -\kappa U^2/2 - \mathcal{B}/24, -\kappa U^3/6 - \mathcal{B}U/24), \\
\tilde{\alpha}_2 &= (-1, -U, \kappa U^2/2 + \mathcal{B}/24, -\kappa U^3/6 - \mathcal{B}U/24), \\
\tilde{\alpha}_3 &= (0, 0, 0, n),
\end{aligned} \tag{3.20}$$

for some integer $U$. In this case we have

$$\alpha_{12} = 4\kappa U^3/3 + \mathcal{B}U/6, \quad \alpha_{23} = \alpha_{31} = n. \tag{3.21}$$

We define $\alpha \equiv \alpha_{23} = \alpha_{31}$, and assume that $\alpha_{12} > 0, \alpha > 0$. Scaling solutions are expected when the triangular inequalities are satisfied, i.e. when $2\alpha \geq \alpha_{12}$.

We choose the following explicit values of the parameters:

$$\kappa = 6, \quad \mathcal{B} = 0, \qquad U = 1, \quad t = B + iJ = 1 + 3i. \tag{3.22}$$

As example of a system which does not allow for scaling solutions we take $n = 3$, while we take $n = 6$ for an example of a system that does. For those values, we have

$$n = 3: \qquad c_1 = \tfrac{135}{2}\sqrt{\tfrac{6}{3697}}, \qquad c_2 = -\tfrac{147}{2}\sqrt{\tfrac{6}{3697}}, \tag{3.23}$$

$$n = 6: \qquad c_1 = \tfrac{33}{10}\sqrt{\tfrac{3}{5}}, \qquad c_2 = -\tfrac{39}{10}\sqrt{\tfrac{3}{5}}. \tag{3.24}$$

For $n = 3$ one finds collinear fixed points for $\rho = 3.41$ with alignment 312 (or its reverse) and $s(p) = 1$, and for $\rho = 11.37$ with alignment 132 (or its reverse) and $s(p) = -1$. A detailed study of the full supergravity solutions shows that they satisfy the regularity condition (3.8) everywhere outside the centers. The contribution of these fixed points gives

$$g_{\mathrm{ref}}(\{\alpha_i\}; y) = \frac{y^8 + y^{-8} - y^2 - y^{-2}}{(y - y^{-1})^2}, \tag{3.25}$$

which is finite for $y \to 1$ as in section 3.2.

For $n = 6$ one finds a single value of $\rho = 3.99$ corresponding to collinear solutions with alignment 312 and $s(p) = 1$ (and its reverse). We have checked that this solutions satisfies (3.8) everywhere. Adding the contributions of these fixed points gives $(y - y^{-1})^{-2}(y^8 + y^{-8})$, which does not have a finite $y \to 1$ limit. Following the procedure described earlier we now take the total contribution to be

$$g_{\mathrm{ref}}(\{\alpha_i\}; y) = (y - y^{-1})^{-2}\left(y^8 + y^{-8} - 2\right), \tag{3.26}$$

which does give a sensible result for the spectrum.

We close this subsection with a comment on the structure of the solution in the regime $0 \le B \ll 1$, $J \gg 1$. For this discussion we parametrize $c_1$ and $c_2$ by $c_1 = \mu - \eta$ and $c_2 = -\mu - \eta$. In the limit described above, $\mu > 0$ and $0 \le \eta \ll 1$. For $2\alpha < \alpha_{12}$, there are two collinear configurations (up to reversal of the $z$ axis) with alignments 132 and 213, parametrized by

$$\rho_{(132)} = \frac{\alpha_{12}\mu}{\alpha_{12} - 2\alpha} + \mathcal{O}(\eta^2) \quad \text{and} \quad \rho_{(213)} = \mu + \sqrt{\frac{2\mu\alpha\eta}{\alpha_{12}}} + \mathcal{O}(\eta), \tag{3.27}$$

respectively. For $2\alpha \ge \alpha_{12}$, only the second solution leads to physical solutions. In the $\eta \to 0$ limit the first solution in (3.27) has a smooth limit, with the center 3 sitting at the middle point between the centers 1 and 2, separated by $r_{12} \sim (\alpha_{12} - 2\alpha)/\mu$. For the second solution, we have instead $r_{23} \simeq r_{13} = \sqrt{\frac{\alpha\alpha_{12}}{2\mu\eta}} + \mathcal{O}(\eta^0)$ with $r_{12} \simeq \alpha_{12}/\mu$. Therefore the point 3 moves off to infinity along the $z$ axis, similarly to what happens at a wall of marginal stability. However, since $\langle \alpha_1 + \alpha_2, \alpha_3 \rangle = 0$, the index does not change across the locus $\eta = 0$. This is an example of a wall of threshold stability [33]. While a non-compact direction opens up in $\mathcal{M}_3$ at $\eta = 0$, the equivariant volume (2.28) stays unchanged. In section 6 we shall see that the locus $\eta = 0$ (or its analogue for more centers) is very convenient to analyze the phase space, even though some of the collinear fixed points sit at infinity. Such 'infrared divergent' configurations should however not be confused with the 'ultraviolet divergent' scaling solutions, which cannot be removed by moving away from $\eta = 0$.

### 3.3.2  Example 2: unphysical scaling solutions

In the example in section 3.3.1 the entropy associated with individual centers vanish as can be easily seen using eq. (3.6) and (3.20). We now consider another example where each center describes a regular black hole with finite entropy. Again for simplicity we take $\kappa = 6$ and $\mathcal{B} = 0$ and choose

$$\alpha_1 = (0, 7, 48, 0), \quad \alpha_2 = (0, 9, 42, 0), \quad \alpha_3 = (0, -8, -48, 6). \tag{3.28}$$

In this case we have

$$\alpha_{12} = 138, \quad \alpha_{23} = 96, \quad \alpha_{31} = 48. \tag{3.29}$$

Thus the triangle inequality is satisfied and we can look for scaling solution. Taking $t = B + iJ = 3i$, we find that collinear solutions to (2.24) exist for the alignment 132 (or its reverse), with $s(p) = 1$ and $\rho = 62.68$. Naively, to the contributions from these collinear fixed points we should add the contribution $-2$ from scaling solutions, so as to obtain an admissible SU(2) character. Numerical analysis shows however that the collinear solutions fail to satisfy (3.8) in some region of the $z$-axis and hence are not valid solutions. Thus the scaling solutions must also be absent. This can be verified explicitly by examining the full solution space parametrized by (2.21): in this case the values of $\rho$ consistent with the triangular inequalities range from $\rho = 62.68$ (corresponding the above collinear solution) to $\rho = \infty$ (corresponding to the scaling regime). We find that throughout this range, the left hand side of (3.8) fails to be positive somewhere in the three dimensional space. Thus, the regularity condition (3.8) rules out the entire phase space $\mathcal{M}_3$ in this case.

## 4  Index from non-scaling multi-centered configurations

In $\mathcal{N} = 2$ supersymmetric string theory, typically the index in the sector of charge $\gamma$ receives contributions both from single centered black holes carrying charge $\gamma$, and from multi-centered black hole solutions, with individual centers carrying charges $\alpha_1, \ldots, \alpha_n$ such that $\sum_i \alpha_i = \gamma$. In this section, we shall consider contributions from those multi-centered configurations which do not allow for scaling solutions, i.e. solutions where three or more centers can come arbitrarily close. As explained in section 2, this requires that there is no subset $A$ of $\{1, \ldots, n\}$ for which we can find vectors $\vec{r}_i$ ($i \in A$) satisfying (2.23). We work at some fixed point in the moduli space, and denote by $Z_\alpha$ the central charge associated with the charge $\alpha$. Using the same logic as in [17] we arrive at the following expression for the (refined) index $\Omega_{\mathrm{ref}}(\gamma; y) \equiv \mathrm{Tr}'(-y)^{2J_3}$ from multi-centered black hole solutions carrying total charge $\gamma$ (assuming that $\gamma$ is a primitive vector of the charge lattice):

$$\sum_{\substack{\{\alpha_i\} \\ \sum_i \alpha_i = \gamma}} \frac{1}{\mathrm{Aut}(\{\alpha_i\})} \, g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n; y) \, \bar{\Omega}^S_{\mathrm{ref}}(\alpha_1; y) \cdots \bar{\Omega}^S_{\mathrm{ref}}(\alpha_n; y). \tag{4.1}$$

Here, $\mathrm{Aut}(\{\alpha_i\})$ is the symmetry factor appropriate for Maxwell-Boltzmann statistics (see footnote 2), while $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n; y)$ given in (2.38) is the contribution to $\mathrm{Tr}'(-y)^{2J_3}$ from

$n$-centered black hole configuration with the centers regarded as distinguishable particles with unit index. $\bar{\Omega}_{\text{ref}}^S(\alpha; y)$ is the 'rational refined index', related to the refined index by

$$\bar{\Omega}_{\text{ref}}^S(\alpha, y) = \sum_{m|\alpha} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} \Omega_{\text{ref}}^S(\alpha/m, y^m) \,. \tag{4.2}$$

It is worthwhile noting that (4.1) and (4.2) exhibit a manifest 'charge conservation property', whereby each power of $\bar{\Omega}_{\text{ref}}^S(\alpha, y)$ carries charge $\alpha$ and each power of $\Omega_{\text{ref}}(\alpha, y^m)$ carries charge $m\alpha$. The usual protected index is obtained by taking the $y \to 1$ limit of (4.1), at the cost of obscuring manifest 'charge conservation'. If $\gamma$ is not primitive, then (4.1) represents $\bar{\Omega}_{\text{ref}}(\gamma, y)$ defined in (1.1) rather than $\Omega_{\text{ref}}(\gamma, y)$.

Using (4.1), (4.2) and the relationship between $\Omega_{\text{ref}}$ and $\bar{\Omega}_{\text{ref}}$ we can express $\Omega_{\text{ref}}(\gamma, y)$ as

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} G(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}) \,, \tag{4.3}$$

for some function $G$. The $i$-th term in the sum represents a contribution from configurations with a total of $\sum_i m_i$ centers, with $m_i$ centers of charge $\beta_i$. If two or more of the $\beta_i$'s are identical, the total number of centers carrying a given charge $\beta_i$ is the sum of the corresponding $m_i$'s. Finally the sum also contains the contribution from a single centered black hole of charge $\gamma$, represented by the term $\Omega_{\text{ref}}^S(\gamma, y)$. In order for the right hand side of (4.3) to be a bona fide SU(2) character whenever the $\Omega_{\text{ref}}^S(\beta_i, y)$'s are, the functions $G$ which appear in (4.3) must be Laurent polynomials in $y$. The reader is referred to appendix B for a proof of this property in the absence of scaling solutions.

Eq. (4.1) (or equivalently (4.3)) gives the net contribution to the total index from all possible single and multi-centered solutions carrying a fixed charge $\gamma$. However it is also useful to identify which terms in (4.1) represent the contribution from a specific multi-centered solution. We shall assume for simplicity that $\gamma$ is primitive so that $\bar{\Omega}_{\text{ref}}(\gamma, y) = \Omega_{\text{ref}}(\gamma, y)$. When all $\alpha_i$'s are different then the summand in (4.1) with $\bar{\Omega}_{\text{ref}}^S$ replaced by $\Omega_{\text{ref}}^S$ represents the contribution to the index from multi-centered black holes carrying charges $\alpha_1, \ldots, \alpha_n$. However when some of the $\alpha_i$'s are equal (say $r_1$ copies of $\beta_1$, $r_2$ copies of $\beta_2$ etc.), then there is additional contribution to this index due to the fact that the $\Omega_{\text{ref}}^S(\alpha/m, y^m)$ term in (4.2) represents the contribution from $m$ identical centers, each of charge $\alpha/m$. Thus, the contribution to the index from a multi-centered configuration with $r_1$ centers of charge $\beta_1$, $r_2$ centers of charge $\beta_2$ etc. is given by

$$\sum_{\substack{\{n_k \geq 1\}, \{s_k^{(a)} \geq 1\} \\ \sum_{a=1}^{n_k} s_k^{(a)} = r_k}} \hat{g}_{\text{ref}}(\{s_k^{(a)} \beta_k\}; y) \prod_k \left[ \prod_{a=1}^{n_k} \left\{ \frac{1}{s_k^{(a)}} \frac{y - y^{-1}}{y^{s_k^{(a)}} - y^{-s_k^{(a)}}} \Omega_{\text{ref}}^S(\beta_k, y^{s_k^{(a)}}) \right\} \right] \,, \tag{4.4}$$

where

$$\hat{g}_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y) = \frac{1}{\text{Aut}(\{\alpha_i\})} \, g_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y) \,. \tag{4.5}$$

After summing (4.4) over all possible choices of $\{\beta_k\}$, $\{r_k\}$ satisfying $\sum_k r_k \beta_k = \gamma$, we recover (4.1). As explained in section 2, the index $g_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y)$ can be computed by

localization. Since we assume that the phase space contains no scaling solutions, all fixed points are isolated and the quantum index is given by (2.38).

In the remainder of this section we shall carry out various consistency tests of our proposal for the index associated with the multi-centered black hole solutions.

## 4.1 Consistency with wall crossing

Let us examine whether our proposal is compatible with wall crossing. For this purpose, assume that the moduli are chosen near a wall of marginal stability where the state carrying total charge $\gamma \equiv \alpha_1 + \cdots + \alpha_n$ becomes marginally unstable against decay into states carrying charges $\gamma_A = \sum_{i \in A} \alpha_i$ and $\gamma_B = \sum_{i \in B} \alpha_i$ where $A$ and $B$ are two complementary subsets of $\{1, \ldots, n\}$. At the wall, the phases of the central charges $Z_{\gamma_A}$ and $Z_{\gamma_B}$ align, and we have

$$\sum_{i \in A} c_i = -\sum_{j \in B} c_j \to 0 \,. \tag{4.6}$$

We shall assume that the moduli are chosen on the side of the wall where

$$\langle \gamma_A, \gamma_B \rangle \sum_{i \in A} c_i > 0 \,. \tag{4.7}$$

In this region of the moduli space, a class of solutions to (2.24) can be constructed by joining two solutions involving the centers in the set $A$ and those in the set $B$ as follows. Let $n_A$ and $n_B$ be the cardinality of the sets $A$ and $B$. Now, choose the relative distances $z_i - z_j$ for $i, j \in A$ according to a particular collinear solution involving the charges $\alpha_{i \in A}$, and the relative distances $z_i - z_j$ for $i, j \in B$ according to the particular collinear solution involving the charges $\alpha_{i \in B}$. Finally, choose the relative separation between the $z_i$'s in the set $A$ and the $z_i$'s in the set $B$ such that

$$\frac{\gamma_{AB}}{|z_A - z_B|} = \sum_{i \in A} c_i \,, \qquad \gamma_{AB} \equiv \langle \gamma_A, \gamma_B \rangle \,, \tag{4.8}$$

where $z_A = (\sum_{i \in A} z_i)/n_A$ and $z_B = (\sum_{i \in B} z_i)/n_B$ are the average positions of the centers in the sets $A$ and $B$ respectively. Near the wall of marginal stability $\sum_{i \in A} c_i \to 0$, and the separation $|z_A - z_B|$ becomes large. We claim that this configuration satisfies eq. (2.24) in the limit (4.6), and can be systematically corrected to an exact solution of eq. (2.24) in the vicinity of the wall.

To see this, we note that if $i$ belong to the set $A$ then eq. (2.24) receives significant contribution only when $j$ also belongs to the set $A$ since the distance $|z_A - z_B|$ computed from (4.8) is large in the limit (4.6). Thus the equations reduce approximately to the equations for collinear multi-centered solutions involving the set $A$ only. By assumption our solution satisfies the latter equations. A similar argument holds when $i$ belongs to the set $B$. There is however a small caveat stemming from the fact that the solutions in the set $A$ are labelled by $n_A - 1$ relative distances while solutions in the set $B$ are labelled by $(n_B - 1) = (n - n_A - 1)$ relative distances. This gives $(n-2)$ parameters, but (2.24) contains $(n-1)$ independent equations (the sum of the equations over all $i$ being trivially satisfied). Since we cannot adjust $(n-2)$ parameters to solve $(n-1)$ independent equations we must

have missed some equation that determines the relative distance between the points in the set $A$ and those in the set $B$. To find the missing equation we sum (2.24) over all $i \in A$ to obtain

$$\sum_{i \in A} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_{ij}}{|z_j - z_i|} = \sum_{i \in A} c_i \,. \tag{4.9}$$

Dividing the sum over $j$ on the left hand side into those for which $j \in A$ and those for which $j \in B$, and noting that the first term vanishes by $i \leftrightarrow j$ symmetry, we get

$$\sum_{i \in A} \sum_{j \in B} \frac{\alpha_{ij}}{|z_j - z_i|} = \sum_{i \in A} c_i \,. \tag{4.10}$$

Approximating $z_j - z_i \simeq z_B - z_A$ for each term in the sum we recognize (4.8). It is then clear that this approximate solution can be extended to an exact solution in the vicinity of the wall, by correcting the locations $z_i$ by a Taylor series in $1/|z_A - z_B|$.

Eq. (4.7) shows that the solution to (4.8) exists as we approach the wall of marginal stability but ceases to exist as we cross the wall since the left hand side of (4.7) and hence $|z_A - z_B|$ computed from (4.8) now becomes negative. Thus the jump in the index will be given by the contribution to (2.38) from this class of fixed points. To evaluate this jump, we first note that

$$y^{\sum_{i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i]} = y^{\sum_{i,j \in A, i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i]} \, y^{\sum_{i,j \in B, i<j} \alpha_{ij} \operatorname{sign}[z_j - z_i]} \, y^{\gamma_{AB} \operatorname{sign}[z_B - z_A]} \,. \tag{4.11}$$

Similarly,

$$(-1)^{\sum_{i<j} \alpha_{ij} + n - 1} = (-1)^{n_A + n_B + \gamma_{AB} - 1} \, (-1)^{\sum_{i,j \in A, i<j} \alpha_{ij}} \, (-1)^{\sum_{i,j \in B, i<j} \alpha_{ij}} \,. \tag{4.12}$$

Finally we need to compute $s(p)$ near the wall of marginal stability. This has been analyzed in detail in appendix A.1. The net result is that for the configuration of the type we are considering, $s(p)$ is given by

$$s(p) = s_A \, s_B \operatorname{sign}(\gamma_{AB}) \operatorname{sign}[z_B - z_A] \,. \tag{4.13}$$

where $s_A$ is the sign which appears in the contribution to $n_A$ centered collinear configurations with centers at $\{z_i, i \in A\}$, and $s_B$ is the sign which appears in the contribution to $n_B$ centered collinear configurations with centers at $\{z_i, i \in B\}$.

Putting (4.11), (4.12) and (4.13) into (2.38) and summing over all $n_A$-centered solutions involving the charges in the set $A$ and all $n_B$-centered solutions involving charges in the set $B$ we see that the net contribution from solutions of this type is given by

$$-\operatorname{sign}(\gamma_{AB}) \operatorname{sign}[z_B - z_A](y - y^{-1})^{-1}(-y)^{\gamma_{AB} \operatorname{sign}[z_B - z_A]} g_{\mathrm{ref}}(\{\alpha_i, i \in A\}; y) g_{\mathrm{ref}}(\{\alpha_j, j \in B\}; y). \tag{4.14}$$

Adding up the contributions from the configurations for which $z_B - z_A > 0$ and those for which $z_B - z_A < 0$, we get the net contribution to the index from the solutions which decay across the wall of marginal stability:

$$-\operatorname{sign}(\gamma_{AB})(y - y^{-1})^{-1} \left[(-y)^{\gamma_{AB}} - (-y)^{-\gamma_{AB}}\right] g_{\mathrm{ref}}(\{\alpha_i, i \in A\}; y) g_{\mathrm{ref}}(\{\alpha_j, j \in B\}; y). \tag{4.15}$$

This is the correct wall crossing formula for primitive decays. Besides the solutions considered here, (2.24) can have other solutions for which the relative distances between the centers remain finite as we approach the wall of marginal stability. They will continue to exist on the other side of the wall and must decay at other walls of marginal stability before we reach the attractor point.

The analysis can be easily generalized to the case of general non-primitive decays. We shall sketch the derivation below. Suppose that we are near a wall of marginal stability where the total charge $\gamma$ can decay into $L$ states carrying charges $\gamma_m = \sum_{i \in A_m} \alpha_i$ for $m = 1, \ldots, L$. For this we need the charges $\gamma_m$ for different $m$ to lie in a two dimensional plane and have $\sum_{i \in A_m} c_i \to 0$ for each $m$. We approach the wall from the side where

$$\langle \gamma_m, \gamma_n \rangle \left( \sum_{i \in A_m} c_i - \sum_{i \in A_n} c_i \right) > 0 \,, \tag{4.16}$$

for any pair $(A_m, A_n)$. In this chamber there exists a class of solutions in which the elements of the set $A_m$ are bunched together for each $m$ within a finite distance and the relative separation between the elements of the set $A_m$ and the elements of the set $A_n$ go to infinity for every pair $(m, n)$. These are the solutions which disappear across the wall of marginal stability; hence the change in the index is given by the index associated with these configurations. We shall order the sets $A_m$ such that $\langle \gamma_m, \gamma_n \rangle > 0$ for $m < n$. Let us assume that for a given collinear solution $p$ of this type, the sets $A_m$ are arranged along the $z$-axis as $A_{\sigma(1)}, \ldots, A_{\sigma(L)}$ for some permutation $\sigma(1), \ldots, \sigma(L)$ of $1, \ldots, L$. Then the contribution to the summand in (2.38) from such a configuration will be given by

$$s(p) \prod_{m=1}^{L} \left( y^{\sum_{i,j \in A_m} \alpha_{ij} \, \mathrm{sign}[z_j - z_i]} \right) \prod_{m<n} \left( y^{\langle \gamma_{\sigma(m)}, \gamma_{\sigma(n)} \rangle} \right) \,. \tag{4.17}$$

The sum over collinear fixed points $p$ which respect the bunching of the centers into the sets $A_1, \ldots, A_L$ will involve independent sum over collinear fixed points inside each set $A_m$, generating the contribution $g_{\mathrm{ref}}(\{i \in A_m\}; y)$ and the sum over permutations of the sets $A_m$, generating a contribution $g_{\mathrm{ref}}(\{\gamma_m\}; y)$. The final result, multiplied by the $\prod_i \bar{\Omega}^S_{\mathrm{ref}}(\{\alpha_i\}, y)$, will be given by

$$g_{\mathrm{ref}}(\{\gamma_m\}; y) \prod_m \left[ g_{\mathrm{ref}}(\{\alpha_i, i \in A_m\}; y) \prod_{i \in A_m} \bar{\Omega}^S_{\mathrm{ref}}(\{\alpha_i\}, y) \right] \,. \tag{4.18}$$

This is precisely what is needed to produce the wall crossing formula given in [17]. The $g_{\mathrm{ref}}(\{\alpha_i, i \in A_m\}; y) \prod_{i \in A_m} \bar{\Omega}^S_{\mathrm{ref}}(\{\alpha_i\}, y)$ factor contribute to the index of the $m$-th 'black hole molecule' introduced in [17], while $g_{\mathrm{ref}}(\{\gamma_m\}; y)$ is the function multiplying the indices of black hole molecules in the wall crossing formula of [17].

## 4.2 Consistency with split attractor flow conjecture

The above analysis also shows that our proposal is consistent with the split attractor flow conjecture under certain assumptions. To see this let us consider the system of black holes

carrying charges $\alpha_1, \ldots, \alpha_n$ and let the moduli flow all the way to the attractor point. On the way we may cross several walls of marginal stability. If at the attractor point there are no multi-centered solutions, the result for the index at the original point in the moduli space can be computed by adding the contributions from the jumps across different walls. Suppose on the $s$-th wall the system is marginally unstable against decay into a pair of states carrying charges $\sum_{i \in A_s} \alpha_i$ and $\sum_{i \in B_s} \alpha_i$. Then the index associated with the states which decay across the wall is given by (4.15) with $(A, B)$ replaced by $(A_s, B_s)$. Thus the net contribution to the index at the original point from this multi-centered configuration is given by

$$g_{\mathrm{ref}}(\{\alpha_i\}; y) = -\sum_s \operatorname{sign}(\gamma_{A_s B_s}) \frac{(-y)^{\gamma_{A_s B_s}} - (-y)^{-\gamma_{A_s B_s}}}{y - y^{-1}}$$
$$g_{\mathrm{ref}}(\{\alpha_i, i \in A_s\}; y) \, g_{\mathrm{ref}}(\{\alpha_j, j \in B_s\}; y) \,. \tag{4.19}$$

In the $y \to 1$ limit this gives

$$g(\{\alpha_i\}) = \sum_s (-1)^{\gamma_{A_s B_s} + 1} |\gamma_{A_s B_s}| \, g(\{\alpha_i, i \in A_s\}) \, g(\{\alpha_j, j \in B_s\}) \,. \tag{4.20}$$

We can now take the multi-centered configurations carrying charges $\{\alpha_i, i \in A_s\}$, $\{\alpha_j, j \in B_s\}$ and calculate their indices by flowing along their attractor flow lines in the same way. Continuing this process till we are left with only single centered black holes, we arrive at the split attractor flow conjecture.

Note however that the above analysis relies on an assumption: that the only jumps in $g_{\mathrm{ref}}$ (and in $g$) take place at the walls of marginal stability. In particular the solutions should not disappear away from the walls of marginal stability (or if they do then they disappear in pairs so that there is no net change in the index).

## 5 Index from scaling multi-centered solutions

The goal of this section is to determine the contribution of scaling solutions to the index of a multi-centered black hole configuration using the Coulomb branch analysis. As shown in appendix B, in the absence of scaling solutions the functions $G$ defined in (4.3) are automatically Laurent polynomials, as is required in order for the result to be a bona fide character of SU(2). This property however does not hold in general when scaling solutions are present, and (4.3) has to be corrected. We shall denote by $G_{\mathrm{ref}}^{\mathrm{coll}}$ the contribution due to regular collinear fixed points only, and determine the corrections to (4.3) by requiring that after adding these corrections the final expression must be a proper SU(2) character, whenever the single-centered refined indices $\Omega^S(\alpha, y)$ are SU(2) characters.

Specifically, we propose to modify (4.3) into

$$\Omega_{\mathrm{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} G_{\mathrm{coll}}(\{\beta_i\}, \{m_i\}; y) \prod_i \left( \Omega_{\mathrm{ref}}^S(\beta_i, y^{m_i}) + \Omega_{\mathrm{scaling}}(\beta_i, y^{m_i}) \right) , \tag{5.1}$$

where $\Omega_{\mathrm{scaling}}(\alpha, y)$ is given by

$$\Omega_{\mathrm{scaling}}(\alpha, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\mathrm{ref}}^S(\beta_i, y^{m_i}) , \tag{5.2}$$

for some function $H(\{\beta_i\}, \{m_i\}; y)$ to be determined. To determine $H$ we substitute (5.2) into (5.1) to express the latter equation as

$$\Omega_{\text{ref}}(\gamma, y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} G(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}), \qquad (5.3)$$

for some functions $G$. We fix $H$ by requiring that $G(\{\beta_i\}, \{m_i\}; y)$ are given by Laurent polynomials in $y$. This leaves open the possibility of adding Laurent polynomials to $H$. This ambiguity is resolved using the minimal modification hypothesis, which says that $H$ must be symmetric under $y \to y^{-1}$ and vanish as $y \to \infty$.

In practice we solve for the functions $H$ using an iterative scheme involving the number of centers. For this suppose we know $H(\{\beta_i\}, \{m_i\}; y)$ in all cases for $\sum_i m_i \leq (n-1)$. Now we can substitute (5.2) into eq. (5.1) and compute the coefficient of $\prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i})$ for all terms with $\sum_i m_i \leq n$. The only unknown term is $\Omega_{\text{scaling}}(\gamma; y)$, originating from the replacement of $\Omega^S(\gamma, y)$ by $\Omega^S(\gamma, y) + \Omega_{\text{scaling}}(\gamma, y)$ in the right hand side of (5.1). This gives

$$\sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \gamma}} H(\{\beta_i\}, \{m_i\}; y) \prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i}). \qquad (5.4)$$

Thus requiring the coefficient of $\prod_i \Omega_{\text{ref}}^S(\beta_i, y^{m_i})$ to be a Laurent polynomial in $y$ we can determine $H(\{\beta_i\}, \{m_i\}; y)$ for $\sum_i m_i = n$. This procedure can then be repeated to find $H(\{\beta_i\}, \{m_i\}; y)$ for $\sum_i m_i = n+1$ and so on.

The algorithm given above gives a prescription for finding the net contribution from the scaling solutions for a fixed total charge $\gamma$. In the rest of the section we shall see how this prescription can be used to determine the contribution of the scaling solutions to a configuration containing a fixed set of centers.

## 5.1 Correction to $g_{\text{ref}}$ for non-identical centers

We shall now show that if the centers carry non-identical charges then the minimal modification hypothesis translates to a simple rule for correcting the function $g_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y)$. First suppose that $\alpha_1, \ldots, \alpha_n$ have been chosen such that there is a scaling solution where all the centers come together, but no scaling solution where a subset of the centers come together. In this case our proposal for the contribution of the scaling solution to $g_{\text{ref}}(\alpha_1, \ldots, \alpha_n; y)$ gives

$$g_{\text{scaling}}(\alpha_1, \ldots, \alpha_n; y) = (-1)^{\sum_{i<j} \alpha_{ij} + n - 1} (y - y^{-1})^{1-n} \sum_{\substack{0 \leq k \leq (n-2) \\ k - \sum_{i<j} \alpha_{ij} \in 2\mathbb{Z}}} a_k \left\{ y^k - (-1)^n y^{-k} \right\}, \quad (5.5)$$

where $a_k$'s are constants to be adjusted so that, after adding (5.5) to the contribution (2.38) of the collinear fixed points, the result has a finite limit as $y \to 1$. It is easy to see that the number of $a_k$'s is precisely equal to the number of divergent terms in the $y \to 1$ limit, so that requiring finiteness as $y \to 1$ uniquely fixes all the $a_k$'s. For example for even $n$ the possible values of $k$ range from 1 to $(n-2)$, with $k$ taking either only even or only odd values depending on the parity of $\sum_{i<j} \alpha_{ij}$. The number of $a_k$'s is then $(n-2)/2$.

On the other hand using $y \leftrightarrow y^{-1}$ symmetry we see that the divergent terms are of the form $(y - y^{-1})^{-2s}$ for $s = 1, 2, \ldots, (n-2)/2$, giving precisely $(n-2)/2$ possible divergent terms. For $n$ odd the allowed values of $k$ are in the range 0 to $(n-2)$ and again $k$ takes either only even values or only odd values. This gives $(n-1)/2$ possible $a_k$'s. On the other hand the possible divergent terms are of the form $(y - y^{-1})^{-2s}$ for $1 \leq s \leq (n-1)/2$, giving precisely $(n-1)/2$ possible divergent terms. It is also worthwhile to note that the condition $k \leq n - 2$ on the powers of $y$ appearing in the numerator of (5.5) is equivalent to the requirement that the correction (5.5) vanishes in the limit $y \to \infty$.

Now consider the more general case where there are scaling solutions in which a subset of the centers come together. As mentioned earlier, we shall proceed by induction, i.e. assume that all the fixed points (including scaling solutions) and their contributions have been determined for any number of centers less or equal to $n - 1$ and then show how this can be used to infer the result for $n$ centered black hole solution. Let us consider an $n$-centered black hole configuration with centers carrying charges $\alpha_1, \ldots, \alpha_n$. Now such configurations will include a set of 'regular' fixed points where all the centers are separated along the $z$-axis. We can determine them using numerical methods. To those we need to add the contribution from fixed points where a (subset of) the centers lie on top of each other. A generic fixed point of this type will have the charges $\{\alpha_{k_1^{(l)}}, \alpha_{k_2^{(l)}}, \ldots, \alpha_{k_{n_l}^{(l)}}\}$ lying on top of each other for $1 \leq l \leq s$, $n_l \geq 1$ such that $\sum_l n_l = n$. To determine its contribution we proceed as follows. We first consider an $l$-centered configuration with individual centers carrying charges $\alpha_{k_1^{(l)}} + \cdots \alpha_{k_{n_l}^{(l)}}$ with $1 \leq l \leq s$ and determine all its regular fixed points. In this computation the effective constant $c_l$ for the $l$-th charge will be given by $2\,\mathrm{Im}(e^{-i\alpha} Z_{\alpha_{k_1^{(l)}} + \cdots \alpha_{k_{n_l}^{(l)}}})$. Now we multiply this contribution by the product of the weight factors of the $s$ scaling solutions, with the $l$'th solution containing centers $\{\alpha_{k_1^{(l)}}, \alpha_{k_2^{(l)}}, \ldots, \alpha_{k_{n_l}^{(l)}}\}$. These are known by induction except for the case $s = 1$. We now add these to the contribution from the regular fixed points of the $n$-centered solution. This procedure leaves out the $s = 1$ term, corresponding to $l = 1$, $n_1 = n$: this is the maximally scaling configuration, where all the centers are on top of each other. Our proposal for the contribution of this term to $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n; y)$ is again given by (5.5), where $a_k$'s are constants to be adjusted so that after adding (5.5) to the other contributions (including the non-maximally scaling ones), the result can be expressed as a Laurent polynomial in $y$.

The procedure just described generalizes the one introduced in section 3.3 in the absence of scaling solutions. It relies on the assumption that the effect of the scaling solutions is to correct $g_{\mathrm{ref}}$ into a proper SU(2) character, with the smallest possible powers of $y$ and $y^{-1}$ in the numerator (reflecting the classical fact that scaling solutions carry zero angular momentum). Eq. (5.5) together with (2.38) gives a complete prescription for computing the functions $g_{\mathrm{ref}}(\{\alpha_i\}; y)$. This construction guarantees that $g_{\mathrm{ref}}(\{\alpha_i\}; y)$ is given by a Laurent polynomial in $y$. As long as the charges $\alpha_i$ are all different, the factor $g_{\mathrm{ref}}$ just described multiplied by $\prod_{i=1}^{n} \Omega^S(\alpha_i; y)$ determine the contribution to the index from multi-centered black hole solutions carrying charges $\alpha_1, \ldots, \alpha_n$. The resulting expression is given by a Laurent polynomial in $y$. As discussed in section 4, even when some of the $\alpha_i$'s are equal but provided there are no scaling solutions, eqs. (4.4), (4.5) give the correct result

for the index of multi-centered black hole solutions. However when some of the $\alpha_i$'s are equal and scaling solutions are allowed, some additional corrections to (4.4) are necessary, which we shall now determine.

## 5.2 Effect of identical particles

We shall now consider the case where some of the centers carry identical charges, e.g. we have $r_1$ centers of charge $\beta_1$, $r_2$ centers of charge $\beta_2$ etc. We shall assume for simplicity that the only allowed scaling solutions involve all the centers coming together, — more general cases may be dealt with using the method of induction as before. In this case our proposal for the index associated with this configuration is the following generalization of (4.4):

$$
\sum_{\substack{\{n_k\},\{s_k^{(a)}\} \\ n_k, s_k^{(a)} \in \mathbb{Z}; n_k, s_k^{(a)} \geq 1, \\ \sum_{a=1}^{n_k} s_k^{(a)} = r_k}} \hat{g}_{\mathrm{ref}}(\{s_k^{(a)}\beta_k\}; y) \prod_l \left[ \prod_{b=1}^{n_l} \left\{ \frac{1}{s_l^{(b)}} \frac{y - y^{-1}}{y^{s_l^{(b)}} - y^{-s_l^{(b)}}} \Omega_{\mathrm{ref}}^S(\beta_l, y^{s_l^{(b)}}) \right\} \right]
$$

$$
+ I_{\mathrm{cor}}(\{\beta_k\}, \{r_k\}; y) ,
$$

(5.6)

where the function $I_{\mathrm{cor}}$ vanishes if all the $r_i$'s are 1, but can be non-zero if some of the $r_i$'s are larger than 1. The function $\hat{g}_{\mathrm{ref}}$ is defined as in (4.5) with $g_{\mathrm{ref}}$ including the corrections due to scaling solutions described in section 5.1. The need for the additional correction terms $I_{\mathrm{cor}}$ can be seen by noting that due to the presence of the $(y^{s_l^{(b)}} - y^{-s_l^{(b)}})$ factor in the denominator it is not guaranteed that the first term in (5.6) can be expressed as a Laurent polynomial in $y$. $I_{\mathrm{cor}}$ is adjusted to compensate for this. We choose

$$
I_{\mathrm{cor}}(\{\beta_k\}, \{r_k\}; y) = \sum_{\substack{\{n_k\},\{s_k^{(a)}\} \\ n_k, s_k^{(a)} \in \mathbb{Z}; n_k, s_k^{(a)} \geq 1, \\ \sum_{a=1}^{n_k} s_k^{(a)} = r_k}} h(\{\beta_k\}; \{s_k^{(a)}\}; y) \prod_l \prod_{b=1}^{n_l} \Omega_{\mathrm{ref}}^S(\beta_l, y^{s_l^{(b)}}) ,
$$

(5.7)

where $h(\{\beta_k\}; \{s_k^{(a)}\}; y)$ is chosen so that

1. It is invariant under $y \to y^{-1}$.

2. $\lim_{y \to \infty} h(\{\beta_k\}; \{s_k^{(a)}\}; y) = 0$.

3. Eq. (5.6) has an expansion of the form $\sum_{m \in \mathbb{Z}} a_m y^m$ with a finite number of terms whenever the $\Omega_{\mathrm{ref}}^S(\beta_i, y)$'s have this property.

Note that the second condition above is another manifestation of the 'minimal modification hypothesis'. These three requirements fix the function $h$ completely in any given situation. Thus this prescription gives a complete algorithm for computing the spectrum of multi-centered black holes given the collinear fixed point solutions to (2.24) satisfying the requirement (2.10).

The term in which all the $s_k^{(a)}$'s in the argument of $h$ are 1 would combine with the term where each $s_k^{(a)} = 1$ in the first term in (5.6). Since the latter terms do not have any

unwanted denominators and are automatically given by Laurent polynomials in $y$, there is no need for any correction terms. Thus $h(\beta_1, \beta_2, \cdots; \{s_k^{(a)}\}; y)$ vanishes if all the $s_k^{(a)}$'s are 1. This shows that if all the $r_i$'s are 1, ı.e. the $\alpha_i$'s are all different, then $I_{\text{cor}}$ vanishes.

## 5.3 Illustration

Since the above discussion has been somewhat abstract we shall now demonstrate this procedure by a hypothetical example (which will map on to a real example in section 6.5). Suppose we have a four centered solution with charges $\alpha_1$, $\alpha_2$, $\alpha_3 = r_1\beta$, $\alpha_4 = r_2\beta$ such that $\langle \alpha_1, \beta \rangle = -\langle \alpha_2, \beta \rangle$. Suppose further that in some region of the moduli space we find that the only collinear configurations involve the order $1, 2, 3, 4$ and its mirror $4, 3, 2, 1$ with $s(p) = \pm 1$. Then the net contribution to $g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y)$ from the collinear fixed points is

$$(-1)^{\alpha_{12}+3}(y - y^{-1})^{-3}(y^{\alpha_{12}} - y^{-\alpha_{12}}). \tag{5.8}$$

Let us suppose further that there are no solutions where a proper subset of the centers are in the scaling configurations, — the only additional contribution comes from the scaling solution where all the centers approach each other. We can determine this term according to the prescription (5.5). This gives the net contribution to $g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y)$ to be

$$\begin{aligned}
g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y) =& (-1)^{\alpha_{12}+3}(y - y^{-1})^{-3} \left\{ y^{\alpha_{12}} - y^{-\alpha_{12}} - \frac{1}{2}\alpha_{12}(y^2 - y^{-2}) \right\} \text{ for } \alpha_{12} \in 2\mathbb{Z}, \\
=& (-1)^{\alpha_{12}+3}(y - y^{-1})^{-3} \left\{ y^{\alpha_{12}} - y^{-\alpha_{12}} - \alpha_{12}(y - y^{-1}) \right\} \text{ for } \alpha_{12} \in 2\mathbb{Z}+1.
\end{aligned} \tag{5.9}$$

After taking out a factor of $y^{-\alpha_{12}}$ we see that the term inside $\{\ \}$ is a polynomial in $y^2$ and has triple zero at $y = 1$. Thus it must have a factor of $(1 - y^2)^3$ cancelling the $(1 - y^2)^3$ factor in the denominator, and the expressions for $g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y)$ given in (5.9) can be expressed as a Laurent polynomial in $y$.

As long as all $\alpha_i$'s are different this ends the discussion for the contribution from this four centered terms. Suppose however $r_1 = r_2 = r$ so that $\alpha_3 = \alpha_4$. In this case the net contribution to the index from this four centered configuration will be given by

$$\begin{aligned}
&\frac{1}{2} g_{\text{ref}}(\alpha_1, \alpha_2, r\beta, r\beta; y) \, \Omega^S_{\text{ref}}(\alpha_1; y) \, \Omega^S_{\text{ref}}(\alpha_2, y) \, \Omega^S_{\text{ref}}(r\beta, y)^2 \\
&+ \frac{1}{2} g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y) \frac{y - y^{-1}}{y^2 - y^{-2}} \, \Omega^S_{\text{ref}}(\alpha_1, y) \, \Omega^S_{\text{ref}}(\alpha_2, y) \, \Omega^S_{\text{ref}}(r\beta, y^2) \\
&+ h(\alpha_1, \alpha_2, r\beta; s_1^{(1)} = 1, s_2^{(1)} = 1, s_3^{(1)} = 2; y) \, \Omega^S_{\text{ref}}(\alpha_1, y) \, \Omega^S_{\text{ref}}(\alpha_2, y) \, \Omega^S_{\text{ref}}(r\beta, y^2).
\end{aligned} \tag{5.10}$$

The second term comes from the term $g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y)\Omega^S_{\text{ref}}(\alpha_1, y)\Omega^S_{\text{ref}}(\alpha_2, y)\bar{\Omega}^S_{\text{ref}}(2r\beta, y)$ and the third term is the correction term given in (5.6), (5.7). To proceed we need to know $g(\alpha_1, \alpha_2, 2r\beta; y)$. Suppose at the same point in the moduli space the collinear three centered configurations carrying charges $\alpha_1$, $\alpha_2$, $\alpha_3 = 2r\beta$ are of the form 123 and 321. In this case we have

$$\begin{aligned}
g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y) =& (-1)^{\alpha_{12}+2}(y - y^{-1})^{-2} \left\{ y^{\alpha_{12}} + y^{-\alpha_{12}} - 2 \right\} \text{ for } \alpha_{12} \in 2\mathbb{Z}, \\
=& (-1)^{\alpha_{12}+2}(y - y^{-1})^{-2} \left\{ y^{\alpha_{12}} + y^{-\alpha_{12}} - (y + y^{-1}) \right\} \text{ for } \alpha_{12} \in 2\mathbb{Z}+1.
\end{aligned} \tag{5.11}$$

Note that we have added the correction terms due to the scaling solutions according to (5.5). We can now substitute (5.9) and (5.11) into (5.10) and determine $h$ by requiring that the resulting expression is given by a Laurent polynomial in $y$ for any choice of $\Omega_{\text{ref}}^S$ satisfying similar properties. Now the first term clearly has this property. In the second term $g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y)$ has this property, but the factor of $(y - y^{-1})/(y^2 - y^{-2}) = y/(1+y^2)$ has a factor of $(1 + y^2)$ in the denominator which could potentially spoil this property unless the numerator has a factor of $(1 + y^2)$. In this case we need a non-vanishing $h$ to cancel the unwanted terms. To proceed we note that if $\alpha_{12}$ is odd ($\alpha_{12} = (2k + 1)$ with $k \in \mathbb{Z}$) then we have

$$\frac{1}{2} \frac{y}{1 + y^2} g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y) = -\frac{1}{2} y^{-2k+2}(1 + y^2)^{-1}(1 - y^2)^{-2}(1 - y^{2k})(1 - y^{2k+2}). \quad (5.12)$$

Each of the two factors $(1 - y^{2k})$ and $(1 - y^{2k+2})$ has a factor of $(1 - y^2)$ canceling the $(1 - y^2)^{-2}$ factor. Furthermore $(1 - y^{2k})$ for even $k$ and $(1 - y^{2k+2})$ for odd $k$ also has a factor of $(1 + y^2)$ that cancels the $(1 + y^2)^{-1}$. Thus in this case (5.10) gives a Laurent polynomial in $y$ without any $h$ term and we can set $h = 0$. On the other hand for even $\alpha_{12}$ ($\alpha_{12} = 2k$ with $k \in \mathbb{Z}$) we have

$$\frac{1}{2} \frac{y}{1 + y^2} g_{\text{ref}}(\alpha_1, \alpha_2, 2r\beta; y) = \frac{1}{2} y^{-2k+3}(1 + y^2)^{-1}(1 - y^2)^{-2}(1 - y^{2k})^2. \quad (5.13)$$

$(1 - y^{2k})^2$ has two factors of $(1 - y^2)$ cancelling the $(1 - y^2)^{-2}$ factor. Furthermore for even $k$ it also has two factors of $(1 + y^2)$ killing the factor of $(1 + y^2)^{-1}$. Thus again in this case there is no need for any correction term and we can set $h$ to 0. Finally for $k$ odd we can express the right hand side of (5.13) as

$$\frac{1}{2} y^{-2k+3}(1 + y^2)^{-1}\{1 + y^2 + \cdots y^{2k-2}\}^2. \quad (5.14)$$

Now as $y^2 \to -1$ this term approaches $\frac{1}{2} y(1 + y^2)^{-1}$. Thus from (5.10) we see that the unwanted terms may be cancelled by choosing $h$ to be negative of this term. This gives

$$\begin{aligned} h(\alpha_1, \alpha_2, r\beta; s_1^{(1)} = 1, s_2^{(1)} = 1, s_3^{(1)} = 2; y) &= 0 \quad \text{for} \quad \alpha_{12} \in 2\mathbb{Z} + 1 \\ &= 0 \quad \text{for} \quad \alpha_{12} \in 4\mathbb{Z} \\ &= -\frac{1}{2}(y + y^{-1})^{-1} \quad \text{for} \quad \alpha_{12} \in 4\mathbb{Z} + 2. \end{aligned} \quad (5.15)$$

In section 6.5 we shall realize this example in the case of dipole halo configurations.

## 5.4 Wall crossing re-examined

Given the modifications due to the scaling solutions, we need to re-examine the analysis in section 4.1 on the compatibility of our prescription with the wall crossing formula. Rather than doing a detailed analysis, we note that our prescription for including contributions from scaling solutions affects the factors $g_{\text{ref}}(\{\alpha_i, i \in A_m\}; y) \prod_{i \in A_m} \bar{\Omega}_{\text{ref}}^S(\alpha_i, y)$ in the square bracket of (4.18), as each set $A_m$ may allow for scaling solutions. Thus, the index associated with individual black hole molecules will change. However, the factor

$g_{\text{ref}}(\{\gamma_m\}; y)$ in front, which determines the jump of the index under wall-crossing, will not be modified since the elements belonging to different sets $A_m$ always remain separated. Thus, the contributions of scaling solutions do not affect the consistency with the wall-crossing formula.

## 6 Dipole halo configurations

In this section, we verify our prescriptions for a class of multi-centered configurations whose phase space and quantization is completely understood, namely dipole halo configurations [16, 29]. These consists of $n$-centered configurations with two distinguished centers carrying charges $\alpha_1$, $\alpha_2$, with $\alpha_{12} \neq 0$, and $n - 2$ centers carrying mutually local charges $\alpha_a$, such that $\alpha_{1a} = -\alpha_{2a}, \alpha_{ab} = 0$. Such configurations were analyzed in detail in [16, 29], whose notations we follow. A particular realization of this system, which we shall use to frame our discussion, is given by a $D6 - \overline{D6}$ pair with $n - 2$ $D0$-branes orbiting around it. After reviewing the main features of such configurations in section 6.1, we describe the 3-centered case in detail in section 6.2 (an example of which was already analyzed in section 3.3.1), and present a general proof of the minimal modification hypothesis in section 6.3, based on recursion relations for the equivariant index. Similar recursion relations for the equivariant volume are presented in section 6.4. Explicit results for 4 and 5 centers can be found in appendix C. In section 6.5 we analyze four centered configurations with two identical centers.

### 6.1 Generalities

We shall consider a system of $n$-centers, the first one of which represents a $D6$ brane with certain U(1) flux, the second one describes a $\overline{D6}$-brane with opposite U(1) flux and the third one onwards corresponds to $D0$-branes carrying $q_3, \ldots, q_n$ units of $D0$-brane charges. Thus the first and the second centers carry opposite $D6$ and $D2$-brane charges but the same $D4$ and $D0$-brane charges. In the language used in section 3.3.1 we have:

$$
\begin{aligned}
\tilde{\alpha}_1 &= (1, -U, -\kappa U^2/2 - \mathcal{B}/24, -\kappa U^3/6 - \mathcal{B}U/24), \\
\tilde{\alpha}_2 &= (-1, -U, \kappa U^2/2 + \mathcal{B}/24, -\kappa U^3/6 - \mathcal{B}U/24), \\
\tilde{\alpha}_a &= (0, 0, 0, q_a),
\end{aligned}
\tag{6.1}
$$

leading to

$$
\alpha_{12} = I \equiv 4\kappa U^3/3 + \mathcal{B}U/6, \quad \alpha_{1a} = -q_a, \quad \alpha_{2a} = q_a, \quad \alpha_{ab} = 0, \quad 3 \leq a, b \leq n, \tag{6.2}
$$

where we assume that $I, q_a$ are positive integers. The system described in (3.20), (3.21) is a special case of this with $n = 3$. We work in the chamber of the moduli space where

$$
c_1 = \mu, \qquad c_2 = -\mu - \eta \sum_{a=3,\ldots,n} q_a, \qquad c_a = \eta \, q_a \quad \text{for } a = 3, \ldots, n, \tag{6.3}
$$

where $\mu$ is a positive constant and $\eta$ is a small positive or negative number. As described at the end of section 3.3.1, the $\eta = 0$ subspace describes a threshold stability wall on which

a subset of the $D0$-branes can get infinitely separated from the rest of the system. But the index does not jump across this wall, since the symplectic product of the charge vector of the expelled $D0$-brane charges with the total charge vector of the $D6-\overline{D6}-D0$ system vanishes.

On the Coulomb branch, the system is described by multi-centered configurations satisfying the equilibrium conditions

$$\frac{1}{r_{1a}} - \frac{1}{r_{2a}} = \eta \,, \qquad -\frac{I}{r_{12}} + \sum_{a=3,\ldots,n} \frac{q_a}{r_{1a}} = -\mu \,. \tag{6.4}$$

In particular, at $\eta = 0$ the $D0$-branes lie either at infinity or on the plane equidistant to the $D6$ and $\overline{D6}$ branes. As shown in [16, 29], for this type of dipole halo configuration the phase space of solutions to (6.4) at $\eta = 0$ is toric and given by a $T^{n-1}$ bundle over the polytope

$$\mathcal{P}(I, \{q_a\}) = \{(m_a, m) \ : \ 0 \le m_a \le q_a \,, -j \le m \le j \,, j > 0\} \quad \subset \mathbb{R}^{n-1} \,, \tag{6.5}$$

where

$$j \equiv I/2 - \sum_{a=3,\ldots,n} m_a \tag{6.6}$$

is a linear function of the $n-2$ variables $m_a$. These variables parametrize the angle $\theta_a$ between $\vec{r}_{1a}$ and $\vec{r}_{12}$ via $m_a = q_a \cos \theta_a$, while $m$ parametrizes the angle $\theta$ between $\vec{r}_{12}$ and the $z$-axis via $m = j \cos \theta$. Physically $m$ represents the component of the angular momentum along $z$-axis. The coordinates $(m, m_a)$ together with coordinates $(\phi, \phi_a) \in [0, 2\pi]^{n-1}$ along the torus fiber provide a set of Darboux coordinates on $\mathcal{M}_n$,

$$\omega = -\mathrm{d}m \wedge \mathrm{d}\phi - \sum_{a=3,\ldots,n} \mathrm{d}m_a \wedge \mathrm{d}\phi_a \,. \tag{6.7}$$

Denoting $\phi_a = \tilde{\phi}_a - \sigma$ where $\sum_a \tilde{\phi}_a = 0$ and using (6.6), it is straightforward to check that (6.7) agrees with (2.14) with $\tilde{\omega} = \sum_a q_a \sin \theta_a \, \mathrm{d}\theta_a \wedge \mathrm{d}\tilde{\phi}_a$. The equivariant volume (2.15) can be rewritten as

$$g_{\mathrm{classical}}(\{\alpha_i\}; y) = (-1)^{I-n+1} \int_{\substack{0 \le m_a \le q_a \\ \sum_a m_a \le I/2}} \mathrm{d}m_3 \cdots \mathrm{d}m_n \int_{-\frac{I}{2}+\sum_a m_a}^{\frac{I}{2}-\sum_a m_a} \mathrm{d}m \, e^{2\nu m} \,, \tag{6.8}$$

$$= (-1)^{I-n+1} \int_{\substack{0 \le m_a \le q_a \\ \sum_a m_a \le I/2}} \mathrm{d}m_3 \cdots \mathrm{d}m_n \frac{\sinh[(I - 2\sum_{a=3}^n m_a)\nu]}{\nu} \,, \quad \nu \equiv \ln y \,.$$

For brevity, we shall denote the r.h.s. of (6.8) also by $S(I; \{q_a\}_{a=3,\ldots,n}; \nu)$. Moreover, we let $S(I; \{q_a\}_{a=3,\ldots,n}; \nu) = 0$ whenever $I < 0$.

For $\sum_a q_a < I/2$, the upper bound in $\sum m_a \le I/2$ is never attained, and the phase space is compact. The equivariant volume can therefore be evaluated rigorously using localization with respect to $J_3$. For $\eta > 0$, the fixed points of $J_3$ are collinear solutions to (6.4), where a (possibly empty) subset $A \subset \{3, \ldots, n\}$ of the $D0$-branes lie on the segment between the $D6$ and $\overline{D6}$ along the $z$ axis, while the $D0$-branes in the complement $B = \{3, \ldots, n\} \backslash A$ lie on the semi-infinite $z$-axis extending from the $D6$-brane to infinity.

In the limit $\eta \to 0$, the centers in $A$ coalesce at the mid-point between the $D6$ branes, while the centers in $B$ run off to infinity. The distance between the $D6$ and $\overline{D6}$ is given by $r_{12} = (I - 2\sum_{a \in A} q_a)/\mu$, which is positive for any subset $A$. The angular momentum carried by this configuration is $J_3 = \left(I - 2\sum_{a \in A} q_a\right)/2$. In appendix A.2 we show that the sign $s(p)$ evaluates to $(-1)^{n_A}$, where $n_A$ is the cardinality of the subset $A$ (the order among the centers inside the clusters $A$ or $B$ is irrelevant since they carry mutually local charges). Thus, the classical phase space integral (6.8) evaluates to

$$S(I; \{q_a\}_{a=3,\dots,n}; \nu) = (2\ln y)^{-n+1} \sum_A (-1)^{n_A + I - n + 1} \left( y^{I - 2\sum_{a \in A} q_a} + (-1)^{n-1} y^{-I + 2\sum_{a \in A} q_a} \right). \tag{6.9}$$

This can of course also be obtained by direct evaluation of (6.8). Since the fixed points are isolated, according to the discussion in section 2.5, the exact refined index is obtained by replacing $2\ln y$ by $y - y^{-1}$, leading to

$$g_{\text{ref}}(\{\alpha_i\}; y) = (-1)^{I + n - 1} (y - y^{-1})^{-n+1} \sum_A (-1)^{n_A} \left( y^{I - 2\sum_{a \in A} q_a} + (-1)^{n-1} y^{-I + 2\sum_{a \in A} q_a} \right). \tag{6.10}$$

In contrast, when $I/2 \le \sum_a q_a$, the phase space $\mathcal{M}_n$ is non-compact, as it has a scaling region represented by the boundary $\sum_a m_a = I/2$ on which $j$ vanishes, and all centers approach each other at arbitrarily small distances.[21] Such configurations are therefore invariant under SO(3). To compactify $\mathcal{M}_n$, we include the boundary $j = 0$ i.e. supplement the open polytope $\mathcal{P}$ with the lower-dimensional polytope

$$\mathcal{Q}(I, \{q_a\}) = \left\{ (m_a, m) \ : \ 0 \le m_a \le q_a, \quad \sum_{a=3}^{n} m_a = I/2, \quad m = 0 \right\}. \tag{6.11}$$

Denoting by $\mathcal{M}_n^{\text{scal}}$ the $(2n-4)$-dimensional compact toric manifold built over the polytope $\mathcal{Q}$, with torus fiber parametrized by $\tilde{\phi}_a$, we define the compactification of $\mathcal{M}_n$ as

$$\hat{\mathcal{M}}_n = \mathcal{M}_n \cup \mathcal{M}_n^{\text{scal}}, \tag{6.12}$$

and require that SO(3) acts trivially on the boundary $\mathcal{M}_n^{\text{scal}}$.

The equivariant volume of $\hat{\mathcal{M}}_n$ can in principle be evaluated by localization. One class of fixed points is given by the same type of collinear configurations as described above (6.9), with the proviso that the subset $A$ must be chosen such that $r_{12} > 0$, i.e.

$$\sum_{a \in A} q_a < \frac{I}{2}. \tag{6.13}$$

The contribution of these isolated fixed points takes the same form as in (6.9), with the restriction (6.13) enforced. We denote this contribution by $S_{\text{coll}}(I; \{q_a\}_{a=3,\dots,n}; \nu)$. In addition to these isolated fixed points, already present in $\mathcal{M}_n$, one also expects a contribution

---

[21]In this example, there are no scaling configurations where only a subset of the centers scale together, since the total charge carried by such a subset would be mutually local with respect to the remaining, non-scaling D0-brane centers.

from the submanifold $\mathcal{M}_n^{\text{scal}}$ inside $\hat{\mathcal{M}}_n$. Although this contribution can in principle be computed using a generalization of the Duistermaat-Heckman formula [38], we shall not attempt to compute it directly, and instead define

$$S(I; \{q_a\}_{a=3,\ldots,n}; \nu) = S_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu) + \Delta S(I; \{q_a\}_{a=3,\ldots,n}; \nu) , \qquad (6.14)$$

where $\Delta S(I; \{q_a\}_{a=3,\ldots,n}; \nu)$ is the contribution of $\mathcal{M}_n^{\text{scal}}$. By a direct computation of the equivariant volume using recursion relations in section 6.4, we shall be able to read off $\Delta S(I; \{q_a\}_{a=3,\ldots,n}; \nu)$. Similarly, the exact refined index $g_{\text{ref}}(\{\alpha_i\}; y)$, which we also denote by $\hat{S}(I; \{q_a\}; \nu)$, decomposes as

$$\hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \hat{S}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu) + \Delta \hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) , \qquad (6.15)$$

where $S_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu)$ denotes the r.h.s. of (6.10), with restriction (6.13) enforced on the subset $A$, and $\Delta \hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu)$ denotes the contribution of the fixed submanifold $\mathcal{M}_n^{\text{scal}}$. As explained in section 5, the minimal modification hypothesis determines the correction $\Delta \hat{S}$ uniquely by requiring that $\hat{S}$ is a SU(2) character and $\Delta \hat{S} \to 0$ as $\nu \to \infty$.

On the other hand, since the phase space of $n$-centered solutions is a toric Kähler manifold, it can also be quantized exactly. We refer to [16, 29] for the details of this procedure in the dipole halo case, and merely quote the result. When all the $q_a$'s are distinct, so that the $D0$-branes are distinguishable, the exact index is given by

$$g_{\text{ref}}(\{\alpha_i\}; y) = (-1)^{I-n+1} \sum_{\substack{m_3,\ldots,m_n \\ m_a \in \mathbb{Z}, 0 \leq m_a \leq q_a-1, \\ \sum_a m_a \leq \lfloor (I-n+1)/2 \rfloor}} \sum_{m=-\frac{1}{2}(I-n+1-2\sum_a m_a)}^{\frac{1}{2}(I-n+1-2\sum_a m_a)} y^{2m} , \qquad (6.16)$$

where $\lfloor x \rfloor$ denotes the largest integer smaller or equal to $x$, and the sum over $m \in \frac{1}{2}\mathbb{Z}$ is such that[22] $m - \frac{1}{2}(I - n + 1 - 2\sum_a m_a) \in \mathbb{Z}$. As already mentioned above (6.15), we denote by $\hat{S}(I; \{q_a\}; \nu)$ the r.h.s. of (6.16), and let $\hat{S}(I; \{q_a\}; \nu) = 0$ whenever $I < n - 1$. Performing the geometric sum over $m$, we arrive at a sum of SU(2) characters

$$\hat{S}(I; \{q_a\}; \nu) = (-1)^{I-n+1} \sum_{\substack{m_3,\ldots,m_n \\ m_a \in \mathbb{Z}, 0 \leq m_a \leq q_a-1, \\ \sum_a m_a \leq \lfloor (I-n+1)/2 \rfloor}} \frac{\sinh\left[(I - n + 2 - 2\sum_{a=3}^{n} m_a)\nu\right]}{\sinh \nu} . \qquad (6.17)$$

If some of the $q_a$'s coincide, and if all $\Omega(\alpha_i)$'s are set to one, the index is still given by (6.16), (6.17), with the additional restriction that the $m_a$'s corresponding to identical particles must be distinct, and the expression must be divided by a symmetry factor $k!$ for every set of $k$ identical $q_a$'s.

In section 6.3, we shall give an inductive proof of the minimal modification prescription described below (6.15) in this class of dipole halo configurations, by establishing recursion relations for the exact index (6.17) and comparing them with the recursion relations obeyed

---

[22]This condition was not stated explicitly in [16, 29] but is necessary for the consistency of the SU(2) action.

by its regular part $\hat{S}_{\text{coll}}(I; \{q_a\}; \nu)$. Using similar methods, in section 6.4 we shall also demonstrate a variant of this minimal modification prescription, which allows to recover the exact equivariant index $\hat{S}(I; \{q_a\}; \nu)$ from the knowledge of the equivariant volume $S(I; \{q_a\}; \nu)$. This prescription goes as follows:

1. In the expression for $S(I; \{q_a\}; \nu)$, replace all $\nu$'s which do not appear as arguments of hyperbolic functions, by $\sinh \nu$. Let us denote the result by $\tilde{S}(I; \{q_a\}; \nu)$.

2. If $\tilde{S}(I; \{q_a\}; \nu)$ can be expressed as Laurent polynomial in $y = e^\nu$, then this is the exact result for $\hat{S}(I; \{q_a\}; \nu)$. Otherwise we add terms which vanish as $\nu \to \pm\infty$ to make the expression into a Laurent polynomial in $y$. This gives the exact result for $\hat{S}(I; \{q_a\}; \nu)$.

Before going to the general proof of these statements, we shall illustrate them in the case of $n = 3$ centers in section 6.2. Explicit computations for $n = 4$ and 5 centers can be found in appendix C. In section 6.5 we shall study four centered configurations with two identical centers and use the minimal modification prescription of section 5.2 to compute the index. This is then compared with the known exact results.

## 6.2 Three-centered solutions

For $n = 3$ we have to consider two cases separately.

### 6.2.1 Non-scaling case: $q_3 < \frac{I}{2}$

For $q_3 < \frac{I}{2}$, the sum over collinear configurations (6.9) reduces to

$$\hat{S}_{\text{coll}}(I; q_3; y) = \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I - y^{I-2q_3} + y^{-I} - y^{-I+2q_3} \right]. \qquad (6.18)$$

This can be expressed as a Laurent polynomial in $y$, and indeed agrees with the exact refined index computed from (6.16).

Alternatively we can begin with the equivariant volume (6.8). This gives

$$S(I; q_3; y) = (-1)^I \int_0^{q_3} \mathrm{d}m_3 \, \frac{\sinh[(I - 2m_3)\nu]}{\nu} = (-1)^I \frac{\cosh(I\nu) - \cosh[(I - 2q_3)\nu]}{2\nu^2}. \qquad (6.19)$$

After the replacement $\nu \to \sinh \nu$ in the denominator, we get

$$\tilde{S}(I; q_3; y) = (-1)^I \frac{\cosh(I\nu) - \cosh[(I - 2q_3)\nu]}{2 \sinh^2 \nu}. \qquad (6.20)$$

It is easy to see that this agrees with (6.18) and hence can be expressed as a Laurent polynomial in $y$. Thus in this case there is no need to add any correction terms and $\tilde{S}(I; q_3; y)$ is the same as $\hat{S}(I; q_3; y)$. In particular, the symplectic volume $S(I; q_3; y = 1)$ agrees with the exact index $\hat{S}(I; q_3; y = 1)$ in the non-scaling regime.

### 6.2.2 Scaling case: $q_3 \geq \frac{I}{2}$

In this case (6.9), with the restriction (6.13) on the set $A$, reduces to

$$\hat{S}_{\text{coll}}(I; q_3; y) = \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I + y^{-I} \right].$$

(6.21)

This diverges as $y \to 1$ and hence we must add corrections described in (5.5). This indeed reproduces the exact index computed from (6.16),

$$\hat{S}(I; q_3; y) = \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I + y^{-I} - (y + y^{-1}) \right], \qquad \text{for } I \text{ odd}$$

$$= \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I + y^{-I} - 2 \right], \qquad \text{for } I \text{ even}.$$

(6.22)

On the other hand the result of the equivariant volume is insensitive to the parity of $I$, and gives

$$S(I; q_3; y) = (-1)^I \int_0^{I/2} dm_3 \, \frac{\sinh[(I - 2m_3)\nu]}{\nu} = (-1)^I \frac{\cosh(I\nu) - 1}{2\nu^2}.$$

(6.23)

The last term in the numerator is recognized as the contribution of the scaling fixed point $\mathcal{Q} = \{m_3 = I/2, m = 0\}$. Replacing $\nu \to \sinh\nu$ in the denominator we arrive at

$$\tilde{S}(I; q_3; y) = (-1)^I \frac{\cosh(I\nu) - 1}{2\sinh^2\nu}.$$

(6.24)

For $I$ even, this agrees with the exact result (6.22). On the other hand if $I$ odd, $\tilde{S}$ differs from the exact result $\hat{S}$ by

$$\hat{S}(I; q_3; y) = \left[ \tilde{S}(I; q_3; y) - (-1)^I \frac{\cosh\nu - 1}{2\sinh^2\nu} \right] \qquad \text{for } I \text{ odd}.$$

(6.25)

Note that the correction term vanishes as $\nu \to \pm\infty$. Thus, had we had started with the expression for $\tilde{S}$ and added corrections following the prescription given below (6.17), we would have arrived at the correct expression for $\hat{S}$. It is also worth noting that when $I$ is odd, the exact index $\hat{S}(I; q_3; y = 1)$ differs from the symplectic volume $S(I; q_3; y = 1)$ by a fraction $(-1)^I/4$, which is necessary to make the result integer. Thus the non-renormalization property which was observed in the non-scaling case breaks down in this case. Finally, we note that we have not discussed the effect of the regularity conditions (3.8). This was discussed in a special case in section 3.3.1, and we expect this to be satisfied also for all the solutions described in this section.

### 6.3 An inductive proof of the minimal modification hypothesis

In this section, we shall prove by induction that $\Delta\hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu)$ appearing in (6.15), giving the difference between the exact index (6.16) and the contribution $S_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu)$ of regular collinear configurations given in (6.10), vanishes as $\nu \to \infty$. This proves the validity of the minimal modification hypothesis for all $n$ for the specific system under consideration.

First, we establish recursion relations for the equivariant index $\hat{S}$, defined in (6.17). For this purpose it is useful to introduce a variant of $\hat{S}$ defined by

$$\hat{C}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = (-1)^{I+n-1} \sum_{\substack{m_3,\ldots,m_n \\ m_a \in \mathbb{Z}, 0 \leq m_a \leq q_a - 1 \\ \sum_a m_a \leq \lfloor (I-n+1)/2 \rfloor}} \frac{\cosh\left[(I-n+2-2\sum_{a=3,\ldots,n} m_a)\nu\right]}{\sinh \nu} \,, \quad (6.26)$$

and another quantity

$$\hat{E}(I; \{q_a\}_{a=3,\ldots n}) = (-1)^{I+n-1} \sum_{\substack{m_3,\ldots,m_n \\ m_a \in \mathbb{Z}, 0 \leq m_a \leq q_a - 1, \\ \sum_a m_a = \lfloor (I-n+1)/2 \rfloor}} 1 \,. \quad (6.27)$$

This is recognized as the number of integer points closest to the polytope $\mathcal{Q}$ in (6.11). We also define $\hat{S}$, $\hat{C}$ and $\hat{E}$ to be zero for $I < (n-1)$. For $n = 2$, i.e. in the absence of $D0$-branes, we let

$$\hat{S}(I, \nu) = (-1)^{I-1} \frac{\sinh(I\nu)}{\sinh \nu}, \quad \hat{C}(I, \nu) = (-1)^{I-1} \frac{\cosh(I\nu)}{\sinh \nu} \,, \quad (6.28)$$

if $I \geq 0$, and zero otherwise. The quantity $\hat{E}(I; \{q_a\})$ can be evaluated from (6.26) using

$$\hat{E}(I; \{q_a\}_{a=3,\ldots n}) = \lim_{\nu \to 0} \left[ \nu \left( \hat{C}(I-1-2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \hat{C}(I-1; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right) \right] \,, \quad (6.29)$$

valid for $n \geq 3$.

Performing the sum (6.17) over the last charge $m_n$ first, we see that the sum is empty if $\sum_{a=3}^{n-1} m_a > \lfloor (I-n+1)/2 \rfloor$, or else runs from 0 to $q-1$ where $q = \min(q_n, \lfloor (I-n+1)/2 \rfloor - \sum_{a=3}^{n-1} m_a + 1)$. Using the geometric sum identities

$$\sum_{m=0}^{q-1} \sinh(A + 2Bm) = \frac{\cosh[A + (2q-1)B] - \cosh(A-B)}{2 \sinh B} \quad (6.30)$$

$$\sum_{m=0}^{q-1} \cosh(A + 2Bm) = \frac{\sinh[A + (2q-1)B] - \sinh(A-B)}{2 \sinh B} \quad (6.31)$$

with $A = (I - n + 2 - 2\sum_{a=3}^{n-1} m_a)\nu$ and $B = -\nu$, we find, for any $n \geq 3$,

$$\hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \hat{C}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \hat{C}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]$$
$$- \frac{\cosh^2 \frac{\nu}{2} + (-1)^{I-n} \sinh^2 \frac{\nu}{2}}{2 \sinh^2 \nu} \hat{E}(I; \{q_a\}_{a=3,\ldots,n})$$

$$\hat{C}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \hat{S}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \hat{S}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]$$
$$- \frac{1 + (-1)^{I-n}}{4 \sinh \nu} \hat{E}(I; \{q_a\}_{a=3,\ldots,n}) \,. \quad (6.32)$$

Similarly, we can derive a recursion relation for the contributions of the regular collinear solutions to the refined index. Given a collinear configuration contributing to the $(n-1)$

particle system with the $D0$-branes carrying charges $q_3, \ldots, q_{n-1}$, we can construct a collinear configuration contributing to the $n$ particle system with an additional $D0$-brane with charge $q_n$ as follows. For simplicity we shall work in the $\eta \to 0$ limit. First of all the additional $D0$-brane can always be placed at infinity for any collinear configuration of the original $(n-1)$ particle system. Also, for any collinear configuration of the original system, if the set $A$ containing the $D0$-branes at the midpoint between $D6$ and $\overline{D6}$ brane satisfies $\sum_{a \in A} q_a < (I - 2q_n)/2$, we can add the $n$-th $D0$-brane at the midpoint. Taking into account the various signs appearing in (6.10) we get the recursion relations, for any $n \geq 3$,

$$
\hat{S}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2\sinh\nu} \left[ \hat{C}_{\text{coll}}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \hat{C}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]
$$
$$
\hat{C}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2\sinh\nu} \left[ \hat{S}_{\text{coll}}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \hat{S}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]
$$
(6.33)

where $\hat{C}_{\text{coll}}$ is defined with an opposite sign compared to $\hat{S}_{\text{coll}}$ in (6.10),

$$
\hat{C}_{\text{coll}}(I; \{q_a\}_{a=3,\ldots,n}; \nu) \equiv (y - y^{-1})^{-n+1} \sum_A (-1)^{n_A + I + n - 1}
$$
$$
\times \left( y^{I - 2\sum_{a \in A} q_a} - (-1)^{n-1} y^{-I + 2\sum_{a \in A} q_a} \right) . \quad (6.34)
$$

Subtracting (6.33) from (6.32), we arrive at

$$
\Delta\hat{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2\sinh\nu} \left[ \Delta\hat{C}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \Delta\hat{C}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]
$$
$$
- \frac{\cosh^2 \frac{\nu}{2} + (-1)^{I-n} \sinh^2 \frac{\nu}{2}}{2\sinh^2\nu} \hat{E}(I; \{q_a\}_{a=3,\ldots,n})
$$
$$
\Delta\hat{C}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2\sinh\nu} \left[ \Delta\hat{S}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \Delta\hat{S}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]
$$
$$
- \frac{1 + (-1)^{I-n}}{4\sinh\nu} \hat{E}(I; \{q_a\}_{a=3,\ldots,n}) \quad (6.35)
$$

Assuming that $\Delta\hat{S}, \Delta\hat{C} \to 0$ as $y \to \infty$ for $n - 1$ particles, it immediately follows from (6.35) that the same statement holds for $n$ particles. The validity of the assumption at $n = 3$ is easily checked using the explicit results in the previous subsection. Thus, the minimal modification prescription is proved for this class of multi-centered configurations. It would be interesting to compute $\Delta\hat{S}$ directly by using the formula (2.35).

## 6.4 Recursion relations for the equivariant volume

In this section we shall derive recursion relations for the equivariant volume (6.8) similar to the ones given in section 6.3 , and then use them to prove the prescription given below (6.17). We first define a variant of the equivariant volume (6.8),

$$
C(I; \{m_a\}_{a=3,\ldots,n}; \nu) = (-1)^{I+n-1} \int_{\substack{0 \leq m_a \leq q_a, \\ \sum_a m_a \leq I/2}} dm_3 \cdots dm_n \frac{\cosh[(I - 2\sum_{a=3}^n m_a)\nu]}{\nu} \quad (6.36)
$$

and

$$E(I; \{q_a\}_{a=3,\dots,n}) = (-1)^{I+n-1} \int_{0 \leq m_a \leq q_a} dm_3 \cdots dm_n \, \delta\Big(\sum_{a=3}^{n} m_a - I/2\Big) \,. \qquad (6.37)$$

This last expression is recognized as the symplectic volume of the submanifold of fixed points based over the polytope $\mathcal{Q}$. For $n = 2$, i.e. in the absence of $D0$-branes, we set

$$S(I,\nu) = (-1)^{I-1}\frac{\sinh(I\nu)}{\nu}\,, \qquad C(I,\nu) = (-1)^{I-1}\frac{\cosh(I\nu)}{\nu} \qquad (6.38)$$

if $I \geq 0$, and zero otherwise. $E$ can be evaluated in terms of (6.36) using

$$E(I; \{q_a\}_{a=3,\dots,n}) = \lim_{\nu \to 0} \left[\nu\left(C(I-2q_n; \{q_a\}_{a=3,\dots,n-1}; \nu) - C(I; \{q_a\}_{a=3,\dots,n-1}; \nu)\right)\right], \quad (6.39)$$

or directly from its Fourier representation,

$$E(I; \{q_a\}_{a=3,\dots,n}) = (-1)^{I+n-1} \int_{\mathbb{R}} \frac{dt}{2\pi t^{n-2}} e^{-i(I - \sum_{a=3,\dots,n} q_a)t/2} \prod_{a=3}^{n} 2\sin\frac{q_a}{2}t \,. \qquad (6.40)$$

In particular, it is a piecewise polynomial in $I, q_a$.

By integrating first with respect to the last charge $m_n$ in (6.8), (6.36), it is straightforward to establish the following relations, valid for $n \geq 3$:

$$S(I; \{q_a\}_{a=3,\dots,n}; \nu) = \frac{1}{2\nu}\left[C(I-2q_n; \{q_a\}_{a=3,\dots,n-1}; \nu) - C(I; \{q_a\}_{a=3,\dots,n-1}; \nu)\right]$$
$$- \frac{1}{2\nu^2} E(I; \{q_a\}_{a=3,\dots,n}) \qquad (6.41)$$

$$C(I; \{q_a\}_{a=3,\dots,n}; \nu) = \frac{1}{2\nu}\left[S(I-2q_n; \{q_a\}_{a=3,\dots,n-1}; \nu) - S(I; \{q_a\}_{a=3,\dots,n-1}; \nu)\right] \quad (6.42)$$

Similarly as in (6.33), the contribution to $S(I; \{q_a\})$ from the regular, collinear fixed points satisfies

$$S_{\mathrm{coll}}(I; \{q_a\}_{a=3,\dots,n}; \nu) = \frac{1}{2\nu}\left[C_{\mathrm{coll}}(I-2q_n; \{q_a\}_{a=3,\dots,n-1}; \nu) - C_{\mathrm{coll}}(I; \{q_a\}_{a=3,\dots,n-1}; \nu)\right]$$
$$C_{\mathrm{coll}}(I; \{q_a\}_{a=3,\dots,n}; \nu) = \frac{1}{2\nu}\left[S_{\mathrm{coll}}(I-2q_n; \{q_a\}_{a=3,\dots,n-1}; \nu) - S_{\mathrm{coll}}(I; \{q_a\}_{a=3,\dots,n-1}; \nu)\right]$$
$$(6.43)$$

where $C_{\mathrm{coll}}$ is defined with an opposite sign compared to $S_{\mathrm{coll}}$ in (6.9). Applying (6.41), (6.43) recursively, the difference $\Delta S = S - S_{\mathrm{coll}}$ evaluates to

$$\Delta S(I; \{q_a\}_{a=3,\dots,n}; \nu) = 2 \sum_{\substack{m=3 \\ n-m\in 2\mathbb{Z}}}^{n} \sum_{B} \frac{(-1)^{n_B+1}}{(2\nu)^{n-m+2}} E\Big(I - 2\sum_{b\in B} q_b; \{q_a\}_{a=3,\dots,m}\Big) \qquad (6.44)$$

where for given $m$, the sum runs over subsets $B \subset \{m+1,\dots n\}$ subject to the restriction that $\sum_{b\in B} q_b < I/2$. This difference can be understood as the contribution of the fixed submanifold $\mathcal{M}_n^{\mathrm{scal}}$ built over the polytope $\mathcal{Q}$. In particular, the first term with $m = n$ in the sum (6.44) is just the symplectic volume $E(I, \{q_a\}_{a=3,\dots,n})$ of $\mathcal{M}_n^{\mathrm{scal}}$, rescaled by a

factor $-1/(2\nu^2)$. The remaining terms in (6.44) should originate from the Euler class of the normal bundle to $\mathcal{M}_n^{\text{scal}}$ which determines the integration mesure on the fixed submanifold, as in (2.35). We give evidence for this claim in two specific examples with $n = 4$ and $n = 5$ in section C.3.

We now turn to the proof of the prescription given below (6.17) for obtaining the exact equivariant index from the equivariant volume. Recall that $\tilde{S}$ was defined as the result of replacing in $S$ all the $\nu$'s outside the argument of hyperbolic functions by $\sinh \nu$. We define $\tilde{C}$ as the result of a similar replacement in $C$. Then the recursion relations for $\tilde{S}$ and $\tilde{C}$ can be obtained by modifying (6.41), (6.42) to:

$$\tilde{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \tilde{C}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \tilde{C}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right],$$
$$- \frac{1}{2 \sinh^2 \nu} E(I; \{q_a\}_{a=3,\ldots,n}) \tag{6.45}$$
$$\tilde{C}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \tilde{S}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \tilde{S}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right].$$

Subtracting (6.33) from (6.45), and defining $\Delta \tilde{S} = \tilde{S} - \hat{S}_{\text{coll}}$, $\Delta \tilde{C} = \tilde{C} - \hat{C}_{\text{coll}}$, we arrive at

$$\Delta \tilde{S}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \Delta \tilde{C}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \Delta \tilde{C}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]$$
$$- \frac{1}{2 \sinh^2 \nu} E(I; \{q_a\}_{a=3,\ldots,n}) \tag{6.46}$$
$$\Delta \tilde{C}(I; \{q_a\}_{a=3,\ldots,n}; \nu) = \frac{1}{2 \sinh \nu} \left[ \Delta \tilde{S}(I - 2q_n; \{q_a\}_{a=3,\ldots,n-1}; \nu) - \Delta \tilde{S}(I; \{q_a\}_{a=3,\ldots,n-1}; \nu) \right]$$

Assuming that $\Delta \tilde{S}, \Delta \tilde{C} \to 0$ as $\nu \to \infty$ for $n-1$ particles, it immediately follows from (6.46) that the same statement holds for $n$ particles. The validity of the assumption is easily checked for $n = 3$. On the other hand we have already seen in section 6.3 that $\hat{S} - \hat{S}_{\text{coll}}$ and $\hat{C} - \hat{C}_{\text{coll}}$ vanishes as $\nu \to \infty$. This shows that $\hat{S} - \tilde{S}$ and $\hat{C} - \tilde{C}$ vanishes as $\nu \to \infty$, thereby confirming the prescription described below (6.17).

## 6.5 Four-centered case with two identical centers

So far we have only considered the cases where the centers carry distinct charges. For the dipole halo configuration analyzed here, the first case of identical charges arise for four centered configurations in which $q_3$ and $q_4$ coincide. We shall now examine this case and compare the results obtained using the minimal modification hypothesis with the exact results. In this case our formula (5.6), (5.7) for the total index associated with this configuration is:

$$\frac{1}{2} g_{\text{ref}}(\alpha_1, \alpha_2, \alpha_3, \alpha_3; y) \Omega_{\text{ref}}^S(\alpha_1, y) \, \Omega_{\text{ref}}^S(\alpha_2, y) \, \Omega_{\text{ref}}^S(\alpha_3, y)^2$$
$$+ \frac{1}{2} g_{\text{ref}}(\alpha_1, \alpha_2, 2\alpha_3; y) \frac{y - y^{-1}}{y^2 - y^{-2}} \Omega_{\text{ref}}^S(\alpha_1, y) \, \Omega_{\text{ref}}^S(\alpha_2, y) \, \Omega_{\text{ref}}^S(\alpha_3, y^2) \tag{6.47}$$
$$+ h(\alpha_1, \alpha_2, r\beta; s_1^{(1)} = 1, s_2^{(1)} = 1, s_3^{(1)} = 2; y) \Omega_{\text{ref}}^S(\alpha_1, y) \, \Omega_{\text{ref}}^S(\alpha_2, y) \, \Omega_{\text{ref}}^S(\alpha_3, y^2).$$

We shall determine the function $h$ using the prescription of section 5.2. Following the convention of [33] and appendix C we shall consider three separate cases **A, B** and **D**. The case **C**, for which $q_3 < I/2 < q_4$ is not relevant here since we have $q_3 = q_4$.

**Case A.** In this case we have $2q_3 < I/2$. Thus according to (6.18) we have

$$
\begin{aligned}
g_{\mathrm{ref}}(\alpha_1, \alpha_2, 2\alpha_3; y) &= \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I - y^{I-4q_3} + y^{-I} - y^{-I+4q_3} \right] \\
&= \frac{(-1)^I (1 - y^{-4q_3})(y^I - y^{-I+4q_3})}{(y - y^{-1})^2} \, .
\end{aligned}
\tag{6.48}
$$

Since the first factor has a factor of $(1 + y^2)$ our prescription of section 5.2 tells us that we have $h = 0$. Thus using (6.47), (C.1) and (6.48) we see that the full index is given by

$$
\begin{aligned}
&\frac{1}{2} \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - 2y^{I-2q_3} + y^{I-4q_3} - y^{-I} + 2y^{-I+2q_3} - y^{-I+4q_3} \right] \\
&\qquad \times \Omega_{\mathrm{ref}}^S(\alpha_1, y) \, \Omega_{\mathrm{ref}}^S(\alpha_2, y) \, \Omega_{\mathrm{ref}}^S(\alpha_3, y)^2 \\
&+ \frac{1}{2} \frac{(-1)^I}{(y - y^{-1})(y^2 - y^{-2})} \left[ y^I - y^{I-4q_3} + y^{-I} - y^{-I+4q_3} \right] \Omega_{\mathrm{ref}}^S(\alpha_1, y) \, \Omega_{\mathrm{ref}}^S(\alpha_2, y) \, \Omega_{\mathrm{ref}}^S(\alpha_3, y^2) \, .
\end{aligned}
\tag{6.49}
$$

It is straightforward to check that eq. (6.49) with $\Omega_{\mathrm{ref}}^S(\alpha, y) = 1$ agrees with the exact refined index computed from (6.16), with the additional restriction $m_3 < m_4$ on the sum to account for the identity of the particles.

**Case B.** In this case we have $q_3 \leq I/2$, $2q_3 \geq I/2$, and hence according to (6.22) we have

$$
\begin{aligned}
g_{\mathrm{ref}}(\alpha_1, \alpha_2, 2\alpha_3; y) &= \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I + y^{-I} - (y + y^{-1}) \right], \qquad \text{for } I \text{ odd} \\
&= \frac{(-1)^I}{(y - y^{-1})^2} \left[ y^I + y^{-I} - 2 \right], \qquad \text{for } I \text{ even} \, .
\end{aligned}
\tag{6.50}
$$

This is the same as in (5.11) with $\alpha_{12}$ replaced by $I$. Since $h$ is determined from $g_{\mathrm{ref}}(\alpha_1, \alpha_2, 2\alpha_3; y)$, it will be given by (5.15):

$$
\begin{aligned}
h(\alpha_1, \alpha_2, r\beta; s_1^{(1)} = 1, s_2^{(1)} = 1, s_3^{(1)} = 2; y) &= 0 \quad \text{for} \quad I \in 2\mathbb{Z} + 1 \\
&= 0 \quad \text{for} \quad I \in 4\mathbb{Z} \\
&= \frac{1}{2} (-1)^{I+3} (y + y^{-1})^{-1} \quad \text{for} \quad I \in 4\mathbb{Z} + 2 \, .
\end{aligned}
\tag{6.51}
$$

Using (6.47), (C.5), (6.50) and (6.51) we now get the total contribution to the index from this configuration to be

$$
\begin{aligned}
&\frac{1}{2} \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - 2y^{I-2q_3} - y^{-I} + 2y^{-I+2q_3} - (y - y^{-1})(4q_3 - I) \right] \\
&\qquad \Omega_{\mathrm{ref}}^S(\alpha_1, y) \, \Omega_{\mathrm{ref}}^S(\alpha_2, y) \, \Omega_{\mathrm{ref}}^S(\alpha_3, y)^2 \\
&+ \frac{1}{2} \frac{(-1)^I}{(y - y^{-1})(y^2 - y^{-2})} \left[ y^I + y^{-I} - (y + y^{-1}) \right] \Omega_{\mathrm{ref}}^S(\alpha_1, y) \, \Omega_{\mathrm{ref}}^S(\alpha_2, y) \, \Omega_{\mathrm{ref}}^S(\alpha_3, y^2) \\
&\hspace{9cm} \text{for } I \in 2\mathbb{Z} + 1
\end{aligned}
\tag{6.52}
$$

$$\frac{1}{2}\frac{(-1)^{I-1}}{(y-y^{-1})^3}\left[y^I - 2y^{I-2q_3} - y^{-I} + 2y^{-I+2q_3} - \frac{1}{2}(y^2-y^{-2})(4q_3-I)\right]$$
$$\Omega^S_{\text{ref}}(\alpha_1,y)\,\Omega^S_{\text{ref}}(\alpha_2,y)\,\Omega^S_{\text{ref}}(\alpha_3,y)^2$$
$$+\frac{1}{2}\frac{(-1)^I}{(y-y^{-1})(y^2-y^{-2})}\left[y^I + y^{-I} - 2\right]\Omega^S_{\text{ref}}(\alpha_1,y)\,\Omega^S_{\text{ref}}(\alpha_2,y)\,\Omega^S_{\text{ref}}(\alpha_3,y^2)$$
$$\text{(6.53)}$$
$$\text{for } I \in 4\mathbb{Z}$$

$$\frac{1}{2}\frac{(-1)^{I-1}}{(y-y^{-1})^3}\left[y^I - 2y^{I-2q_3} - y^{-I} + 2y^{-I+2q_3} - \frac{1}{2}(y^2-y^{-2})(4q_3-I)\right]$$
$$\Omega^S_{\text{ref}}(\alpha_1,y)\,\Omega^S_{\text{ref}}(\alpha_2,y)\,\Omega^S_{\text{ref}}(\alpha_3,y)^2$$
$$+\frac{1}{2}\frac{(-1)^I}{(y-y^{-1})(y^2-y^{-2})}\left[y^I + y^{-I} - 2\right]\Omega^S_{\text{ref}}(\alpha_1,y)\,\Omega^S_{\text{ref}}(\alpha_2,y)\,\Omega^S_{\text{ref}}(\alpha_3,y^2)$$
$$+\frac{1}{2}(-1)^{I+3}(y+y^{-1})^{-1}\Omega^S_{\text{ref}}(\alpha_1,y)\,\Omega^S_{\text{ref}}(\alpha_2,y)\,\Omega^S_{\text{ref}}(\alpha_3,y^2)\,,$$
$$\text{(6.54)}$$
$$\text{for } I \in 4\mathbb{Z}+2\,.$$

Again, one may check that eq. (6.52)–(6.54) with $\Omega^S_{\text{ref}}(\alpha,y)=1$ agree with the exact refined index computed from (6.16) with the restriction $m_3 < m_4$.

**Case D.** In this case we have $q_3 > I/2$ and as a result $g_{\text{ref}}(\alpha_1,\alpha_2,2\alpha_3,y)$ and hence $h$ will be given by (6.50) and (6.51) respectively. Thus the only difference from case **B** is in the expression for $g_{\text{ref}}(\alpha_1,\alpha_2,\alpha_3,\alpha_3;y)$ given in (C.14). Thus using (6.47), (C.14), (6.50) and (6.51) we now get the total contribution to the index from this configuration to be

$$\frac{1}{2}\frac{(-1)^{I-1}}{(y-y^{-1})^3}\left[y^I - y^{-I} - I(y-y^{-1})\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y)^2$$
$$+\frac{1}{2}\frac{(-1)^I}{(y-y^{-1})(y^2-y^{-2})}\left[y^I + y^{-I} - (y+y^{-1})\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y^2)$$
$$\text{(6.55)}$$
$$\text{for } I \in 2\mathbb{Z}+1$$

$$\frac{1}{2}\frac{(-1)^{I-1}}{(y-y^{-1})^3}\left[y^I - y^{-I} - \frac{I}{2}(y^2-y^{-2})\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y)^2$$
$$+\frac{1}{2}\frac{(-1)^I}{(y-y^{-1})(y^2-y^{-2})}\left[y^I + y^{-I} - 2\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y^2)$$
$$\text{(6.56)}$$
$$\text{for } I \in 4\mathbb{Z}$$

$$\frac{1}{2}\frac{(-1)^{I-1}}{(y-y^{-1})^3}\left[y^I - y^{-I} - \frac{I}{2}(y^2-y^{-2})\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y)^2$$
$$+\frac{1}{2}\frac{(-1)^I}{(y-y^{-1})(y^2-y^{-2})}\left[y^I + y^{-I} - 2\right]\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y^2)$$
$$+\frac{1}{2}(-1)^{I+3}(y+y^{-1})^{-1}\Omega^S_{\text{ref}}(\alpha_1,y)\Omega^S_{\text{ref}}(\alpha_2,y)\Omega^S_{\text{ref}}(\alpha_3,y^2)\,,$$
$$\text{(6.57)}$$
$$\text{for } I \in 4\mathbb{Z}+2\,.$$

Again, one may check that eq. (6.55)–(6.57) with $\Omega^S_{\text{ref}}(\alpha,y)=1$ agree with the exact refined index computed from (6.16) with the restriction $m_3 < m_4$.

## Acknowledgments

## A  Sign rules for collinear fixed points

In this appendix, we provide some details on the computation of the sign of the contributions from collinear fixed points near a wall of marginal stability (section 4.1) and for dipole halo configurations (section 6.1). Recall that $s(p)$ is given by (2.29), where $\hat{M}(p)$ is the Hessian of the 'superpotential' (2.25) at the critical point $p$.

### A.1  Sign rules near a wall of marginal stability

First we shall compute the sign $s(p)$ associated with a collinear configuration $p$ near a wall of marginal stability, if the collinear configuration breaks up into two widely separated clusters as we approach the wall. Such a configuration has been described in section 4.1 where the two sets into which the centers split have been called $A$ and $B$. We follow the notation of section 4.1, and work with the configuration for which $z_B > z_A$, i.e. $z_j - z_i > 0$ for $i \in A$, $j \in B$. Near $R \equiv |z_A - z_B| \to \infty$, the superpotential (2.25) decomposes as

$$\hat{W}(z_i, \lambda) \sim \hat{W}_A\left(z_{i \in A}, \lambda_A = \frac{n_A}{n}\lambda\right) + \hat{W}_B\left(z_{j \in B}, \lambda_B = \frac{n_B}{n}\lambda\right) - \sum_{i \in A, j \in B} \alpha_{ij} \ln|z_i - z_j|. \quad \text{(A.1)}$$

For such a configuration the Hessian of $\hat{W}$ with respect to $\lambda, z_{i \in A}, z_{j \in B}$ takes the form

$$\hat{M} = \hat{M}_0 + \hat{M}_1, \quad \text{(A.2)}$$

where $\hat{M}_0$ is the Hessian in the strict $R \to \infty$ limit and $M_1$ is of order $1/R^2$. We have

$$\hat{M}_0 = \begin{pmatrix} 0 & \frac{n_A}{n}u_A^T & \frac{n_B}{n}u_B^T \\ \frac{n_A}{n}u_A & M_A & 0 \\ \frac{n_B}{n}u_B & 0 & M_B \end{pmatrix} \quad \text{(A.3)}$$

where $M_A$ is the Hessian of $W_A$ with respect to $z_{i \in A}$, $u_A$ is the $n_A$-dimensional column vector with entries $1/n_A$, and similarly for $M_B, u_B$. In particular, we note that $M_A u_A = M_B u_B = 0$. $\hat{M}_1$ is given by

$$(\hat{M}_1)_{00} = (\hat{M}_1)_{0i} = (\hat{M}_1)_{i0} = 0 \quad \text{for } 1 \leq i \leq n,$$

$$(\hat{M}_1)_{ij} = \frac{1}{R^2}\begin{cases} \delta_{ij}\sum_{k \in B}\alpha_{ik} & \text{for } i, j \in A \\ \delta_{ij}\sum_{k \in A}\alpha_{ki} & \text{for } i, j \in B \\ -\alpha_{ij} & \text{for } i \in A, \ j \in B \\ -\alpha_{ji} & \text{for } i \in B, \ j \in A \end{cases} \quad \text{(A.4)}$$

On the other hand, the Hessian of $\hat{W}_A$ (respectively, $\hat{W}_B$) with respect to $\lambda_A, z_a$ ($a \in A$) (respectively, $\lambda_B, z_b$, $b \in B$) is given by

$$\hat{M}_A = \begin{pmatrix} 0 & u_A^T \\ u_A & M_A \end{pmatrix}, \qquad \hat{M}_B = \begin{pmatrix} 0 & u_B^T \\ u_B & M_B \end{pmatrix}. \tag{A.5}$$

To compare the signs of $\det \hat{M}$ and $\det \hat{M}_A \det \hat{M}_B$, we shall construct an eigensystem of $\hat{M}$ in terms of the eigensystems of $\hat{M}_A$ and $\hat{M}_B$, in the limit $R \to \infty$. First, we note that an eigensystem of $\hat{M}_A$ is given by the $n_A + 1$ (eigenvectors,eigenvalues)

$$\left( [\pm\frac{1}{\sqrt{n_A}}, u_A], \pm\frac{1}{\sqrt{n_A}} \right), \quad \left( [0, v_A^{(i)}], \lambda_A^{(i)} \right) \tag{A.6}$$

where $v_A^{(i)}$, $i = 1, \ldots, n_A - 1$ are eigenvectors of $M_A$ in the subspace orthogonal to the null eigenvector $u_A$. Similarly, let $v_B^{(j)}$ be a system of eigenvectors of $M_B$ in the subspace orthogonal to the null eigenvector $u_B$, with eigenvalues $\lambda_B^{(j)}$. In the strict $R \to \infty$ limit, $\hat{M}$ reduces to $\hat{M}_0$ and an eigensystem of $\hat{M}_0$ is given by the $n_A + n_B + 1$ (eigenvectors,eigenvalues)

$$\left( [\pm\frac{1}{\sqrt{n}}, \frac{n_A}{n} u_A, \frac{n_B}{n} u_B], \pm\frac{1}{\sqrt{n}} \right), \left( [0, v_A^{(i)}, 0_B], \lambda_A^{(i)} \right), \left( [0, 0_A, v_B^{(j)}], \lambda_B^{(j)} \right), \left( [0, u_A, -u_B], 0 \right) \tag{A.7}$$

In particular, the last eigenvector, corresponding to a change in the relative separation between the two clusters keeping their inner structure fixed, yields a zero-mode of $\hat{M}_0$. Since the eigenvalues $\lambda_A^{(i)}, \lambda_B^{(j)}$ are generically distinct and non-zero, the structure of the spectrum will retain its form at large but finite $R$, except that the last eigenvalue will be lifted to a non-zero eigenvalue $\lambda(R)$. To compute $\lambda(R)$, it suffices to use non-degenerate perturbation theory and calculate the expectation value of $\hat{M}_1$ on the unperturbed eigenvector $U = [0, u_A, -u_B]$. This gives

$$\lambda(R) \sim \frac{U^T \cdot \hat{M}_1 \cdot U}{U^T \cdot U} \sim \frac{\gamma_{AB}(n_A + n_B)}{n_A n_B R^2}. \tag{A.8}$$

As a result,

$$\det \hat{M} \sim -\frac{1}{n}\lambda(R) \prod_i \lambda_A^{(i)}) \prod_j \lambda_B^{(j)} \sim -\frac{1}{R^2} \gamma_{AB} \det \hat{M}_A \det \hat{M}_B \tag{A.9}$$

Using (2.29), we arrive at (4.13) for $z_B > z_A$. The result for $z_B < z_A$ follows by exchanging $A$ and $B$ in the above analysis.

## A.2 Sign rules for collinear dipole halos

In this section, we establish the sign rule $s(p) = (-1)^{n_A}$ used in (6.9) for collinear dipole halo configurations in the non-scaling regime. For small positive $\eta$, the solutions described above (6.9) satisfy

$$z_a \to \frac{1}{2}(z_1 + z_2) - \frac{(r_{12})^3}{8R^2} \quad \forall a \in A, \quad z_b \to \frac{1}{2}(z_1 + z_2) - R \quad \forall b \in B, \tag{A.10}$$

where $R \simeq \sqrt{r_{12}/\eta}$. In this limit, the Hessian $\hat{M}$ takes the following form to order $R^{-3}$:

$$\hat{M}_{00} = 0, \quad \hat{M}_{0i} = \hat{M}_{i0} = \frac{1}{n} \quad \text{for } 1 \leq i,j \leq n, \quad \hat{M}_{12} = \hat{M}_{21} = -\frac{I}{(r_{12})^2},$$

$$\hat{M}_{11} = \frac{I - 4\sum_{a \in A} q_a}{r_{12}^2} + \frac{1}{R^2}\Big(\sum_{b \in B} q_b - 2\sum_{a \in A} q_a\Big) + \frac{r_{12}}{R^3}\sum_{b \in B} q_b,$$

$$\hat{M}_{22} = \frac{I - 4\sum_{a \in A} q_a}{r_{12}^2} + \frac{1}{R^2}\Big(2\sum_{a \in A} q_a - \sum_{b \in B} q_b\Big) + \frac{r_{12}}{R^3}\sum_{b \in B} q_b,$$

$$\hat{M}_{1a} = \hat{M}_{a1} = \frac{4}{(r_{12})^2}q_a + \frac{2}{R^2}q_a \quad \text{for } a \in A, \quad \hat{M}_{1b} = \hat{M}_{b1} = -\frac{1}{R^2}q_b - \frac{r_{12}}{R^3}q_b \quad \text{for } b \in B,$$

$$\hat{M}_{2a} = \hat{M}_{a2} = \frac{4}{(r_{12})^2}q_a - \frac{2}{R^2}q_a \quad \text{for } a \in A, \quad \hat{M}_{2b} = \hat{M}_{b2} = \frac{1}{R^2}q_b - \frac{r_{12}}{R^3}q_b \quad \text{for } b \in B,$$

$$\hat{M}_{aa'} = -\frac{8}{(r_{12})^2}q_a\,\delta_{aa'} \quad \text{for } a,a' \in A, \qquad \hat{M}_{bb'} = 2\frac{r_{12}}{R^3}q_b\,\delta_{bb'} \quad \text{for } b,b' \in B,$$

$$\hat{M}_{ab} = \hat{M}_{ba} = 0 \qquad \text{for } a \in A, b \in B. \tag{A.11}$$

To evaluate the determinant of this matrix in the large $R$ limit we proceed as follows. First we define a new matrix $\tilde{M}$ by dropping the first two rows and columns of $\hat{M}$; we have seen earlier that $\det \hat{M} = -\widehat{\det M} = -\det \tilde{M}$. To evaluate $\det \tilde{M}$, we can add half of the the second to last row of the matrix to the first row and then add half of the second to last columns of the matrix to the first column. This does not change the determinant but simplifies the matrix. Let us denote the resulting matrix by $\tilde{M}_0 + \tilde{M}_1$ where $\tilde{M}_0$ is the limit of the matrix as $R \to \infty$ and $\tilde{M}_1$ is the remainder. It is straightforward to check that $\tilde{M}_0$ is diagonal and has eigenvalues

$$\frac{(I - 2\sum_{a \in A} q_a)}{r_{12}^2}, \quad \Big\{-\frac{8q_a}{(r_{12})^2}\Big\}, \quad \{0_{n_B}\}, \tag{A.12}$$

where $0_{n_B}$ denotes that the eigenvalue 0 is repeated $n_B$ times. When we take into account the effect of $\tilde{M}_1$, the non-zero eigenvalues are not affected appreciably but the zero eigenvalues are lifted and can be obtained using first order degenerate perturbation theory. This gives the approximate eigenvalues of $\tilde{M}_0 + \tilde{M}_1$ to be:

$$\frac{(I - 2\sum_{a \in A} q_a)}{r_{12}^2}, \quad \Big\{-\frac{8q_a}{(r_{12})^2}\Big\}, \quad \Big\{\Big(\frac{2r_{12}}{R^3}\Big)_{n_B}\Big\}. \tag{A.13}$$

To leading order in the limit $R \to \infty$, $\det \hat{M} = -\det \tilde{M}$ is therefore given by

$$\det \hat{M} \simeq 2^{n+2n_A-2}\,(-1)^{n_A+1}\frac{(I - 2\sum_{a \in A} q_a)\,(\prod_{a \in A} q_a)\,(\prod_{b \in B} q_b)}{R^{3(n-2-n_A)}r_{12}^{3n_A-n+4}}. \tag{A.14}$$

In particular, the sign of $\det \hat{M}$ is $(-1)^{n_A+1}$. Note that to obtain this result, it is important to keep all subleading terms through order $1/R^3$ as indicated in (A.11), since the determinant at lower order vanishes. Using (2.29) we now arrive at

$$s(p) = (-1)^{n_A}. \tag{A.15}$$

## B   Laurent polynomial property in absence of scaling solutions

Eq. (4.1), (4.2) (or equivalently (4.3)) gives the index associated with a multi-centered black hole solution when the charges of the components do not allow for scaling solutions. Let us restrict to the case where $\gamma$ is primitive so that (4.1) directly gives the refined index $\Omega_{\mathrm{ref}}(\gamma, y)$ rather than its rational counterpart $\bar{\Omega}_{\mathrm{ref}}(\gamma, y)$. In this case the right hand side of (4.1) must be a Laurent polynomial in $y$ (i.e. a finite linear combination of $y^{\pm m}$) whenever the $\Omega_{\mathrm{ref}}^S(\alpha_i, y)$'s are since otherwise the result cannot be interpreted as an SU(2) character. Our goal in this appendix will be to prove this property of (4.1).[23]

Clearly this will not be true for an arbitrary choice of the functions $g_{\mathrm{ref}}$, but we shall use (4.19) — valid when at the attractor point only single centered black holes contribute to the index — to restrict the form of $g_{\mathrm{ref}}$. It is clear from this equation that the right hand side of (4.19) will be a Laurent polynomial in $y$ if the $g_{\mathrm{ref}}$'s appearing on the right hand side have this property. Thus by iterative application of (4.19) we can establish that $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n, y)$ is given by a Laurent polynomial in $y$. It now follows from (4.1) that when the $\alpha_i$'s are all primitive then the right hand side of this equation is a Laurent polynomial in $y$ since the individual $\bar{\Omega}_{\mathrm{ref}}^S(\alpha_i, y) = \Omega_{\mathrm{ref}}^S(\alpha_i, y)$'s and the $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n, y)$ have this property. This argument fails when some of the $\alpha_i$'s are not primitive: in this case $\bar{\Omega}_{\mathrm{ref}}^S(\alpha_i, y)$ defined in (4.2) have extra factors of $m(y^m - y^{-m})$ in the denominator, which must cancel in order that the final expression is a Laurent polynomial in $y$. We shall now demonstrate that this cancellation does take place.

By iterative application of (4.19) we can express $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_n)$ as a sum over attractor flow trees in which a total charge $\gamma$ decays via an appropriate tree to the charges $\alpha_1, \ldots, \alpha_n$. At a vertex at which a charge $\beta_1$ decays into two clusters of charge $\beta_2$ and $\beta_3$, we get a multiplicative factor of

$$(-1)^{\langle \beta_2, \beta_3 \rangle + 1} \mathrm{sign}\langle \beta_2, \beta_3 \rangle \frac{\sinh(\langle \beta_2, \beta_3 \rangle \nu)}{\sinh \nu} . \tag{B.1}$$

Let us focus on the vertex at which a non-primitive charge $\alpha_i$ gets attached to the tree. Suppose at this vertex an internal line carrying charge $\beta + \alpha_i$ decays into another internal line of charge $\beta$ and the external line of charge $\alpha_i$. Our goal will be to show that the contribution from the vertex factor cancels the $(y^m - y^{-m})$ factors in the denominator appearing in (4.2). The product of the vertex factor and the $\bar{\Omega}^S(\alpha_i, y)$ factor is given by

$$\begin{aligned}
&(-1)^{\langle \beta, \alpha_i \rangle + 1} \mathrm{sign}\langle \beta, \alpha_i \rangle \frac{\sinh(\langle \beta, \alpha_i \rangle \nu)}{\sinh \nu} \bar{\Omega}^S(\alpha_i, y) \\
&= (-1)^{\langle \beta, \alpha_i \rangle + 1} \mathrm{sign}\langle \beta, \alpha_i \rangle \sum_{m | \alpha_i} m^{-1} \frac{\sinh(\langle \beta, \alpha_i \rangle \nu)}{\sinh(m \nu)} \Omega^S(\alpha_i/m, y^m) .
\end{aligned} \tag{B.2}$$

Since $m | \alpha_i$, $\langle \beta, \alpha_i \rangle$ is an integral multiple of $m$. In this case the unwanted denominator factors cancel and $\sinh(\langle \beta, \alpha_i \rangle \nu)/\sinh(m\nu)$ is a Laurent polynomial in $y$.

---

[23]In order to prove that the result is indeed an SU(2) character, one must also show that the coefficients of $y^{\pm m}$ are integers. This is indeed true, but we shall omit the proof.

This leaves the end vertices of the attractor flow tree, at which a charge $\alpha_i + \alpha_j$ decays into a pair of charges $\alpha_i$ and $\alpha_j$. The contribution from such a vertex will be of the form

$$
\begin{aligned}
(-1)^{\langle \alpha_i, \alpha_j \rangle + 1} &\operatorname{sign}\langle \alpha_i, \alpha_j \rangle \frac{\sinh(\langle \alpha_i, \alpha_j \rangle \nu)}{\sinh \nu} \, \bar{\Omega}^S(\alpha_i, y) \bar{\Omega}^S(\alpha_j, y) \\
&= (-1)^{\langle \alpha_i, \alpha_j \rangle + 1} \operatorname{sign}\langle \alpha_i, \alpha_j \rangle \sum_{m|\alpha_i} \sum_{p|\alpha_j} m^{-1} p^{-1} \frac{\sinh(\langle \alpha_i, \alpha_j \rangle \nu) \sinh \nu}{\sinh(m\nu) \sinh(p\nu)} \\
&\quad \times \Omega^S(\alpha_i/m, y^m) \, \Omega^S(\alpha_j/p, y^p) \,.
\end{aligned}
\tag{B.3}
$$

Let us define $z = y^2 = e^{2\nu}$. Since $m|\alpha_i$ and $p|\alpha_j$, $\langle \alpha_i, \alpha_j \rangle$ is an integral multiple of $mp$. In this case $\sinh(\langle \alpha_i, \alpha_j \rangle \nu)$ will have zeroes at $z^{mp} = 1$. On the other hand the denominator $\sinh(m\nu)\sinh(p\nu)$ has zeroes at $z^m = 1$ and also at $z^p = 1$. If $m$ and $p$ are relatively prime then the locations of these zeroes are distinct except for a common zero of both factors at $z = 1$. Furthermore they coincide with the zeroes of the numerator at $z^{mp} = 1$. Thus all the zeroes of the denominator $\sinh(m\nu)\sinh(p\nu)$ cancel against the zeroes of the numerator $\sinh(\langle \alpha_i, \alpha_j \rangle \nu) \sinh \nu$. Thus as long as $m$ and $p$ are relatively prime, (B.3) is a Laurent polynomial in $y$.

Now suppose that $q \equiv \gcd(m, p) > 1$. Then $\sinh(m\nu)\sinh(p\nu)$ will have double zeroes at each of the $q$ solutions to $z^q = 1$. In contrast the $\sinh(\langle \alpha_i, \alpha_j \rangle \nu)$ factor in the numerator will generically have only single zeroes at each solution of $z^q = 1$. Combining this with the extra factor of $\sinh \nu$ in the numerator, we see that there is effectively a left-over factor of $\sinh \nu / \sinh(q\nu)$ from the vertex, multiplied by factors which are Laurent polynomials in $y$. We now show that the factor $\sinh \nu / \sinh(q\nu)$ is cancelled by other vertex factors in the same tree.

To see this, note that since $q$ divides both $\alpha_i, \alpha_j$, it divides their sum $\alpha_i + \alpha_j$. We can now repeat the analysis at the next vertex where say a line carrying charge $\alpha_i + \alpha_j + \beta$ splits into charges $\beta$ and $\alpha_i + \alpha_j$. As long as $\beta$ and $q$ do not have a common factor, the analysis of the previous paragraph shows that the vertex factor will cancel all the unwanted denominators, including the left-over factor of $\sinh(q\nu)/\sinh \nu$ from the previous vertex. If on the other hand $\beta$ and $q$ have a common factor $s$ then we shall have a left-over factor of $\sinh \nu / \sinh(s\nu)$ besides factors containing Laurent polynomials in $y$. Furthermore $\alpha_i + \alpha_j + \beta$ will have the same common factor $s$. The analysis can now be repeated for the next vertex. Proceeding this way, and using the fact that the initial charge $\gamma$ is taken to be primitive, one can prove that at the end all the denominator factors cancel, and we are left with a Laurent polynomial in $y$, proving the desired result.

## C Equivariant volumes and indices in dipole halos: $n = 4, 5$

In this section, we provide explicit results for the equivariant volume and equivariant index of the moduli space of $n$-centered dipole halo configurations with 4 and 5 centers. This serves as a check on our minimal modification hypothesis and on the recursion relations derived in section 6.3, section 6.4, and provides useful insight of the fixed points responsible for these contributions.
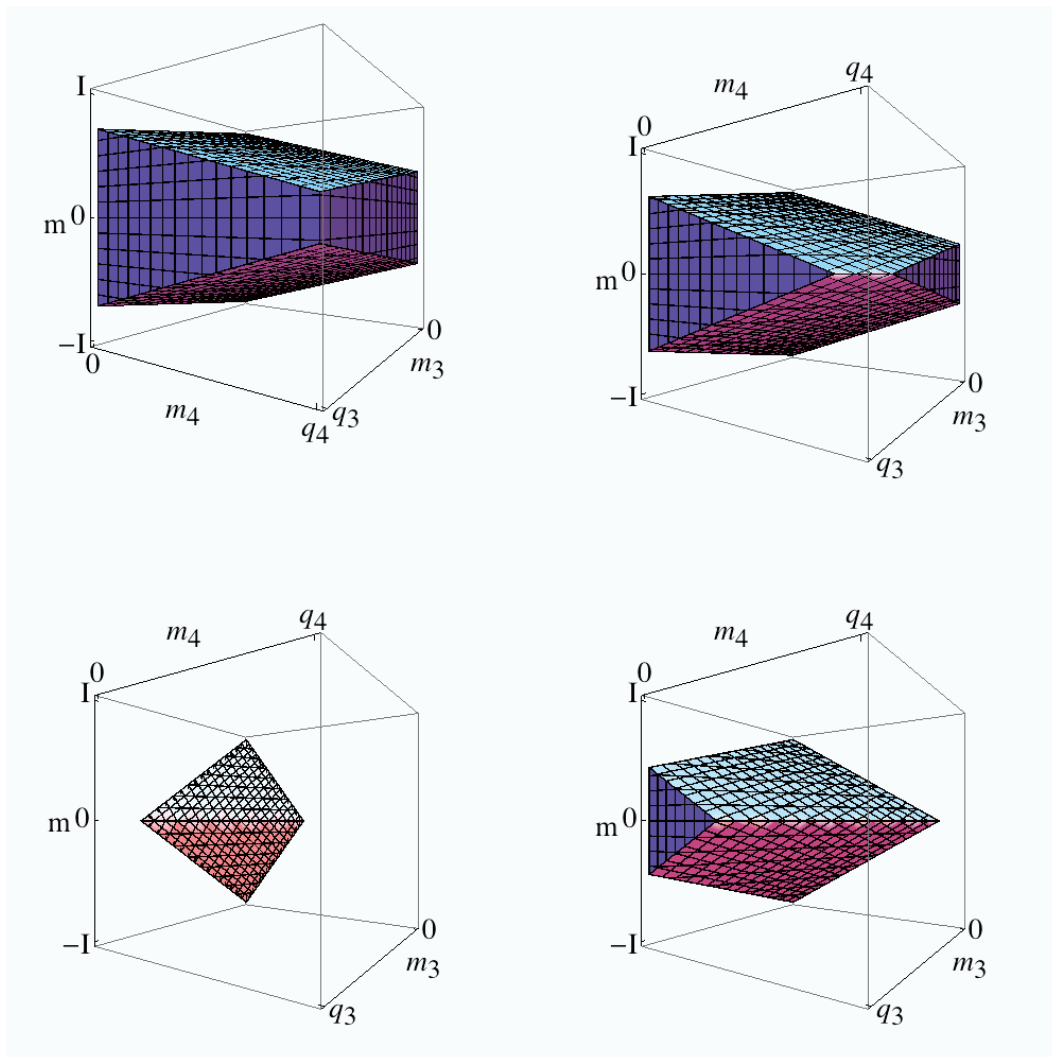
**Figure 2**. Polytopes associated to 4-center dipole halos in charge regimes $A, B, C, D$ respectively, in clockwise order starting from top-left corner.

### C.1 Four-centered case with distinct centers

For $n = 4$ we have to treat several different cases, depending on the value of $I/2$ relative to $q_3$, $q_4$ and $q_3 + q_4$. We label the four possible cases as in [29], see figure 2.

**Case A: $q_3 + q_4 < \frac{I}{2}$.** In this case, the contribution to $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_4; y)$ from collinear solutions, given in (6.10), is:

$$\hat{S}_{\mathrm{coll}}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{I-2q_3} - y^{I-2q_4} + y^{I-2q_3-2q_4} \right.$$
$$\left. -y^{-I} + y^{-I+2q_3} + y^{-I+2q_4} - y^{-I+2q_3+2q_4} \right]. \quad \text{(C.1)}$$

This has finite $y \to 1$ limit and hence, according to our proposal in section 5, should be the complete answer. Indeed, it can be checked that (C.1) agrees with the exact result (6.16).

On the other hand, the equivariant volume evaluates to

$$S(I; q_3, q_4; \nu) = (-1)^{I-1} \int_0^{q_3} \mathrm{d}m_3 \int_0^{q_4} \mathrm{d}m_4 \, \frac{\sinh[(I - 2(m_3 + m_4)\nu]}{\nu}$$

$$= \frac{(-1)^{I-1}}{4\nu^3} \left( \sinh(I\nu) + \sinh[(2q_3 - I)\nu] + \sinh[(2q_4 - I)\nu] + \sinh[(I - 2q_3 - 2q_4)\nu] \right) . \tag{C.2}$$

Replacing $\nu \to \sinh \nu$ in the denominator we get

$$\tilde{S}(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\sinh^3 \nu} \Big\{ \sinh(I\nu) + \sinh[(2q_3 - I)\nu] + \sinh[(2q_4 - I)\nu] \\ + \sinh[(I - 2q_3 - 2q_4)\nu] \Big\}. \tag{C.3}$$

Thus already agrees with the exact result $\hat{S}$ given in (C.1). Hence the prescription below (6.17) and the minimal modification hypothesis gives the same result.

**Case B: $q_3, q_4 \leq \frac{I}{2} \leq q_3 + q_4$.** In this case (6.10) leads to the following contribution to $g_{\mathrm{ref}}(\alpha_1, \ldots, \alpha_4; y)$:

$$\hat{S}_{\mathrm{coll}}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{I-2q_3} - y^{I-2q_4} - y^{-I} + y^{-I+2q_3} + y^{-I+2q_4} \right]. \tag{C.4}$$

This diverges in the $y \to 1$ limit. Hence we must add extra contributions of the form given in (5.5) to have a Laurent polynomial. This leads to

$$\hat{S}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \Big[ y^I - y^{I-2q_3} - y^{I-2q_4} - y^{-I} + y^{-I+2q_3} + y^{-I+2q_4} \\ - (y - y^{-1})(2q_3 + 2q_4 - I) \Big], \qquad \text{for } I \text{ odd}$$

$$= \frac{(-1)^{I-1}}{(y - y^{-1})^3} \Big[ y^I - y^{I-2q_3} - y^{I-2q_4} - y^{-I} + y^{-I+2q_3} + y^{-I+2q_4} \\ - \frac{1}{2}(y^2 - y^{-2})(2q_3 + 2q_4 - I) \Big], \quad \text{for } I \text{ even}. \tag{C.5}$$

The need for adding correction terms shows that in this case there are scaling solutions. It can be checked that (C.5) indeed agrees with the exact result (6.16).

On the other hand, the equivariant volume evaluates to

$$S(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\nu^3} \left\{ \sinh(I\nu) + \sinh[(2q_3 - I)\nu] + \sinh[(2q_4 - I)\nu] - (2q_3 + 2q_4 - I)\nu \right\}. \tag{C.6}$$

The last term is recognized as the contribution of the submanifold of fixed points $\mathcal{M}_n^{\mathrm{scal}}$ in (6.12), with symplectic volume $E(I, q_3, q_4) = (-1)^I (q_3 + q_4 - I/2)$. Replacing $\nu$ by $\sinh \nu$ in the denominator and in the last term of the numerator, we arrive at

$$\tilde{S}(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\sinh^3 \nu} \Big\{ \sinh(I\nu) + \sinh[(2q_3 - I)\nu] + \sinh[(2q_4 - I)\nu] \\ - (2q_3 + 2q_4 - I) \sinh \nu \Big\}. \tag{C.7}$$

Again, this differs from $\hat{S}$ given in (C.5) by a term that vanishes as $\nu \to \pm\infty$:

$$\hat{S}(I; q_3, q_4; y) = \tilde{S}(I; q_3, q_4; y) + (-1)^{I-1} \frac{2q_3 + 2q_4 - I}{4 \sinh^3 \nu} \left\{ \begin{array}{c} 0 : I \text{ odd} \\ \sinh\nu - \frac{1}{2}\sinh 2\nu : I \text{ even} \end{array} \right\} . \quad \text{(C.8)}$$

Thus the minimal modification hypothesis and the prescription given below (6.17) agree with each other and the exact result.

**Case C: $q_3 \leq \frac{I}{2} \leq q_4$.** In this case (6.10) leads to the following contribution to $g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y)$:

$$\hat{S}_{\text{coll}}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{I-2q_3} - y^{-I} + y^{-I+2q_3} \right] . \quad \text{(C.9)}$$

This diverges in the $y \to 1$ limit. Hence we must add extra contributions of the form given in (5.5) to have a Laurent polynomial. This leads to

$$\hat{S}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{I-2q_3} - y^{-I} + y^{-I+2q_3} - 2(y - y^{-1})q_3 \right],$$

$$\text{for } I \text{ odd}$$

$$= \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{I-2q_3} - y^{-I} + y^{-I+2q_3} - (y^2 - y^{-2})q_3 \right], \quad \text{(C.10)}$$

$$\text{for } I \text{ even},$$

in agreement with the exact result (6.16).

On the other hand, the equivariant volume leads to

$$S(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\nu^3} \left\{ \sinh(I\nu) + \sinh[(2q_3 - I)\nu] - 2q_3\nu \right\} . \quad \text{(C.11)}$$

Again, the last term in (C.11) is recognized as the contribution from the fixed submanifold $\mathcal{M}_n^{\text{scal}}$, with symplectic volume $E(I, q_3, q_4) = (-1)^I q_3$. Following the rules described earlier, we get

$$\tilde{S}(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4 \sinh^3 \nu} \left\{ \sinh(I\nu) + \sinh[(2q_3 - I)\nu] - 2q_3 \sinh\nu \right\} . \quad \text{(C.12)}$$

It is easy to see that this differs from $\hat{S}$ given in (C.10) by terms which vanish as $\nu \to \pm\infty$.

**Case D: $q_3, q_4 \geq \frac{I}{2}$.** In this case (6.10) leads to the following contribution to $g_{\text{ref}}(\alpha_1, \ldots, \alpha_4; y)$:

$$\hat{S}_{\text{coll}}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{-I} \right] . \quad \text{(C.13)}$$

This reduces to the case discussed in section 5.3. Therefore eq. (5.9) gives

$$\hat{S}(I; q_3, q_4; y) = \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{-I} - I(y - y^{-1}) \right], \qquad \text{for } I \text{ odd}$$

$$= \frac{(-1)^{I-1}}{(y - y^{-1})^3} \left[ y^I - y^{-I} - \frac{I}{2}(y^2 - y^{-2}) \right], \qquad \text{for } I \text{ even}. \quad \text{(C.14)}$$

Again, the results agree with the exact refined index (6.16).

The equivariant volume gives

$$S(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\nu^3} \left(\sinh(I\nu) - I\nu\right) = S_{\text{coll}}(I; q_3, q_4; \nu) - \frac{(-1)^{I-1}I}{4\nu^2}, \tag{C.15}$$

and hence

$$\tilde{S}(I; q_3, q_4; \nu) = \frac{(-1)^{I-1}}{4\sinh^3 \nu} \left(\sinh(I\nu) - I\sinh\nu\right). \tag{C.16}$$

Again this differs from the exact result (C.14) by a term that vanishes as $\nu \to \pm\infty$. The last term in (C.15) is moreover recognized as the contribution from the fixed submanifold $\mathcal{M}_n^{\text{scal}}$, with symplectic volume $E(I, q_3, q_4) = (-1)^I I/2$. We further comment on this contribution in section C.3.

## C.2 Five-centered case with distinct centers

In this section we compute the equivariant volume for 5 distinct centers. Without loss of generality, we assume that $q_3 < q_4 < q_5$. If in addition $q_3 + q_4 \geq q_5$, then we have the ordering

$$i) \qquad 0 \leq q_3 \leq q_4 \leq q_5 \leq q_3 + q_4 \leq q_3 + q_5 \leq q_4 + q_5 \leq q_3 + q_4 + q_5 \tag{C.17}$$

If instead $q_5 \geq q_3 + q_4$, then

$$ii) \qquad 0 \leq q_3 \leq q_4 \leq q_3 + q_4 \leq q_5 \leq q_3 + q_5 \leq q_4 + q_5 \leq q_3 + q_4 + q_5 \tag{C.18}$$

We now split the discussion according to the position of $I/2$ relative to these values, starting with case i) and then discussing the appropriate change in case ii). The polytopes arising in case i) are depicted in figure 3.

For $q_3 + q_4 + q_5 < I/2$,

$$\begin{aligned}
S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} &\left(\cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] - \cosh[(I - 2q_5)\nu]\right. \\
&+ \cosh[(I - 2q_3 - 2q_4)\nu] + \cosh[(I - 2q_3 - 2q_5)\nu] \\
&\left.+ \cosh[(I - 2q_4 - 2q_5)\nu] - \cosh[(I - 2q_3 - 2q_4 - 2q_5)\nu]\right).
\end{aligned} \tag{C.19}$$

For $q_4 + q_5 < I/2 < q_3 + q_4 + q_5$,

$$\begin{aligned}
S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} &\left(\cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] - \cosh[(I - 2q_5)\nu]\right. \\
&+ \cosh[(I - 2q_3 - 2q_4)\nu] + \cosh[(I - 2q_3 - 2q_5)\nu] \\
&\left.+ \cosh[(I - 2q_4 - 2q_5)\nu]\right) - \frac{(-1)^I}{16\nu^2}(I - 2q_3 - 2q_4 - 2q_5)^2 - \frac{(-1)^I}{8\nu^4}.
\end{aligned} \tag{C.20}$$

For $q_3 + q_5 < I/2 < q_4 + q_5$,

$$\begin{aligned}
S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} &\left(\cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] - \cosh[(I - 2q_5)\nu]\right. \\
&\left.+ \cosh[(I - 2q_3 - 2q_4)\nu] + \cosh[(I - 2q_3 - 2q_5)\nu]\right] \\
&+ \frac{(-1)^I}{4\nu^2} q_3(I - q_3 - 2q_4 - 2q_5)
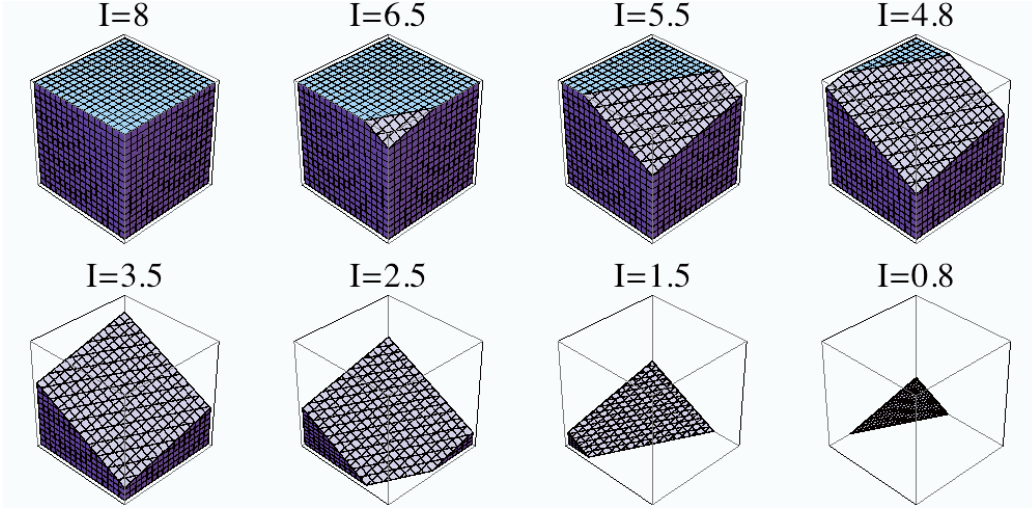\end{aligned} \tag{C.21}$$

**Figure 3**. Polytopes associated to 5-center dipole halos with $(q_3, q_4, q_5) = (1, 2, 4)$ and varying values of $I$. The vertical, left and right axes correspond to $m_5, m_3, m_4$. The dimension associated to $m \in [-j, j]$ with $j = \frac{1}{2}I - \sum_a m_a$ is suppressed.

For $q_3 + q_4 < I/2 < q_3 + q_5$,

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} \left( \cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] - \cosh[(I - 2q_5)\nu] \right.$$
$$\left. + \cosh[(I - 2q_3 - 2q_4)\nu]] + \frac{(-1)^I}{16\nu^2}(I^2 - 8q_3 q_4 - 4Iq_5 + 4q_5^2) + \frac{(-1)^I}{8\nu^4} \right. \tag{C.22}$$

For $q_5 < I/2 < q_3 + q_4$,

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} \left( \cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] - \cosh[(I - 2q_5)\nu] \right)$$
$$+ \frac{(-1)^I}{8\nu^2}(I^2 - 2I(q_3 + q_4 + q_5) + 2(q_3^2 + q_4^2 + q_5^2)) + \frac{(-1)^I}{4\nu^4} . \tag{C.23}$$

For $q_4 < I/2 < q_5$,

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} \left( \cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] \right)$$
$$+ \frac{(-1)^I}{16\nu^2}(I^2 - 4I(q_3 + q_4) + 4(q_3^2 + q_4^2)) + \frac{(-1)^I}{8\nu^4} . \tag{C.24}$$

For $q_3 < I/2 < q_4$,

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} \left( \cosh(I\nu) - \cosh[(I - 2q_3)\nu] \right) - \frac{(-1)^I}{4\nu^2} q_3(I - q_3) . \tag{C.25}$$

For $0 \leq I/2 \leq q_3$,

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{16\nu^4} \left[ 2\cosh(I\nu) - \nu^2 I^2 - 2 \right] . \tag{C.26}$$

This case is further discussed in section C.3.

In case (ii), the region $q_4 < I/2 < q_3 + q_5$ instead splits in three regions: $q_5 < I/2 < q_3 + q_5$, where $S$ is still given by (C.22), $q_3 + q_4 < I/2 < q_5$, where

$$S(I; q_3, q_4, q_5; \nu) = \frac{(-1)^I}{8\nu^4} (\cosh(I\nu) - \cosh[(I - 2q_3)\nu] - \cosh[(I - 2q_4)\nu] \\ + \cosh[(I - 2q_3 - 2q_4)\nu]) - \frac{(-1)^I}{2\nu^2} q_3 q_4, \tag{C.27}$$

and $q_4 < I/2 < q_3 + q_4$, where $S$ is still given by (C.24).

## C.3 Multi-equivariant volumes

To interpret the above results as a sum over isolated and non-isolated fixed points, it is useful to compute the equivariant volume for the most general torus action on $\mathcal{M}_n$,

$$S(I, \{q_a\}; \nu, \{\nu_a\}) = (-1)^{I-n+1} \int_{\substack{0 \le m_a \le q_a \\ \sum_a m_a \le I/2}} \mathrm{d}m_3 \cdots \mathrm{d}m_n \, e^{2\sum \nu_a m_a} \frac{\sinh[(I - 2\sum_{a=3}^n m_a)\nu]}{\nu}. \tag{C.28}$$

and compare it to the corresponding equivariant volume of the fixed submanifold $\mathcal{M}_n^{\text{scal}}$

$$E(I, \{q_a\}; \{\nu_a\}) = (-1)^{I-n+1} \int_{0 \le m_a \le q_a} \mathrm{d}m_3 \cdots \mathrm{d}m_n \, e^{2\sum \nu_a m_a} \delta\left(\sum_a m_a - \frac{I}{2}\right). \tag{C.29}$$

We shall refer to (C.28) and (C.29) as the multi-equivariant' volume of $\mathcal{M}_n$ and $\mathcal{M}_n^{\text{scal}}$, respectively. We shall compute these equivariant volumes in two simple cases with $n = 4$ and $n = 5$ centers, which demonstrate that the non-isolated fixed point contribution to $S(I, \{q_a\}; \nu; \{\nu_a\})$ is closely related to $E(I, \{q_a\}; \nu; \{\nu_a\})$, though not identical.

For $n = 4$ and $I/2 < q_3, q_4$, eq. (C.28) evaluates to

$$S(I, q_3, q_4; \nu, \nu_3, \nu_4) = (-1)^{I-1} \left( \frac{e^{\nu I}}{8\nu(\nu - \nu_3)(\nu - \nu_4)} - \frac{e^{-\nu I}}{8\nu(\nu + \nu_3)(\nu + \nu_4)} \right. \\ \left. + \frac{e^{\nu_3 I}}{4(\nu^2 - \nu_3^2)(\nu_4 - \nu_3)} + \frac{e^{\nu_4 I}}{4(\nu^2 - \nu_4^2)(\nu_3 - \nu_4)} \right). \tag{C.30}$$

These four contributions correspond to the four vertices of the polytope $\mathcal{P}$ (see figure 2, bottom-left graph): the first two arise from collinear configurations ($m_{3,4} = 0, m = \pm I/2$) while the last two are of scaling type, with $j = 0$ ($m_3 = I/2, m_4 = m = 0$ or $m_4 = I/2, m_3 = m = 0$). Rescaling $\nu_a$ by a common factor $\epsilon$ and taking the limit $\epsilon \to 0$ (as we shall always do when taking the limit $\nu_a \to 0$), eq. (C.30) reduces to (C.15). On the other hand, the multi-equivariant volume of $\mathcal{M}_n^{\text{scal}}$ is given by

$$E(I, q_3, q_4; \nu_3, \nu_4) = (-1)^{I-1} \left( \frac{e^{\nu_3 I}}{2(\nu_3 - \nu_4)} - \frac{e^{\nu_4 I}}{2(\nu_3 - \nu_4)} \right). \tag{C.31}$$

This differs from the second line in (C.30) for general values of $\nu_3, \nu_4$, although it agree with it in the limit $\nu_3, \nu_4 \to 0$, after rescaling by $-1/(2\nu^2)$. The difference between (C.31) and the second line of (C.30) should originate from the Euler class of the normal bundle

of $\mathcal{M}_n^{\text{scal}}$ inside $\mathcal{M}_n$, which appears in the denominator of the localization formula. The comparison of the two formulae shows that this Euler class should contribute a factor of $\nu^2/(\nu^2 - \nu_a^2)$ at each of the fixed points of the toric action.

To see that such corrections can be important even in the limit $\nu \to 0$, let us consider the case $n = 5$ and $I/2 < q_3, q_4, q_5$, and identify the fixed points contributing to the equivariant volume computed by direct integration in (C.26). In this case, eq. (C.28) evaluates to

$$S(I, q_3, q_4, q_5; \nu; \nu_3, \nu_4, \nu_5) = (-1)^I \left( \frac{e^{\nu I}}{16\nu(\nu - \nu_3)(\nu - \nu_4)(\nu - \nu_5)} \right.$$
$$+ \frac{e^{-\nu I}}{16\nu(\nu + \nu_3)(\nu + \nu_4)(\nu + \nu_5)} - \frac{e^{\nu_3 I}}{8(\nu^2 - \nu_3^2)(\nu_3 - \nu_4)(\nu_3 - \nu_5)}$$
$$\left. - \frac{e^{\nu_4 I}}{8(\nu^2 - \nu_4^2)(\nu_4 - \nu_3)(\nu_4 - \nu_5)} - \frac{e^{\nu_5 I}}{8(\nu^2 - \nu_5^2)(\nu_5 - \nu_3)(\nu_5 - \nu_4)} \right)$$
(C.32)

These five contributions correspond to the five vertices of the polytope $\mathcal{P}$, displayed on the bottom-right corner of figure 3 (after restoring the direction along $m$): the first two arise from collinear configurations ($m_{3,4,5} = 0, m = \pm I/2$) while the last three are of scaling type, with $j = 0$ ($m_3 = I/2, m_4 = m_5 = m = 0$ and permutations thereof). Rescaling $\nu_a$ by a common factor $\epsilon$ and taking the limit $\epsilon \to 0$, (C.32) reduces to (C.26). In particular, the first two terms in (C.32) have a smooth limit at $\nu_a \to 0$ and reproduce the first term in (C.26). The second, $\mathcal{O}(I^2/\nu^2)$ term in (C.26) arises by expanding $e^{\nu_a I}$ to second order in $\nu_a$, while the last, $\mathcal{O}(1/\nu^4)$ term in (C.26) arises by expanding $1/(\nu^2 - \nu_a^2)$ to second order in $\nu_a$. In contrast, the multi-equivariant volume of $\mathcal{M}_n^{\text{scal}}$ is given by

$$E(I, q_3, q_4, q_5; \nu_3, \nu_4, \nu_5) = (-1)^I \left( \frac{e^{\nu_3 I}}{4(\nu_3 - \nu_4)(\nu_3 - \nu_5)} \right.$$
$$\left. + \frac{e^{\nu_4 I}}{4(\nu_4 - \nu_3)(\nu_4 - \nu_5)} + \frac{e^{\nu_5 I}}{4(\nu_5 - \nu_3)(\nu_3 - \nu_4)} \right) .$$
(C.33)

This reduces to $E(I, q_3, q_4, q_5) = (-1)^I I^2/8$ in the limit $\nu_a \to 0$. Thus, after rescaling by a factor $-1/(2\nu^2)$, the multi-equivariant volume $-E(I, q_3, q_4, q_5; \nu_3, \nu_4, \nu_5)/(2\nu^2)$ correctly accounts for the $\mathcal{O}(I^2/\nu^2)$ term in (C.26), but fails to reproduce the $\mathcal{O}(1/\nu^4)$. Again, this indicates that the Euler class of the normal bundle of $\mathcal{M}_n^{\text{scal}}$ which appears in the denominator of the localization formula should produce an additional factor $\nu^2/(\nu^2 - \nu_a^2)$ at each fixed point.

More generally, we expect that the linear combination of equivariant volumes $E(I, \{q_a\})$ appearing in (6.44) can be interpreted as the integral of $\text{Ch}(\mathcal{L}, \nu)/\text{Eu}(N\mathcal{M}_n^{\text{scal}})$ over the fixed submanifold $\mathcal{M}_n^{\text{scal}}$. Similarly, we expect that the analog linear combination of equivariant indices $\hat{E}(I, \{q_a\})$ which would appear in a similar formula for $\Delta \hat{S}$ corresponds to the equivariant integral (2.35). It would be interesting to carry this out in detail.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

# References

[1] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [`hep-th/0005049`] [SPIRES].

[2] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [`hep-th/0206072`] [SPIRES].

[3] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, `hep-th/0304094` [SPIRES].

[4] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D 7** (1973) 2333 [SPIRES].

[5] S.W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199 [SPIRES].

[6] S. Ferrara, R. Kallosh and A. Strominger, *N = 2 extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [`hep-th/9508072`] [SPIRES].

[7] S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, *Phys. Rev.* **D 54** (1996) 1525 [`hep-th/9603090`] [SPIRES].

[8] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [`hep-th/9702103`] [SPIRES].

[9] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [`gr-qc/9307038`] [SPIRES].

[10] K. Behrndt et al., *Classical and quantum N = 2 supersymmetric black holes*, *Nucl. Phys.* **B 488** (1997) 236 [`hep-th/9610105`] [SPIRES].

[11] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [`hep-th/9812082`] [SPIRES].

[12] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Supersymmetric black hole solutions with $R^2$ interactions*, `hep-th/0003157` [SPIRES].

[13] A. Sen, *Entropy Function and $AdS_2/CFT_1$ Correspondence*, *JHEP* **11** (2008) 075 [`arXiv:0805.0095`] [SPIRES].

[14] R.K. Gupta and A. Sen, *$AdS_3/CFT_2$ to $AdS_2/CFT_1$*, *JHEP* **04** (2009) 034 [`arXiv:0806.0053`] [SPIRES].

[15] A. Sen, *Quantum Entropy Function from $AdS_2/CFT_1$ Correspondence*, *Int. J. Mod. Phys.* **A 24** (2009) 4225 [`arXiv:0809.3304`] [SPIRES].

[16] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, *Quantizing N = 2 Multicenter Solutions*, *JHEP* **05** (2009) 002 [`arXiv:0807.4556`] [SPIRES].

[17] J. Manschot, B. Pioline and A. Sen, *Wall-Crossing from Boltzmann Black Hole Halos*, `arXiv:1011.1258` [SPIRES].

[18] B. Pioline, *Four ways across the wall*, proceedings of the workshop *Algebra, Geometry and Mathematical Physics*, Tjärnö Sweden, October 2010 `arXiv:1103.0261` [SPIRES].

[19] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, `arXiv:0811.2435`.

[20] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, `arXiv:0810.5645` [SPIRES].

[21] J. Manschot, *The Betti numbers of the moduli space of stable sheaves of rank 3 on $P^2$*, `arXiv:1009.1775` [SPIRES].

[22] J.J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, *Invent. Math.* **69** (1982) 259.

[23] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, `hep-th/0702146` [SPIRES].

[24] E. Andriyash, F. Denef, D.L. Jafferis and G.W. Moore, *Wall-crossing from supersymmetric galaxies*, `arXiv:1008.0030` [SPIRES].

[25] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. I*, *Ann. Math.* **86** (1967) 374.

[26] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, *Ann. Math.* **88** (1968) 451.

[27] I. Bena, C.-W. Wang and N.P. Warner, *Mergers and Typical Black Hole Microstates*, *JHEP* **11** (2006) 042 [`hep-th/0608217`] [SPIRES].

[28] I. Bena, C.-W. Wang and N.P. Warner, *Plumbing the Abyss: Black Ring Microstates*, *JHEP* **07** (2008) 019 [`arXiv:0706.3786`] [SPIRES].

[29] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, *A bound on the entropy of supergravity?*, *JHEP* **02** (2010) 062 [`arXiv:0906.0011`] [SPIRES].

[30] A. Sen, *Arithmetic of Quantum Entropy Function*, *JHEP* **08** (2009) 068 [`arXiv:0903.1477`] [SPIRES].

[31] A. Dabholkar, J. Gomes, S. Murthy and A. Sen, *Supersymmetric Index from Black Hole Entropy*, *JHEP* **04** (2011) 034 [`arXiv:1009.3226`] [SPIRES].

[32] M. Vergne, *Applications of equivariant cohomology*, in *International Congress of Mathematicians. Vol. I*, Eur. Math. Soc., Zürich Switzerland (2007) [`math/0607389`].

[33] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, *Black hole bound states in $AdS_3 \times S^2$*, *JHEP* **11** (2008) 050 [`arXiv:0802.2257`] [SPIRES].

[34] N. Berline, E. Getzler and M. Vergne, *Fundamental Principles of Mathematical Sciences. Vol. 298: Heat kernels and Dirac operators*, Springer-Verlag, Berlin Germany (1992).

[35] J.J. Duistermaat, *Progress in Nonlinear Differential Equations and their Applications. Vol. 18: The heat kernel Lefschetz fixed point formula for the spin-c Dirac operator*, Birkhäuser Boston Inc., Boston U.S.A. (1996).

[36] D. Gaiotto, G.W. Moore and A. Neitzke, *Framed BPS States*, `arXiv:1006.0146` [SPIRES].

[37] S. Lee and P. Yi, *Framed BPS States, Moduli Dynamics and Wall-Crossing*, *JHEP* **04** (2011) 098 [`arXiv:1102.1729`] [SPIRES].

[38] N. Berline and M. Vergne, *The equivariant index and Kirillov's character formula*, *Am. J. Math.* **107** (1985) 1159.

[39] S. Alexandrov, D. Persson and B. Pioline, *Fivebrane instantons, topological wave functions and hypermultiplet moduli spaces*, *JHEP* **03** (2011) 111 [`arXiv:1010.5792`] [SPIRES].

[40] M. Shmakova, *Calabi-Yau Black Holes*, *Phys. Rev.* **D 56** (1997) 540 [`hep-th/9612076`] [SPIRES].

[41] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Examples of stationary BPS solutions in $\mathcal{N} = 2$ supergravity theories with $R^2$-interactions*, *Fortsch. Phys.* **49** (2001) 557 [`hep-th/0012232`] [SPIRES].