JOURNAL OF FUNCTIONAL ANALYSIS 56, 29–78 (1984)

Applications of the Complex Interpolation Method to a von Neumann Algebra: Non-commutative L^p-Spaces*

HIDEKI KOSAKI†

Department of Mathematics, University of Kansas, Lawrence, Kansas, 66045

Communicated by A. Connes

Received September 1, 1981

Non-commutative L^p-spaces, $1 < p < \infty$, associated with a von Neumann algebra are considered. The paper consists of two parts. In part I, by making use of the complex interpolation method, non-commutative L^p-spaces are defined as interpolation spaces between the von Neumann algebra in question and its predual. Also, all expected properties (such as duality and uniform convexity) are proved in the frame of interpolaton theory and relative modular theory. In part II, these L^p-spaces are compared with Haagerup's L^p-spaces. Based on this comparison, a non-commutative analogue of the classical Stein–Weiss interpolation theorem is obtained.

Contents.

0. INTRODUCTION

This paper is devoted to a new construction of non-commutative L^p-spaces, $1 < p < \infty$, from a von Neumann algebra (which is not necessarily semi-finite). The construction is based on the complex interpolation method (due to Calderón [7]). All expected properties are proved by complex interpolation theory and (relative) modular theory (Tomita–Takesaki theory

* This work is supported in part by the National Science Foundation Grant MCS-8102158.

† Present adress: Department of Mathematics, University of Pennsylvania, Philadelphia, Pa. 19104.

0022-1236/84 $3.00

Copyright © 1984 by Academic Press, Inc.
All rights of reproduction in any form reserved.

[37]). Also, a non-commutative analogue of the classical Stein–Weiss interpolation theorem is obtained.

Certain applications of the theory of operator algebras to other fields are carried out through theory of non-commutative L^p-spaces. Thus, starting from a pair (M, ϕ_0) consisting of a semi-finite von Neumann algebra M and a trace ϕ_0 on it, many authors studied non-commutative L^p-spaces, [13, 26, 29, 33]. After the development of the Tomita–Takesaki theory, Haagerup [17] generalized them to the case when ϕ_0 is a weight (so that M need not be semi-finite; see also [4, 9, 19, 21, 22, 40]). His L^p-spaces are based on crossed products [39]. Roughly speaking, crossed product technique is "Fourier analysis" for von Neumann algebras. Instead of using it, we will employ the complex interpolation method which is a product of deep classical analysis (including Fourier analysis). This complex interpolation method itself is quite an abstract method, however, when it is applied to a von Neumann algebra it is not abstract at all and actually fitting. Namely, it provides us a nice interpretation of "relative KMS functions." It may be said (see Remark 3.5) that the study of non-commutative L^p-spaces is the study of the behavior of these functions inside of the strip.

We now describe the origanization of the paper. It consists of two parts. Part I is devoted to non-commutative L^p-spaces $L^p(M; \phi_0)$, $1 < p < \infty$, for a distinguished faithful normal state ϕ_0 on a given von Neumann algebra M. After some preliminaries (Sections 1, 2), in Section 3 we imbed M into its predual M_* via

$$x \to x\phi_0, \qquad (*)$$

and $L^p(M; \phi_0)$, $1 < p < \infty$, is defined as the complex interpolation space $C_{\theta=1/p}(M, M_*)$, $0 < \theta < 1$. Also we show the equivalence between $L^2(M, \phi_0)$ and a (standard) Hilbert space. Based on this fact, in Section 4 we prove certain properties of our L^p-spaces. Among other results, we prove the uniform convexity of the L^p-spaces, $1 < p < \infty$ (Theorem 4.2) which gives rise to the affirmative answer to Dixmier's question in [13]. Actually we give two proofs. The one in Section 4 is based on quite a general result [11] on the complex interpolation method, whose proof is presented in the Appendix for the reader's convenience. The other is based on the Clarkson and McCarthy inequalities in Section 5. In Section 4 we also state the uniqueness theorem (Theorem 4.5) of the L^p-spaces. However, its proof is based on arguments used repeatedly in Part II so that it will be proved in Section 13 (the last section in Part II). In Section 6, we consider an M-bimodule structure of the L^p-spaces.

Part II deals with a non-commutative analogue of the classical Stein–Weiss interpolation theorem. Fixing two faithful normal states ψ_0, ϕ_0, in Section 7 we consider a one-parameter family of imbeddings of M into

M_*, parametrized by $0 \leqslant \eta \leqslant 1$. When $\eta = 0$ (resp. $\eta = 1$), this reduces to the "left" injection (∗) (resp. the "right" injection $x \to \psi_0 x$) so that we obtain the left L^p-space $L^p(M; \phi_0) = L^p(M; \phi_0)_L$ (resp. right L^p-space $L^p(M; \psi_0)_R$). In Section 9, we investigate relations between these spaces and Haagerup's L^p-spaces (Section 8). After preparing some technical lemmas in Section 10, in Section 11 we prove the main result (Theorem 11.1) of Part II which characterizes the complex interpolation spaces between the left and right L^p spaces. This characterization can be considered as a non-commutative analogue of the classical Stein–Weiss interpolation theorem. (see the beginning of Section 12) In Section 12, we collect some remarks and related topics on this theorem.

I. Non-commutative L^p-Spaces

1. *The Complex Interpolation Method*

In this section we briefly recall the complex interpolation method due to Calderón. It will be helpful to make the present article readable to operator algebraists who (like the author) have little knowledge of interpolation theory, and partly to fix our notations. Full details of the materials until Lemma 1.3 can be found in [6, 7, 32, 42]. However, Theorem 1.5 seems to be new.

Let $X = (X_0, X_1)$ be a pair of two *compatible* Banach spaces with norms $\| \ \|_{X_0} = \| \ \|_0$ and $\| \ \|_{X_1} = \| \ \|_1$, respectively. Namely, there should exist a larger space $\tilde{X}$ so that both of X_0 and X_1 can be considered as subspaces. (In usual applications of interpolation theory to function spaces, the space $\tilde{X}$ is sometimes unspecified. In our case it will be the predual of a given von Neumann algebra.) Then the algebraic sum $\Sigma (= \Sigma(X) = \Sigma(X_0, X_1)) = X_0 + X_1$ (in $\tilde{X}$) is a Banach space under the norm

$$\|x\|_\Sigma = \inf\{\|x_0\|_0 + \|x_1\|_1 ; x = x_0 + x_1, x_j \in X_j\}$$

due to the completeness of X_0 and X_1. One then defines a space of certain $\Sigma(X)$-valued functions on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ as

$$\begin{aligned} F\ (=F(X) = F(X_0, X_1)) &\\ = \{f : 0 \leqslant \operatorname{Re} z \leqslant 1 \to \Sigma \text{ satisfying} &\\ \text{(i)}\ & \text{bounded and continuous, and analytic} \\ & \text{in the interior (with respect to } \| \ \|_\Sigma), \\ \text{(ii)}\ & f(j+it) \in X_j,\ t \in \mathbb{R},\ j = 0, 1, \\ \text{(iii)}\ & \text{for } j = 0, 1, \text{ the map: } t \in \mathbb{R} \to f(j+it) \in X_j \\ & \text{is } \| \ \|_j\text{-continuous, and } \lim_{t \to \pm\infty} \|f(j+it)\|_j = 0\}. \end{aligned}$$

It follows from the Phragmén–Lindelöf theorem that the space F is a Banach space under the norm

$$|||f||| \ (=|||f|||_F) = \mathrm{Max}(\sup_{t\in\mathbb{R}} \|f(it)\|_0, \sup_{t\in\mathbb{R}} \|f(1+it)\|_1).$$

DEFINITION 1.1 [6, 7]. For each $0<\theta<1$, *the complex interpolation space* C_θ $(=C_\theta(X)=C_\theta(X_0, X_1))$ is the set of all $f(\theta)$, $f\in F$, equipped with *the complex interpolation norm*

$$\|x\|_\theta \ (=\|x\|_{C_\theta}) = \inf\{|||f|||; f\in F, f(\theta)=x\}.$$

We remark that $\| \ \|_0$ and $\| \ \|_1$ majorize $\| \ \|_\Sigma$ on X_0 and X_1, respectively. Therefore, the Phragmén–Lindelöf theorem implies

$$\|f(\theta)\|_\Sigma \leqslant |||f|||.$$

In particular, $C_\theta(X)$ is continuously included in $\Sigma(X)$. Also, being the quotient space F/K_θ (equipped with the quotient space norm), $C_\theta(X)$ is a Banach space. Here, K_θ is the *closed* subspace consisting of all $f\in F$ satisfying $f(\theta)=0$.

We now state two results which will be repeatedly used later. The first theorem is considered as an abstract version of the classical Riesz–Thorin interpolation theorem, while the second density result is a consequence of Fourier analysis.

THEOREM 1.2 [6, Theorem 4.1.2]. *Let* $X=(X_0, X_1)$ *and* $Y=(Y_0, Y_1)$ *be two pairs of compatible Banach spaces, and* T *be a linear operator from* $\Sigma(X)$ *to* $\Sigma(Y)$. *If* T *maps* X_0 *(resp.* X_1*) into* Y_0 *(resp.* Y_1*) with*

$$\|Tx_0\|_{Y_0} \leqslant M_0 \|x_0\|_{X_0}, \qquad x_0\in X_0$$

(resp. $\|Tx_1\|_{Y_1} \leqslant M_1 \|x_1\|_{X_1}$, $x_1\in X_1$,*) then* T *maps* $C_\theta(X)$ *into* $C_\theta(Y)$ *for each* $0<\theta<1$ *with*

$$\|Tx\|_{C_\theta(Y)} \leqslant M_0^{1-\theta} M_1^\theta \|x\|_{C_\theta(X)}, \qquad x\in C_\theta(X).$$

(In other words, C_θ is an exact interpolation functor of exponent θ [6].)

LEMMA 1.3 [6, Lemma 4.2.3]. *Let* $F_0(X)$ *be the set of all* $X_0\cap X_1$-*valued functions of the form*

$$f(z) = \exp(\lambda z^2) \sum_{n=1}^{N} \exp(\lambda_n z) x_n,$$

$\lambda>0$; $N\in\mathbb{N}_+$; $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N\in\mathbb{R}$; $x_1, x_2, x_3, \ldots, x_N\in X_0\cap X_1$. *Then* $F_0(X)$ *is dense in* $F(X)$.

So far we have been employing the standard definitions of the complex interpolation method found in [6, 7]. We now replace condition (iii) in $F(X)$ by a slightly weaker condition so that we will obtain a slightly larger F space. (Some discussions concerning them can be found in [42, Sect. 1.9].) We will consider the complex interpolation space constructed from this new F-space which is apparently larger than the space $C_\theta(X)$. We will show that this apparently larger space actually coincides with $C_\theta(X)$ with equal norms under mild assumptions. This fact will turn out to be our powerful tool later.

DEFINITION 1.4. Let F' $(=F'(X)=F'(X_0, X_1))$ be the set of all functions f: $0 \leqslant \operatorname{Re} z \leqslant 1 \to \Sigma$ satisfying conditions (i) and (ii) in $F(X)$, and

$$\text{(iii)}' \quad |||f||| \; (=\operatorname{Max}(\sup_{t\in\mathbb{R}} \|f(it)\|_0, \sup_{t\in\mathbb{R}} \|f(1+it)\|_1)) < \infty.$$

The complex interpolation space and its complex interpolation norm constructed (in the same way as explained in Definition 1.1) by using $F'(X)$ instead of $F(X)$ are denoted by $C'_\theta(X)$ and $\| \, \|'_\theta$, respectively.

As before, $C'_\theta(X)$ equipped with $\| \, \|'_\theta$ is a Banach space continuously included in $\Sigma(X)$. Since (iii)$'$ is weaker than (iii), one obviously has

$$C_\theta(X) \subseteq C'_\theta(X), \qquad \|x\|'_\theta \leqslant \|x\|_\theta, \quad x \in C_\theta(X) \tag{1}$$

for each $0 < \theta < 1$. The next result is an abstract version of some arguments in [18, 40] combined.

THEOREM 1.5. *Assume that the unit balls in X_0 and X_1 are closed in $\Sigma(X)$. Let Y $(\subseteq \Sigma(X)$ as a linear space) be a reflexive Banach space, and* $0 < \theta < 1$.

If we have

$$\begin{aligned} &C_\theta(X) \subseteq Y \subseteq C'_\theta(X), \\ &\|x\|'_\theta \leqslant \|x\|_Y, \qquad x \in Y, \\ &\|x\|_Y \leqslant \|x\|_\theta, \qquad x \in C_\theta(X), \end{aligned} \tag{2}$$

then $C_\theta(X) = C'_\theta(X) = Y$ with all equal norms. In particular, when at least one of X_0 and X_1 is reflexive (so that $C_\theta(X)$ is reflexive [7]), the $C_\theta(X) = C'_\theta(X)$ with equal norms.

Without any assumption, it is known that $C_\theta(X) = C'_\theta(X)$ with *equivalent* norms (see [42, Sect. 1.9]). However, the above result tells more. As far as our application is concerned, the interpolation space C'_θ constructed in Definition 1.4 is more natural than that in Definition 1.1 (see Remark 3.5). However, we cannot disregard the space C_θ, the reason being that $F_0(X)$ (in Lemma 1.3) is not dense in $F'(X)$.

To prove the theorem, we need the following "smoothing lemma" due to Haagerup [18].

LEMMA 1.6. *Assume that the unit balls of* X_0 *and* X_1 *are closed in* $\Sigma(X)$. *For any* f' *in* $F'(X)$, *there exists a sequence* $\{f'_n\}$ *of* $\Sigma(X)$-*valued functions on* $0 \leqslant \operatorname{Re} z \leqslant 1$ *such that each* f'_n *satisfies conditions* (i), (ii), *and*

(iii)″ *for* $j = 0, 1$, *the map*: $t \in \mathbb{R} \to f'_n(j + it) \in X_j$ *is* $\| \ \|_j$-*cotinuous*,

(iv) $|||f'_n||| \leqslant |||f'|||$,

and, furthermore, for each $0 < \theta < 1$ *the sequence* $\{f'_n(\theta)\}_n$ *satisfies*

(v) $\lim_{n \to \infty} \|f'_n(\theta) - f'(\theta)\|_\Sigma = 0$.

Proof. For each $n = 1, 2, \ldots$, we set

$$f'_n(z) = (n/\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-nt^2) f'(z - it)\, dt, \qquad 0 \leqslant \operatorname{Re} z \leqslant 1,$$

as a $\Sigma(X)$-valued Bochner integral. (Reall that $f'(z)$ satisfies (i).) The unit balls of X_0 and X_1 being closed in $\Sigma(X)$, each f'_n satisfies (ii). Also, (i) and (iv) are easily checked for f'_n because $\{(n/\pi)^{1/2} \exp(-nt^2)\}_n$ is an approximate unit in $L^1(\mathbb{R}; dt)$. Easy computations show that

$$f'_n(z) - f'_n(z') = (n/\pi)^{1/2} \int_{-\infty}^{\infty} [\exp\{-n(t - i(z - z'))^2\} - \exp(-nt^2)]\, f'(z' - it)\, dt \qquad (\operatorname{Re} z = \operatorname{Re} z')$$

$$f'_n(\theta) - f'(\theta) = (n/\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-nt^2)\{f'(\theta - it) - f'(\theta)\}\, dt.$$

The first (resp. second) equality guarantees that each f'_n satisfies (iii)′ (resp. (v)). Q.E.D.

COROLLARY 1.7. *Assume that the unit balls of* X_0 *and* X_1 *are closed in* $\Sigma(X)$ *and that* $s > r \geqslant 0$. *The closure* (*with respect to* $\| \ \|_\Sigma$) *of the ball* $(C_\theta)_s = \{x \in C_\theta;\ \|x\|_\theta \leqslant s\}$ *in* $\Sigma(X)$ *is larger than the ball* $(C'_\theta)_r = \{x \in C'_\theta;\ \|x\|'_\theta \leqslant r\}$.

Proof. Choose and fix an $x \in C'_\theta$ with $\|x\|'_\theta \leqslant r$. We then pick up an $f' \in F'(X)$ such that

$$x = f'(\theta), \qquad |||f'||| \leqslant s.$$

Applying the previous lemma to f', we obtain $\{f'_n\}$. We then set

$$f_n(z) = \exp((z^2 - \theta^2 - 1)/n)\, f'_n(z), \qquad n = 1, 2, \ldots, 0 \leqslant \operatorname{Re} z \leqslant 1.$$

Since $|\exp((z^2-\theta^2-1)/n)| \to 0$ as $\operatorname{Im} z \to \pm\infty$, each f_n belongs to $F(X)$ so that the sequence $\{f_n(\theta)\}_n$ is in $C_\theta(X)$. The above exponential factor is always majorized by 1 in modulus so that one estimates

$$\begin{aligned}\|f_n(\theta)\|_\theta &\leqslant |||f_n||| \\ &\leqslant |||f'_n||| \\ &\leqslant |||f||| \qquad \text{(due to (iv))} \\ &\leqslant s.\end{aligned}$$

Finally, $f_n(\theta) = \exp(-1/n)\, f'_n(\theta)$ and $\exp(-1/n)$ tends to 1 as $n \to \infty$ so that (v) implies

$$\lim_{n\to\infty} \|f_n(\theta) - x\|_\Sigma = \lim_{n\to\infty} \|f'_n(\theta) - f'(\theta)\|_\Sigma = 0. \qquad \text{Q.E.Q.}$$

Proof of Theorem 1.5. At first we prove $Y = C'_\theta$ with equal norms. We choose and fix an $x \in C'_\theta$ with $\|x\|'_\theta = r$. Because of Corollary 1.7, for any $s > r$, there exists a sequence $\{x_n\}$ in $(C_\theta)_s$ ($\subseteq Y_s$ due to (2)) satisfying

$$\lim_{n\to\infty} \|x_n - x\|_\Sigma = 0. \tag{3}$$

Since Y is reflexive, Alaoglu's theorem asserts that $\{x_n\}$ admits a $\sigma(Y, Y^*)$-accumulation point $y \in Y$, $\|y\|_Y \leqslant s$. Passing to a suitable subsequence, we may and do assume that $\{x_n\}$ tends to y in the $\sigma(Y, Y^*)$-topology. Since $Y \subseteq C'_\theta \subseteq \Sigma$ continuously (because of (2)), $\{x_n\}$ tends to y in the $\sigma(\Sigma, \Sigma^*)$-topology as well. Thus (3) implies $x = y$, that is, $x \in Y$ and $\|x\|_Y \leqslant s$. The arbitrariness of $s > r$ shows that

$$\|x\|_Y \leqslant r = \|x\|'_\theta.$$

Second, we prove $C_\theta = C'_\theta$ with equal norms. Again, we start from an $x \in C'_\theta$, $\|x\|'_\theta = r$. For each $s = \varepsilon + r > r$, we repeat the arguments in the first half and obtain a sequence $\{x_n\}$ in $(C_\theta)_s$ (a suitable subsequence of which) converges to $y \in (C'_\theta)_s = Y_s$ in the $\sigma(C'_\theta, C'^*_\theta)$-topology and $x = y$ (i.e., x is the single $\sigma(C'_\theta, C'^*_\theta)$-accumulation point of $\{x_n\}$). Thus, $\{x_n\}$ tends to x in the $\sigma(C'_\theta, C'^*_\theta)$-topology. In other words, $(C'_\theta)_r$ is included in the $\sigma(C'_\theta, C'^*_\theta)$-closure of $(C_\theta)_s$. However, $(C_\theta)_s$ being convex, the Hahn–Banach theorem implies

$$(C'_\theta)_r \subseteq \overline{(C_\theta)_{s=\varepsilon+r}},$$

where the closure is taken with respect to the norm $\|\ \|'_\theta$. Then Lemma 17.2, [41] yields

$$(C'_\theta)_r \subseteq (C_\theta)_{r+2\varepsilon}.$$

We thus conclude

$$C'_\theta \subseteq C_\theta, \qquad \|x\|_\theta \leqslant r + 2\varepsilon \quad (\varepsilon > 0),$$

that is,

$$\|x\|_\theta \leqslant r = \|x\|'_\theta. \qquad \text{Q.E.D.}$$

Finally, we combine Theorem 1.5 and Lemma 1.3 to obtain the next result.

THEOREM 1.8. *As before we assume that the unit balls of X_0 and X_1 are closed in $\Sigma(X)$. Let Y be a reflexive Banach space satisfying $X_0 \cap X_1 \subseteq Y \subseteq \Sigma(X)$ (as a linear space). Then $C_\theta(X) = Y = C'_\theta(X)$ with all equal norms provided that the following two conditions are fulfilled*:

(a) *for each y in Y there exists an $f \in F'(X)$ such that*

$$f(\theta) = y, \qquad |||f||| = \|y\|_Y,$$

(b) *each $g \in F_0(X)$ (described in Lemma* 1.3) *satisfies*

$$\|g(\theta)\|_Y \leqslant |||g|||.$$

Proof. Condition (a) immediately implies

$$Y \subseteq C'_\theta(X), \qquad \|y\|'_\theta \leqslant \|y\|_Y, \quad y \in Y.$$

One then considers the evaluation map $e_\theta \colon g \in F(X) \to g(\theta) \in \Sigma(X)$, which is obviously continuous (see the paragraph after Definition 1.1). Condition (b), the density of $F_0(X)$ in $F(X)$ (Lemma 1.3), and the completeness of Y imply

$$g(\theta) \in Y \qquad (\subseteq C'_\theta(X) \subseteq \Sigma(X) \text{ continuously}),\ g \in F(X),$$
$$\|g(\theta)\|_Y \leqslant |||g|||,$$

that is,

$$C_\theta(X) \subseteq Y,$$
$$\|x\|_Y \leqslant \inf\{|||g|||;\ g \in F(X),\ g(\theta) = x\}$$
$$= \|x\|_\theta, \qquad x \in C_\theta(X).$$

Thus the theorem follows from Theorem 1.5. Q.E.D.

2. *Relative Modular Theory*

In this section we collect some basic notations and results on relative modular operators [8, 10, 12]. Although these operators are usually defined and studied for faithful normal positive functionals (or rather weights) on von Neumann algebras, we will have to deal with these associated with non-faithful functionals as well. However, almost all known properties remain valid for non-faithful ones under the natural modification concerning the supports of functionals in question. Full details are found in [22, Chap. I].

Let M be a (σ-finite) von Neumann algebra. We fix an arbitrary ϕ and a faithful ϕ_0 in the positive part M_*^+ of the predual M_*. We take a standard form $(M, H, J, P^\natural)$ [2, 14], and let ξ, ξ_0 be unique implementing vectors in $P^\natural$ for ϕ and ϕ_0 respectively, that is, $\phi_0 = (\cdot\, \xi_0 \mid \xi_0)$ and $\phi = (\cdot\, \xi \mid \xi)$. Let p (resp. p') be the projection onto the closure of $M'\xi$ (resp. $M\xi$) so that $p = Jp'J \in M$ is the support projection of ϕ. Here, of course M' $(=JMJ)$ is the commutant of M being considered to act on the Hilbert space H. These are fixed throughout the section.

We now consider the four operators

$$a\xi_0 \in M\xi_0 \to a^*\xi_0 \in M\xi_0,$$
$$c\xi_0 \in pM\xi_0 \to c^*\xi \in Mp\xi,$$
$$b\xi \in Mp\xi \to b^*\xi_0 \in pM\xi_0,$$
$$d\xi \in pMp\xi \to d^*\xi \in pMp\xi.$$

They are (well-defined) densely defined closable (conjugate linear) operators form H to H, from pH to $p'H$, from $p'H$ to pH, and from $pp'H$ to $pp'H$, respectively. The first operator is exactly the usual S-operator determined by the pair (M, ϕ_0), [37]. Also the last operator is the S-operator determined by the *faithful* ϕ on the reduced von Neumann algebra pMp, which is isomorphic to $p'pMp$ acting standardly on $pp'H$.

Definition 2.1. The absolute value parts of the polar decompositions of the closures of the above four operators are denoted by $\Delta_{\phi_0}^{1/2}$, $\Delta_{\phi\phi_0}^{1/2}$, $\Delta_{\phi_0\phi}^{1/2}$, $\Delta_{\phi}^{1/2}$, respectively. The positive self-adjoint operator $\Delta_{\phi\phi_0}$ is called the relative modular operator (of ϕ relative to ϕ_0).

Remark 2.2. Obviously, Δ_{ϕ_0}, $\Delta_{\phi\phi_0}$, $\Delta_{\phi_0\phi}$, Δ_{ϕ} are non-singular positive self-adjoint operators on H, pH, $p'H$, and $pp'H$, respectively. However, in what follows, we will regard $\Delta_{\phi\phi_0}, \Delta_{\phi_0\phi}, \Delta_{\phi}$ as operators on H whose supports are pH, $p'H$, and $pp'H$, respectively.

We note that the phase parts of the polar decompositions considered in Definition 2.1 are all J. In fact, this follows from the fact that ξ and ξ_0 are in $P^\natural$ (see [2, Theorem 1]) Using the 2×2-matrix argument [8], one can easily prove

$$J\Delta_{\phi\phi_0}J=\Delta^{-1}_{\phi_0\phi},$$
$$\Delta_{\phi_0}\xi_0=\xi_0,\qquad \Delta_\phi\xi=\xi,$$
$$J\Delta^{1/2}_{\phi\phi_0}x\xi_0=x^*\xi,\qquad J\Delta^{1/2}_{\phi_0\phi}x\xi=x^*\xi_0,\quad x\in M, \tag{4}$$
$$M\xi_0 \text{ is a core for } \Delta^{1/2}_{\phi\phi_0}\quad (\text{as well as } \Delta^{1/2}_{\phi_0}),$$
$$\Delta_{\phi\phi_0}=\Delta_{\phi_0}\qquad \text{if}\quad \phi=\phi_0.$$

Also, the 2×2-matrix argument shows

LEMMA 2.3. (i) *For* $c\in M$, $t\in\mathbb{R}$,

$$\Delta^{it}_{\phi\phi_0}c\Delta^{-it}_{\phi_0}=\Delta^{it}_{\phi}c\Delta^{-it}_{\phi_0\phi}\in M.$$

(ii) *For* $d\in M$, $t\in\mathbb{R}$,

$$\Delta^{it}_{\phi\phi_0}d\Delta^{-it}_{\phi\phi_0}=\Delta^{it}_{\phi}d\Delta^{-it}_{\phi}\in M.$$

DEFINITION 2.4 [8, 10]. *For* $t\in\mathbb{R}$, *we set*

$$(D\phi;D\phi_0)_t=\Delta^{it}_{\phi\phi_0}\Delta^{-it}_{\phi_0},$$

the *Radon–Nikodym cocycle* (of ϕ relative to ϕ_0). More generally, for $c\in M$, $t\in\mathbb{R}$, we set

$$\sigma^{\phi\phi_0}_t(c)=\Delta^{it}_{\phi\phi_0}c\Delta^{-it}_{\phi_0}=(D\phi;D\phi_0)_t\sigma_t(c).$$

Here, $\sigma_t=A\ d\Delta^{it}_{\phi_0}$ is the modular automorphism on M determined by (M,ϕ_0) [37].

Due to the previous lemma, $(D\phi;D\phi_0)_t$, $t\in\mathbb{R}$, is a partial isometry in M with the initial (resp. final) projection $\sigma_t(p)$ (resp. p), and $\sigma^{\phi\phi_0}_t$ maps M into itself. The following relations are easily checked:

$$(D\phi;D\phi_0)_{t+s}=(D\phi;D\phi_0)_t\,\sigma_t((D\phi;D\phi_0)_s),\qquad t,s\in\mathbb{R},$$
$$(D\phi;D\phi_0)_t\sigma_t(x)(D\phi;D\phi_0)^*_t=\sigma^{\phi}_t(x)(=\Delta^{it}_{\phi}x\Delta^{-it}_{\phi}),\qquad x\in M.$$

We now state a "predual version" of the relative KMS condition [8, 10]. Certainly the next result is known, however, we present its proof for the sake of completeness and because of the fact that this result will play an important role throughout.

THEOREM 2.5. *For each* $x\in M$, *the map*: $t\in\mathbb{R}\to\sigma^{\phi\phi_0}_t(x)\phi_0$ $(=\phi_0(\cdot\ \sigma^{\phi\phi_0}_t(x)))\in M_*$ *extends to a bounded and continuous* (M_*-*valued*) *function* $f_x(z)$ *on* $-1\leqslant \operatorname{Im} z\leqslant 0$, *analytic in the interior. Here, the continuity and analyticity are understood with respect to the predual norm.*

Furthermore, for $z = -i + t$, $t \in \mathbb{R}$, *we have*

$$f_x(-i+t) = \phi\sigma_t^{\phi\phi_0}(x) \qquad (=\phi(\sigma_t^{\phi\phi_0}(x)) \cdot\,)).$$

Proof. We set

$$g(z) = (\cdot\, \Delta_{\phi\phi_0}^{iz} x\xi_0 \mid \xi_0) \in M_*, \qquad -\tfrac{1}{2} \leqslant \operatorname{Im} z \leqslant 0,$$
$$h(z) = (\cdot\, \xi \mid \Delta_{\phi_0\phi}^{1+i\bar{z}} x^*\xi) \in M_*, \qquad -1 \leqslant \operatorname{Im} z \leqslant -\tfrac{1}{2}.$$

Because of $x\xi_0 \in D(\Delta_{\phi\phi_0}^{1/2})$ and $x^*\xi \in D(\Delta_{\phi_0\phi}^{1/2})$ (see (4)), $g(z)$ and $h(z)$ are bounded and continuous functions, analytic in the interior. For $t \in \mathbb{R}$, $y \in M$, we compute

$$\begin{aligned}
(g(t))(y) &= (y\Delta_{\phi\phi_0}^{it} x\xi_0 \mid \xi_0)\\
&= (y\Delta_{\phi\phi_0}^{it} x\Delta_{\phi_0}^{-it}\xi_0 \mid \xi_0) \qquad (\Delta_{\phi_0}^{-it}\xi_0 = \xi_0)\\
&= (y\sigma_t^{\phi\phi_0}(x)\,\xi_0 \mid \xi_0) = (\sigma_t^{\phi\phi_0}(x)\,\phi_0)(y),\\
(h(-i+t))(y) &= (y\xi \mid \Delta_{\phi_0\phi}^{it} x^*\xi)\\
&= (y\xi \mid \Delta_{\phi_0\phi}^{it} x^*\Delta_{\phi}^{-it}\xi) \qquad (\Delta_{\phi}^{-it}\xi = \xi)\\
&= (y\xi \mid (\Delta_{\phi}^{it} x\Delta_{\phi_0\phi}^{-it})^*\xi)\\
&= (y\xi \mid (\Delta_{\phi\phi_0}^{it} x\Delta_{\phi_0}^{-it})^*\xi) \qquad \text{(Lemma 2.3(i))}\\
&= (\sigma_t^{\phi\phi_0}(x)\,y\xi \mid \xi) = (\phi\sigma_t^{\phi\phi_0}(x))(y),\\
(h(-\tfrac{1}{2}i+t))(y) &= (y\xi \mid \Delta_{\phi_0\phi}^{1/2+it} x^*\xi)\\
&= (\Delta_{\phi_0\phi}^{1/2}\, y\xi \mid \Delta_{\phi_0\phi}^{it} x^*\xi)\\
&= (Jy^*\xi_0 \mid \Delta_{\phi_0\phi}^{it} x^*\xi) \qquad \text{(due to (4))}\\
&= (J\Delta_{\phi_0\phi}^{it} x^*\xi \mid y^*\xi_0)\\
&= (J\Delta_{\phi_0\phi}^{it} J\Delta_{\phi\phi_0}^{1/2} x\xi_0 \mid y^*\xi_0) \qquad \text{(due to (4))}\\
&= (\Delta_{\phi\phi_0}^{it}\Delta_{\phi\phi_0}^{1/2} x\xi_0 \mid y^*\xi_0) \qquad \text{(due to (4))}\\
&= (y\Delta^{1/2+it} x\xi_0 \mid \xi_0)\\
&= (g(-\tfrac{1}{2}i+t))(y).
\end{aligned}$$

It thus follows from Morera's theorem that

$$\begin{aligned}
f_x(z) &= g(z) \qquad \text{if} \quad -\tfrac{1}{2} \leqslant \operatorname{Im} z \leqslant 0,\\
&= h(z) \qquad \text{if} \quad -1 \leqslant \operatorname{Im} z \leqslant -\tfrac{1}{2},
\end{aligned}$$

enjoys the properties stated in the theorem. Q.E.D.

Remark 2.6. Keeping the above result in mind, we may and do write $\sigma_z^{\phi\phi_0}(x)\phi_0$, $\sigma_z^{\phi_0}(x)\phi_0$, and $(D\phi; D\phi_0)_z\phi_0$, for $-1 \leqslant \operatorname{Im} z \leqslant 0$, as elements in M_* (although $\sigma_z^{\phi\phi_0}(x)$, $\sigma_z^{\phi_0}(x)$, and $(D\phi; D\phi_0)_z$ make no sense as elements in M). Of course we then have

$$\sigma_{-i+t}^{\phi\phi_0}(x)\,\phi_0 = \phi\sigma_t^{\phi\phi_0}(x),$$
$$\sigma_{-i+t}^{\phi_0}(x)\,\phi_0 = \phi_0\sigma_t^{\phi_0}(x),$$
$$(D\phi; D\phi_0)_{-i+t}\phi_0 = \phi(D\phi; D\phi_0)_t,$$

for each $t \in \mathbb{R}$.

3. *Non-commutative L^p-Spaces*

We define L^p-spaces, $1 < p < \infty$, associated with a given von Neumann algebra by using the complex interpolation method explained in Section 1. We then show that the L^2-space is a (standard) Hilbert space.

From now on, let ϕ_0 be a distinguished *faithful normal state* on a (σ-finite) von Neumann algebra M. We will define our L^p-spaces, $1 < p < \infty$, as complex interpolation spaces between the algebra M (="L^∞-space") and its predual M_* (="L^1-space"). Since we are dealing with a "non-commutative *probability* measure" ϕ_0, the L^∞-space M must be included in the L^1-space M_*. In other words, we have to imbed M into M_*.

As the above motivation suggests, we now imbed M into M_* via

$$x \in M \to x\phi_0 \in M_*, \tag{5}$$

and keep this imbedding throughout Part I. Shortly we will observe that this imbedding is fitting to the (relative) KMS condition which has been playing important roles in the recent development of the theory of operator algebras (see Remark 3.5). Also, another possibility of natural imbeddings will be studied in Part II.

Thus, M *is* a subspace of M_*, and we obtain the pair (M, M_*). An element $x = x\phi_0$ in M has the two norms

$$\|x\|_\infty = \|x\phi_0\|_\infty, \qquad \text{the uniform norm of } M,$$
$$\|x\|_1 = \|x\phi_0\|_1, \qquad \text{the predual norm of } M_*.$$

Of course, the imbedding does depend on a choice of ϕ_0. When there is possibility of confusion (Section 13), we write

$$\|x\phi_0\|_\infty^{\phi_0} \qquad (=\|x\|_\infty),$$
$$\|x\|_1^{\phi_0} \qquad (=\|x\phi_0\|_1).$$

Also, in such a case, we denote the pair (M, M_*) by (M^{ϕ_0}, M_*), that is, $M^{\phi_0} = M\phi_0$ is the imbedded image of M under (5).

The subspace M is dense in M_*, and we have

$$\|x\|_1 = \|x\phi_0\|_1 \leqslant \|x\|_\infty \|\phi_0\|_1 = \|x\|_\infty = \|x\phi_0\|_\infty \tag{6}$$

so that the pair (M, M_*) is compatible. Because M is included in M_*, the general construction of complex interpolation spaces explained in Section 1 is somewhat simplified. In fact, we have

$$\begin{aligned} \Sigma(M, M_*) &= M_*, \\ M \cap M_* &= M, \\ \|\ \|_\Sigma &= \|\ \|_1. \end{aligned} \tag{7}$$

DEFINITION 3.1. The complex interpolation space $C_{\theta = 1/p}(M; M_*)$ $(=C_{1/p}(M^{\phi_0}, M_*))$, $1 < p < \infty$, is denoted by $L^p(M; \phi_0)$, *the non-commutative L^p-space* associated with M (with respect to ϕ_0). For each $a \in L^p(M; \phi_0)$, the complex interpolation norm $\|a\|_{\theta = 1/p}$ is denoted by $\|a\|_p$, *the L^p-norm* of a $(L^1(M; \phi_0) = M_*$ and $L^\infty(M; \phi_0) = M)$.

Remark 3.2. Non-commutative L^p-spaces $L^p(M; \phi_0)$, $1 < p < \infty$, will be referred to as "left" L^p-spaces in Part II because we considered the "left" injection defined by (5). Since $\Sigma(M, M_*) = M_*$ ((7)), $L^p(M; \phi_0)$, $1 < p < \infty$, are realized inside of M_*. Furthermore, as a consequence of (6), we have, for $1 < p' < p < \infty$,

$$\begin{gathered} M\ (=M\phi_0) \subseteq L^p(M; \phi_0) \subseteq L^{p'}(M; \phi_0) \subseteq M_*, \\ \|x\|_\infty \geqslant \|x\|_p \geqslant \|x\|_{p'} \geqslant \|x\|_1, \qquad x \in M \end{gathered}$$

(see [6, Theorem 4.2.1(a), (b)]). We also note that M is dense in each $L^p(M; \phi_0)$, $1 < p < \infty$, due to [6, Theorem 4.1.2], while the intersection of L^p-spaces considered in [17] with different p's consists of zero alone.

The rest of the section will be devoted to prove the equivalence between the L^2-space and a standard Hilbert space. This fact will be crucially used in the next section. As in Section 2, let $(M, H, J, P^\natural)$ be a standard form and ξ_0 be the unique (unit) cyclic and separating vector in $P^\natural$ satisfying $\phi_0 = (\cdot\ \xi_0 \mid \xi_0) = \omega_{\xi_0}$. We then imbed M into H and H into M_* via

$$\begin{aligned} x &\to x\xi_0, \\ \zeta &\to (\cdot\ \zeta \mid \xi_0), \end{aligned}$$

respectively. If one combines these two, one obtains

$$x \to x\xi_0 \to (\cdot\ x\xi_0 \mid \xi_0) = x\phi_0.$$

which is exactly the imbedding (5). We clearly have

$$\|x\|_\infty \geqslant \|x\|_H = \|x\xi_0\|_H \geqslant \|x\|_1 = \|x\xi_0\|_1 = \|(\cdot\, x\xi_0 \mid \xi_0)\|_1. \tag{8}$$

THEOREM 3.3. *The non-commurative L^2-space $L^2(M; \phi_0)$ is the standard Hilbert space H with equal norms. Here, H is being imbedded into M_* via $\zeta \to (\cdot\, \zeta \mid \xi_0)$.*

Proof. The unit ball of M is closed in $\Sigma(M, M_*) = M_*$. (This can be proved easily. However, a more general fact will be proved later, Lemma 7.4.) The Hilbert space H being reflexive and satisfying $M = M \cap M_* \subseteq H \subseteq M_* = \Sigma(M, M_*)$, we have to check just conditions (a), (b) in Theorem 1.8.

We begin with (a). Choose and fix an element ζ in H with the polar decomposition $\zeta = u|\zeta|$ in the sense of [2, 14]; $(u \in M, |\zeta| \in P^\natural)$. We consider the M_*-valued function on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ defined by

$$f_\zeta(z) = \phi(1)^{(1/2)-z} u(D\phi; D\phi_0)_{-iz}\phi_0,$$

where $\phi = \omega_{|\zeta|} = (\cdot\, |\zeta| \,\big|\, |\zeta|)$. Obviously, f_ζ is a bounded and continuous M_*-valued function, which is analytic in the interior (see Theorem 2.5). Also, for $z = it$, $t \in \mathbb{R}$, we compute

$$\begin{aligned} f_\zeta(it) &= \phi(1)^{(1/2)-it} u(D\phi; D\phi_0)_t \phi_0 \in M(=M\phi_0), \\ \|f_\zeta(it)\|_\infty &= \phi(1)^{(1/2)} \|u(D\phi; D\phi_0)_t\|_\infty \\ &= \phi(1)^{(1/2)} \\ &= \||\zeta|\|_H = \|\zeta\|_H, \end{aligned}$$

while, for $z = 1 + it$, $t \in \mathbb{R}$, we compute

$$\begin{aligned} f_\zeta(1+it) &= \phi(1)^{-(1/2)-it} u(D\phi; D\phi_0)_{-i+t}\phi_0 \\ &= \phi(1)^{-(1/2)-it} u\phi(D\phi; D\phi_0)_t \qquad \text{(Theorem 2, 5)}, \\ \|f_\zeta(1+it)\|_1 &= \phi(1)^{-1/2} \|u\phi(D\phi; D\phi_0)_t\|_1 \\ &= \phi(1)^{-1/2} \|\phi\|_1 \\ &= \phi(1)^{1/2} = \|\zeta\|_H. \end{aligned}$$

Thus f_ζ belongs to $F'(M, M_*)$ (Definition 1.4) with

$$|||f_\zeta||| = \|\zeta\|_H. \tag{9}$$

Also we have

$$f_\zeta(\tfrac{1}{2}) = (\cdot\, \zeta \mid \xi_0),$$

which is exactly ζ imbedded into M_*. In fact, for any $x \in M$, one computes

$$\begin{aligned}(f_\zeta(it))(x) &= \phi(1)^{(1/2)-it}(U(D\phi; D\phi_0)_t \phi_0)(x)\\ &= \phi(1)^{(1/2)-it}(xu(D\phi; D\phi_0)_t \xi_0 \mid \xi_0)\\ &= \phi(1)^{(1/2)-it}(xu\, \Delta^{it}_{\phi\phi_0} \xi_0 \mid \xi_0)\end{aligned}$$

so that $(f_\zeta(\frac{1}{2}))(x)$ is computed by

$$\begin{aligned}(f_\zeta(\tfrac{1}{2}))(x) &= (xu\, \Delta^{1/2}_{\phi\phi_0} \xi_0 \mid \xi_0)\\ &= (xu\, |\zeta| \mid \xi_0) \qquad (\phi = \omega_{|\zeta|})\\ &= (x\zeta \mid \xi_0).\end{aligned}$$

To check (b), we take an arbitrary element g in $F_0(M, M_*)$ (Lemma 1.3). Namely, $g(z)$ is of the form $g(z) = g'(z)\, \phi_0$,

$$g'(z) = \exp(\lambda z^2) \sum_{n=1}^{N} \exp(\lambda_n z)\, x_n, \tag{10}$$

$\lambda > 0$; $\lambda_n \in \mathbb{R}$; $x_n \in M$ (not imbedded into M_*). For each $\zeta \in H$, we consider $f_\zeta(z)$ as in the first half of the proof, and set

$$H(z) = (f_\zeta(z))(g'(1-\bar{z})^*), \qquad 0 \leqslant \operatorname{Re} z \leqslant 1$$

$(f_\zeta(z) \in M_*,\ g'(1-\bar{z})^* \in M)$. We have already known that f_ζ is an M_*-valued bounded and continuous function, which is analytic in the interior. Using (10), we compute

$$H(z) = \exp(\lambda(1-z)^2) \sum_{n=1}^{N} \exp(\lambda_n(1-z))(f_\zeta(z))(x_n^*)$$

so that the numerical function $H(z)$ is bounded and continuous, analytic in the interior. To estimate its bound, we will use the Phragmén–Lindelöf theorem by considering its boundary values. For $z = 1 + it$, $t \in \mathbb{R}$, we estimate

$$\begin{aligned}H(1+it) &= (f_\zeta(1+it))(g'(it)^*),\\ |H(1+it)| &\leqslant \|f_\zeta(1+it)\|_1\, \|g'(it)^*\|_\infty\\ &= \|f_\zeta(1+it)\|_1\, \|g'(it)\|_\infty\\ &= \|f_\zeta(1+it)\|_1\, \|g'(it)\phi_0\|_\infty\\ &= \|f_\zeta(1+it)\|_1\, \|g(it)\|_\infty\\ &\leqslant |||f_\zeta|||\ |||g|||.\end{aligned}$$

For each $z = it$, $t \in \mathbb{R}$, we have

$$H(it) = (f_\zeta(it))(g'(1+it)^*).$$

Since $f_\zeta \in F'(M, M_*)$, $f_\zeta(it)$ is of the form

$$f_\zeta(it) = f'(it)\phi_0, f'(it) \in M \qquad \text{(not imbedded into } M_*\text{)},$$

we estimate

$$\begin{aligned} |H(it)| &= |(f'(it)\,\phi_0)(g'(1+it)^*)| \\ &= |\phi_0(g'(1+it)^* f'(it))| \\ &= |\phi_0(f'(it)^* g'(1+it))| \qquad (\phi_0(x^*) = \overline{\phi_0(x)}) \\ &= |(g'(1+it)\,\phi_0)(f'(it)^*)| \\ &\leqslant \| g'(1+it)\,\phi_0\|_1 \, \|f'(it)^*\|_\infty \\ &= \| g(1+it)\|_1 \, \|f'(it)\|_\infty \\ &\leqslant ||| g||| \, |||f_\zeta|||. \end{aligned}$$

We thus have

$$|H(z)| \leqslant ||| g||| \, |||f_\zeta||| = \|\zeta\|_H \, ||| g|||$$

for each $0 \leqslant \operatorname{Re} z \leqslant 1$ (due to (9)). In particular, with $z = \frac{1}{2}$, we have

$$|H(\tfrac{1}{2})| = |(f_\zeta(\tfrac{1}{2}))(g'(\tfrac{1}{2})^*)| \leqslant \|\zeta\|_H \, ||| g|||.$$

We recall that $f_\zeta(\frac{1}{2}) = (\cdot\, \zeta \mid \xi_0)$ so that

$$(f_\zeta(\tfrac{1}{2}))(g'(\tfrac{1}{2})^*) = (g'(\tfrac{1}{2})^*\zeta \mid \xi_0) = (\zeta \mid g'(\tfrac{1}{2})\xi_0),$$

that is,

$$|(\zeta \mid g'(\tfrac{1}{2})\xi_0)| \leqslant \|\zeta\|_H \, ||| g|||.$$

Since this is valid for each $\zeta \in H$, we have

$$\| g'(\tfrac{1}{2})\xi_0\|_H \leqslant ||| g|||.$$

However, since $g'(\frac{1}{2})\xi_0 \in H$ is identified with

$$(\cdot\, g'(\tfrac{1}{2})\xi_0 \mid \xi_0) = g'(\tfrac{1}{2})\phi_0 = g(\tfrac{1}{2}),$$

we have

$$\| g(\tfrac{1}{2})\|_H \leqslant ||| g|||. \qquad \text{(Q.E.D.)}$$

We close the section by stating the next remarks.

Remark 3.4. Theorem 1.8 actually shows that

$$C_{1/2}(M, M_*) \qquad (=L^2(M; \phi_0)) = C'_{1/2}(M, M_*) = H \ (=(\cdot\ H \mid \xi_0)).$$

For C_θ, $0 < \theta < 1$, many nice properties are known [6, 7], as we shall see in the next section. For example, the reiteration theorem [6, Theorem 4.6.1] implies

$$\begin{aligned} L^p(M; \phi_0) &= C_{2/p}(M, L^2(M, \phi_0)), && 2 < p < \infty, \\ L^p(M; \phi_0) &= C_{2/p-1}(L^2(M; \phi_0), M_*), && 1 < p < 2. \end{aligned} \tag{11}$$

In both interpolations, the "boundary" space $L^2(M; \phi_0)$ is reflexive. Thus a result of Calderón [7] implies that all $L^p(M; \phi_0)$, $1 < p < \infty$, are reflexive Banach spaces. In particular, Theorem 1.5 implies that one actually has

$$L^p(M; \phi_0) = C_{1/p}(M, M_*) = C'_{1/p}(M, M_*)$$

so that we can use either C_θ or C'_θ to deal with the L^p-spaces.

Remark 3.5. Usually, the complex interpolation method is used to identify some concrete function spaces with complex interpolation spaces between another such spaces. Therefore, the complex interpolation spaces themselves are regarded as quite abstract spaces. However, the reader might observe that, when the complex interpolation method is applied to (M, M_*), it is not abstract at all and actually fitting. For example, in the proof of $H = L^2(M, \phi_0)$ $(=C_{1/2} = C'_{1/2})$, we saw that any $\zeta = u\,|\zeta|$ $(u \in M, |\zeta| \in P^\natural$, $\phi = \omega_{|\zeta|})$ with $\|\zeta\|_H = \phi(1) = 1$ admits a "representing" function

$$f_\zeta(z) = u(D\phi; D\phi_0)_{-iz}\,\phi_0$$

in $F'(M, M_*)$ (but not in $F(M, M_*)$) satisfying

$$f_\zeta(\tfrac{1}{2}) = \zeta \ (=(\cdot\ \zeta \mid \xi_0)).$$

Furthermore, this f_ζ attains the norm $\|\zeta\|_H = \|\zeta\|'_{1/2} = \|\zeta\|_{1/2}$. (This situation remains valid for any $0 < \theta = 1/p < 1$ as we will see in Part II.) We note that this f_ζ is exaly a "relative KMS function," operator algebraists' favorite object. We thus come to the conclusion: The complex interpolation method C'_θ gives us a nice interpretation of relative KMS functions. The study of their behavior *inside of the strip* is exactly the study of non-commutative L^p-states $L^p(M; \phi_0) = C'_{1/p}(M, M_*)$, $1 < p < \infty$.

4. *Properties of Non-commutative L^p-spaces*

We exhibit some properties of our L^p-spaces. Having established Theorem 3.3, many of them are direct consequences of complex interpolation theory [6, 7].

We begin with the following standard result:

PROPOSITION 4.1. *Let α be an automorphism of M satisfying $\phi_0 \cdot \alpha = \phi_0$. Then, for each $1 \leqslant p < \infty$, α induces the (surjective) isometry $\alpha = \alpha_p$ on $L^p(M, \phi_0)$. Also, let ε be a normal projection of norm 1 from M onto its von Neumann subalgebra N satisfying $\phi_0 \circ \varepsilon = \phi_0$ (see [38]). Then, for each $1 \leqslant p < \infty$, ε induces the projection $\varepsilon = \varepsilon_p$ from $L^p(M; \phi_0)$ onto $L^p(N; \phi_0 = \phi_0|_N)$.*

Proof. Due to the invariance $\phi_0 \circ \alpha = \phi_0$, the map: $x\phi_0 \to \alpha(x)\phi_0$ induces the surjective isometry α_1 from $M_* = \Sigma(M, M_*)$ onto itself, which sends M into itself isometrically. Thus the result follows from Theorem 1.2.

The second assertion can be proved by similar arguments. Q.E.D.

Theorem 4.2.1(a) [6] and (11) in Remark 3.4 show

$$\begin{aligned} L^p(M; \phi_0) &= C_{2/p}(M, L^2(M, \phi_0)), && (2 < p < \infty), \\ L^p(M; \phi_0) &= C_{2/p-1}(L^2(M; \phi_0), M_*) && (1 < p < 2) \qquad (12) \\ &= C_{2(1-1/p)}(M_*, L^2(M; \phi_0)). \end{aligned}$$

The next result is the affirmative answer to Dixmier's question in [13], for which we will give an alternative proof in Section 5.

THEOREM 4.2. *For each $1 < p < \infty$, $L^p(M; \phi_0)$ is a uniformly convex Banach space.*

Proof. Being a Hilbert space, $L^2(M; \phi_0)$ is uniformly convex. Thus the result follows from (12) and Theorem A in the Appendix. Q.E.D.

THEOREM 4.3. *If $1/p + 1/q = 1$ and $1 < p < \infty$, then $L^p(M; \phi_0)$ is the dual Banach space of $L^q(M; \phi_0)$. Thus $L^p(M; \phi_0)$ is also uniformly smooth.*

Proof. Due to the reflexivity obtained in Remark 3.4 (or rather the previous theorem), we may and do assume $2 < p < \infty$. Since $M = (M_*)^*$ and $L^2(M; \phi_0)^* = L^2(M; \phi_0)$, the result, follows from (12) and the duality theorem [6, Theorem 4.5.1] together with the reflexivity of $L^2(M; \phi_0)$.

Q.E.D.

We finally state the following uniqueness theorem which will be proved in the last section (in Part II) by using the method employed in Part II.

THEOREM 4.4. *The space $L^p(M; \phi_0)$, $1 < p < \infty$, does not depend on a choice of ϕ_0 in the sense that, for another faithful $\phi_1 \in M_*$, $L^p(M; \phi_0)$ is isometrically isomorphic to $L^p(M; \phi_1)$.*

We remark that the corresponding result for Haagerup's L^p-spaces [17, Sect. 8] follows from a certain universality of a crossed product and the dual action on it [39]. On the other hand, our proof in Section 13 is based on relative modular theory (Section 2) and complex interpolation theory.

5. *L^p-Norm Inequalities*

Using the abstract Riesz–Thorin theorem (Theorem 1.2), we obtain certain L^p-norm inequalities. For each $1 \leqslant p,\ p' \leqslant \infty$, we consider the direct product $L^p(M;\phi_0) \times L^p(M;\phi_0)$ equipped with the norm

$$\|(a,b)\|_{pp'} = (\|a\|_p^{p'} + \|b\|_p^{p'})^{1/p'}.$$

As usual $\|(a,b)\|_{p\infty}$ should be understood as

$$\mathrm{Max}(\|a\|_p, \|b\|_p).$$

Throughout the section, this Banach space will be denoted by $E_p^{p'}$. Having defined our L^p-spaces as complex interpolation spaces, we immediately have the next useful result by [6, Theorem 5.1.6].

LEMMA 5.1. *For $1 \leqslant p,\ p',\ q,\ q' \leqslant \infty$ and $0 < \theta < 1$. The space $E_r^{r'}$ is exactly the complex interpolation space $C_\theta(E_p^{p'}, E_q^{q'})$ with equal norms. Here, r and r' are determined by*

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}, \quad \frac{1}{r'} = \frac{1-\theta}{p'} + \frac{\theta}{q'}.$$

PROPOSITION 5.2. *For, $a, b \in L^p(M;\phi_0)$, $2 \leqslant p \leqslant \infty$,*

$$(\|a+b\|_p^p + \|a-b\|_p^p)^{1/p} \leqslant \sqrt{2}\,(\|a\|_p^2 + \|b\|_p^2)^{1/2}.$$

In particular, with the classical Hölder's inequality, the following Clarkson's inequality holds:

$$(\|a+b\|_p^p + \|a-b\|_p^p)^{1/p} \leqslant 2^{1-1/p}(\|a\|_p^p + \|b\|_p^p)^{1/p}.$$

Proof. The first inequality means that the map $T\colon (a,b) \in E_p^2 \to (a+b, a-b) \in E_p^p$ has the norm less than $\sqrt{2}$. For the two extreme values, this is certainly the case. In fact, for $p = 2,\ \infty$, one computes

$$(\|a+b\|_2^2 + \|a-b\|_2^2)^{1/2} = \sqrt{2}\,(\|a\|_2^2 + \|b\|_2^2)^{1/2} \qquad \text{(Theorem 3.3)},$$

$$\mathrm{Max}(\|a+b\|_\infty, \|a-b\|_\infty) \leqslant \|a\|_\infty + \|b\|_\infty \leqslant \sqrt{2}\,(\|a\|_\infty^2 + \|b\|_\infty^2)^{1/2}.$$

Since the previous lemma shows

$$E_p^p = C_\theta(E_\infty^\infty, E_2^2), \qquad E_p^2 = C_\theta(E_\infty^2, E_2^2), \quad \theta = 2/p,$$

the result follows from Theorem 1.2. Q.E.D.

PROPOSITION 5.3. *For $a, b \in L^p(M; \phi_0)$, $1 \leqslant p \leqslant 2$, with $1/p + 1/q = 1$, the following McCarthy's inequality* [28, 34] *holds*:

$$(\|a+b\|_p^q + \|a-b\|_p^q)^{1/q} \leqslant 2^{1/q}(\|a\|_p^p + \|b\|_p^p)^{1/p}.$$

Proof. Lemma 5.1 shows

$$E_p^p = C_\theta(E_1^1, E_2^2), \qquad E_p^q = C_\theta(E_1^\infty, E_2^2), \quad \theta = 2/q.$$

On the other hand, the inequality is easily checked for the two extreme values $p = 2, \infty$. Thus, the map $T: E_p^p \to E_p^q$ considered in the proof of the preceding theorem has the norm less than

$$(2^0)^{1-2/q}(2^{1/2})^{2/q} = 2^{1/q}. \qquad \text{Q.E.D.}$$

Remark 5.4. The uniform convexity established in Theorem 4.2 is also a consequence of the above two inequalities. Certain strengthenings of Clarkson's inequality and applications of the uniform convexity will be obtained in [24].

We close the section by stating the next inequalities, which are reversed Clarkson–McCarthy inequalities. After replacing a, b by $a+b$, $a-b$, respectively, they can be proved by the same arguments as in the above two propositions.

PROPOSITION 5.5. *For $a, b \in L^p(M; \phi_0)$, $1 \leqslant p \leqslant 2$,*

$$(\|a+b\|_p^p + \|a-b\|_p^p)^{1/p} \geqslant \sqrt{2}\,(\|a\|_p^2 + \|b\|_p^2)^{1/2} \geqslant 2^{1-1/p}(\|a\|_p^p + \|b\|_p^p)^{1/p}.$$

For $a, b \in L^p(M; \phi_0)$, $2 \leqslant p \leqslant \infty$, with $1/p + 1/q = 1$,

$$(\|a+b\|_p^q + \|a-b\|_p^q)^{1/q} \geqslant 2^{1/q}(\|a\|_p^p + \|b\|_p^p)^{1/p}.$$

6. *Bimodule Structure*

In this section, we shall let M act on L^p-spaces from the left and the right so that the L^p-spaces turn out to be M-bimodules. A left action is easier because we imbedded M into M_* via the "left" action (5) which is consistent with the natural left actions of M on M and M_*.

We begin with a left action. W temporarily denote the natural left action of M on M_* by T. Namely, for a pair $(x, \phi) \in M \times M_*$, we set

$$T(x, \phi) = x\phi (= \phi(\cdot\, x)).$$

When $\phi = y\phi_0 \in M\phi_0$, we obviously have

$$x(y\phi_0) = (xy)\phi_0.$$

In other words, the restriction of T to $M \times M$ ($\subseteq M \times M_*$) is exactly the usual left multiplication in M. Furthermore, we have

$$\begin{aligned} &\|x\phi\|_1 \leqslant \|x\|_\infty \|\phi\|_1, \\ &\|x(y\phi_0)\|_\infty = \|xy\|_\infty \leqslant \|x\|_\infty \|y\|_\infty = \|x\|_\infty \|y\phi_0\|_\infty. \end{aligned} \tag{13}$$

Thus, a bilinear version of Theorem 1.2 [6, Theorem 4.1.1] implies that, for each $1 < p < \infty$, T induces the bilinear map T_p from $M \times L^p(M; \phi_0)$ to $L^p(M; \phi_0)$. In what follows, we will write $x \cdot a$ instead of $T_p(x, a)$. Of course we have

$$\|x \cdot a\|_p \leqslant \|x\|_\infty \|a\|_p$$

due to (13).

Remark 6.1. Let j_p denote the inclusion map from M into $L^p(M; \phi_0)$ ($\subseteq M_*$), that is, $j_p(y) = y\phi_0$ considered as an element in $L^p(M; \phi_0)$. If $a = j_p(y)$, $y \in M$, one obviously has

$$x \cdot a = x \cdot j_p(y) = j_p(xy)$$

from the construction. Clearly, this M-left action gives an M-left module structure on $L^p(M; \phi_0)$.

We now try to define a right action of $x \in M$. To avoid certain technical difficulties, we assume smoothness of this x for the modular automorphism group $\sigma_t = \sigma_t^{\phi_0}$. More precisely, let x be an element in M such that the map: $t \in \mathbb{R} \to \sigma_t(x) \in M$ extends to an entire function. We, however, remark that a right action of an arbitrary $x \in M$ can be constructed by the method used in Sections 11 and 13.

For this smooth $x \in M$ and $f \in F(M, M_*)$, we set

$$(\pi_x f)(z) = f(z)\, \sigma_{-i(z-1)}(x), \qquad 0 \leqslant \operatorname{Re} z \leqslant 1.$$

For $z = it$, $t \in \mathbb{R}$, $f(it)$ is of the form

$$f(it) = f'(it)\, \phi_0 \in M\phi_0,$$

and one computes

$$(\pi_x f)(it) = (f'(it)\,\phi_0)\,\sigma_{i+t}(x)$$
$$= f'(it)\,\sigma_t(x)\,\phi_0 \qquad \text{(KMS condition)},$$

which belongs to $M(=M\phi_0)$ and $\|\ \|_\infty$-continuous on $t \in \mathbb{R}$ due to the norm (=weak) analyticity of $\sigma_z(x)$. Furthermore, for $z = 0$, we have

$$(\pi_x f)(0) = f'(0)\,x\phi_0,$$

which is the imbedded image of the product $f'(0)x$ in M under (5). Also, for $z = 1 + it$, $t \in \mathbb{R}$, one computes

$$(\pi_x f)(1 + it) = f(1 + it)\,\sigma_t(x),$$
$$(\pi_x f)(1) = f(1)x,$$

the second of which means that x is acting on $f(1) \in M_*$ from the right. The above considerations show that $\pi_x f$ belongs to $F(M, M_*)$ and

$$|||\pi_x f||| \leqslant \|x\|_\infty\,|||f|||, \qquad f \in F(M, M_*).$$

Also, $f(\theta) = 0$ implies $(\pi_x f)(\theta) = 0$, $0 < \theta < 1$. Thus, passing to the quotient spaces, we have the induced map

$$\pi_x^p : a \in L^p(M; \phi_0) \to \pi_x^p(a) \in L^p(M; \phi_0),$$

and we rather write

$$a \cdot x = \pi_x^p(a).$$

The fact $(a \cdot x) \cdot y = a \cdot (xy)$ $(a \in L^p(M; \phi_0); x, y \in M)$ is proved from the automorphism property $\sigma_t(xy) = \sigma_t(x)\,\sigma_t(y)$.

Remark 6.2. For a (smooth) $x \in M$ and $a = j_p(y)$, $y \in M$ (see Remark 6.1), we have

$$a \cdot x = j_p(y) \cdot x = j_p(y\sigma_{-i/p}(x)).$$

(Notice that $y\sigma_{-i/p}(x)\phi_0 = (y\phi_0)\sigma_{-i(1/p-1)}(x)$.)

In Section 3 we established the equivalence between $L^2(M; \phi_0)$ and the standard Hilbert space H. We now check what the above left and right actions are for the L^2-space. When $L^2(M; \phi_0)$ and H are identified (Theorem 3.3), $x\phi_0 = (\cdot\, x\xi_0 \mid \xi_0) = j_2(x)$ (see Remark 6.1) in $L^2(M; \phi_0)$ $(\subseteq M_*)$ is identified with $x\xi_0 \in H$ so that we may write

$$j_2(y) = y\xi_0.$$

For this $\zeta = j_2(y) = y\xi_0$, the left and right actions of an $x \in M$ are computed by

$$\begin{aligned} x \cdot \zeta &= x \cdot j_2(y) \\ &= j_2(xy) \qquad \text{(Remark 6.1)} \\ &= xy\xi_0 = x\zeta, \\ \zeta \cdot x &= j_2(y) \cdot x \\ &= j_2(y\sigma_{-i/2}(x)) \qquad \text{(Remark 6.2)} \\ &= y\sigma_{-i/2}(x)\,\xi_0 \\ &= y\,\Delta_{\phi_0}^{1/2}\,x\xi_0 \\ &= yJx^*J\xi_0 \qquad \text{(see (4) in Section 2)} \\ &= Jx^*Jy\xi_0 = Jx^*J\zeta. \end{aligned}$$

Because $M\phi_0 = j_2(M)$ is dense in $H = L^2(M; \phi_0)$, the preceding computations remain valid for an arbitrary $\zeta \in H$. Thus, the left action of M on $L^2(M; \phi_0)$ is the usual action as operators, while the right action corresponds to the action of the commutant $M' = JMJ$ as operators.

Remark 6.3. The above facts suggest that the map J on (a dense subspace in) $L^2(M; \phi_0)$ should be defined by

$$J(x \cdot j_2(1)) = j_2(1) \cdot x^*,$$

or equivalently,

$$Jj_2(x) = j_2(\sigma_{-i/2}(x^*)).$$

Also the closure of the set of all $x \cdot j_2(1) \cdot x^* = j_2(x\sigma_{-i/2}(x^*))$ gives rise to the natural cone $L^2(M; \phi_0)_+$. In other words, the quadruple $(M, L^2(M; \phi_0), J, L^2(M; \phi_0)_+)$ is a standard form, where the action of M on $L^2(M; \phi_0)$ should be understood as the left action.

Before going further we note that the right action of M is not consistent with the inclusion

$$L^p(M; \phi_0) \subseteq L^{p'}(M; \phi_0), \qquad p \geqslant p'$$

(see Remark 3.2). In fact, for $p = \infty$, $p' = 2$, $x, y \in M$,

$$\begin{aligned} j_2(xy) &= xy\xi_0, \\ j_2(x) \cdot y &= Jy^*Jx\xi_0 = xJy^*\xi_0. \end{aligned}$$

Unless ξ_0 is a trace vector, these two vectors are different.

Remark 6.4. The right and left actions are dual to the each other, which will be made precise shortly. By checking how the duality theorem [6, Theorem 4.5.1] was applied to prove Theorem 4.3, one knows that, for $1/p + 1/q = 1$, the duality between $j_p(x) \in L^p(M; \phi_0)$ and $j_q(y) \in L^q(M; \phi_0)$ is realized by the bilinear form

$$\langle j_p(x), j_q(y) \rangle = (\sigma_{-i/p}(y)\, \phi_0)(x) = \phi_0(x\sigma_z(y))|_{z=-i/p}.$$

For simplicity, let us assume that $x, y, z \in M$ are smooth for σ_t. Then we have

$$\begin{aligned}
\langle j_p(x), j_q(y) \cdot z \rangle &= \langle j_p(x), j_q(y\sigma_{-i/q}(z)) \rangle \\
&= \phi_0(x\sigma_{-i/p}(y\sigma_{-i/q}(z))) \\
&= \phi_0(x\sigma_{-i/p}(y)\, \sigma_{-i}(z)) \\
&= \phi_0(zx\sigma_{-i/p}(y)) \qquad \text{(relative KMS condition)} \\
&= \langle j_p(zx), j_q(y) \rangle \\
&= \langle z \cdot j_p(x), j_q(y) \rangle.
\end{aligned}$$

It is possible to justify the above comutations for arbitrary $x, y, z \in M$ by using the multiple KMS condition [1].

II. Non-commutative Stein–Weiss Interpolation Theorem

7. *Another Imbeddings of M into M_**

In Section 3, we considered the "left" injection: $x \to x\phi_0$ (see (5)). In this section we investigate some another imbeddings of M into M_* so that we will obtain different complex interpolation spaces from those considered in Part I.

In Part II we fix two distinguished faithful normal states ϕ_0, ψ_0 on a von Neumann algebra M. For each $0 \leqslant \eta \leqslant 1$, we consider the imbedding of M into M_* defined by

$$x \in M \to \sigma_{-i\eta}^{\psi_0\phi_0}(x)\, \phi_0 \in M_*. \tag{14}$$

Definition 7.1. Keeping the imbedding (14) in mind (and fixing ϕ_0, ψ_0 throughout), we denote the pair consisting of $M(\hookrightarrow \sigma_{-i\eta}^{\psi_0\phi_0}(M)\phi_0 = M^\eta \subseteq M_*)$ and M_* by (M^η, M_*).

Thus $(M, M_*) = (M^{\phi_0}, M_*)$ in Section 3 is now (M^0, M_*). As in Section 3, we sometimes write

$$\|\sigma_{-i\eta}^{\psi_0\phi_0}(x)\,\phi_0\|_\infty^\eta = \|x\|_\infty,$$
$$\|x\|_1^\eta = \|\sigma_{-i\eta}^{\psi_0\phi_0}(x)\,\phi_0\|_1,$$

so that $\|\ \|_\infty^{\phi_0}$, $\|\ \|_1^{\phi_0}$ in Section 3 are $\|\ \|_\infty^0$, $\|\ \|_1^0$, respectively. In what follows, we never consider a power of $\|\ \|_\infty$, $\|\ \|_1$ so that the above notations will never make confusion. Since ϕ_0 and ψ_0 are states, one obtains

$$\|x\|_1^\eta = \|\sigma_{-i\eta}^{\psi_0\phi_0}(x)\,\phi_0\|_1 \leqslant \|x\|_\infty = \|\sigma_{-i\eta}^{\psi_0\phi_0}(x)\,\phi_0\|_\infty^\eta,$$

that is, $\|\ \|_1 \leqslant \|\ \|_\infty^\eta$ on the subspace M^η of M_*. Therefore, the pair (M^η, M_*) is compatible and satisfies

$$\Sigma(M^\eta, M_*) = M_*, \qquad M^\eta \cap M_* = M^\eta, \qquad \|\ \|_\Sigma = \|\ \|_1. \tag{15}$$

For the two extreme values $\eta = 0, 1$, we have

$$\sigma_0^{\psi_0\phi_0}(x)\,\phi_0 = x\phi_0 \qquad \text{(same as (5) in Section 3)},$$
$$\sigma_{-i}^{\psi_0\phi_0}(x)\,\phi_0 = \psi_0 x \qquad \text{(Theorem 2.5)},$$

which are the "left" and "right" injections, respectively.

DEFINITION 7.2. For each $1 < p < \infty$, the complex interpolation space $C_{\theta=1/p}(M^0, M_*)$ (resp. $C_{\theta=1/p}(M^1, M_*)$) equipped with the complex interpolation norm is denoted by $L^p(M; \phi_0)_L$ (resp. $L^p(M; \psi_0)_R$) and we call it *the left L^p-space* with respect to ϕ_0 (resp. *the right L^p-space* with respect to ψ_0). (Thus, $L^p(M; \phi_0)_L$ is exactly $L^p(M; \phi_0)$ defined in Section 3. However, the notation $L^p(M; \phi_0)_L$ will be mainly used in Part II.)

Therefore, all left and right L^p-spaces are subspaces of the predual M_*. The right L^p-spaces are assentially the left L^p-spaces of the commutant M'. More precisely, we have

PROPOSITION 7.3. *Let $(M, H, J, P^\natural)$ be a standard form, and ψ_0' be the state on $M' = JMJ$ defined by*

$$\psi_0'(x') = \psi_0(Jx'^*J), \qquad x' \in M'.$$

Then the right L^p-space $L^p(M; \psi_0)_R$ is isometrically isomorphic to the left L^p-space $L^p(M'; \psi_0')_L$ of the commutant M'.

Its proof is straightforward and we will not use this result later so that full details are left to the reader.

To use Theorem 1.8 in Section 1, we prepare the following result:

LEMMA 7.4. *The unit ball of* M^{η} (*with respect to* $\| \ \|_{\infty}^{\eta}$) *is closed in* M_* (*recall* (15)).

Before proving it, we prepare the next lemma on the harmonic measure (Poisson integral) for the strip $0 \leqslant \operatorname{Re} z \leqslant 1$, which will be used in later sections and in the Appendix.

LEMMA 7.5 [36]. *For* $z = \alpha + i\beta$ $(0 < \alpha < 1, \beta \in \mathbb{R})$, *and* $t \in \mathbb{R}$, *we set*

$$P_j(z, t) = (\tfrac{1}{2}) \sin(\pi\alpha)\{\cosh \pi(t - \beta) - (-1)^{-j} \cos \pi\alpha\}^{-1}, \qquad j = 0, 1.$$

If $f(z)$ *is a bounded and continuous function on* $0 \leqslant \operatorname{Re} z \leqslant 1$, *analytic in the interior, then we have the integral expression*

$$f(z) = \int_{-\infty}^{\infty} f(it)\, P_0(z, t)\, dt + \int_{-\infty}^{\infty} f(1 + it)\, P_1(z, t)\, dt, \qquad 0 < \operatorname{Re} z < 1.$$

Proof of Lemma 7.4. Assume that $\|x_n\|_{\infty} \leqslant 1$ $(x_n \in M)$ and

$$\lim_{n \to \infty} \|\sigma_{-i\eta}^{\psi_0\phi_0}(x_n)\, \phi_0 - \phi\|_1 = 0 \qquad \text{for} \quad \phi \in M_*.$$

Due to the σ-weak compactness of the unit ball of M (by passing to a suitable subsequence) we may and do assume $\{x_n\}$ tends to some $x \in M$ in the σ-weak topology. To complete the proof, it suffices to show $\phi = \sigma_{-i\eta}^{\psi_0\phi_0}(x)\phi_0$. Thus, it suffices to show that

$$\lim_{n \to \infty} \phi_0(y\sigma_{-i\eta}^{\psi_0\phi_0}(x_n)) = \phi_0(y\sigma_{-i\eta}^{\psi_0\phi_0}(x))$$

for each $y \in M$.

By the relative KMS condition (Theorem 2.5) and the above lemma, we have

$$\phi_0(y\sigma_{-i\eta}^{\psi_0\phi_0}(x_n)) = \int_{-\infty}^{\infty} \phi_0(y\sigma_t^{\psi_0\phi_0}(x_n))\, P_0(\eta, t)\, dt$$
$$+ \int_{-\infty}^{\infty} \psi_0(\sigma_t^{\psi_0\phi_0}(x_n)y)\, P_1(\eta, t)\, dt.$$

The integrands are estimated by

$$|\phi_0(y\sigma_t^{\psi_0\phi_0}(x_n))| \leqslant \|\phi_0\|_1 \|y\|_{\infty} \|x_n\|_{\infty} \leqslant \|y\|_{\infty},$$
$$|\psi_0(\sigma_t^{\psi_0\phi_0}(x_n)y)| \leqslant \|\psi_0\|_1 \|x_n\|_{\infty} \|y\|_{\infty} \leqslant \|y\|_{\infty}.$$

Also, for each fixed $t \in \mathbb{R}$, as $n \to \infty$ we have

$$\phi_0(y\sigma_t^{\psi_0\phi_0}(x_n)) \to \phi_0(y\sigma_t^{\psi_0\phi_0}(x)),$$
$$\psi_0(\sigma_t^{\psi_0\phi_0}(x_n)y) \to \psi_0(\sigma_t^{\psi_0\phi_0}(x)y),$$

because $\sigma_t^{\psi_0\phi_0}$ is σ-weakly continuous. We now recall

$$\int_{-\infty}^{\infty} P_0(\eta, t)\, dt = 1 - \eta,$$
$$\int_{-\infty}^{\infty} P_1(\eta, t)\, dt = \eta. \tag{16}$$

Lebesgue's dominated convergence theorem applied to these two finite measures implies

$$\lim_{n\to\infty} \phi_0(y\sigma_{-i\eta}^{\psi_0\phi_0}(x_n))$$
$$= \int_{-\infty}^{\infty} \phi_0(y\sigma_t^{\psi_0\phi_0}(x))\, P_0(\eta, t)\, dt + \int_{-\infty}^{\infty} \psi_0(\sigma_t^{\psi_0\phi_0}(x)y)\, P_1(\eta, t)\, dt$$
$$= \phi_0(y\sigma_{-i\eta}^{\psi_0\phi_0}(x)).$$ Q.E.D.

8. *Haagerup's L^p-Spaces*

As mentioned in the Introduction, there are several theories of non-commutative L^p-spaces. In this section, we briefly recall Haagerup's pioneering theory of L^p-spaces [17]. His L^p-spaces are equivalent to other L^p-spaces described in [4, 9, 19, 21, 40]. Also, when an algebra is semi-finite, they reduce to the classical L^p-spaces [13, 26, 29, 33], based on theory of traces.

Fixing ϕ_0 as usual and its associated modular automorphism group $\sigma_t = \sigma_t^{\phi_0}$, we consider the crossed product $R = M \times_\sigma \mathbb{R}$ [39]. (Actually, R and Haagerup's L^p-spaces described shortly do not depend on a choice of ϕ_0 due to [39, Theorem 8.1].) The crossed product R admits the distinguished faithful semi-finite normal trace τ (so that R is semi-finite) and the dual action θ_s, $s \in \hat{\mathbb{R}} = \mathbb{R}$ (scaling automophisms) on R satisfying $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$. For each semi-finite normal weight ϕ on M, we denote its dual weight on R by $\hat{\phi}$, [12, 15, 16, 39]. Let h_ϕ be the Radon–Nikodym derivative $d\hat{\phi}/d\tau$ of $\hat{\phi}$ relative to τ. Proving that h_ϕ is τ-measureble [29, 33], if and only if $\phi \in M_*^+$, Haagerup [17] defined his L^p-space, $1 \leqslant p \leqslant \infty$, by

$L^p = \{k$; τ-measurable operator (affiliated with R) satisfying
$\theta_s(k) = \exp(-s/p)k$, $s \in \mathbb{R}.\}$
$= \{k$; closed operator affiliated with R whose polar decomposition
$k = u\,|k|$ satisfies $u \in M$, $|k| = h_\phi^{1/p}$ for some (unique) $\phi \in M_*^+\}$.

To avoid confusion (by keeping M throughout) we shall denote the above Haagerup's L^p-space simply by L^p in the rest of the paper. (This notation is legitimate because his L^p-space does not depend on a choice of ϕ_0 as pointed out earlier.) Elements in L^p are added and multiplied freely by using the concept of strong sums and products [33].

The original algebra M (imbedded into R) is exactly the fixed point subalgebra R^θ of R under the dual action θ_s, $s \in \mathbb{R}$. Since all τ-measurable θ-invariant operators are bounded, one obtains $L^\infty = M$ as expected. Besices L^∞, L^1 and L^2 are of special importance. At first, L^1 is order isomorphic to the predual M_* via

$$k = u\,|k| = uh_\phi \to u\phi \qquad (|k| = h_\phi \text{ and } \phi \in M_*^+),$$

and the positive linear form

$$\text{tr}\colon k = uh_\phi \in L^1 \to (u\phi)(1) = \phi(u)$$

possesses the "tracial" property

$$\text{tr}(k_1 k_2) = \text{tr}(k_2 k_1), \qquad k_1 \in L^p,\, k_2 \in L^q,\ 1/p + 1/q = 1.$$

The positive functional tr is used to define a Banach space norm on L^p. Namely, we set

$$\|k\|_p = \text{tr}(|k|^p)^{1/p} \qquad (= \phi(1)^{1/p} \text{ if } k = uh_\phi^{1/p}), \tag{17}$$

and the duality between L^p and L^q $(1/p + 1/q = 1)$ is realized by the bilinear form

$$(k_1, k_2) \to \text{tr}(k_1 k_2) = \langle k_1, k_2 \rangle.$$

Second, L^2 is a Hilbert space under the inner product

$$(k_1, k_2) \to (k_1 \mid k_2) = \text{tr}(k_1 k_2^*).$$

Furthermore $(M, L^2, J = *, L^2_+)$ is a standard form. Here, L^2_+ is the "natural" cone consisting of all positive (as an operator) elements in L^2 and M is understood to act on L^2 as left multiplications.

Because of the universality of a standard form [2, 14], we may identify $(M, L^2, *, L^2_+)$ with $(M, H, J, P^\natural)$ used in Section 2. Then the unique implementing vector for $\phi = h_\phi \in M_* \cong L^1$ in the natural cone $L^2_+ = P^\natural$ is exactly $h_\phi^{1/2}$. Also, modular bjects in Section 2 are easily described as

$$\begin{aligned} (D\phi_1 ; D\phi_2)_t &= h_{\phi_1}^{it} h_{\phi_2}^{-it}, && t \in \mathbb{R}, \\ \sigma_t^{\phi_1\phi_2}(x) &= h_{\phi_1}^{it} x h_{\phi_2}^{-it}, && t \in \mathbb{R},\, x \in M. \end{aligned} \tag{18}$$

In many parts in Part II (exceptions are Theorem 11, 1 and Section 13) we identify $(M, L^2, *, L^2_+) = (M, H, J, P^\natural)$ and $M_* = L^1$. It is thus convenient for us to describe (M^η, M_*) introduced in Section 7 in terms of Haagerup's L^p-spaces L^p, $1 \leqslant p \leqslant \infty$. To do so, we denote h_{ϕ_0}, h_{ψ_0} simply by h_0, k_0, respectively.

At first, the imbedding (14) is

$$x \in M \to k_0^\eta x h_0^{1-\eta} \in L^1$$

(recall (18)) so that we have

$$\begin{aligned} M^\eta &= k_0^\eta M h_0^{1-\eta}, \\ \| k_0^\eta x h_0^{1-\eta} \|_\infty^\eta &= \|x\|_\infty, \qquad (19) \\ \|x\|_1^\eta &= \| k_0^\eta x h_0^{1-\eta} \|_1. \end{aligned}$$

Therefore, $L^p(M; \phi_0)_L = L^p(M; \phi_0)$ (resp. $L^p(M; \psi_0)_R$) is exactly $C_{1/p}(Mh_0, L^1)$ (resp. $C_{1/p}(k_0 M, L^1)$), where Mh_0(resp. $k_0 M$) is equipped with the norm

$$\| x h_0 \|_\infty^0 = \|x\|_\infty$$

(resp. $\| k_0 x \|_\infty^1 = \|x\|_\infty$).

9. *Certain Complex Interpolation Spaces*

We consider the pair (M^η, M_*), $0 \leqslant \eta \leqslant 1$, described in Section 7, and characterize the complex interpolation spaces $C_\theta(M^\eta, M_*)$, $0 < \theta < 1$, in terms of Haagerup's L^p-spaces L^p, $1 \leqslant p \leqslant \infty$.

For each $0 \leqslant \eta \leqslant 1$ we imbed L^p, $1 \leqslant p \leqslant \infty$, into L^1 via

$$i_p^\eta : a \in L^p \to k_0^{\eta/q} a g_0^{(1-\eta)/q} \in L^1 \qquad (20)$$

with the corresponding conjugate exponent q so that

$$i_p^\eta(L^p) = k_0^{\eta/q} L^p h_0^{(1-\eta)/q}.$$

We also write

$$\| i_p^\eta(a) \|_p = \| k_0^{\eta/q} a h_0^{(1-\eta)/q} \|_p = \|a\|_p \qquad (21)$$

(unless confusion occurs), where $\|a\|_p$ is defined by (17).

The rest of the section will be devoted to the proof of the following characterization of $C_\theta(M^\eta, M_*)$:

THEOREM 9.1. *For each $0 \leqslant \eta \leqslant 1$ and $1 < p < \infty$, the complex interpolation space $C_{\theta = 1/p}(M^\eta, M_*)$ of the pair defined in Definition 7.1 is (with* *equal norms) exactly* $i_p^\eta(L^p)$ $(\subseteq M_* \cong L^1)$ *with the norm* (21). *Or more precisely, when* L^1 *and* M_* *are identified,* $i_p^\eta(L^p)$ *is exactly the complex interpolation space* $C_\theta(k_0^{\eta/q}Mh_0^{(1-\eta)/q}, L^1)$, *where* $k_0^{\eta/q}Mh_0^{(1-\eta)/q}$ *is equipped with the norm* $\| \|_\infty^\eta$ *defined by* (19).

Remark 9.2. When $\eta = 0$ and $\eta = 1$, $C_{1/p}(M^\eta, M_*)$ reduces to $L^p(M; \phi_0)_L$ $(=L^p(M; \phi_0)$ in Section 3) and $L^p(M; \psi_0)_R$, respectively. Thus Theorem 9.1 shows that

$$L^p(M; \phi_0)_L = L^p h_0^{1/q}, \qquad L^p(M; \psi_0)_R = k_0^{1/q} L^p$$

with norms given by

$$\| a h_0^{1/q} \|_p = \| a \|_p, \qquad \| k_0^{1/q} a \|_p = \| a \|_p, \tag{21$'$}$$

respectively. Especially, when $\eta = 0$ and $p = 2$, the theorem reduces to Theorem 3.3.

Proof of Theorem 9.1. Thanks to Lemma 7.4 and the known reflexivity of L^p $(=i_p^\eta(L^p))$, one can use Theorem 1.8. The proof we shall give is a generalized version of that of Theorem 3.3. We will thus sketch arguments. However, for the reader's convenience, we repeat the definitions of $F'(k_0^\eta M h_0^{1-\eta}, L^1)$ (Defintion 1.4) and $F_0(k_0^\eta M h_0^{1-\eta}, L^1)$ (Lemma 1.3) in the present set up. Namely, we set

$F'(k_0^\eta M h_0^{1-\eta}, L^1)$

$= \{f: 0 \leqslant \operatorname{Re} z \leqslant 1 \to L^1$ satisfying

(i) bounded and continuous, analytic in the interior,

(ii) $f(it) = k_0^\eta f'(it)\, h_0^{1-\eta} \in k_0^\eta M h_0^{1-\eta}$, $t \in \mathbb{R}$,

(iii) $|||f||| = \operatorname{Max}(\sup_{t \in \mathbb{R}} \| f'(it) \|_\infty = \sup_{t \in \mathbb{R}} \| f(it) \|_\infty^\eta$, $\sup_{t \in \mathbb{R}} \| f(1+it) \|_1) < \infty \}$,

and

$F_0(k_0^\eta M h_0^{1-\eta}, L^1)$

$$= \left\{ g: 0 \leqslant \operatorname{Re} z \leqslant 1 \to L^1 \text{ of the form } g(z) = k_0^\eta g'(z)\, h_0^{1-\eta}, \right.$$
$$\left. g'(z) = \exp(\lambda z^2) \sum_{n=1}^{N} \exp(\lambda_n z)\, x_n, \lambda > 0; \lambda_n \in \mathbb{R}; x_n \in M \right\}.$$

We begin with checking condition (a) in Theorem 1.8. Choose and fix $a = u\,|a| = u h_\phi^{1/p} \in L^p$, which is identified with

$$i_p^\eta(a) = k_0^{\eta/q} u h_\phi^{1/p} h_0^{(1-\eta)/q} = k_0^{\eta/q} a h_0^{(1-\eta)/q}$$

(see (20)), and we set

$$f(z) = \phi(1)^{1/p - z} k_0^{\eta(1-z)} u h_\phi^z h_0^{(1-\eta)(1-z)}, \qquad 0 \leqslant \operatorname{Re} z \leqslant 1.$$

One then easily computes

$$f_a(it) \in k_0^\eta M h_0^{1-\eta}, \qquad \| f_a(it) \|_\infty^\eta = \| a \|_p,$$
$$f_a \in F'(k_0^\eta M h_0^{1-\eta}, L^1), \qquad \||f_a\|| = \| a \|_p.$$
$$f_a(1/p) = k_0^{\eta/q} a h_0^{(1-\eta)/q}.$$

Second, we check (b). However, we check it after replacing $\theta = 1/p$ by $1 - \theta = 1/q$. Obviously we have

$$k_0^\eta M h_0^{1-\eta} \subseteq i_q^\eta(L^q) = k_0^{\eta/p} L^q h^{(1-\eta)/p},$$

and, for each $g(z) = k_0^\eta g'(z) h_0^{1-\eta}$ in $F_0(M^\eta, M_*)$, we will prove

$$\| g(1/q) \|_q \leqslant \||g\||. \tag{22}$$

To do so, we take an arbitrary $a \in L^p$ and the corresponding $f_a(z)$ as in the first half of the proof. We then consider the bounded and continuous function

$$H(z) = \operatorname{tr}(f_a(z)\, g'(1 - \bar{z})^*)$$

on $0 \leqslant \operatorname{Re} z \leqslant 1$, which is analytic in the interior. As in the proof of Theorem 3.3, one computes

$$|H(z)| \leqslant \| a \|_p \||g\||.$$

In particular, with $z = 1/p$, we have

$$\begin{aligned} &|\operatorname{tr}(f_a(1/p)\, g'(1/q)^*)| \\ &\quad = |\operatorname{tr}(k^{\eta/q} a h_0^{(1-\eta)/q} g'(1/q)^*)| \\ &\quad = |\operatorname{tr}(a(k_0^{\eta/q} g'(1/q)\, h_0^{(1-\eta)/q})^*)| \leqslant \| a \|_p \||g\||. \end{aligned}$$

Thus, the duality between L^p and L^q implies

$$\|(k_0^{\eta/q} g'(1/q)\, h_0^{(1-\eta)/q})^*\|_q = \| k_0^{\eta/q} g'(1/q)\, h_0^{(1-\eta)/q} \|_q \leqslant \||g\||.$$

Since $k_0^{\eta/q} g'(1/q)\, h_0^{(1-\eta)/q}$ in L^q is identified with

$$i_q^\eta(k_0^{\eta/q} g'(1/q)\, h_0^{(1-\eta)/q}) = k_0^\eta\, g'(1/q)\, h_0^{1-\eta} = g(1/q),$$

the above inequality is precisely (22). Q.E.D.

We recall that Araki [2] introduced a one-parameter family $\{P^\alpha = P^\alpha_{\phi_0}\}$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, of positive cones in a standard Hilbert space. (Namely, P^α is the closure of $\Delta^\alpha M_+ \xi_0$ in the standard Hilbert space, where $\phi_0 = \omega_{\xi_0}$ and Δ is the corresponding modular operator.) We know (Theorem 3.3) that $L^2(M; \phi_0) = L^2(M; \phi_0)_L$ is a standard Hilbert space so that the cones are inside of $L^2(M; \phi_0)_L = L^2 h_0^{1/2}$ (see Remark 9.2). Let L^p_+ denote the positive part of the Haagerup's L^p-space. When $2 \leqslant p \leqslant \infty$,

$$L^p_+ h_0^{1/q} = L^p_+ h_0^{1/q - 1/2} h_0^{1/2} = L^p_+ h_0^{(1/2) - 1/p} h_0^{1/2}, \qquad \tfrac{1}{2} - 1/p \geqslant 0,$$

corresponds to $L^p_+ h_0^{(1/2) - 1/p}$ in L^2. It is easily checked that

$$(L^p_+ h_0^{(1/2) - 1/p})^- = \{k \in L^2;\, h_0^{(1/2) - 1/p} k \geqslant 0\}, \tag{23}$$

where the closure is taken in L^2. (For full details, see [20].) On the other hand, when $1 \leqslant p < 2$, we formally compute

$$L^p_+ h_0^{1/q} = L^p_+ h_0^{(1/2) - 1/p} h_0^{1/2} \qquad (\text{note } \tfrac{1}{2} - 1/p < 0)$$

(although $h_0^{(1/2) - 1/p}$ is not τ-measurable and this product does not generally make sense either). However, for a "smooth" part in L^p_+, the computation is justified and the closure of "$L^p_+ h_0^{(1/2) - 1/p}$" is characterized by

$$\{k \in L^2;\, k h_0^{1/p - (1/2)} \geqslant 0\}. \tag{24}$$

According to [20, Proposition 2.2], (23) and (24) are exactly P^α, $\alpha = 1/2p$, realized in L^2. Thus, we state

Remark 9.3. Let $P^{1/2p}_{\phi_0}$, $1 \leqslant p \leqslant \infty$, be the cones realized in the standard Hilbert space $L^2(M; \phi_0) = L^2(M; \phi_0)_L$. Then

$$\begin{aligned} L^p(M; \phi_0)_+ &= (P^{1/2p}_{\phi_0} \cap L^p(M; \phi_0))^- && \text{if} \quad 1 \leqslant p < 2, \\ &= P^{1/2p}_{\phi_0} \cap L^p(M; \phi_0) && \text{if} \quad 2 \leqslant p < \infty, \end{aligned}$$

gives us a reasonable definition of a positive part of our L^p-space. Here, the closure is taken with respect to $\| \; \|_p$ and $L^p(M; \phi_0) \subseteq L^2(M; \phi_0)$ (if $p \geqslant 2$) or $L^2(M; \phi_0) \subseteq L^p(M; \phi_0)$ (if $p \leqslant 2$) as explained in Remark 3.2. Full details and closely related subjects are found in [4, 20, 21, 23]. Later we shall treat the cones from a different viewpoint (Remark 12.4).

10. *Technical Lemmas*

We collect some technical lemmas which will be used in the next section. Especially we try to extend $\sigma_t^{\psi_0 \phi_0}$ ($k_0^{it} \cdot h_0^{-it}$ on L^∞) to a strongly continuous one-parameter group of isometries on L^p, $1 \leqslant p < \infty$.

LEMMA 10.1. *For each* $k \in L^p$, $1 \leqslant p \leqslant \infty$, *we have*

$$\|k_0^{it} k h_0^{-it}\|_p = \|k\|_p, \qquad t \in \mathbb{R}.$$

Proof. The space L^p is isometrically isomorphic to $L^p(M; \phi_0)$ via

$$i_p^0 : k \to k h_0^{1/q} \qquad (\in L^p(M; \phi_0) \subseteq M_* = L^1)$$

(see (19)). Furthermore, i_p^0 is compatible with $k_0^{it} \cdot h_0^{-it}$ in the sense that

$$k_0^{it} i_p^0(k)\, h_0^{-it} = i_p^0(k_0^{it} k h_0^{-it}).$$

Thus, thanks to Theorem 1.2, it suffices to check the equality for the two extreme values $p = 1, \infty$. However this is obvious because $\sigma_t^{\psi_0 \phi_0}$ is an isometry. Q.E.D.

LEMMA 10.2. *For each* $k \in L^p$, $1 \leqslant p < \infty$, *the map*: $t \in \mathbb{R} \to k_0^{it} k h_0^{-it} \in L^p$ *is norm-continuous.*

Proof. *Case* 1 $(p = 2)$. The space L^2 being a standard Hilbert space, the result is obvious because of

$$k_0^{it} k h_0^{-it} = \Delta_{\psi_0 \phi_0}^{it} k.$$

Case 2 $(1 \leqslant p < 2)$. Obviously the subspace $L^2 h_0^{1/p - (1/2)}$ is dense in L^p. Thus, thanks to Lemma 10.1, we may and do assume that k is of the form

$$k = k_1 h_0^{1/p - (1/2)}, \qquad k_1 \in L^2.$$

Then the result follows from Case 1 and Hölder's inequality.

Case 3 $(2 < p < \infty)$. Due to Lemma 10.1 and the uniform convexity of L^p (Clarkson's inequality), it suffices to check the weak continuity. However, this follows immediatedly from Case 2 with ϕ_0 and ψ_0 interchanged. Q.E.D.

Keeping the above two lemmas in mind, one can prove the next result by using the triangle inequality for $\| \ \|_p$.

COROLLARY 10.3. *If a map*: $t \in \mathbb{R} \to k(t) \in L^p$ *is norm-continuous, then so is the map*: $t \in \mathbb{R} \to k_0^{it} k(t)\, h_0^{-it} \in L^p$ $(1 \leqslant p < \infty)$.

For each $k \in L^p$, we can "regularize" k as

$$k_n = (n/\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-nt^2)\, k_0^{it} k h_0^{-it}\, dt, \qquad n = 1, 2, \ldots.$$

Lemmas 10.1, 10.2 then assert that $\|k_n\|_p \leqslant \|k\|_p$ and $\lim_{n\to\infty} \|k_n - k\|_p = 0$. Therefore we obtain the next two results. Actually, the second result is stronger than the first, however, we state both of them for later reference.

LEMMA 10.4. *The set of all analytic elements in L^p for $k_0^{it} \cdot h_0^{-it}$, that is,*

$$\{k \in L^p;\ a\ map\colon t \in \mathbb{R} \to k_0^{it} k h_0^{-it} \in L^p \text{ extends to an entire function}\}$$

is dense in $L^p (1 \leqslant p < \infty)$.

LEMMA 10.5. *Let N be the σ-weakly dense subspace in M of all analytic $x \in M$ for $\sigma_t^{\psi_0 \phi_0}$. Then $N\phi_0$ $(\subseteq L^p(M;\phi_0) \subseteq M_*)$ is dense in $L^p(M;\phi_0)$.*

To prove Lemma 10.5, one has to notice that $N\phi_0$ is dense is $M\phi_0$ with respect to the $\| \ \|_p$-norm and that $M\phi_0$ is dense in $L^p(M;\phi_0)$ (Remark 3.2).

11. *Non-commutative Stein–Weiss Interpolation Theorem*

In this section, we prove a non-commutative analogue of the classical Stein–Weiss interpolation theorem [35] which is our main result in Part II. Comments on the theorem will be collected in the next section.

We defined the left L^p-space $L^p(M;\phi_0)_L$ and the right L^p-space $L^p(M;\psi_0)_R$ is Section 7. We now characterize complex interpolation spaces between them. We feel that the following "Haagerup's L^p-space-free" statement is preferable.

THEOREM 11.1 (Non-commutative Stein–Weiss interpolation theorem). *For $0 < \eta < 1$ and $1 < p < \infty$, the complex interpolation space $C_\eta(L^p(M;\phi_0)_L, L^p(M;\psi_0)_R)$ is (with equal norms) the complex interpolation space $C_{1/p}(M^\eta, M_*)$. Here the pair (M^η, M_*) was defined in Definition* 7.2.

Relation between the theorem and the classical Stein–Weiss interpolation theorem will be explained in the next section. Every space involved in the theorem is a subspace of M_*. In the rest of the section, we identify $L^1 = M_*$ so that $\phi_0 = h_0$ and $\phi_0 = k_0$. Due to Theorem 9.1, Theorem 11.1 follows from the following result to the proof of which the rest of the section will be devoted.

THEOREM 11.2. *The complex interpolation space $C_\eta(L^p h_0^{1/q},\ k_0^{1/q} L^p)$ $(= C_\eta(L^p(M;\phi_0)_L,\ L^p(M\colon \psi_0)_R)$—see Remark 9.2) is (with equal norms) $i_p^\eta(L^p) = k_0^{\eta/q} L^p h_0^{(1-\eta)/q}$ ((20)) equipped with the norm (21). Here, the norms of $L^p h_0^{1/q}$ and $k_0^{1/q} L^p$ are defined by (21)$'$.*

Our strategy is to construct a natural isometric surjective mapping from $F(L^p h_0^{1/q}, L^p h_0^{1/q})$ onto $F(L^p h_0^{1/q}, k_0^{1/q} L^p)$ which (passing to respective quotient spaces) induces the surjective isometry from $L^p h_0^{1/q}$ onto $C_\eta = C_\eta(L^p h_0^{1/q}, k_0^{1/q} L^p)$. Then we will be able to compute C_η by using the explicit form of the above surjective isometry.

We use the following simplified terminologies only in the rest of the section:

$$\Sigma = \Sigma(L^p h_0^{1/q}, k_0^{1/q} L^p) = L^p h_0^{1/q} + k_0^{1/q} L^p$$

with the norm $\| \ \|_\Sigma$,

$F_1 = F(L^p h_0^{1/q}, L^p h_0^{1/q})$
$= \{ f(z) = f(z)' h_0^{1/q} \colon 0 \leqslant \operatorname{Re} z \leqslant 1 \to L^p h_0^{1/q}$ satisfying

(i) $f'(z)$ is a bounded and continuous L^p-valued function, analytic in the interior,

(iii) $\lim_{t \to \pm\infty} \|f'(z)\|_p = 0.\}$

with the norm

$$|||f|||_1 = \operatorname{Max}(\sup_{t \in \mathbb{R}} \|f'(it)\|_p, \sup_{t \in \mathbb{R}} \|f'(1+it)\|_p)$$

$(= \sup\{\|f'(z)\|_p; 0 \leqslant \operatorname{Re} z \leqslant 1\})$,

$F_2 = F(L^p h_0^{1/q}, k_0^{1/q} L^p)$
$= \{ g(z) \colon 0 \leqslant \operatorname{Re} z \leqslant 1 \to \Sigma$ satisfying

(i) bounded and continuous, analytic in the interior (with respect to $\| \ \|_\Sigma$),

(ii) $g(it) = g_0(it) h_0^{1/q} \in L^p h_0^{1/q}, t \in \mathbb{R}$,
$g(1+it) = k_0^{1/q} g_1(1+it) \in k_0^{1/q} L^p, t \in \mathbb{R}$,

(iii) For $j = 0, 1$, $g_j(j+it)$ is $\| \ \|_p$-continuous in $t \in \mathbb{R}$, and $\lim_{t \to \pm\infty} \| g_j(j+it)\|_p = 0.\}$

with the norm

$$|||g|||_2 = \operatorname{Max}(\sup_{t \in \mathbb{R}} \| g_0(it)\|_p, \sup_{t \in \mathbb{R}} \| g_1(1+it)\|_p).$$

Also, we will use letters f, f', g, g_0, g_1 in the way that they appear in the above definitions.

Let us start proving the theorem. The first step is to construct a linear mapping from F_1 to F_2. Using the harmonic measure $\{P_j(z,t)\}_{j=0,1}$ in Lemma 7.5, for each

$$f(z) = f'(z)\, h_0^{1/q} \in F_1,$$

we set

$$(\pi f)(z) = \int_{-\infty}^{\infty} k_0^{it/q} f'(it)\, h_0^{-it/q} h_0^{1/q} P_0(z, t)\, dt,$$

$$+ \int_{-\infty}^{\infty} k_0^{1/q} k_0^{it/q} f'(1+it)\, h_0^{-it/q} P_1(z, t)\, dt, \qquad \text{if} \quad 0 < \operatorname{Re} z < 1,$$

$$= k_0^{it/q} f'(it)\, h_0^{-it/q} h_0^{1/q} \qquad \text{if} \quad z = it,$$

$$= k_0^{1/q} k_0^{it/q} f'(1+it)\, h_0^{-it/q} \qquad \text{if} \quad z = 1 + it,$$

so that $(\pi f)(z)$ belongs to Σ. (Corollary 10.3 guarantees that the above two vector valued integrals make sense.)

Remark 11.3. Intuitively (when $f(z)$ is "smooth" enough), we have

$$(\pi f)(z) = k_0^{z/q} f'(z)\, h_0^{(1-z)/q} = k_0^{z/q} f(z)\, h_0^{-z/q}.$$

However, from this form it is difficult to observe $(\pi f)(z) \in \Sigma$.

Lemma 10.1 implies

$$|||\pi f|||_2 = |||f|||_1 \tag{25}$$

(although we have not yet known if πf is in F_2).

To show $f \in F_2$, we at first notice

$$\|(\pi f)(z)\|_\Sigma \leqslant |||f|||_1, \qquad f \in F_1. \tag{26}$$

(This follows from the above integral expression and the fact that on the boundaries of the strip $\| \;\|_\Sigma$ is majorized by $\| \;\|_p$.) We also have

LEMMA 11.4. *For each $f \in (F_1)_0$ described in Lemma 1.3, πf belongs to F_2.*

Proof. Due to the linearity of π, we may and do assume

$$f(z) = f'(z)\, h_0^{1/q},$$

$$f'(z) = \exp(\lambda z^2 + \mu z) h, \qquad \lambda > 0, \mu \in \mathbb{R}, h \in L^p.$$

Furthermore, due to Lemma 10.5 and (26) (and the completeness of F_2), we may assume that $h = x h_0^{1/p}$ with an $x \in N$ described in Lemma 10.5. For this smooth $f'(z)$, easy computation (see Remark 11.3) shows

$$(\pi f)(z) = \exp(\lambda z^2 + \mu z)\, \sigma_{-iz/q}^{\psi_0 \phi_0}(x)\, h_0.$$

From this, conditions (i)–(iii) in F_2 are easily checked. For example, since $z \to \sigma_{-iz/q}^{\psi_0\phi_0}(x) \in M$ is uniformly ($=\sigma$-weakly) analytic, (i) is fulfilled. Q.E.D.

LEMMA 11.5. *The map π is an isometry from F_1 to F_2.*

Proof. Since the uniform limit of continuous (resp. analytic) functions is continuous (resp. analytic), Lemma 11.4 and (26) (together with Lemma 1.3) show that πf, $f \in F_1$, satisfies (i) in F_2. Also, Lemma 10.1 and Corollary 10.3 guarantee that πf satisfies (ii) and (iii). Finally, because of (25), π is an isometry. Q.E.D.

The second step is to show the surjectivity of π. In fact, we construct its inverse mapping. Namely, for each $g(z)$ in F_2, we set

$$
\begin{aligned}
(\pi' g)(z) &= \int_{-\infty}^{\infty} k_0^{-it/q} g_0(it)\, h_0^{it/q} h_0^{1/q} P_0(z, t)\, dt \\
&\quad + \int_{-\infty}^{\infty} k_0^{-it/q} g_1(1+it)\, h_0^{it/q} h_0^{1/q} P_1(z, t)\, dt \qquad \text{if} \quad 0 < \operatorname{Re} z < 1, \\
&= k_0^{-it/q} g_0(it)\, h_0^{it/q} h_0^{1/q} \qquad \text{if} \quad z = it, \\
&= k_0^{-it/q} g_1(1+it)\, h_0^{it/q} h^{1/q} \qquad \text{if} \quad z = 1 + it
\end{aligned}
$$

so that $(\pi' g)(z) \in L^p h_0^{1/q}$. Again we have

$$
|||\pi' g|||_1 = |||g|||_2,
$$
$$
\|(\pi' g)(z)\|_p \leqslant |||g|||_2.
$$

Also, as a counterpart of Lemma 11.4, we have

LEMMA 11.6. *For each $g \in (F_2)_0$ (Lemma 1.3), $\pi' g$ belongs to F_1.*

Proof. As before we may assume

$$
\begin{aligned}
g(z) &= \exp(\lambda z^2 + \mu z) k, \qquad \lambda > 0; \mu \in \mathbb{R}, \\
k &= x_0 h_0^{1/q} = k_0^{1/q} x_1, \qquad x_0, x_1 \in L^p
\end{aligned}
\tag{27}
$$

so that we have

$$
\begin{aligned}
g_0(it) &= \exp(\lambda (it)^2 + \mu (it)) x_0, \\
g_1(1+it) &= \exp(\lambda (1+it)^2 + \mu (1+it)) x_1.
\end{aligned}
$$

Therefore, one computes $(\pi' g)(z) = f'(z)\, h_0^{1/q}$ with

$$f'(z) = \int_{-\infty}^{\infty} \exp(\lambda(it)^2 + \mu(it))\, k_0^{-it/q} x_0 h_0^{it/q} P_0(z, t)\, dt$$
$$+ \int_{-\infty}^{\infty} \exp(\lambda(1+it)^2 + \mu(1+it))\, k_0^{-it} x_1 h_0^{it/q} P_1(z, t)\, dt \qquad \text{if} \quad 0 < \operatorname{Re} z < 1,$$
$$= \exp(\lambda(it)^2 + (it))\, k_0^{it/q} x_0 h_0^{it/q} \qquad \text{if} \quad z = it,$$
$$= \exp(\lambda(1+it)^2 + \mu(1+it))\, k_0^{-it/q} x_1 h_0^{it/q} \qquad \text{if} \quad z = 1 + it.$$

For each $h \in L^q$ such that $h_0^z h k_0^{-z}$ is entire (those h's form a dense subspace in L^q due to Lemma 10.4), the mapping

$$z \to \langle f'(z), h\rangle$$

is entire. In fact,

$$H(it) = \langle f'(it), h\rangle = \exp(\lambda(it)^2 + \mu(it))\, \operatorname{tr}(k_0^{-it/q} x_0 h_0^{it/q} h)$$
$$= \exp(\lambda(it)^2 + \mu(it))\, \operatorname{tr}(x_0 h_0^{it/q} h k_0^{-it/q})$$

so that it extends to an entire function. Furthermore, for $z = 1 + it$, we have

$$H(1+it) = \exp(\lambda(1+it)^2 + \mu(1+it))\, \operatorname{tr}(x_0 h_0^{(1+it)/q} h k_0^{-(1+it)/q})$$
$$= \exp(\lambda(1+it)^2 + \mu(1+it))\, \operatorname{tr}(x_1 h_0^{it/q} h k_0^{-it/q}) \quad \text{(because of (27))}$$
$$= \exp(\lambda(1+it)^2 + \mu(1+it))\, \operatorname{tr}(k_0^{-it/q} x_1 h_0^{it/q} h)$$
$$= \langle f'(1+it), h\rangle.$$

Since $\|f'(z)\|_p \leqslant |||g|||_p$, the above computations show that $f'(z)$ is $\|\ \|_p$ (=weakly) entire and $\pi' g$ satisfies (i) in F_1. Also (iii) follows from Lemma 10.1. Q.E.D.

Thus, the same arguments as in the proof of Lemma 11.5 show

LEMMA 11.7. *The map π' is an isometry from F_2 to F_1.*

Obviously, $k_0^{it/q} \cdot h_0^{-it/q}$ in the definition of π and $k_0^{-it/q} \cdot h_0^{it/q}$ in that of π' cancel out with the each other so that $\pi' \circ \pi$ (resp. $\pi \circ \pi'$) is the identity mapping of F_1 (resp. F_2). More strongly, we have

LEMMA 11.8. *The map π is a surjective isometry from F_1 onto F_2. Furthermore, for each $0 < \theta < 1$ and $f \in F_1$, $f(\theta) = 0$ if and only if $(\pi f)(\theta) = 0$.*

Proof. We just prove that $f(\theta)=0$ implies $(\pi f)(\theta)=0$. Similar arguments imply this implification for π', which together with $\pi'\circ\pi=Id$ will yield the other direction.

Take and fix an $f(z)=f'(z)\,h_0^{1/q}\in F_1$ and assume that

$$f(\theta)=0, \text{ that is, } f'(\theta)=0 \text{ in } L^p.$$

For each $x\in M$ such that $h_0^z x k_0^{-z}$ is entire, easy computation shows

$$((\pi f)(z))(x)=\operatorname{tr}(f'(z)\,h_0^{1/q}\sigma_{iz/q}^{\phi_0\omega_0}(x))$$

so that $f'(\theta)=0$ implies $(\pi f)(\theta)=0$. Q.E.D.

End of the proof of Theorem 11.2. Having obtained the above lemma, (by passing to the quotient spaces) π induces the surjective isometry π_η from $L^p h_0^{1/q}$ onto C_η. For a smooth $h\in L^p$ for $k_0^{it}\cdot h_0^{-it}$, we consider

$$f(z)=\exp(z^2-\eta^2)\,hh_0^{1/q}\in F_1,$$
$$(f(\eta)=hh_0^{1/q}).$$

We then have

$$\begin{aligned}\pi_\eta(hh_0^{1/q})&=\pi_\eta(f(\eta))\\&=(\pi f)(\eta)\\&=\exp(z^2-\eta^2)\,k_0^{z/q}hh_0^{-z/q}|_{z=\eta}\qquad\text{(recall Remark 11.3)}\\&=k_0^{\eta/q}hh_0^{(1-\eta)/q}\end{aligned}$$

so that

$$k_0^{\eta/q}hh_0^{(1-\eta)/q}\in C_\eta,$$
$$\|k_0^{\eta/q}hh_0^{(1-\eta)/q}\|_{C_\eta}=\|hh_0^{1/q}\|_p=\|h\|_p,$$

the map π_η being isometric. Thus the density of smooth h's (Lemma 10.4) shows that

$$C_\eta=k_0^{\eta/q}L^p h_0^{(1-\eta)/q},$$
$$\|k_0^{\eta/q}hh_0^{(1-\eta)/q}\|_{C_\eta}=\|h\|_p,\qquad h\in L^p.$$

In other words, C_η is exactly $i_p^\eta(L^p)$. Q.E.D.

12. *Remarks*

To understand Theorem 11.1 better, we specialize ourselves to the abelian von Neumann algebra $M=L^\infty(\mathbb{R};dt)$ (acting standardly on the Hilbert space $L^2(\mathbb{R}; dt)$) with the predual $M_* = L^1(\mathbb{R}; dt)$. We fix strictly positive functions $h_0(t)$ and $k_0(t)$ in $L^1(\mathbb{R}; dt)$ so that

$$\phi_0(\cdot) = \int_{-\infty}^{\infty} \cdot\, h_0(t)\, dt, \qquad \psi_0(\cdot) = \int_{-\infty}^{\infty} \cdot\, k_0(t)\, dt,$$

give rise to faithful normal linear functionals on M. Then, for each $0 \leqslant \eta \leqslant 1$, $\sigma_{-i\eta}^{\psi_0\phi_0}(\cdot)\phi_0$, (in (14)) is computed by

$$\begin{aligned} f(t) \in L^\infty(\mathbb{R}; dt) \to k_0(t)^{i(-i\eta)} f(t)\, h_0(t)^{i(i\eta)} h_0(t) \\ = f(t)\, h_0(t)^{(1-\eta)} k_0(t)^\eta \in L^1(\mathbb{R}; dt). \end{aligned}$$

Therefore, easy computation (or Theorem 9.1) shows that the complex interpolation space $C_{1/p}(M^\eta, M_*)$ considered in Theorem 9.1 is $L^p(R; dt)\, h_0^{(1-\eta)/q} k_0^{\eta/q}$. A function

$$f(t) = g(t)\, h_0(t)^{(1-\eta)/q} k_0(t)^{\eta/q}$$

belongs to this space if and only if

$$\left\{\int_{-\infty}^{\infty} |g(t)|^p\, dt\right\}^{1/p} = \left\{\int_{-\infty}^{\infty} |f(t)|^p h_0(t)^{-p(1-\eta)/q} k_0(t)^{-p\eta/q}\, dt\right\}^{1/p} < \infty.$$

In other words, we have

$$C_{1/p}(M^\eta, M_*) = L^p(\mathbb{R}; h_0(t)^{-p(1-\eta)/q} k_0(t)^{-p\eta/q}\, dt).$$

Also, considering the two special values $\eta = 0, 1$, we know

$$\begin{aligned} L^p(M; \phi_0)_L = L^p(\mathbb{R}; h_0(t)^{-p/q}\, dt), \\ L^p(M; \psi_0)_R = L^p(\mathbb{R}; k_0(t)^{-p/q}\, dt). \end{aligned}$$

Therefore, introducing

$$w_0(t) = h_0(t)^{-p/q}, \qquad w_1(t) = k_0(t)^{-p/q},$$

we conclude that

$$C_\eta(L^p(\mathbb{R}; w_0(t)\, dt), L^p(\mathbb{R}; w_1(t)\, dt)) = L^p(\mathbb{R}; w_0(t)^{1-\eta} w_1(t)^\eta\, dt).$$

This is known as the Stein–Weiss interpolation theorem [35].

We now return to a general von Neumann algebra M and give some comments.

Remark 12.1. As an analogue of a general form of the classical Stein–Weiss interpolation theorem, we believe that one can also characterize

$C_\eta(L^p(M;\phi_0)_L, L^{p'}(M,\psi_0)_R)$ with different ϕ_0, ψ_0 and different p, p'. However, the case $p = p'$ seems to be more interesting as subsequent remarks in the section show.

Remark 12.2. One might equally be interested in characterizing interpolation spaces between two left (or two right) L^p-spaces. It is likely that a reasonable result is available only when ϕ_0 and ψ_0 commute in the sense of [37, Sect. 15]. We thus believe that this is an ill-posed question. We would like to remind the reader that Pusz and Woronowicz [30] obtained a certain interesting object (purification) by "interpolating" left and right sesquilinear forms (instead of two left sesquilinear forms; see Remark 12.5).

Remark 12.3. When $\eta = \frac{1}{2}$ and $\phi_0 = \psi_0$ in Theorem 11.1, one obtains the "averages" $C_{1/2}(L^p(M;\phi_0)_L, L^p(M;\phi_0)_R) = i_p^{1/2}(L^2) = h_0^{1/2q} L^2 h_0^{1/2q}$ between left and right L^p-spaces. These are exactly Terp's L^p-spaces [40] constructed by using the "symmetric" injection: $x \to (\phi_0)_x$. (in her case, ϕ_0 can be a weight.) In fact, if π is the GNS representation induced by ϕ_0 (and ξ_0 is the corresponding cyclic and separating vector and $\Lambda(x) = \pi(x)\xi_0 = \pi(x)\Lambda(1)$), the injection $M \hookrightarrow M^{1/2} \subseteq M_*$ for $\eta = \frac{1}{2}$ is given by

$$x \to \sigma_{-i/2}(x)\phi_0,$$

and one computes

$$\begin{aligned}(\sigma_{-i/2}(x)\,\phi_0)(z^*y) &= \phi_0(z^*y\sigma_{-i/2}(x))\\ &= (\pi(z^*y)\,\Delta_{\phi_0}^{1/2}\pi(x)\,\xi_0 \mid \xi_0)\\ &= (\pi(y)\,J\pi(x^*)\,J\xi_0 \mid \pi(z)\,\xi_0)\\ &= (J\pi(x^*)\,J\Lambda(y) \mid \Lambda(z)), \qquad y, z \in M,\end{aligned}$$

which is exactly $(\phi_0)_x(z^*y)$ (see [40, p. 49]).

Remark 12.4. When $p = 2$ and $\psi_0 = \phi_0$, we have a one-parameter family $C_\eta(L^2(M;\phi_0)_L, L^2(M;\phi_0)_R)$, $0 \leqslant \eta \leqslant 1$, of Hilbert spaces. All of them are standard Hilbert spaces. In fact, the proof of Theorem 11.1 shows that

$$\begin{aligned}\pi_\eta : kh_0^{1/2} \in L^2(M;\phi_0)_L = L^2(M;\phi_0) = L^2 h_0^{1/2}\\ \to k_0^{\eta/2} k h_0^{(1-\eta)/2} \in C_\eta\end{aligned}$$

gives a surjective isometry, and the isomorphism between $L^2(M;\phi_0)$ and L^2 is given by

$$i_2^0 : k \in L^2 \to kh_0^{1/2} \in L^2(M;\phi_0)$$

(see (19)). Therefore,

$$\pi_\eta \circ i_2^0 : k \in L^2 \to k_0^{\eta/2} k h_0^{(1-\eta)/2} \in L^2(M;\phi_0) \tag{28}$$

is an isomorphism. We also recall that when we constructed $C_\eta(M^\eta, M_*)$ we considered the injection

$$x \in M \to \sigma_{-i\eta}(x)\,\phi_0 = h_0^\eta x h_0^{1-\eta} \in L^1 = M_*.$$

Considering $h_0^\eta x h_0^{1-\eta}$ as an element in $C_{1/2}(M^\eta, M_*)$, we now write

$$j_2^\eta : x \in M \to h_0^\eta x h_0^{1-\eta} \in C_{1/2}(M^\eta, M_*),$$

and Theorem 11.1 asserts

$$C_{1/2}(M^\eta, M_*) = C_\eta(L^2(M; \phi_0)_L, L^2(M; \psi_0)_R).$$

The image $j_2^\eta(M_+) = h_0^\eta M_+ h_0^{1-\eta}$ corresponds (by (28)) to

$$(\pi_\eta \circ i_2^0)^{-1}(h_0^\eta M_+ h_0^{1-\eta}) = h_0^{\eta/2} M_+ h_0^{(1-\eta)/2} = \Delta_{\phi_0}^{\eta/2} M_+ h_0^{1/2}$$

in the standard Hilbert space L^2 so that the closure of $j_2^\eta(M_+)$ in the standard Hilbert space C_η is $P_{\phi_0}^{\eta/2}$ (recall Remark 9.3). Thus, Theorem 11.1 asserts that the cones $P_{\phi_0}^{\eta/2}$, $0 \leqslant \eta \leqslant 1$, are obtained by "interpolating" the two extreme cones $P_{\phi_0}^0$ (in $L^2(M; \phi_0)_L$) and $P_{\phi_0}^{1/2}$ (in $L^2(M; \phi_0)_R$). A more precise meaning of the above "interpolating" seems to deserve further investigation.

Remark 12.5. Finally we consider the case $p = 2$, $\phi_0 \neq \psi_0$. As in the previous remark, we write

$$\begin{aligned} j_2^\eta : x \in M \to k_0^\eta x h_0^{1-\eta} &\in C_\eta(L^2(M; \phi_0)_L, L^2(M; \psi_0)_R) \\ &= k_0^{\eta/2} L^2 h_0^{(1-\eta)/2} = i_2^\eta(L^2) \qquad \text{(see (19))}. \end{aligned}$$

Since $j_2^\eta(x) = k_0^\eta x h_0^{1-\eta} = i_2^\eta(k_0^{\eta/2} x h_0^{(1-\eta)/2})$, the norm of $j_2^\eta(x)$ is given by (see (21)),

$$\begin{aligned} \| k_0^{\eta/2} x h_0^{(1-\eta)/2} \|_2 &= \mathrm{tr}(| k_0^{\eta/2} x h_0^{(1-\eta)/2} |^2) \\ &= \mathrm{tr}(k_0^\eta x h_0^{1-\eta} x^*). \end{aligned}$$

Thus, the Hilbert space C_η may be regarded as the completion of M equipped with the quadratic form

$$x \in M \to \mathrm{tr}(k_0^\eta x h_0^{1-\eta} x^*) \in [0, \infty) \qquad (= \| \Delta_{\psi_0 \phi_0}^{\eta/2} x \xi_0 \|^2). \tag{29}$$

In particular, with $\eta = 0, 1$, $L^2(M; \phi_0)_L$ and $L^2(M; \psi_0)_R$ are the completions of M with respect to

$$x \in M \to \mathrm{tr}(h_0 x^* x) = \phi_0(x^* x), \tag{29$'$}$$

$$x \in M \to \mathrm{tr}(k_0 x x^*) = \psi_0(x x^*), \tag{29$''$}$$

respectively. The reader may notice that (29)′ and (29)″ are left and right forms considered by Pusz and Woronowicz [30, 31, 43], and also that (29) is a form appearing typically in the Wigner–Yanase–Dyson–Lieb concavity [3, 27, 31, 34, 43]. Namely, this concavity states that

$$\alpha\phi_1 + \beta\phi_2 \leqslant \phi_3, \qquad \alpha\psi_1 + \beta\psi_2 \leqslant \psi_3 \quad (\alpha, \beta \geqslant 0; \phi_i, \psi_i \in M_*^+)$$

imply

$$\alpha\,\mathrm{tr}(h_{\psi_1}^{\eta} x h_{\phi_1}^{1-\eta} x^*) + \beta\,\mathrm{tr}(h_{\psi_2}^{\eta} x h_{\phi_2}^{1-\eta} x^*) \leqslant \mathrm{tr}(h_{\psi_3}^{\eta} x h_{\phi_3}^{1-\eta} x^*). \tag{30}$$

We know (Theorem 11.1) that the form (29) (or more precisely, its associated quadratic norm) can be obtained as the complex interpolation form between (29)′ and (29)″. Also, for the "boundary" forms (29)′ and (29)″, (30) is obviously valid (with the equality). Hence, the Wigner–Yanase–Dyson–Lieb concavity may be interpreted as "concave dependence of interpolated norms on boundary datas." This viewpoint is being taken by Uhlmann [43] and he showed that the above described concave dependence for a certain quadratic interpolation functor (IQ_θ in [3]), which is actually the complex interpolation functor C_θ as the above discussion shows. Further analysis (including certain generalizations and simplifications) will appear elsewhere [25].

13. *Proof of Uniqueness Theorem*

This section is devoted to the proof of Theorem 4.4 so that we return to terminologies used in Part I. Of course, the result follows from Theorem 9.1 (with $\eta = 0$) and the fact that Haagerup's L^p does not depend on a choice of ϕ_0. However, "L^p-free" (i.e., "crossed product-free") proof is much more desirable. We give such a proof in this section. Our proof is based on complex interpolation theory itself (and relative modular theory in Section 2). The proof involves arguments used repeatedly in Section 11.

Easy computations suggest that an obvious candidate of a "nice" map (satisfying properties similar to those in Proposition 11.8) from $F(M^{\phi_0}, M_*)$ onto $F(M^{\phi_1}, M_*)$ (see the beginning of Section 3) is

$$\begin{aligned}(\pi f)(z) &= f'(z)(D\phi_0; D\phi_1)_{-iz}\phi_1\\ &= f(z)(D\phi_0; D\phi 1)_{-i(z-1)},\\ f(z) &= f'(z)\,\phi_0 \in F(M^{\phi_0}, M_*)\end{aligned}$$

(and $f'(z)$ is "smooth" enough). Notice that this is similar to the map which was used to construct the right action $a \cdot x$ in Section 6. However, the above πf does not belong to $F(M^{\phi_1}, M_*)$ unfortunately because $t \to (D\phi_0; D\phi_1)_t$ fails to be $\| \|_\infty$-continuous generally. This difficulty is removed if one uses the duality (Theorem 4.3).

Proof of Theorem 4.4. Because of the Theorems 4.3 and 3.3, and (11), it suffices to construct an isometric isomorphism from $C_\theta(H^{\phi_0}, M_*)$ onto $C_\theta(H^{\phi_1}, M_*)$. Here, H is a standard Hilbert space, and in the first (resp. second) interpolation, H is being imbedded into the predual M_* via

$$\zeta \to (\cdot\, \zeta \mid \xi_0)$$

(resp. $\zeta \to (\cdot\, \zeta \mid \xi_1)$) with the unique implementing vector ξ_0 for ϕ_0 (resp. ξ_1 for ϕ_1) in the natural cone $P^\natural$. For each $f(z)$ in $F(H^{\phi_0}, M_*)$ with

$$f(it) = (\cdot\, \zeta(it) \mid \xi_0), \qquad \zeta(it) \in H,$$

we set

$$\begin{aligned}
(\pi f)(z) &= \int_{-\infty}^{\infty} (\cdot\, \zeta(it) \mid \Delta_{\phi_1\phi_0}^{it/2} \xi_1)\, P_0(z, t)\, dt \\
&\quad + \int_{-\infty}^{\infty} f(1 + it)(D\phi_1 ; D\phi_0)_{t/2}^* P_1(z, t)\, dt, \quad \text{if } 0 < \operatorname{Re} z < 1, \\
&= (\cdot\, \zeta(it) \mid \Delta_{\phi_1\phi_0}^{it/2} \xi_1) \qquad \text{if } z = it, \\
&= f(1 + it)(D\phi_1 ; D\phi_0)_{t/2}^* \qquad \text{if } z = 1 + it
\end{aligned}$$

($(\pi f)(z) \in M_*$). The following observation is crucial in our proof: We notice that (recall (4)),

$$\begin{aligned}
\Delta_{\phi_1\phi_0}^{it/2} \xi_1 &= \Delta_{\phi_1\phi_0}^{it/2} \Delta_{\phi_1}^{-it/2} \xi_1 \\
&= J \Delta_{\phi_0\phi_1}^{it/2} \Delta_{\phi_1}^{-it} J \xi_1 \\
&= J (D\phi_0 ; D\phi_1)_{t/2} J \xi_1,
\end{aligned}$$

and that $J(D\phi_0 ; D\phi_1)_{t/2} J$ is a unitary operator in the commutant $JMJ = M'$.

For $z = it$, $x \in M$, we thus have

$$\begin{aligned}
((\pi f)(it))(x) &= (x\zeta(it) \mid J(D\phi_0 ; D\phi_1)_{t/2} J \xi_1) \\
&= (xJ(D\phi_0 ; D\phi_1)_{t/2}^* J \zeta(it) \mid \xi_1).
\end{aligned}$$

Therefore, $(\pi f)(it)$ certainly belongs to H^{ϕ_1} and

$$\begin{aligned}
\|(\pi f)(it)\|_{H^{\phi_1}} &= \|J(D\phi_1 ; D\phi_0)_{t/2}^* J \zeta(it)\|_H \\
&= \|\zeta(it)\|_H \\
&= \|f(it)\|_{H^{\phi_0}}.
\end{aligned}$$

Also, we obviously have

$$\|(\pi f)(1+it)\|_1 = \|f(1+it)\|_1$$

so that $|||\pi f||| = |||f|||$. Next, for

$$f(z) = (\cdot\, \zeta(z) \mid \xi_0) \in F_0(H^{\phi_0}, M_*),$$
$$\zeta(z) = \exp(\lambda z^2 + \mu z)\zeta, \qquad \lambda > 0, \mu \in \mathbb{R}, \zeta \in H,$$

we try to compute $(\pi f)(z)$. For $z = 1 + it$, we have

$$\begin{aligned}((\pi f)(1+it))(x) &= \exp(\lambda(1+it)^2 + \mu(1+it))((D\phi_1 ; D\phi_0)^*_{t/2} x\zeta \mid \xi_0)\\ &= \exp(\lambda(1+it)^2 + \mu(1+it))(x\zeta \mid \Delta^{it/2}_{\phi_1\phi_0}\xi_0).\end{aligned}$$

Since ξ_0 belongs to $D(\Delta^{1/2}_{\phi_1\phi_0})$, this function extends to a bounded continuous function

$$\exp(\lambda z^2 + \mu z)(x\zeta \mid \Delta^{(1-\bar{z})/2}_{\phi_1\phi_0}\xi_0)$$

on $0 \leqslant \operatorname{Re} z \leqslant 1$, which is analytic in the interior. Furthermore, for $z = it$, it gives rise to

$$\begin{aligned}\exp(\lambda(it)^2 + \mu(it))(x\zeta \mid \Delta^{it/2}_{\phi_1\phi_0}\Delta^{1/2}_{\phi_1\phi_0}\xi_0) &= \exp(\lambda(it)^2 + \mu(it))(x\zeta \mid \Delta^{it/2}_{\phi_1\phi_0}\xi_1)\\ &= ((\pi f)(it))(x).\end{aligned}$$

Therefore, πf, $f \in F_0(H^{\phi_0}, M_*)$, is an M_*-valued bounded and continuous function on $0 \leqslant \operatorname{Re} z \leqslant 1$, which is analytic in the interior. Actually, πf belongs to $F(H^{\phi_1}, M_*)$ as checked easily. Thus, arguments in the proof of Lemma 11.5 show that the map π sends $F(H^{\phi_0}, M_*)$ into $F(H^{\phi_1}, M_*)$ isometrically.

To prove that $f(\theta) = 0$ implies $(\pi f)(\theta) = 0$, we choose an x in M such that

$$t \in \mathbb{R} \to (D\phi_0 ; D\phi_1)_t x = (D\phi_1 ; D\phi_0)^*_t\, x \in M$$

extends to an entire function $x(z)$. (Such x's are known to form a σ-weakly dense subspace in M because of the usual regularization method.) For this x and any $f \in F(H^{\phi_0}, M_*)$, we compute

$$((\pi f)(1+it))(x) = (f(1+it))((D\phi_0 ; D\phi_1)_{t/2} x).$$

By the uniqueness of analytic continuation, we have

$$((\pi f)(z))(x) = f(z)(x(-i(z-1)/2)).$$

Thus, $f(\theta) = 0$ implies $((\pi f)(\theta))(x) = 0$, that is, $(\pi f)(\theta) = 0$ as we desired.

If one changes roles of ϕ_0 and ϕ_1 (also ξ_0 and ξ_1), one obtains the inverse isometric mapping because of the chain rule

$$(D\phi_0 ; D\phi_1)_t (D\phi_1 ; D\phi_0)_t = (D\phi_1 ; D\phi_0)_t (D\phi_0 ; D\phi_1)_t = 1.$$

Therefore we conclude that π is a surjective isometry from $F(H^{\phi_0}, M_*)$ onto $F(H^{\phi_1}, M_*)$ satisfying

$$f(\theta) = 0 \text{ if and only if } (\pi f)(\theta) = 0.$$

Passing to the quotient spaces, we obtain the surjective isometry from $C_\theta(H^{\phi_0}, M_*)$ onto $C_\theta(H^{\phi_1}, M_*)$, $0 < \theta < 1$. Q.E.D.

APPENDIX: Uniform Convexity of complex Interpolation Spaces

Calderón showed in [7] that all complex interpolation spaces $C_\theta(X_0, X_1)$, $0 < \theta < 1$, are reflexive if at least one of X_0 and X_1 is. This result remains valid if one replaces the reflexivity by the uniform convexisty. (The corresponding result is known for a certain real interpolation method [5].) Further analysis can be found in [11].

Theorem A. *Let $X = (X_0, X_1)$ be a compatible couple of Banach spaces. If at least one of X_0 and X_1 is uniformly convex, then so is the complex interpolation space $C_\theta(X)$, $0 < \theta < 1$.*

We need the harmonic measure $\{P_j(z, t)\}_{j=0,1}$ for the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ (see Lemma 7.5). We state the following powerful inequalities:

Lemma A.1 [6, Lemma 4.3.2]. *For each f in $F(X)$, we have*

$$\text{(i)} \quad \log\|f(\theta)\|_\theta \leqslant \int_{-\infty}^{\infty} \log\|f(it)\|_0 P_0(\theta, t)\, dt + \int_{-\infty}^{\infty} \log\|f(1+it)\|_1 P_1(\theta, t)\, dt,$$

$$\text{(ii)} \quad \|f(\theta)\|_\theta \leqslant \left\{\int_{-\infty}^{\infty} \|f(it)\|_0 P_0(\theta, t) \frac{dt}{1-\theta}\right\}^{1-\theta} \times \left\{\int_{-\infty}^{\infty} \|f(1+it)\|_1 P_1(\theta, t) \frac{dt}{\theta}\right\}^{\theta}.$$

Because of (16), $P_0(\theta, t)(dt/(1-\theta))$ is a probability measure on $\mathbb{R}$ (for each θ). By $L^2(X_0)$, we denote the Banach space consisting of all X_0-valued square integrable functions on $\mathbb{R}$ with respect to the above probability measure (for a fixed θ). The following result is known [5, p. 71]:

LEMMA A.2. *If X_0 is uniformly convex, then so is $L^2(X_0)$.*

We denote the modulus of convexity of $L^2(X_0)$ by $\delta(\cdot)$. Namely, for a small $\varepsilon > 0$, we set

$$\delta(\varepsilon) = \inf\{1 - \|(\tfrac{1}{2})(x + y)\|; x, y \in L^2(X_0), \qquad \|x\|, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\}.$$

(If X_0 is uniformly convex, then $\delta(\varepsilon)$ is strictly positive due to the above lemma.) We notice that in our situation (Theorem 4.3), X_0 was a Hilbert space, for which Lemma A.2 is obvious.

Proof of Theorem A. We may and do assume that X_0 is uniformly convex [6, Theorem 4.2.1(a)]. We first fix a number $\alpha > 2$.

Choose and fix a small $\varepsilon > 0$ throughout. Let x, y be elements in $C_\theta(X)$, $0 < \theta < 1$, such that $\|x\|_\theta \leqslant 1$, $\|y\|_\theta \leqslant 1$, and $\|x - y\|_\theta \geqslant \varepsilon$.

We then take $\eta > 0$ such that $0 < \eta < (\alpha/2)^\theta - 1$. By the definition of the interpolation norm, there exist two $\Sigma(X)$-valued functions f and g in $F(X)$ satisfying

$$x = f(\theta), \qquad |||f||| \leqslant 1 + \eta,$$
$$y = g(\theta), \qquad |||g||| \leqslant 1 + \eta.$$

Using Lemma A.1(i), we estimate

$$\begin{aligned}\log \varepsilon &\leqslant \log \|x - y\|_\theta \\ &\leqslant \int_{-\infty}^{\infty} \log \|f(it) - g(it)\|_0 P_0(\theta, t)\, dt + \theta \log\{2(1 + \eta)\}\end{aligned}$$

because of (16) and

$$\|f(1 + it) - g(1 + it)\|_1 \leqslant |||f||| + |||g||| \leqslant 2(1 + \eta).$$

Therefore, we have

$$\varepsilon\{2(1 + \eta)\}^{-\theta} \leqslant \exp \int_{-\infty}^{\infty} \log \|f(it) - g(it)\|_0 P_0(\theta, t)\, dt.$$

The measure $P_0(\theta, t)(dt/(1 - \theta))$ being probability, we then estimate

$$\begin{aligned}&[\varepsilon\{2(1 + \eta)\}^{-\theta}]^{1/(1-\theta)} \\ &\quad\leqslant \exp \int_{-\infty}^{\infty} \log \|f(it) - g(it)\|_0 P_0(\theta, t) \frac{dt}{1 - \theta}\end{aligned}$$

$$\leqslant \int_{-\infty}^{\infty} \|f(it) - g(it)\|_0 P_0(\theta, t) \frac{dt}{1-\theta} \qquad \text{(Jensen's inequality)}$$

$$\leqslant \left\{ \int_{-\infty}^{\infty} \|f(it) - g(it)\|_0^2 P_0(\theta, t) \frac{dt}{1-\theta} \right\}^{1/2}$$

(Cauchy–Schwarz inequality).

In the auxiary Banach space $L^2(X_0)$, we consider the two elements

$$F(t) = \frac{f(t)}{1+\eta}, \qquad G(t) = \frac{g(t)}{1+\eta}.$$

Then the above estimate shows that

$$\begin{aligned} \|F - G\| &\geqslant \frac{1}{1+\eta} [\varepsilon\{2(1+\eta)\}^{-\theta}]^{1/(1-\theta)} \\ &= \varepsilon^{1/(1-\theta)} 2^{-\theta/(1-\theta)} (1+\eta)^{-1/(1-\theta)} \\ &\geqslant \varepsilon^{1/(1-\theta)} 2^{-\theta/(1-\theta)} (\alpha/2)^{-\theta/(1-\theta)} \qquad \text{(recall } 0 < \eta < (\alpha/2)^\theta - 1) \\ &= (\varepsilon\alpha^{-\theta})^{1/(1-\theta)}. \end{aligned}$$

Also, F and G belong to the unit ball of $L^2(X_0)$. It follows from Lemma A.2 that

$$\|(\tfrac{1}{2})(F+G)\| \leqslant 1 - \delta((\varepsilon\alpha^{-\theta})^{1/(1-\theta)}).$$

Thus, Lemma A.1(ii) implies

$$\begin{aligned} \|(\tfrac{1}{2})(x+y)\|_\theta &\leqslant \left\{ \int_{-\infty}^{\infty} \left\| \left(\frac{1}{2}\right) (f(it) + g(it)) \right\|_0 P_0(\theta, t) \frac{dt}{1-\theta} \right\}^{1-\theta} \\ &\quad \times \left\{ \int_{-\infty}^{\infty} \left\| \left(\frac{1}{2}\right) (f(1+it) + g(1+it)) \right\|_1 P_1(\theta, t) \frac{dt}{\theta} \right\}^{\theta} \\ &\leqslant \left\{ \int_{-\infty}^{\infty} \left\| \left(\frac{1}{2}\right) (f(it) + g(it)) \right\|_0^2 P_0(\theta, t) \frac{dt}{1-\theta} \right\}^{(1-\theta)/2} \\ &\quad \times \left\{ \int_{-\infty}^{\infty} \left\| \left(\frac{1}{2}\right) (f(1+it) + g(1+it)) \right\|_1 P_1(\theta, t) \frac{dt}{\theta} \right\}^{\theta} \end{aligned}$$

(Cauchy–Schwarz inequality)

$$\begin{aligned} &\leqslant \left\{ (1+\eta) \left\| \left(\frac{1}{2}\right) (F+G) \right\| \right\}^{1-\theta} (1+\eta)^\theta \\ &\leqslant (1+\eta)\{1 - \delta((\alpha\varepsilon^{-\theta})^{1/(1-\theta)})\}^{1-\theta}. \end{aligned}$$

Letting $\eta \downarrow 0$, we obtain

$$\|(\tfrac{1}{2})(x+y)\|_\theta \leqslant \{1-\delta((\varepsilon\alpha^{-\theta})^{1/(1-\theta)})\}^{1-\theta}.$$

Since the right-hand side is a positive number strictly less than 1 and depends only on ε and α (not x, y), $C_\theta(X)$ is uniformly convex as desired.

Q.E.D.

Acknowledgments

The author would like to express his most sincere gratitude to Professor M. Takesaki for constant encouragement and enthusiasm on the present materials; especially, a starting point of the work in Part II was his question to the author. The author also thanks Professors M. Cwikel and S. Reisner for a result in the Appendix, and Professors U. Haagerup and S. Krantz for fruitful correspondence [18] and stimulating discussions.

References

1. H. Araki, Multiple time analysis of a quantum statistical state satisfying the KMS boundary condition, *Publ. Res. Instr. Math. Sci.* **A-4** (1968), 361–371.
2. H. Araki, Some properties of modular conjugation operator of von Neumann algebras and a non-kcommutative Radon–Nikodym theorem with a chain rule, *Pacific J. Math.* **50** (1974), 309–354.
3. H. Araki, Relative entropy of states of von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **11** (1975–1976), 809–833.
4. H. Araki and T. Masuda, Positive cones and L^p-spaces for von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **18** (1982), 339–411.
5. B. Beauzamy, Espaces d'interpolations réels: Topologie et géometrie, Lecture Notes in Math., No. 666, Springer-Verlag, Berlin, 1978.
6. J. Berg and J. Löfström, "Interpolation Spaces: An Introduction," Springer-Verlag, Berlin, 1976.
7. A. P. Calderón, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24** (1964), 113–190.
8. A. Connes, Une classifications des facteurs de type III, *Ann. Sci. École Norm. Sup.* (4) **6** (1973), 113–252.
9. A. Connes, Spatial theory of von Neumann algebras, *J. Funct. Anal.* **35** (1980), 153–164.
10. A. Connes and M. Takesaki, Flow of weights on factors of type III, *Tôhoku Math. J.* **29** (1977), 473–575.
11. M. Cwikel and S. Reisner, Interpolation of uniformly convex Banach spaces, *Proc. Amer. Math. Soc.* **84** (1982), 555–559.
12. T. Digernes, "Duality for Weights on Covariant Systems and Its Applications," Thesis, UCLA, 1975.
13. J. Dixmier, Formes linéares sur un anneau d'opérateurs, *Bull. Soc. Math. France* **81** (1953), 9–39.
14. U. Haagerup, The standard form of von Neumann algebras, *Math . Scand.* **37** (1975), 271–283.
15. U. Haagerup, Operator valued weights in von Neumann algebras. I, *J. Funct. Anal* **32** (1979), 175–206.
16. U. Haagerup, On the dual weights for crossed products on von Neumann algebras. II, *Math. Scand.* **43** (1978), 119–140.

17. U. HAAGERUP, L^p-spaces associated with an arbitrary von Neumann algebra, Colloques Internationaux CNRS, No. 274, pp. 175–184.
18. U. HAAGERUP, Letters to the author.
19. M. HILSUM, Les espaces L^p d'une algèbre de von Neumann (théorie spatiale), *J. Funct. Anal.* **40** (1980), 151–169.
20. H. KOSAKI, Positive cones associated with a von Neumann algebra, *Math. Scand.* **48** (1980), 295–307.
21. H. KOSAKI, Positive cones and L^p-spaces associated with a von Neumann algebra, *J. Operator Theory* **6** (1981), 13–23.
22. H. KOSAKI, Canonical L^p-Spaces Associated with an Arbitrary Abstract von Neumann Algebra," Thesis, UCLA, 1980.
23. H. KOSAKI, T-theorem for L^p-associated with a von Neumann algebra, *J. Operator Theory* **7** (1982), 267–277.
24. H. KOSAKI, Applications of uniform convexity of non-commutative L^p-spaces, *Trans. Amer. Math. Soc.*, in press.
25. H. KOSAKI, Wigner–Yanase–Dyson–Lieb concavity and interpolation theory, *Comm. Math. Phys.* **87** (1982), 315–329.
26. R. KUNZE, *Lp* Fourier transforms on locally compact unimodular groups, *Trans. Amer. Math. Soc.* **89** (1958), 519–540.
27. E. LIEB, Convex trace functions and the Wigner–Yanase–Dyson conjecture, *Advan. in Math.* **11** (1973), 267–288.
28. C. A. MCCARTHY, C_p, *Israel J. Math.* **5** (1967), 249–271.
29. E. NELSON, Notes on non-commutative integration, *J. Funct. Anal.* **15** (1974), 103–116.
30. W. PUSZ AND S. WORONOWICZ, Functional calculus for sesquilinear forms and the purification map, *Rep. Math. Phys.* **8** (1975), 159–170.
31. W. PUSZ, AND S. WORONOWICZ, Form convex functions and the WYDL and other inequalities, *Lett. Math. Phys.* **2** (1978), 505–512.
32. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics. II," Academic Press, New York, 1975.
33. I. SEGAL, A non-commutative extension of abstract integration, *Ann. of Math.* **37** (1953), 401–457.
34. B. SIMON, Trace ideals and their applications, Math. Soc., Lecture Note Series, No. 35, London Cambridge Univ. Press, London, 1979.
35. E. STEIN AND G. WEISS, Interpolation of operators with changes of measures, *Trans. Amer. Math. Soc.* **87** (1958), 159–172.
36. E. STEIN AND G. WEISS, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, N. J., 1971.
37. M. TAKESAKI, Tomita's theory of modular Hilbert algebras and its applications, Lecture Notes in Math., No. 128, Springer-Verlag, Berlin, 1970.
38. M. TAKESAKI, Conditional expectations in von Neumann algebras, *J. Funct. Anal.* **9** (1972), 306–321.
39. M. TAKESAKI, Duality for crossed products and atructure of von Neumann algebras of type III, *Acta Math.* **131** (1973), 249–310.
40. M. TERP, Interpolation spaces between a von Neumann algebra and its predual, *J. Operator Theory* **8** (1982), 327–360.
41. F. TREVES, "Topological Vector Spaces, Distributions, and Kernels," Academic Press, New York, 1967.
42. H. TRIEBEL, "Interpolation Theory, Function Spaces, Differential Operators," Vol. 18, North-Holland Mathematical Library, Amsterdam, 1978.
43. A. UHLMANN, Relative entropy and the Wigner–Yanase–Dyson–Lieb concavity in an interpolation theory, *Comm. Math. Phys.* **54** (1977), 21–32.