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On the blow-up structure for the generalized periodic Camassa–Holm and Degasperis–Procesi equations ☆

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Abstract

Considered herein are the generalized Camassa–Holm and Degasperis–Procesi equations in the spatially periodic setting. The precise blow-up scenarios of strong solutions are derived for both of equations. Several conditions on the initial data guaranteeing the development of singularities in finite time for strong solutions of these two equations are established. The exact blow-up rates are also determined. Finally, geometric descriptions of these two integrable equations from non-stretching invariant curve flows in centro-equiaffine geometries, pseudo-spherical surfaces and affine surfaces are given.

Keywords: Camassa–Holm equation; Degasperis–Procesi equation; Hunter–Saxton equation; Blow-up; Wave breaking

1. Introduction

In this paper, we are concerned with the initial-value problem associated with the generalized periodic Camassa–Holm (μCH) equation [29]

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$$\begin{cases} \mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(t,x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u(t,x)\,dx$ denotes its mean. Obviously, if $\mu(u) = 0$, which implies that $\mu(u_t) = 0$, then this equation reduces to the Hunter–Saxton (HS) equation [26], which is a short wave limit of the Camassa–Holm (CH) equation [4].

We also consider the initial-value problem associated with the generalized periodic Degasperis–Procesi (μDP) equation [33]

$$\begin{cases} \mu(u_t) - u_{xxt} = -3\mu(u)u_x + 3u_x u_{xx} + u u_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $u(t,x)$ and $\mu(u)$ are the same as in the above. Setting $\mu(u) = 0$, this equation becomes the short wave limit of the Degasperis–Procesi (DP) equation [17] or the μBurgers equation [33].

It is known that the Camassa–Holm equation and the Degasperis–Procesi equation are the cases $\lambda = 2$ and $\lambda = 3$, respectively, of the following family of equations

$$m_t + u m_x + \lambda u_x m = 0, \tag{1.3}$$

with $m = Au$ and $A = 1 - \partial_x^2$, where each equation in the family admits peakons [18] although only $\lambda = 2$ and $\lambda = 3$ are believed to be integrable [4,17].

It is observed that the corresponding μ-version of the family is also given by (1.3) with $m = Au$, $A = \mu - \partial_x^2$, where the choices $\lambda = 2$ and $\lambda = 3$ yield the generalized equations, i.e. the μCH and μDP equations, respectively.

It is clear that the closest relatives of the μCH equation are the Camassa–Holm equation with $A = 1 - \partial_x^2$

$$u_t - u_{txx} + 3u u_x = 2u_x u_{xx} + u u_{xxx}, \tag{1.4}$$

and the Hunter–Saxton equation with $A = -\partial_x^2$

$$-u_{txx} = 2u_x u_{xx} + u u_{xxx}. \tag{1.5}$$

Both of the CH equation and the HS equation have attracted much attention among the integrable systems and the communities of the partial differential equations. The Camassa–Holm equation was introduced in [4] as a shallow water approximation and has a bi-Hamiltonian structure [23]. The Hunter–Saxton equation firstly appeared in [26] as an asymptotic equation for rotators in liquid crystals.

The Camassa–Holm equation is a completely integrable system with a bi-Hamiltonian structure and hence it possesses an infinite sequence of conservation laws, see [16] for the periodic case. It admits soliton-like solutions (called peakons) in both periodic and non-periodic setting [4]. The Camassa–Holm equation describes geodesic flows on the infinite dimensional group $\mathcal{D}^s(\mathbb{S})$ of orientation-preserving diffeomorphisms of the unit circle $\mathbb{S}$ of Sobolev class H^s and endowed with a right-invariant metric by the H^1 inner product [31,36]. The Hunter–Saxton equation also describes the geodesic flow on the homogeneous space of the group $\mathcal{D}^s(\mathbb{S})$ modulo the subgroup of rigid rotations $Rot(\mathbb{S}) \simeq \mathbb{S}$ equipped with the $\dot{H}^1$ right-invariant metric [32], which at the identity is

$$\langle u, v\rangle_{\dot{H}^1} = \int_{\mathbb{S}} u_x v_x \, dx.$$

This equation possesses a bi-Hamiltonian structure and is formally integrable (see [27]).

Another remarkable property of the Camassa–Holm equation is the presence of breaking waves (i.e. the solution remains bounded while its slope becomes unbounded in finite time [4,10, 11,14,16,37,39]). Wave breaking is one of the most intriguing long-standing problems of water wave theory [39].

Another important integrable equation admitting peakon solitons is the Degasperis–Procesi equation [17] and it takes the form

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}.$$

It is regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm shallow water equation, and it can also be obtained from the governing equations for water waves [15]. The Degasperis–Procesi equation is a geodesic flow of a rigid invariant symmetric linear connection on the diffeomorphism group of the circle [19]. More interestingly, it admits the shock peakons in both periodic [21] and non-periodic settings [35]. Wave breaking phenomena and global existence of solutions of the Degasperis–Procesi equation were investigated in [9,21,22,34], for example.

The μCH ($\lambda = 2$) was introduced by Khesin, Lenells and Misiolek [29] (also called μHS equation). Similar to the HS equation [26], the μCH equation describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic field and self-interaction. Here, the solution $u(t, x)$ of the μCH equation is the director field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocity, and t is a slow time variable. Nematic liquid crystals are fields consisting of long rigid molecules. The μCH equation is an Euler equation on $\mathcal{D}^s(\mathbb{S})$ (the set of circle diffeomorphism of the Sobolev class H^s) and it describes the geodesic flow on $\mathcal{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product [29]

$$\langle u, v\rangle = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x \, dx.$$

It was shown in [29] that the μCH equation is formally integrable and can be viewed as the compatibility condition between

$$\psi_{xx} = \xi m \psi$$

and

$$\psi_t = \left(\frac{1}{2\xi} - u\right)\psi_x + \frac{1}{2}u_x\psi,$$

where $\xi \in \mathbb{C}$ is a spectral parameter and $m = \mu(u) - u_{xx}$.

On the other hand, the μCH equation admits bi-Hamiltonian structure and infinite hierarchy of conservation laws. The first few conservation laws in the hierarchy are

$$H_0 = \int_{\mathbb{S}} m\,dx, \qquad H_1 = \frac{1}{2}\int_{\mathbb{S}} mu\,dx, \qquad H_2 = \int_{\mathbb{S}} \left(\mu(u)u^2 + \frac{1}{2}uu_x^2\right)dx.$$

Whereas the Hunter–Saxton equation does not have any bounded traveling-wave solutions at all, the μCH equation admits traveling waves that can be regarded as the appropriate candidates for solitons. Moreover, the μCH equation admits not only periodic one-peakon solution $u(t,x) = \varphi(x - ct)$ where

$$\varphi(x) = \frac{c}{26}\left(12x^2 + 23\right)$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and φ is extended periodically to the real line, but also the multi-peakons of the form

$$u = \sum_{i=1}^{N} p_i(t) g\left(x - q_i(t)\right),$$

where

$$g(x) = \frac{1}{2}x(x-1) + \frac{13}{12}$$

is the Green's function of the operator $(\mu - \partial_x^2)^{-1}$.

The μDP equation ($\lambda = 3$) was firstly introduced by Lenells, Misiolek and Tiğlay in [33]. It can be formally described as an evolution equation on the space of tensor densities over the Lie algebra of smooth vector fields on the circle $\mathbb{S}$. As mentioned in [33], such geometric interpretation is not completely satisfactory. Recently, Escher, Kohlmann and Kolev [20] verified that the periodic μDP equation describes the geodesic flow of a right-invariant affine connection on the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ of all smooth and orientation-preserving diffeomorphisms of the circle $\mathbb{S}$. The μDP equation admits the Lax pair

$$\psi_{xxx} = -\xi m\psi,$$
$$\psi_t = -\frac{1}{\xi}\psi_{xx} - u\psi_x + u_x\psi,$$

where $\xi \in \mathbb{C}$ is a spectral parameter and $m = \mu(u) - u_{xx}$. Similar to the μCH equation, the μDP equation also admits bi-Hamiltonian structure and infinite hierarchy of conservation laws, and it is formally integrable [33]. The first few conservation laws in the hierarchy are

$$\tilde{H}_0 = -\frac{9}{2}\int_{\mathbb{S}} m\,dx, \qquad \tilde{H}_1 = \frac{1}{2}\int_{\mathbb{S}} u^2\,dx, \qquad \tilde{H}_2 = \int_{\mathbb{S}} \left(\frac{3}{2}\mu(u)\left(A^{-1}\partial_x u\right)^2 + \frac{1}{6}u^3\right)dx.$$

In addition to the peakon solutions same as those of the μCH equation, the μDP equation admits the shock-peakon solutions

$$u = \sum_{i=1}^{N}\left[p_i(t)g\left(x - q_i(t)\right) + s_i(t)g'\left(x - q_i(t)\right)\right],$$

where

$$g'(x) = \begin{cases} 0, & x = 0, \\ x - \frac{1}{2}, & x \in (0,1) \end{cases}$$

is the derivative of $g(x)$ assigning the value zero to the $g'(0)$.

The goal of the present paper is to derive some better conditions of blow-up solutions and determine blow-up rate for the μCH and μDP equations as well as give new geometric descriptions of these two equations through invariant curve flows in centro-equiaffine geometries and pseudo-spherical surfaces or affine surfaces.

To establish blow-up results in view, we shall use the Lyapunov function $V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx$ introduced in [12] to find some sufficient conditions of blow-up solutions for the μCH equation (*Theorem* 3.3 and *Theorem* 3.5) and the μDP equation (*Theorem* 3.7). Based on the conservation laws H_0, H_1 and H_2 with the best constant in the Sobolev imbedding $H^1(\mathbb{S}) \subset L^\infty(\mathbb{S})$, we are able to improve significantly blow-up results shown in [29] and [33].

It is noted that the norm $\|u(t)\|_{L^\infty}$ of the μDP equation is not uniformly bounded. To determine a better condition of blow-up solutions, we can employ the method of characteristics along a proper choice of a trajectory $q(t,x)$ defined in (2.15) which captures some zero of the flow $u(t,x)$. Using this new method of characteristics together with the conservation laws, we can derive an improved blow-up result to guarantee the slope of the flow tends to negative infinity for the μDP equation (*Theorem* 3.6) (see also *Theorem* 3.4 for the μCH equation). This method is also expected to have further applications in other nonlinear dispersive equations with a part of the Burgers equation.

The rest of the paper is organized as follows. In Section 2, we present some properties and estimates of the solutions for the μCH and μDP equations, which will be used for establishing blow-up results. The main part of the paper, Section 3 is to derive some precise blow-up scenarios of strong solutions and establish various results of blow-up solutions with certain initial profiles. The exact blow-up rate of solutions for these two equations will be determined in Section 4. Finally in Appendix A, we obtain the μCH and μDP equations again from non-stretching invariant curve flows in the two-dimensional and three-dimensional centro-equiaffine geometries, respectively. We also show that both of equations describe pseudo-spherical surfaces and affine surfaces, respectively.

Notation. Throughout this paper, we identify all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Since all spaces of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. Throughout the paper, for a given Banach space W, we denote its norm by $\|\cdot\|_W$.

2. Preliminaries

In this section, we first present the Sobolev-type inequalities which play a key role to obtain blow-up results for the initial-value problem (1.1) and (1.2) in the sequel. Then based on the first few conservation laws, we will prove some a priori estimates.

Lemma 2.1. *(See [11].) If $f \in H^3$ is such that $\int_{\mathbb{S}} f(x)\,dx = a_0/2$, then for every $\varepsilon > 0$, we have*

$$\max_{x\in\mathbb{S}} f^2(x) \leqslant \frac{\varepsilon+2}{24}\int_{\mathbb{S}} f_x^2(x)\,dx + \frac{\varepsilon+2}{4\varepsilon}a_0^2.$$

Remark 2.1. Since H^3 is dense in H^1, Lemma 2.1 also holds for every $f \in H^1$. Moreover, if $\int_{\mathbb{S}} f(x)\,dx = 0$, from the deduction of this lemma we arrive at the following inequality

$$\max_{x\in\mathbb{S}} f^2(x) \leqslant \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)\,dx, \quad f \in H^1. \tag{2.1}$$

Lemma 2.2. *(See [3].) For every $f(x) \in H^1(a,b)$ periodic and with zero average, i.e. such that $\int_a^b f(x)\,dx = 0$, we have*

$$\int_a^b f^2(x)\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b \left|f'(x)\right|^2 dx,$$

and equality holds if and only if

$$f(x) = A\cos\left(\frac{2\pi x}{b-a}\right) + B\sin\left(\frac{2\pi x}{b-a}\right).$$

Lemma 2.3. *(See [13].) Let $T > 0$ and $u \in C^1([0,T); H^2)$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ with $w(t) := \inf_{x\in\mathbb{S}} u_x(t,x) = u_x(t,\xi(t))$. Moreover, the function $w(t)$ is absolutely continuous on $(0,T)$, and it satisfies*

$$\frac{dw}{dt} = u_{xt}\big(t,\xi(t)\big), \quad \text{a.e. on } (0,T).$$

Lemma 2.4. *(See [28].) If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\|fg\|_{H^r} \leqslant c\big(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}\big),$$

where c is a constant depending only on r.

Lemma 2.5. *(See [28].) If $r > 0$, then*

$$\big\|[\Lambda^r, f]g\big\|_{L^2} \leqslant c\big(\|\partial_x f\|_{L^\infty}\big\|\Lambda^{r-1}g\big\|_{L^2} + \big\|\Lambda^r f\big\|_{L^2}\|g\|_{L^\infty}\big),$$

where c is a constant depending only on r.

In the following, we verify some a priori estimates for the μCH equation. Recall that the first two conserved quantities of the μCH equation are

$$H_0 = \int_{\mathbb{S}} m\,dx = \int_{\mathbb{S}} \big(\mu(u) - u_{xx}\big)\,dx = \mu(u),$$

and

$$H_1 = \frac{1}{2}\int_{\mathbb{S}} mu\,dx = \frac{1}{2}\mu^2\big(u(t)\big) + \frac{1}{2}\int_{\mathbb{S}} u_x^2(t,x)\,dx.$$

It is easy to see that $\mu(u)$ and $\int_{\mathbb{S}} u_x^2(t,x)\,dx$ are conserved in time [29]. Thus

$$\mu(u_t) = 0. \tag{2.2}$$

In what follows we denote

$$\mu_0 = \mu(u_0) = \mu(u) = \int_{\mathbb{S}} u(t,x)\,dx \tag{2.3}$$

and

$$\mu_1 = \bigg(\int_{\mathbb{S}} u_x^2(0,x)\,dx\bigg)^{\frac{1}{2}} = \bigg(\int_{\mathbb{S}} u_x^2(t,x)\,dx\bigg)^{\frac{1}{2}}. \tag{2.4}$$

Then μ_0 and μ_1 are constants and independent of time t. Note that

$$\int_{\mathbb{S}} \big(u(t,x) - \mu_0\big)\,dx = \mu_0 - \mu_0 = 0.$$

By Lemma 2.1, we find that

$$\max_{x\in\mathbb{S}}\big[u(t,x) - \mu_0\big]^2 \leqslant \frac{1}{12}\int_{\mathbb{S}} u_x^2(t,x)\,dx = \frac{1}{12}\int_{\mathbb{S}} u_x^2(0,x)\,dx = \frac{1}{12}\mu_1^2.$$

From the above estimate, we find that the amplitude of the wave remains bounded in any time, that is,

$$\|u(t,\cdot)\|_{L^\infty} - |\mu_0| \leqslant \|u(t,\cdot) - \mu_0\|_{L^\infty} \leqslant \frac{\sqrt{3}}{6}\mu_1,$$

and so

$$\|u(t,\cdot)\|_{L^\infty} \leqslant |\mu_0| + \frac{\sqrt{3}}{6}\mu_1. \tag{2.5}$$

On the other hand, we have

$$\|u(t,x)\|_{L^2}^2 = \int_{\mathbb{S}} u^2(t,x)\,dx \leqslant \|u(t,\cdot)\|_{L^\infty}^2 \leqslant \left(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1\right)^2. \tag{2.6}$$

It then follows that

$$\begin{aligned}\|u(t,\cdot)\|_{H^1} &= \|u(t,\cdot)\|_{L^2} + \|u_x(t,\cdot)\|_{L^2} = \left(\int_{\mathbb{S}} u^2(t,x)\,dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{S}} u_x^2(t,x)\,dx\right)^{\frac{1}{2}} \\ &\leqslant |\mu_0| + \frac{\sqrt{3}}{6}\mu_1 + \mu_1 = |\mu_0| + \left(1 + \frac{\sqrt{3}}{6}\right)\mu_1. \end{aligned} \tag{2.7}$$

Similar to the conservation law H_0 for the μCH equation, it is easy to see that $\mu(u_t) = 0$ and $\int_{\mathbb{S}} u(t,x)\,dx$ is also conserved in time for the μDP equation. Since $\tilde{H}_1$ is a conserved quantity for the μDP equation, we define

$$\mu_2 = \left(\int_{\mathbb{S}} u^2(0,x)\,dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}} u^2(t,x)\,dx\right)^{\frac{1}{2}}. \tag{2.8}$$

Then μ_2 is independent of time t.

Recall in [29] that the mean of any solution $u(t,x)$ is conserved by the flow and hence the initial-value problem (1.1) and (1.2) can be recast in the following.

$$\begin{cases} u_t + uu_x + A^{-1}\partial_x\left(\lambda\mu_0 u + \frac{3-\lambda}{2}u_x^2\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{2.9}$$

with $\lambda = 2$ and $\lambda = 3$, respectively, where $A = \mu - \partial_x^2$ is an isomorphism between H^s and H^{s-2} with the inverse $v = A^{-1}w$ given explicitly by

$$\begin{aligned} v(x) &= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(s)\,ds\,dy \\ &\quad - \int_0^x\int_0^y w(s)\,ds\,dy + \int_0^1\int_0^y\int_0^s w(r)\,dr\,ds\,dy. \end{aligned} \tag{2.10}$$

Since A^{-1} and ∂_x commute, the following identities hold

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)\,dx - \int_0^x w(y)\,dy + \int_0^1\int_0^x w(y)\,dy\,dx, \tag{2.11}$$

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)\,dx. \tag{2.12}$$

We have the following estimates

$$\begin{aligned}\left\|A^{-1}\partial_x u\right\|_{H^s} &\leqslant \left\|A^{-1}\partial_x u\right\|_{L^2} + \left\|\partial_x A^{-1}\partial_x u\right\|_{H^{s-1}} \\ &\leqslant 3\|u\|_{L^2} + \left\|-u + \int_{\mathbb{S}} u\,dx\right\|_{H^{s-1}} \\ &\leqslant 3\|u\|_{L^2} + 2\|u\|_{H^{s-1}} \leqslant 5\|u\|_{H^{s-1}},\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\left\|A^{-1}\partial_x u_x^2\right\|_{H^s} &\leqslant \left\|A^{-1}\partial_x u_x^2\right\|_{L^2} + \left\|\partial_x A^{-1}\partial_x u_x^2\right\|_{H^{s-1}} \\ &\leqslant 3\|u_x\|_{L^2}^2 + \left\|-u_x^2 + \int_{\mathbb{S}} u_x^2\,dx\right\|_{H^{s-1}} \\ &\leqslant 3\|u_x\|_{L^2}^2 + 2\left\|u_x^2\right\|_{H^{s-1}} \leqslant 5\|u_x\|_{L^\infty}\|u\|_{H^s},\end{aligned} \tag{2.14}$$

where in the last step we used Lemma 2.4 with $r = s - 1$.

The following local well-posedness results of the initial-value problems (2.9) were already established in [29] and [33] by transforming them into the initial-value problems of certain ordinary differential equations through the Euler coordinates.

Proposition 2.1. *Let $u_0 \in H^s$, $s > 3/2$. Then there exist a maximal life span $T > 0$ and a unique solution u to* (2.9) *such that*

$$u \in C\big([0,T); H^s\big) \cap C^1\big([0,T); H^{s-1}\big)$$

which depends continuously on the initial data u_0.

What is more, we can prove that the maximal time $T > 0$ is independent of the choice of the index s, which is formulated by the following proposition.

Proposition 2.2. *The maximal T in Proposition* 2.1 *can be chosen independent of s in the following sense. If*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s\big) \cap C^1\big([0,T); H^{s-1}\big)$$

is a solution to (2.9)*, and if $u_0 \in H^{s'}$ for some $s' \neq s$, $s' > 3/2$, then*

$$u \in C\big([0,T); H^{s'}\big) \cap C^1\big([0,T); H^{s'-1}\big)$$

with the same value of T. In particular, if $u_0 \in H^\infty = \bigcap_{x \geqslant 0} H^s$, then $u \in C([0,T); H^\infty)$.

Proof. If $s' < s$, the result follows from the uniqueness of the solution guaranteed by Proposition 2.1. So it suffices to consider the case $s' > s$. Setting $m(t) = Au(t) = (\mu - \partial_x^2)u(t)$ and recalling $\mu(u) = \int_{\mathbb{S}} u\,dx = \mu(u_0)$, it follows from the first equation in (2.9) that

$$\frac{dm}{dt} + E(t)m + F(t)m = 0, \qquad m(0) = Au(0) = \big(\mu - \partial_x^2\big)u(0),$$

where $E(t)m = \partial_x(um)$ and $F(t)m = (\lambda - 1)u_x m$.

Since $u \in C([0,T); H^s)$ and $u_0 \in H^{s'}$, then $m \in C([0,T); H^{s-2})$ and $m_0 = m(0) = Au(0) \in H^{s'-2}$. It is our purpose to prove $m \in C([0,T); H^{s'-2})$. Notice that $A = \mu - \partial_x^2$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$, which means that $u \in C([0,T); H^{s'})$.

It is observed that $u \in C([0,T); H^s)$, $u_x \in H^{s-1}$, and H^{s-1} with $s > 3/2$ is a Banach algebra. This then implies that $F(t) \in L(H^{s-1})$.

Following the argument in [40], it is easy to see that the family $E(t)$ generates a unique operator $\{U(t,\tau)\}$ associated with the spaces $X = H^l$ and $Y = H^k$, where $-s \leqslant l \leqslant s-2$, $1-s \leqslant k \leqslant s-1$, and $k \geqslant l+1$. Accordingly, an evolution operator $\{U(t,\tau)\}$ for the family $E(t)$ exists and is unique. In particular, for $-s \leqslant r \leqslant s-1$, $\{U(t,\tau)\}$ maps H^r into itself.

Next, choosing $X = H^{s-3}$ and $Y = H^{s-2}$, and noting that $m \in C([0,T); H^{s-1}) \cap C^1([0,T); H^{s-2})$, we obtain

$$\frac{d}{d\tau}\big(U(t,\tau)m(\tau)\big) = U(t,\tau)\big({-F(\tau)m(\tau)}\big).$$

If $\lambda = 1$, then $F(\tau) = 0$. An integration over $\tau \in [0,t]$ yields $m(t) = U(t,0)m(0)$. Since $-s < s-2 < s'-2 \leqslant s-1$, the family $\{U(t,\tau)\}$ is a strongly continuous map from the space $H^{s'-2}$ into itself. Noting that $m(0) \in H^{s'-2}$ and $u = A^{-1}m$, we immediately get the result of Proposition 2.2 for the case $s < s' \leqslant s+1$.

If $\lambda \neq 1$, integrating the above differential equation over $\tau \in [0,t]$ we obtain

$$m(t) = U(t,0)m(0) - \int_0^t U(t,\tau)F(\tau)m(\tau)\,d\tau.$$

If $s < s' \leqslant s+1$, then $F(t) = (\lambda - 1)u_x(t) \in L(H^{s'-2})$ is strong continuous on $[0,t)$, and $H^{s-1}H^{s'-2} \subset H^{s'-2}$ since $s - 1 > 1/2$. As an integral equation of Volterra type, the above equation can be solved for m by successive approximation. Moreover, for $-s < s-2 < s'-2 \leqslant$

$s-1$, the operator $\{U(t,\tau)\}$ is a strongly continuous map from $H^{s'-2}$ into itself. Therefore, the assertion of Proposition 2.2 for the case $s < s' \leqslant s+1$ is established.

In the case $s' > s+1$, the result follows by a repeated application of the above argument. This completes the proof of the proposition. □

In [29] and [33], the authors also showed that the μCH and μDP equations admit global (in time) solutions and a blow-up mechanism. It is our purpose here to derive the precise scenarios and initial conditions guaranteeing the blow-up of strong solutions to the initial-value problem (1.1) and (1.2), which will significantly improve the results in [29] and [33]. In the case of $\mu_0 = 0$, these two equations reduce to the Hunter–Saxon equation and μBurgers equation respectively. Since these two special cases have recently been the object of intensive study ([2,26,27,33,41], for example), we only focus on the case of $\mu_0 \neq 0$ in the rest of the paper.

Given a solution $u(t,x)$ of the initial-value problem (2.9) with initial data u_0, we let $t \to q(t,x)$ be the flow of $u(t,x)$, that is

$$\begin{cases} \dfrac{dq(t,x)}{dt} = u\big(t, q(t,x)\big), & t > 0,\ x \in \mathbb{R}, \\ q(0,x) = x. \end{cases} \tag{2.15}$$

A direct calculation shows that $q_{tx}(t,x) = u_x(t,q(t,x))q_x(t,x)$. Hence for any $t>0$, $x \in \mathbb{R}$, we find that

$$q_x(t,x) = \mathrm{e}^{\int_0^t u_x(\tau, q(\tau,x))\,d\tau} > 0,$$

which implies that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0,T)$. This is inferred that the L^∞-norm of any function $v(t,\cdot) \in L^\infty$, $t \in [0,T)$ is preserved under the family of diffeomorphisms $q(t,\cdot)$ with $t \in [0,T)$, that is,

$$\big\|v(t,\cdot)\big\|_{L^\infty} = \big\|v\big(t,q(t,\cdot)\big)\big\|_{L^\infty}, \quad t \in [0,T).$$

Consider the μ-version (2.9) with $m = Au$, $A = \mu - \partial_x^2$. It is easy to verify that at each point of the circle, the solution $u(t,x)$ satisfies a local conservation law

$$m\big(t,q(t,x)\big)\big(\partial_x q(t,x)\big)^\lambda = m_0(x) = \mu\big(u_0(x)\big) - u_0''(x),$$

where $u_0''(x)$ is the second derivative of $u_0(x)$ with respective to x.

Unlike the μCH equation, $\|u(t)\|_{L^\infty}$ of the solution $u(t,x)$ of the μDP equation is not uniformly bounded for t. However, we are able to establish an important estimate in the following.

Lemma 2.6. *Assume $u_0 \in H^s, s > 3/2$. Let T be the maximal existence time of the solution $u(t,x)$ to the initial-value problem* (1.2) *associated with the μDP equation. Then we have*

$$\big\|u(t,x)\big\|_{L^\infty} \leqslant \left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)t + \|u_0\|_{L^\infty}, \quad \forall t \in [0,T].$$

Proof. Since the existence time T is independent of the choice of s by Proposition 2.2, applying a simple density argument, we only need to consider the case $s = 3$. Let T be the maximal existence time of the solution $u(t,x)$ to the initial-value problem (1.2) with the initial data $u_0 \in H^3$. By (2.9) with $\lambda = 3$, the first equation of the initial-value problem (1.2) is equivalent to the following equation

$$u_t + uu_x = -3\mu_0 A^{-1}\partial_x u.$$

In view of (2.11), we have

$$\left|A^{-1}\partial_x u\right| \leqslant \frac{1}{2}|\mu_0| + 2\mu_2.$$

On the other hand, it follows from (2.15) that

$$\frac{du(t,q(t,x))}{dt} = u_t\big(t,q(t,x)\big) + u_x\big(t,q(t,x)\big)\frac{dq(t,x)}{dt} = (u_t + uu_x)\big(t,q(t,x)\big).$$

Combining the above two estimates yields

$$-\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right) \leqslant \frac{du(t,q(t,x))}{dt} \leqslant \frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2.$$

Integrating the above inequality with respect to $t < T$ on $[0,t]$, one easily finds

$$-\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)t + u_0(x) \leqslant u\big(t,q(t,x)\big) \leqslant \left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)t + u_0(x).$$

This thus implies that

$$\left|u\big(t,q(t,x)\big)\right| \leqslant \left\|u\big(t,q(t,\cdot)\big)\right\|_{L^\infty} \leqslant \left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)t + \left\|u_0(x)\right\|_{L^\infty}.$$

In view of the diffeomorphism property of $q(t,\cdot)$, we obtain

$$\left\|u(t,\cdot)\right\|_{L^\infty} = \left\|u\big(t,q(t,\cdot)\big)\right\|_{L^\infty} \leqslant \left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)t + \left\|u_0(x)\right\|_{L^\infty}.$$

This completes the proof of Lemma 2.6. □

3. Blow-up solutions

In this section, we establish the precise blow-up scenarios and give sufficient conditions for blow-up of solutions to the initial-value problem (1.1) and (1.2). Indeed, we determine the precise blow-up scenarios for the problem (2.9) in the following.

Theorem 3.1. *Suppose that* $\lambda \in \mathbb{R}$. *Let* $u_0 \in H^s$, $s > 3/2$ *be given and assume that* T *is the maximal existence time of the corresponding solution* $u(t,x)$ *to the initial-value problem* (2.9) *with the initial data* u_0. *If there exists* $M > 0$ *such that*

$$\|u_x(t)\|_{L^\infty} \leqslant M, \quad t \in [0,T),$$

then the H^s*-norm of* $u(t,\cdot)$ *does not blow up on* $[0,T)$.

Proof. We always assume that c is a generic positive constant. Let $\Lambda = (1-\partial_x^2)^{1/2}$. Applying the operator Λ^s to the first equation in (2.9), then multiplying by $\Lambda^s u$ and integrating over $\mathbb{S}$ with respect to x lead to

$$\frac{d}{dt}\|u\|_{H^s}^2 = -2(uu_x,u)_{H^s} - 2\left(u, A^{-1}\partial_x\left(\lambda\mu_0 u + \frac{3-\lambda}{2}u_x^2\right)\right)_{H^s}.$$

Let us estimate the right-hand side of the above equation,

$$\begin{aligned}\left|(uu_x,u)_{H^s}\right| &= \left|\left(\Lambda^s(uu_x), \Lambda^s u\right)_{L^2}\right| = \left|\left(\left[\Lambda^s,u\right]u_x, \Lambda^s u\right)_{L^2} + \left(u\Lambda^s u_x, \Lambda^s u\right)_{L^2}\right| \\ &\leqslant \left\|\left[\Lambda^s,u\right]u_x\right\|_{L^2}\left\|\Lambda^s u\right\|_{L^2} + \frac{1}{2}\left|\left(u_x\Lambda^s u, \Lambda^s u\right)_{L^2}\right| \\ &\leqslant c\|u_x\|_{L^\infty}\|u\|_{H^s}^2.\end{aligned}$$

In the above inequality, we used Lemma 2.5 with $r=s$.

It then follows from (2.13) and (2.14) that

$$\left|\left(u, A^{-1}\partial_x u\right)_{H^s}\right| \leqslant c\|u\|_{H^s}\left\|A^{-1}\partial_x u\right\|_{H^s} \leqslant c\|u\|_{H^s}^2$$

and

$$\left|\left(u, A^{-1}\partial_x u_x^2\right)_{H^s}\right| \leqslant c\|u\|_{H^s}\left\|A^{-1}\partial_x u_x^2\right\|_{H^s} \leqslant c\|u_x\|_{L^\infty}\|u\|_{H^s}^2.$$

Combining the above three estimates, we arrive at

$$\frac{d}{dt}\|u\|_{H^s}^2 \leqslant c\left(1+\|u_x\|_{L^\infty}\right)\|u\|_{H^s}^2.$$

An application of Gronwall's inequality and the assumption of the theorem leads to

$$\|u\|_{H^s}^2 \leqslant \exp\left(c(1+M)t\right)\|u_0\|_{H^s}^2.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let* $u_0 \in H^s$, $s > 3/2$, *and* $u(t,x)$ *be the solution of the initial-value problem* (2.9) *with life-span* T. *Then* T *is finite if and only if*

$$\liminf_{t\uparrow T}\left\{\inf_{x\in\mathbb{S}}\left[(2\lambda-1)u_x(t,x)\right]\right\} = -\infty.$$

Proof. Since the existence time T is independent of the choice of s in view of Proposition 2.2, we only need to consider the case $s = 3$ by utilizing a simple density argument. Multiplying the μ-version of the family (1.3) by m and integrating over $\mathbb{S}$ with respect to x yield

$$\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx = -2\lambda\int_{\mathbb{S}} u_x m^2\,dx - 2\int_{\mathbb{S}} umm_x\,dx$$
$$= (1-2\lambda)\int_{\mathbb{S}} u_x m^2\,dx.$$

If $(2\lambda-1)u_x$ is bounded from below on $[0,T)\times\mathbb{S}$, i.e., there exists $N>0$ such that $(2\lambda-1)u_x \geqslant -N$ on $[0,T)\times\mathbb{S}$, then it is thereby inferred from the above estimate that

$$\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx \leqslant N\int_{\mathbb{S}} m^2\,dx.$$

Applying Gronwall's inequality then yields for $t\in[0,T)$

$$\int_{\mathbb{S}} m^2\,dx \leqslant e^{Nt}\int_{\mathbb{S}} m_0^2\,dx.$$

Note that $\mu(u)$ is independent of t to the μ-version of the family (1.3) for any $\lambda\in\mathbb{R}$. Thus we have

$$\int_{\mathbb{S}} m^2(t,x)\,dx = \mu^2(u) + \int_{\mathbb{S}} u_{xx}^2\,dx = \mu_0^2 + \|u_{xx}\|_{L^2}^2.$$

It then follows from Sobolev's embedding $H^1 \hookrightarrow L^\infty$ and Remark 2.1 that for $t\in[0,T)$

$$\|u_x\|_{L^\infty} \leqslant \frac{1}{2\sqrt{3}}\|u_{xx}\|_{L^2} \leqslant \frac{1}{2\sqrt{3}}\|m\|_{L^2} \leqslant \frac{1}{2\sqrt{3}}e^{\frac{1}{2}NT}\|m_0\|_{L^2}.$$

As a consequence of Theorem 3.1, we deduce that the solution exists globally in time.

On the other hand, if the slope of the solution becomes unbounded from below, by the existence of the local strong solution Proposition 2.1 and Sobolev's embedding theorem, we infer that the solution will blow-up in finite time. The proof of Theorem 3.2 is then completed. □

Remark 3.1. In particular, for $\lambda = 1/2$, the solution u to (2.9) exists in H^s, $s > 3/2$ globally in time.

In the following, by means of the blow-up scenarios we establish some sufficient conditions guaranteeing the development of singularities.

3.1. Blow-up for the μCH equation

We are now in a position to give the first blow-up result for the μCH equation.

Theorem 3.3. *Let $u_0 \in H^s$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t,x)$ to* (1.1) *with the initial data u_0. If $(\sqrt{3}/\pi)|\mu_0| < \mu_1$, where μ_0 and μ_1 are defined in* (2.3) *and* (2.4)*, then the corresponding solution $u(t,x)$ to* (1.1) *associated with the* μCH *equation must blow up in finite time T with*

$$0 < T \leqslant \inf_{\alpha \in I}\left(\frac{6}{1-6\alpha|\mu_0|} + 4\pi^2\alpha\frac{1+|\int_{\mathbb{S}} u_{0x}^3(x)\,dx|}{6\pi^2\alpha\mu_1^4 - 3|\mu_0|\mu_1^2}\right),$$

where $I = (\frac{|\mu_0|}{2\pi^2\mu_1^2}, \frac{1}{6|\mu_0|})$, such that

$$\liminf_{t\uparrow T}\left(\inf_{x\in\mathbb{S}} u_x(t,x)\right) = -\infty.$$

Proof. As discussed above, it suffices to consider the case $s = 3$. Since the case $\mu_0 = 0$ was proved in [29], we only need to discuss the case $\mu_0 \neq 0$. In this case, $\mu_1 > 0$. Differentiating the μCH equation with respect to x yields

$$u_{tx} + u_x^2 + uu_{xx} + A^{-1}\partial_x^2\left(2u\mu_0 + \frac{1}{2}u_x^2\right) = 0.$$

In view of (2.3), (2.4) and (2.12), we have

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + 2u\mu_0 - 2\mu_0^2 - \frac{1}{2}\mu_1^2. \tag{3.1}$$

Multiplying (3.1) by $3u_x^2$ and integrating on $\mathbb{S}$ with respect to x, we obtain for any $t \in [0,T)$ that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= \int_{\mathbb{S}} 3u_x^2u_{xt}\,dx \\
&= -\frac{3}{2}\int_{\mathbb{S}} u_x^4\,dx - \int_{\mathbb{S}} 3uu_x^2u_{xx}\,dx + 6\mu_0\int_{\mathbb{S}} uu_x^2\,dx \\
&\quad - 6\mu_0^2\int_{\mathbb{S}} u_x^2\,dx - \frac{3}{2}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2 \\
&= -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - \frac{3}{2}\mu_1^4 + 6\mu_0\int_{\mathbb{S}} (u-\mu_0)u_x^2\,dx.
\end{aligned} \tag{3.2}$$

On the other hand, it follows from Lemma 2.2 for any $\alpha > 0$ that

$$\int_{\mathbb{S}} (u-\mu_0)u_x^2\,dx \leqslant \left(\int_{\mathbb{S}} (u-\mu_0)^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}} u_x^4\,dx\right)^{\frac{1}{2}}$$
$$\leqslant \frac{\alpha}{2}\int_{\mathbb{S}} u_x^4\,dx + \frac{1}{2\alpha}\int_{\mathbb{S}} (u-\mu_0)^2\,dx$$
$$\leqslant \frac{\alpha}{2}\int_{\mathbb{S}} u_x^4\,dx + \frac{1}{8\pi^2\alpha}\int_{\mathbb{S}} u_x^2\,dx.$$

Therefore we deduce that

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leqslant \left(3\alpha|\mu_0| - \frac{1}{2}\right)\int_{\mathbb{S}} u_x^4\,dx - \frac{3}{2}\mu_1^4 + \frac{3}{4\pi^2\alpha}|\mu_0|\mu_1^2.$$

By the assumption of the theorem, we know that $|\mu_0|/(2\pi^2\mu_1^2) < 1/(6|\mu_0|)$. Let $\alpha > 0$ satisfy

$$\frac{|\mu_0|}{2\pi^2\mu_1^2} < \alpha < \frac{1}{6|\mu_0|}.$$

This in turn implies that $6\alpha|\mu_0| - 1 < 0$ and $2\pi^2\alpha\mu_1^2 - |\mu_0| > 0$. Define c_1 and c_2 by

$$c_1 = \frac{1}{2} - 3\alpha|\mu_0| > 0, \qquad c_2 = \frac{3}{2}\mu_1^4 - \frac{3}{4\pi^2\alpha}|\mu_0|\mu_1^2 > 0.$$

It is then clear that

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leqslant -c_1\int_{\mathbb{S}} u_x^4\,dx - c_2 \leqslant -c_1\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}} - c_2.$$

Let $V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx$ with $t \in [0,T)$. Then the above inequality can be rewritten as

$$\frac{d}{dt}V(t) \leqslant -c_1\big(V(t)\big)^{\frac{4}{3}} - c_2 \leqslant -c_2 < 0, \quad t \in [0,T).$$

This implies that $V(t)$ decreases strictly in $[0,T)$. Let $t_1 = (1 + |V(0)|)/c_2$. One can assume $t_1 < T$. Otherwise, $T \leqslant t_1 < \infty$ and the theorem is proved. Now integrating the above inequality over $[0,t_1]$ yields

$$V(t_1) = V(0) + \int_0^{t_1} \frac{d}{dt}V(t)\,dt \leqslant \big|V(0)\big| - c_2 t_1 \leqslant -1.$$

It is also found that

$$\frac{d}{dt}V(t) \leqslant -c_1\big(V(t)\big)^{\frac{4}{3}}, \quad t \in [t_1,T),$$

which leads to

$$-3\frac{d}{dt}\left(\frac{1}{(V(t))^{\frac{1}{3}}}\right) = \big(V(t)\big)^{-\frac{4}{3}}\frac{d}{dt}V(t) \leqslant -c_1, \quad t \in [t_1, T).$$

Integrating both sides of the above inequality and applying $V(t_1) \leqslant -1$ yield

$$-\frac{1}{(V(t))^{\frac{1}{3}}} - 1 \leqslant -\frac{1}{(V(t))^{\frac{1}{3}}} + \frac{1}{(V(t_1))^{\frac{1}{3}}} \leqslant -\frac{c_1}{3}(t - t_1), \quad t \in [t_1, T).$$

Recall that $V(t) \leqslant V(t_1) \leqslant -1$ in $[t_1, T)$. It follows that

$$V(t) \leqslant \left[\frac{3}{c_1(t - t_1) - 3}\right]^3 \to -\infty, \quad \text{as } t \to t_1 + \frac{3}{c_1}.$$

On the other hand, we have

$$\int_{\mathbb{S}} u_x^3\,dx \geqslant \inf_{x\in\mathbb{S}} u_x(t, x)\int_{\mathbb{S}} u_x^2\,dx = \mu_1^2 \inf_{x\in\mathbb{S}} u_x(t, x).$$

This then implies that $0 < T \leqslant t_1 + 3/c_1$ such that

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{S}} u_x(t, x)\Big) = -\infty.$$

This completes the proof of Theorem 3.3. □

Remark 3.2. Note that in [29], the initial condition of blow-up mechanism is $4|\mu_0| \leqslant \mu_1$. Therefore, Theorem 3.3 improves the blow-up result in [29].

In the case $(\sqrt{3}/\pi)|\mu_0| \geqslant \mu_1$, we have the following blow-up result.

Theorem 3.4. *Let $u_0 \in H^s$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to* (1.1) *with the initial data u_0. If $(\sqrt{3}/\pi)|\mu_0| \geqslant \mu_1$ and*

$$\inf_{x\in\mathbb{S}} u_0'(x) < -\sqrt{2\mu_1\left(\frac{\sqrt{3}}{3}|\mu_0| - \frac{1}{2}\mu_1\right)} :\equiv -K,$$

where $u_0'(x)$ is the derivative of $u_0(x)$ with respective to x, then the corresponding solution $u(t, x)$ to (1.1) *blows up in finite time T with*

$$0 < T \leqslant \left(-\frac{2}{\inf_{x\in\mathbb{S}} u_0'(x) + \sqrt{-K\inf_{x\in\mathbb{S}} u_0'(x)}}\right),$$

such that

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big) = -\infty.$$

Proof. As discussed above, it suffices to consider the case $s = 3$. Note that the assumption $(\sqrt{3}/\pi)|\mu_0| \geqslant \mu_1$ implies that $(2/\sqrt{3})|\mu_0| > \mu_1$. Therefore the non-negative constant K is well defined. In view of (3.1), we have

$$u_{tx} + uu_{xx} = -\frac{1}{2}u_x^2 + 2u\mu_0 - 2\mu_0^2 - \frac{1}{2}\mu_1^2.$$

By Lemma 2.3, there is $x_0 \in \mathbb{S}$ such that $u_0'(x_0) = \inf_{x\in\mathbb{S}} u_0'(x)$. Define $w(t) = u_x(t, q(t, x_0))$, where $q(t, x_0)$ is the flow of $u(t, q(t, x_0))$. Then

$$\frac{d}{dt}w(t) = (u_{tx} + u_{xx}q_t)\big(t, q(t, x_0)\big) = (u_{tx} + uu_{xx})\big(t, q(t, x_0)\big).$$

Substituting $(t, q(t, x_0))$ into the above equation and using (2.1), we obtain

$$\begin{aligned}\frac{d}{dt}w(t) &= -\frac{1}{2}w^2(t) + 2u\mu_0\big(t, q(t, x_0)\big) - 2\mu_0^2 - \frac{1}{2}\mu_1^2\\ &\leqslant -\frac{1}{2}w^2(t) + 2\mu_0\big|u\big(t, q(t, x_0)\big) - \mu_0\big| - \frac{1}{2}\mu_1^2\\ &\leqslant -\frac{1}{2}w^2(t) + \mu_1\left(\frac{\sqrt{3}}{3}|\mu_0| - \frac{1}{2}\mu_1\right)\\ &= -\frac{1}{2}w^2(t) + \frac{1}{2}K^2.\end{aligned}$$

By the assumption $w(0) < -K$, it follows that $w'(0) < 0$ and $w(t)$ is strictly decreasing on $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{K}{-u_0'(x_0)}} \in \left(0, \frac{1}{2}\right).$$

And so

$$\big(u_0'(x_0)\big)^2 = \frac{K^2}{(1-2\delta)^4} < w^2(t),$$

which is to say

$$K^2 < (1-2\delta)^4 w^2(t).$$

Therefore

$$\frac{d}{dt}w(t) \leqslant -\frac{1}{2}w^2(t)\big[1 - (1-2\delta)^4\big] \leqslant -\delta w^2(t), \quad t \in [0, T),$$

which leads to

$$-\frac{d}{dt}\frac{1}{w(t)} = \frac{1}{w^2(t)}\frac{d}{dt}w(t) \leqslant -\delta, \quad t \in [0, T).$$

Integrating both sides over $[0, t)$ yields

$$-\frac{1}{w(t)} + \frac{1}{u_0'(x_0)} \leqslant -\delta t, \quad t \in [0, T).$$

So

$$w(t) \leqslant \frac{u_0'(x_0)}{1 + \delta t u_0'(x_0)} \to -\infty, \quad \text{as } t \to -\frac{1}{\delta u_0'(x_0)}.$$

This implies

$$T \leqslant -\frac{1}{\delta u_0'(x_0)} < +\infty.$$

In consequence, we have

$$\liminf_{t \uparrow T}\Big(\inf_{x \in \mathbb{S}} u_x(t, x)\Big) = -\infty.$$

This completes the proof of Theorem 3.4. □

Remark 3.3. We can apply Lemma 2.3 to verify the above theorem under the same conditions. In fact, if we define $w(t) = u_x(t, \xi(t)) = \inf_{x \in \mathbb{S}}[u_x(t, x)]$, then for all $t \in [0, T)$, $u_{xx}(t, \xi(t)) = 0$. Thus if $(\sqrt{3}/\pi)|\mu_0| \geqslant \mu_1$, one finds that

$$\frac{d}{dt}w(t) \leqslant -\frac{1}{2}w^2(t) + \frac{1}{2}K^2,$$

where K is the same as Theorem 3.4. Then applying the assumptions of Theorem 3.4 and following the lines of the proof of Theorem 3.4, we see that if

$$w(0) < -\sqrt{2\mu_1\left(\frac{\sqrt{3}}{3}|\mu_0| - \frac{1}{2}\mu_1\right)},$$

then T is finite and $\liminf_{t \uparrow T}(\inf_{x \in \mathbb{S}} u_x(t, x)) = -\infty$.

Using the conserved quantity H_2, we can derive the following blow-up result.

Theorem 3.5. *Let $u_0 \in H^s$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to* (1.1) *with the initial data u_0. If $\mu_1^4 + 4\mu_0^2\mu_1^2 > 8\mu_0 H_2$ (in particular, $\mu_0 H_2 \leqslant 0$), where μ_0, μ_1 are defined in* (2.3) *and* (2.4)*. Then the corresponding solution $u(t, x)$ to* (1.1) *blows up in finite time T with*

$$0 < T \leqslant 6 + \frac{1 + |\int_{\mathbb{S}} u_{0x}^3(x)\,dx|}{\frac{3}{2}\mu_1^4 + 6\mu_0^2\mu_1^2 - 12\mu_0 H_2},$$

such that

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big) = -\infty.$$

Proof. Again it suffices to consider the case $s = 3$. Recall that

$$H_2 = \int_{\mathbb{S}} \left(\mu_0 u^2 + \frac{1}{2} u u_x^2\right) dx$$

is independent of time t. In view of (3.2), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - \frac{3}{2}\mu_1^4 + 6\mu_0\int_{\mathbb{S}} u u_x^2\,dx - 6\mu_0^2\int_{\mathbb{S}} u_x^2\,dx \\
&= -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - \frac{3}{2}\mu_1^4 + 12\mu_0 H_2 - 12\mu_0^2\int_{\mathbb{S}} u^2\,dx - 6\mu_0^2\mu_1^2 \\
&\leqslant -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - \frac{3}{2}\mu_1^4 + 12\mu_0 H_2 - 6\mu_0^2\mu_1^2.
\end{aligned}$$

By the assumption of the theorem, we have that $\mu_1^4 + 4\mu_0^2\mu_1^2 > 8\mu_0 H_2$. Let

$$c_1 = \frac{1}{2} > 0, \qquad c_2 = \frac{3}{2}\mu_1^4 + 6\mu_0^2\mu_1^2 - 12\mu_0 H_2 > 0.$$

It then follows that

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leqslant -c_1\int_{\mathbb{S}} u_x^4\,dx - c_2 \leqslant -c_1\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}} - c_2.$$

Define $V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx$ with $t \in [0,T)$. It is clear that

$$\frac{d}{dt}V(t) \leqslant -c_1\big(V(t)\big)^{\frac{4}{3}} - c_2 \leqslant -c_2 < 0, \quad t \in [0,T).$$

Let $t_1 = (1 + |V(0)|)/c_2$. Then following the proof of Theorem 3.3, we have

$$T \leqslant t_1 + \frac{3}{c_1} < +\infty.$$

This implies the desired result as in Theorem 3.5. □

3.2. Blow-up for the μDP equation

The first blow-up result for the μDP equation is given in the following.

Theorem 3.6. *Let $u_0 \in H^s$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t,x)$ to* (1.2) *with the initial data u_0. If $\mu_0 \tilde{H}_2 \leqslant 0$, i.e. $\int_{\mathbb{S}}(\frac{3}{2}\mu_0^2(A^{-1}\partial_x u_0)^2 + \frac{\mu_0}{6}u_0^3)\,dx \leqslant 0$, then the corresponding solution $u(t,x)$ to* (1.2) *associated with the* μDP *equation must blow up in finite time $T > 0$. If $\mu_0 \neq 0$, then*

$$0 < T \leqslant 1 + \frac{1 + |u_0'(\xi_0)|}{3\mu_0^2},$$

such that

$$\liminf_{t \uparrow T}\Bigl(\inf_{x \in \mathbb{S}} u_x(t,x)\Bigr) = -\infty$$

where $\mu_0 u_0(\xi_0) \leqslant 0$.

Proof. Since the case $\mu_0 = 0$ was proved in [33], we only need to show the case $\mu_0 \neq 0$. Again as discussed previously, it suffices to prove the theorem only with $s = 3$. To this end, by the assumption $\mu_0 \tilde{H}_2 \leqslant 0$, we firstly want to show there exists some $\xi(t) \in \mathbb{S}$ for any fixed $t \in [0, T)$ with the maximum existence time $T > 0$ and $\xi(0) = \xi_0$ such that $\mu_0 u(t, \xi(t)) \leqslant 0$ for any fixed $t \in [0, T)$. If not, then $\mu_0 u(t,x) > 0$ with any $x \in \mathbb{S}$ and some $t \in [0,T)$. Recall that

$$\tilde{H}_2 = \int_{\mathbb{S}} \left(\frac{3}{2}\mu(u)\bigl(A^{-1}\partial_x u\bigr)^2 + \frac{1}{6}u^3\right) dx = \int_{\mathbb{S}} \left(\frac{3}{2}\mu_0\bigl(A^{-1}\partial_x u_0\bigr)^2 + \frac{1}{6}u_0^3\right) dx$$

and $\mu(u(t)) = \mu(u_0) = \mu_0$ are independent of time t. It then follows that

$$\mu_0 \tilde{H}_2 = \int_{\mathbb{S}} \left(\frac{3}{2}\mu_0^2\bigl(A^{-1}\partial_x u\bigr)^2 + \frac{1}{6}(\mu_0 u)u^2\right) dx > 0$$

which contradicts the assumption $\mu_0 \tilde{H}_2 \leqslant 0$.

Next we take the trajectory $q(t,x)$ defined in (2.15). Since $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. It is then inferred that there exists $x_0(t) \in \mathbb{R}$ such that

$$q\bigl(t, x_0(t)\bigr) = \xi(t), \quad t \in [0, T)$$

with $x_0(0) = \xi_0$. Define $w(t) = u_x(t, q(t, x_0(t)))$, where $q(t, x_0(t))$ is the flow of $u(t, q(t, x_0(t)))$. Differentiating the μDP equation with respect to x yields

$$u_{tx} + u_x^2 + u u_{xx} + 3\mu_0 A^{-1}\partial_x^2 u = 0.$$

From (2.12) we deduce that

$$u_{tx} = -u_x^2 - uu_{xx} + 3\mu_0(u - \mu_0). \tag{3.3}$$

However

$$\frac{d}{dt}w(t) = (u_{tx} + u_{xx}q_t)\big(t, q\big(t, x_0(t)\big)\big) = (u_{tx} + uu_{xx})\big(t, q\big(t, x_0(t)\big)\big).$$

Substituting $(t, q(t, x_0(t)))$ into Eq. (3.3), we obtain

$$\frac{d}{dt}w(t) = -w^2(t) + 3\mu_0 u\big(t, q\big(t, x_0(t)\big)\big) - 3\mu_0^2.$$

By the assumption of the theorem, $\mu_0 u(t, q(t, x_0(t))) = \mu_0 u(t, \xi(t)) \leqslant 0$. This implies that

$$\frac{d}{dt}w(t) \leqslant -w^2(t) - 3\mu_0^2, \quad t \geqslant 0.$$

Let $t_1 = (1 + |u_0'(\xi_0)|)/(3\mu_0^2)$. Then similar to the proof of Theorem 3.3, one finds that $w(t_1) \leqslant -1$ and

$$-\frac{1}{w(t)} - 1 \leqslant -\frac{1}{w(t)} + \frac{1}{w(t_1)} \leqslant -(t - t_1), \quad t \in [t_1, T).$$

Therefore

$$w(t) \leqslant \frac{1}{t - t_1 - 1} \to -\infty, \quad \text{as} \quad t \to t_1 + 1,$$

which implies that $T \leqslant t_1 + 1 < +\infty$ with

$$\liminf_{t \uparrow T}\Big(\inf_{x \in \mathbb{S}} u_x(t, x)\Big) = -\infty.$$

This completes the proof of Theorem 3.6. □

Remark 3.4. It is observed in the proof of Theorem 3.6 that we may control the sign of the conserved quantity $\tilde{H}_2$ to guarantee the blow-up solutions of the μDP equation. This new method is expected to have further applications to other nonlinear wave equations with a part of the Burgers equation.

We are now in a position to give another blow-up result for the μDP equation.

Theorem 3.7. *Let $u_0 \in H^s$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to* (1.2) *with the initial data u_0. If*

$$|\mu_0| < \sqrt{\frac{32\pi^2 - 9}{32\pi^2}}\mu_2,$$

where μ_0, μ_2 *are defined in* (2.3) *and* (2.8)*, then the corresponding solution* $u(t,x)$ *to* (1.2) *must blow up in finite time* $T > 0$ *with*

$$0 < T \leqslant \inf_{\alpha \in I}\left(\frac{6}{4 - 9\alpha|\mu_0|} + 2\alpha \frac{1 + |\int_{\mathbb{S}} u_{0x}^3(x)\,dx|}{72\pi^2\alpha\mu_0^2(\mu_2^2 - \mu_0^2) - 9|\mu_0|\mu_2^2}\right),$$

where $I = (\frac{\mu_2^2}{8\pi^2|\mu_0|(\mu_2^2-\mu_0^2)}, \frac{4}{9|\mu_0|})$ *such that*

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big) = -\infty.$$

Proof. Since the case $\mu_0 = 0$ was proved in [33], we only need to show the case $\mu_0 \neq 0$. Similar to the proof of above theorem, it suffices to consider the case $s = 3$. Recall (3.3), i.e.,

$$u_{tx} + u_x^2 + uu_{xx} = 3u\mu_0 - 3\mu_0^2.$$

Multiplying the above equation by $3u_x^2$ and integrating on $\mathbb{S}$ with respect to x, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= \int_{\mathbb{S}} 3u_x^2 u_{xt}\,dx \\
&= -3\int_{\mathbb{S}} u_x^4\,dx - \int_{\mathbb{S}} 3uu_x^2u_{xx}\,dx + 9\mu_0\int_{\mathbb{S}} uu_x^2\,dx - 9\mu_0^2\int_{\mathbb{S}} u_x^2\,dx \\
&= -2\int_{\mathbb{S}} u_x^4\,dx + 9\mu_0\int_{\mathbb{S}} uu_x^2\,dx - 9\mu_0^2\int_{\mathbb{S}} u_x^2\,dx.
\end{aligned}$$

On the other hand, it follows from Lemma 2.2 that

$$\int_{\mathbb{S}} (u - \mu_0)^2\,dx \leqslant \frac{1}{4\pi^2}\int_{\mathbb{S}} u_x^2\,dx.$$

Or, what is the same,

$$\int_{\mathbb{S}} u_x^2\,dx \geqslant 4\pi^2\int_{\mathbb{S}} (u - \mu_0)^2\,dx.$$

It is also easy to see that

$$\int_{\mathbb{S}} (u - \mu_0)^2\,dx = \int_{\mathbb{S}} \big(u^2 - 2\mu_0 u + \mu_0^2\big)\,dx = \mu_2^2 - \mu_0^2.$$

This implies in turn that

$$\int_{\mathbb{S}} u_x^2\,dx \geqslant 4\pi^2\big(\mu_2^2 - \mu_0^2\big).$$

In addition, it is noted that

$$\int_{\mathbb{S}} uu_x^2\,dx \leqslant \Big(\int_{\mathbb{S}} u^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{S}} u_x^4\,dx\Big)^{\frac{1}{2}} \leqslant \frac{\alpha}{2}\int_{\mathbb{S}} u_x^4\,dx + \frac{1}{2\alpha}\int_{\mathbb{S}} u^2\,dx$$
$$= \frac{\alpha}{2}\int_{\mathbb{S}} u_x^4\,dx + \frac{1}{2\alpha}\mu_2^2.$$

Therefore, combining the above inequalities yields

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leqslant \Big(\frac{9}{2}\alpha|\mu_0| - 2\Big)\int_{\mathbb{S}} u_x^4\,dx + \frac{9}{2\alpha}|\mu_0|\mu_2^2 - 36\pi^2\mu_0^2\big(\mu_2^2 - \mu_0^2\big).$$

Set

$$\frac{9}{2}\alpha|\mu_0| - 2 < 0,$$

which is to say, $\alpha < 4/(9|\mu_0|)$. If $\alpha > 0$ also satisfies

$$36\pi^2\mu_0^2\big(\mu_2^2 - \mu_0^2\big) - \frac{9}{2\alpha}|\mu_0|\mu_2^2 > 0,$$

then one finds that

$$\alpha > \frac{\mu_2^2}{8\pi^2|\mu_0|(\mu_2^2 - \mu_0^2)}.$$

By the assumption of this theorem, we know that

$$|\mu_0| < \sqrt{\frac{32\pi^2 - 9}{32\pi^2}}\,\mu_2.$$

Therefore one can choose $\alpha > 0$ such that

$$\frac{\mu_2^2}{8\pi^2|\mu_0|(\mu_2^2 - \mu_0^2)} < \alpha < \frac{4}{9|\mu_0|}.$$

Let

$$c_1 = 2 - \frac{9}{2}\alpha|\mu_0| > 0, \qquad c_2 = 36\pi^2\mu_0^2\big(\mu_2^2 - \mu_0^2\big) - \frac{9}{2\alpha}|\mu_0|\mu_2^2 > 0.$$

This then follows that

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leqslant -c_1\int_{\mathbb{S}} u_x^4\,dx - c_2 \leqslant -c_1\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}} - c_2.$$

Again, define $V(t)=\int_{\mathbb{S}} u_x^3(t,x)\,dx$ with $t\in[0,T)$. Then we have

$$\frac{d}{dt}V(t) \leqslant -c_1\big(V(t)\big)^{\frac{4}{3}} - c_2 \leqslant -c_2 < 0, \quad t\in[0,T).$$

Similar to the proof of Theorem 3.3, we define $t_1=(1+|V(0)|)/c_2$ and conclude $T\leqslant t_1+\frac{3}{c_1}<+\infty$. In consequence of Theorem 3.2, we obtain

$$\liminf_{t\uparrow T}\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big) = -\infty.$$

This completes the proof of Theorem 3.7. □

4. Blow-up rate

Our attention is now turned to the question of the blow-up rate of the slope to a breaking wave for the initial-value problem (1.1) and (1.2).

Theorem 4.1. *Let $u(t,x)$ be the solution to the initial-value problem* (1.1) *associated with the* μCH *equation with initial data $u_0\in H^s$, $s>3/2$. Let $T>0$ be the maximal time of existence of the solution $u(t,x)$. If $T<\infty$, we have*

$$\lim_{t\uparrow T}\Big\{\inf_{x\in\mathbb{S}}\big[u_x(t,x)(T-t)\big]\Big\} = -2$$

while the solution remains uniformly bounded.

Proof. The uniform boundedness of the solution can be easily obtained by the a priori estimate (2.5). By Lemma 2.3, we can see that the function

$$w(t)=\inf_{x\in\mathbb{S}} u_x(t,x)=u_x\big(t,\xi(t)\big)$$

is locally Lipschitz with $w(t)<0$, $t\in[0,T)$. Note that $u_{xx}(t,\xi(t))=0$ for a.e. $t\in(0,T)$.

It follows from Theorem 3.4 that

$$\frac{d}{dt}w(t) \leqslant -\frac{1}{2}w^2(t)+N, \quad t\in[0,T), \tag{4.1}$$

where $N=\frac{1}{2}K^2$ and $K=\sqrt{2\mu_1(|\frac{\sqrt{3}}{3}|\mu_0|-\frac{1}{2}\mu_1|)}$.

Now fix any $\varepsilon\in(0,1/2)$. From Theorem 3.2, there exists $t_0\in(0,T)$ such that $w(t_0)<-\sqrt{2N+\frac{N}{\varepsilon}}$. Notice that $w(t)$ is locally Lipschitz so that it is absolutely continuous on $[0,T)$. It follows from the above inequality that $w(t)$ is decreasing on $[t_0, T)$ and satisfies that

$$w(t) < -\sqrt{2N + \frac{N}{\varepsilon}} < -\sqrt{\frac{N}{\varepsilon}}, \quad t \in [t_0, T).$$

Since $w(t)$ is decreasing on $[t_0, T)$, it follows that

$$\lim_{t \uparrow T} w(t) = -\infty.$$

From (4.1), we obtain

$$\frac{1}{2} - \varepsilon \leqslant \frac{d}{dt}\big(w(t)^{-1}\big) = -\frac{w'(t)}{w^2(t)} \leqslant \frac{1}{2} + \varepsilon.$$

Integrating the above equation on (t, T) with $t \in (t_0, T)$ and noticing that $\lim_{t \uparrow T} w(t) = -\infty$, we get

$$\left(\frac{1}{2} - \varepsilon\right)(T - t) \leqslant -\frac{1}{w(t)} \leqslant \left(\frac{1}{2} + \varepsilon\right)(T - t).$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, in view of the definition of $w(t)$, the above inequality implies the desired result of Theorem 4.1. □

Theorem 4.2. *Let $u(t, x)$ be the solution to the initial-value problem* (1.2) *associated with the* μDP *equation with initial data $u_0 \in H^s$, $s > 3/2$. Let $T > 0$ be the maximal time of existence of the solution $u(t, x)$. If $T < \infty$, we have*

$$\lim_{t \uparrow T} \Big\{ \inf_{x \in \mathbb{S}} \big[u_x(t, x)(T - t)\big] \Big\} = -1$$

while the solution remains uniformly bounded.

Proof. Again we may assume $s = 3$ to prove this theorem. The uniform boundedness of the solution can be easily obtained by the a priori estimate in Lemma 2.6. By (3.3), we have

$$u_{tx} + u_x^2 + u u_{xx} = 3u\mu_0 - 3\mu_0^2.$$

It is inferred from Lemma 2.3 that the function

$$w(t) = \inf_{x \in \mathbb{S}} u_x(t, x) = u_x\big(t, \xi(t)\big)$$

is locally Lipschitz with $w(t) < 0$, $t \in [0, T)$. Note that $u_{xx}(t, \xi(t)) = 0$ for a.e. $t \in (0, T)$. Then we deduce that

$$w'(t) = -w^2(t) + 3u\big(t, \xi(t)\big)\mu_0 - 3\mu_0^2, \quad t \in [0, T). \tag{4.2}$$

It follows from Lemma 2.6 that

$$\begin{aligned}\left|3u\big(t,\xi(t)\big)\mu_0 - 3\mu_0^2\right| &\leqslant 3|\mu_0|\left\|u(t,x)\right\|_{L^\infty} + 3\mu_0^2\\ &\leqslant 3|\mu_0|\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)T + 3|\mu_0|\|u_0\|_{L^\infty} + 3\mu_0^2, \quad t \in [0,T].\end{aligned}$$

Set

$$N(T) = 3|\mu_0|\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2\right)T + 3|\mu_0|\|u_0\|_{L^\infty} + 3\mu_0^2.$$

Combining above estimates, we deduce that

$$w'(t) \leqslant -w^2(t) + N(T), \quad t \in [0,T). \tag{4.3}$$

Now fix any $\varepsilon \in (0,1)$. In view of Theorem 3.2, there exists $t_0 \in (0,T)$ such that $w(t_0) < -\sqrt{N(T) + \frac{N(T)}{\varepsilon}}$. Notice that $w(t)$ is locally Lipschitz so that it is absolutely continuous on $[0,T)$. It then follows from the above inequality that $w(t)$ is decreasing on $[t_0,T)$ and satisfies that

$$w(t) < -\sqrt{N(T) + \frac{N(T)}{\varepsilon}} < -\sqrt{\frac{N(T)}{\varepsilon}}, \quad t \in [t_0,T).$$

Since $w(t)$ is decreasing on $[t_0,T)$, it follows that

$$\lim_{t\uparrow T} w(t) = -\infty.$$

It is found from (4.3) that

$$1 - \varepsilon \leqslant \frac{d}{dt}\big(w(t)^{-1}\big) = -\frac{w'(t)}{w^2(t)} \leqslant 1 + \varepsilon.$$

Integrating the above equation on (t,T) with $t \in (t_0,T)$ and noticing that $\lim_{t\uparrow T} w(t) = -\infty$, we get

$$(1-\varepsilon)(T-t) \leqslant -\frac{1}{w(t)} \leqslant (1+\varepsilon)(T-t).$$

Since $\varepsilon \in (0,1)$ is arbitrary, in view of the definition of $w(t)$, the above inequality implies the desired result of Theorem 4.2. □

Appendix A. Geometric descriptions of these two equations

Integrable equations solved by the inverse scattering transformation method have elegant geometric interpretations. Several different geometric frameworks have been used to provide geometric interpretations to integrable systems. Besides the Arnold's approach to Euler equations on Lie groups [1] (see also more recent exposition [30]), other two geometric descriptions are important in the study of integrable systems. In an intriguing work due to Chern and Tenenblat [5], they provide a classification of a class of nonlinear evolution equations describing pseudo-spherical surfaces. As a consequence, many integrable equations are shown to be geometrically integrable. The approach also provides a direct approach to compute conservation laws of integrable systems. Another interesting geometric interpretation is provided through invariant geometric curve or surface flows. For instance, the mKdV equation, the Schrödinger equation, the KdV equation and the Sawada–Kotera equation arise naturally from non-stretching invariant curve flows in Klein geometries (see [7,8,24,25] and references therein). It is noticed that the celebrated CH equation arises from a non-stretching invariant plane curve flow in centro-equiaffine geometry [7] and describes pseudo-spherical surfaces [38].

In this appendix, we show that the μCH equation and μDP equation also arise from non-stretching invariant curve flows respectively in plane centro-equiaffine geometry CA^2 and three-dimensional centro-equiaffine geometry CA^3. Furthermore, we show that the μCH equation describes pseudo-spherical surfaces, and the μDP equation describes affine surfaces.

First, we study non-stretching invariant plane curve flows in centro-equiaffine geometry CA^2. Consider a star-shaped plane curve $\gamma(p)$ with a parameter p, i.e., the curve satisfies $[\gamma, \gamma_p] \neq 0$, here $[\gamma_1, \gamma_2]$ denotes the determinant of two vectors γ_1 and γ_2. Along the curve one may represent it by a special parameter σ satisfying

$$[\gamma, \gamma_\sigma] = 1. \tag{A.1}$$

In terms of the free parameter p, the centro-equiaffine arc-length is given by

$$d\sigma = [\gamma, \gamma_p]\, dp.$$

It follows from (A.1) that there exists a function ϕ such that

$$\gamma_{\sigma\sigma} + \phi\gamma = 0,$$

where

$$\phi = [\gamma_\sigma, \gamma_{\sigma\sigma}]$$

is the centro-equiaffine curvature of $\gamma(\sigma)$. It is easily to verify that ϕ is invariant with respect to the linear transformation

$$\gamma' = A\gamma, \quad A \in SL(2, \mathbb{R}).$$

The centro-equiaffine tangent and normal vectors of γ are defined to be

$$\mathbf{T} = \gamma_\sigma, \qquad \mathbf{N} = -\gamma.$$

Hence we have the Serret–Frenet formula for γ

$$\mathbf{T}_\sigma = \phi\mathbf{N}, \qquad \mathbf{N}_\sigma = -T.$$

Consider the plane curve flow for $\gamma(\sigma, t)$ in CA^2

$$\gamma_t = f\mathbf{N} + g\mathbf{T}, \tag{A.2}$$

where f and g are respectively the normal and tangent velocities, which depend on the centro-equiaffine curvature ϕ and its derivatives with respect to arc-length σ.

Let $d\sigma = \xi\, dp$, and $L = \oint d\sigma$ be the centro-equiaffine perimeter for a closed curve. Assume that the centro-equiaffine arc-length does not depend on time and L is invariant along the flow, then the velocities f and g satisfy

$$g_\sigma - 2f = 0, \qquad \oint_\gamma f\, d\sigma = 0. \tag{A.3}$$

A direct computation shows that the curvature satisfies the equation

$$\phi_t = \left(D_\sigma^2 + 4\phi + 2\phi_\sigma \partial_\sigma^{-1}\right) f \tag{A.4}$$

after using (A.3).

Setting $\phi \equiv m = \delta u - u_{\sigma\sigma}$ and $f = -u_\sigma$, we obtain the CH and HS equations

$$m_t + u_{\sigma\sigma\sigma} + 4mu_\sigma + 2um_\sigma = 0,$$

respectively for $\delta = 1$ and $\delta = 0$. After the change of variables

$$t \to t, \quad \sigma \to x - t, \quad u \to \frac{1}{2}u, \tag{A.5}$$

we get the CH and HS equations in the standard form

$$m_t + 2mu_x + um_x = 0, \tag{A.6}$$

where $m = \delta u - u_{xx}$.

Now we assume that ϕ is periodic, i.e., $\phi(\sigma + 1) = \phi(\sigma)$. Let $\phi = \int_0^1 u\, d\sigma + h_\sigma$, where $h(t, \sigma)$ is also a periodic function of σ. It implies that

$$\int_0^1 u\, d\sigma = \int_0^1 \phi\, d\sigma = \mu(u).$$

Namely, $\mu(u)$ is the mean curvature of γ. Taking

$$h = -u_\sigma, \qquad f = -u_\sigma,$$

we arrive at the equation

$$m_t + u_{\sigma\sigma\sigma} + 4mu_\sigma + 2um_\sigma = 0.$$

The change of variables (A.5) leads to the μCH equation

$$m_t + 2mu_x + um_x = 0, \quad m = \mu(u) - u_{xx}. \tag{A.7}$$

Here we make a remark about the geometric description of the μCH equation.

Remark A1. It was shown by Reyes [38] that CH and HS equations describe pseudo-spherical surfaces. Similarly, we can show that the μCH equation also describes pseudo-spherical surfaces, i.e., there exist the one-forms

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\left(\lambda m - \frac{1}{2}\lambda^2 + 2\right)dx + \frac{1}{2}\left[\frac{1}{2}\lambda^2 u - \lambda\left(u_x + um + \frac{1}{2}\right) + \mu(u) - 2u + \frac{2}{\lambda}\right]dt,\\
\omega_2 &= \lambda\, dx + (1 - \lambda u + u_x)\, dt,\\
\omega_3 &= \frac{1}{2}\left(\lambda m - \frac{1}{2}\lambda^2 - 2\right)dx + \frac{1}{2}\left[\frac{1}{2}\lambda^2 u - \lambda\left(u_x + um + \frac{1}{2}\right) + \mu(u) + 2u - \frac{2}{\lambda}\right]dt,
\end{aligned}$$

which satisfy the structure equations for pseudo-spherical surface

$$d\omega_1 = \omega_3 \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_3, \qquad d\omega_3 = \omega_1 \wedge \omega_2.$$

Based on the structure equations, using the equations for pseudo-potential, we are able to obtain two sets of conservation laws of the μCH equation.

Next, we consider non-stretching invariant space curve flows in three-dimensional centro-equiaffine geometry CA^3. The isometries of the centro-equiaffine geometry consists of the linear transformations $x' = Ax$, $A \in SL(3, \mathbb{R})$. For a general curve $\gamma(p)$ with a parameter p, satisfying $[\gamma, \gamma_p, \gamma_{pp}] \neq 0$, along the curve one may reparametrize it by a special parameter σ satisfying

$$[\gamma, \gamma_\sigma, \gamma_{\sigma\sigma}] = 1, \tag{A.8}$$

everywhere, where $[\gamma_1, \gamma_2, \gamma_3]$ denotes the determinant of the vectors γ_1, γ_2 and γ_3. In terms of a free parameter p, the centro-equiaffine arc-length σ is defined to be

$$d\sigma = [\gamma, \gamma_p, \gamma_{pp}]^{\frac{1}{3}}\, dp.$$

It follows from (A.8) that there exist two functions α and β such that

$$\gamma_{\sigma\sigma\sigma} = \alpha\gamma + \beta\gamma_\sigma,$$

where

$$\alpha = [\gamma_\sigma, \gamma_{\sigma\sigma}, \gamma_{\sigma\sigma\sigma}], \qquad \beta = [\gamma, \gamma_{\sigma\sigma\sigma}, \gamma_{\sigma\sigma}].$$

It is readily to verify that α and β are invariant with respect to centro-equiaffine linear transformations. We define them to be the curvatures of γ. Whence we have the Serret–Frenet formula

$$\begin{pmatrix} \gamma \\ \gamma_\sigma \\ \gamma_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & 0 \end{pmatrix}. \tag{A.9}$$

Now consider the invariant curve flow in CA^3 governed by

$$\gamma_t = F\gamma + G\gamma_\sigma + H\gamma_{\sigma\sigma} \tag{A.10}$$

where F, G and H are velocities, depending on the centro-equiaffine curvatures α and β and their derivatives with respect to σ.

Assume that the arc-length σ does not depend on time t, namely $[\frac{\partial}{\partial\sigma}, \frac{\partial}{\partial t}] = 0$, which implies that the velocities satisfy

$$F + G_\sigma + \frac{2}{3}\beta H + \frac{1}{3}H_{\sigma\sigma} = 0. \tag{A.11}$$

Similar to the plane case, for a closed curve γ, one requires that the centro-equiaffine perimeter is invariant under the curve flow. It turns out that

$$\oint\left(F + \frac{2}{3}\beta H\right) d\sigma = 0. \tag{A.12}$$

The evolution for the curvatures is

$$\begin{aligned} \alpha_t &= \left[F_{\sigma\sigma} + \alpha(G + 2H_\sigma) + \alpha_\sigma H\right]_\sigma + 2\alpha G_\sigma - \beta F_\sigma + \alpha H_{\sigma\sigma}, \\ \beta_t &= \left[3F_\sigma + G_{\sigma\sigma} + \beta(G + 2H_\sigma) + (\alpha + \beta_\sigma)H\right]_\sigma + 2\alpha H_\sigma \\ &\quad + \beta H_{\sigma\sigma} + \beta G_\sigma + \alpha_\sigma H. \end{aligned} \tag{A.13}$$

Now we consider two possibilities. First, we set $\beta = 1$, $F = u_\sigma + 2/3$, $G = -u$, $H = -1$ and $\alpha = -(u - u_{\sigma\sigma})$, so that (A.11), (A.12) and the second one of (A.13) hold. Thus the first one of (A.13) becomes

$$m_t + 3mu_\sigma + um_\sigma = 0, \tag{A.14}$$

which is exactly the DP equation.

Second, for the periodic case, we choose $\beta = 0$, $F = u_\sigma + 2/3$, $G = -u$, $H = -1$ and $\alpha = -(\mu(u) - u_{\sigma\sigma})$. Then the second one of (A.13) holds identically, and the first one of (A.13) becomes

$$m_t + 3mu_\sigma + um_\sigma = 0, \tag{A.15}$$

where $m = \mu(u) - u_{xx}$, which is the μDP equation.

In the following, we show that the μDP equation describes affine surfaces. Let A^3 be the unimodular affine space of dimension three and the unimodular affine group G be generated by the following transformations

$$\gamma' = A\gamma + B,$$

where $A \in SL(3, \mathbb{R})$, $B \in \mathbb{R}^3$. Let γ be the position vector of an affine surface M in A^3 and e_1, e_2, e_3 be an affine frame on M such that e_1, e_2 are tangent to M at γ, and

$$[e_1, e_2, e_3] = 1.$$

We write

$$d\gamma = \sum_j \omega^j e_j, \qquad de_j = \sum_k \omega_j^k e_k,$$

where ω^j and ω_j^k are the Maurer–Cartan forms of G, which satisfy

$$\sum_j \omega_j^j = 0, \qquad d\omega_j^l = \sum_j^3 \omega_j^k \omega_k^l,$$
$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0, \qquad \omega^3 = 0,$$
$$d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge \omega_2^2, \qquad d\omega^1 = \omega^1 \wedge \omega_1^1 + \omega^2 \wedge \omega_2^1, \tag{A.16}$$

for $j, l = 1, 2, 3$.

Definition A.1. (See [6].) A partial differential equation for a function $u(t, x)$ describes affine surfaces if there exist smooth functions f_j^k, g_j^k, h_{pq}, $1 \leqslant j, k \leqslant 3$, $1 \leqslant p, q \leqslant 2$, depending only on u and their derivatives such that the 1-forms $\omega_j^k = f_j^k\,dx + g_j^k\,dt$, $\omega^p = h_{p1}\,dx + h_{p2}\,dt$ satisfy the structure equations (A.16) for affine surfaces M.

We have the following theorem.

Theorem A.1. *The* μDP *equation describes affine surfaces.*

Proof. Indeed, we take

$$f_1^1 = 0, \qquad g_1^1 = \frac{1}{2\lambda} - u_x, \qquad f_1^2 = \lambda m, \qquad g_1^2 = -\frac{1}{4\lambda}\big(4\lambda^2 um - 4\lambda u_x + 1\big),$$
$$f_1^3 = \frac{1}{2\lambda}, \qquad g_1^3 = -\frac{1}{2\lambda}\big(2\mu(u) + u\big), \qquad f_2^1 = 0, \qquad g_2^1 = \frac{1}{\lambda}, \qquad f_2^2 = 0,$$
$$g_2^2 = u_x - \frac{1}{2\lambda}, \qquad f_2^3 = \frac{1}{\lambda}, \qquad g_2^3 = -\frac{u}{\lambda}, \qquad f_3^1 = -\lambda, \qquad g_3^1 = \lambda u, \qquad f_3^2 = \frac{\lambda}{2},$$
$$g_3^2 = \lambda\left(\mu(u) - \frac{1}{2}u\right), \qquad f_3^3 = 0, \qquad g_3^3 = 0,$$
$$h_{11} = 1, \qquad h_{12} = -u, \qquad h_{21} = -\frac{1}{2}, \qquad h_{22} = \frac{1}{2}u - \mu(u).$$

A straightforward computation shows that the Maurer–Cartan forms defined by these functions satisfy the structure equations for affine surfaces. □

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