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# A symbolic calculus and $L^2$-boundedness on nilpotent Lie groups

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**Abstract**

We work on a general nilpotent Lie group

$$\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \ldots \oplus \mathscr{G}_r,$$

where $r \geqslant 1$ and $\mathscr{G}^{(k)} = \oplus_{j=k}^{r}$ is the descending central series of $\mathscr{G}$. A composition theorem and an $L^2$ boundedness theorem for convolution operators $f \to f \star A$ are proved. The composition theorem holds for symbols $a = A^\wedge$ satisfying the estimates

$$|D^\alpha a(\xi)| \leqslant C_\alpha \mathbf{m}(\xi) g(\xi)^{-\alpha},$$

where $\mathbf{m}$ is a weight and

$$g(\xi)^\alpha = g_1(\xi)^{\alpha_1} \ldots g_r(\xi)^{\alpha_r},$$

where

$$g_k(\xi) = \left(1 + \left(\sum_{j=k+1}^{r} ||\xi_j||^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

The class of weights admissible for the calculus is considerably larger than those of the existing calculi. For the $L^2$-boundedness it is sufficient that

$$|D^\alpha a(\xi)| \leqslant C_\alpha g(\xi)^{-\alpha}.$$

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This goes in the direction of Howe's conjecture and improves the results of Howe and Manchon. It is very likely that our methods could also be used to extend the calculus of Melin to general homogeneous groups.
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## 1. Introduction

Although the idea is in a way present in Strichartz [8], it seems that it was Howe [4] who first argued convincingly for the possibility of a symbolic calculus for convolution operators on nilpotent Lie groups as a convenient replacement for the highly efficient operational calculus provided by the Fourier transform in the abelian case. The essence of such a calculus would be to describe the product

$$a\#b = (a^\vee \bigstar b^\vee)^\wedge, \quad a, b \in C_c^\infty(\mathscr{G}^\bigstar),$$

on a (connected simply connected) nilpotent Lie group $\mathscr{G}$, where $f^\wedge$ and $f^\vee$ denote the abelian Fourier transforms on the Lie algebra (identified with $\mathscr{G}$) and its dual, and its continuity in terms of, e.g., asymptotic expansions controlled by suitable norms similar to those used in the theory of pseudodifferential operators. In addition, such a calculus should also provide some sufficient conditions for $L^2$-boundedness of convolution operators $f \to f \bigstar A$ on $\mathscr{G}$ in terms of their symbols $a = A^\wedge$. This idea gets much support from the remarkable relationship between the convolution structure of the Heisenberg group and the Weyl calculus for pseudodifferential operators, as explained in, e.g., Howe [3]. What Howe actually proves is the boundedness theorem for convolution operators whose symbols satisfy the estimates

$$|D^\alpha a(\xi)| \leqslant C_\alpha (1 + ||\xi||)^{-\rho|\alpha|}$$

with $\rho > \frac{1}{2}$ (cf. Howe [4]). Here $|| \cdot ||$ stands for a linear space norm on $\mathscr{G}$ and $|\alpha|$ is the length of a multi-index $\alpha$. He also makes a conjecture on relaxing the estimates by requiring only certain derivatives to vanish in certain directions so that the theorem could apply not only to pseudolocal operators.

With the Weyl calculus as developed and clarified by Hörmander [1,2], a certain translation-invariant unitary operator was brought to focus and shown to be crucial for the calculus. Let $W = V \oplus V^\bigstar$ be the phase space for symbols of pseudodifferential operators on the vector space $V$. The starting point for Hörmander's theory is the observation that the Weyl product of symbols $a, b \in C_c^\infty(W)$ can be written down as

$$a\#b(w) = e^{iB(w,D)}(a \otimes b)(w, w),$$

where $B$ is the natural symplectic bilinear form on $W = W^\bigstar$.

This has inspired Melin [6] to look for an analogue to the Hörmander operator in the nilpotent group context and he came up with the following formula:

$$a\#b(\xi) = \mathbf{U}(a\otimes b)(\xi,\xi),$$

where

$$\mathbf{U}(F)^\vee(x,y) = F^\vee\left(\frac{x-y+xy}{2}, \frac{y-x+xy}{2}\right), \quad x,y\in\mathscr{G}.$$

Melin shows that the unitary operator $\mathbf{U}$ can be imbedded in a one-parameter unitary group $U_t$ with the infinitesimal generator $\Gamma$ which is a differential operator on $\mathscr{G}^\star \times \mathscr{G}^\star$ with polynomial coefficients and he thoroughly investigates the properties of $\Gamma$ under the assumption that $\mathscr{G}$ is a homogeneous stratified group. As a result he obtains a composition formula for classes of symbols satisfying the estimates

$$|D^\alpha a(\xi)| \leqslant C_\alpha(1+|\xi|)^{m-\mathrm{d}(\alpha)}, \tag{1.1}$$

where $|\cdot|$ is the homogeneous norm on $\mathscr{G}$ and $\mathrm{d}(\alpha)$ is a homogeneous length of a multi-index $\alpha$. He also proves an $L^2$-boundedness theorem for symbols satisfying (1.1) with $m=0$.

Subsequently, Manchon [5] takes over Melin's starting point and sets up to investigate the operator $\Gamma$ and the unitary group $U_t$ in the context of a general nilpotent group to come up with a composition formula for classes of symbols satisfying the estimates

$$|D^\alpha a(\xi)| \leqslant C_\alpha(1+||\xi||)^{m-\rho|\alpha|}$$

with $\rho > \frac{1}{2}$ and Howe's $L^2$-boundedness theorem. It has to be stressed that the techniques of Howe, Manchon, and Melin are pairwise different.

What makes the whole matter difficult is that the Melin operator $\mathbf{U}$ unlike the Hörmander operator is not translation-invariant. However, it can be represented as a composition,

$$\mathbf{U}f(x,\lambda) = P_\lambda \circ \mathbf{U}'f(\cdot,\lambda)(\xi), \quad (\xi,\lambda)\in\mathscr{G} = \mathscr{G}'\oplus\mathscr{G}_r,$$

of a translation-invariant operator $P_\lambda$ on a quotient group $\mathscr{G}' = \mathscr{G}/\mathscr{G}_r$, where $\mathscr{G}_r$ is the centre of $\mathscr{G}$, and the Melin operator $\mathbf{U}'$ on $\mathscr{G}'$, the central variable $\lambda$ playing the rôle of a parameter. This is our substitute for the missing translation invariance. It allows us to stay much closer to Hörmander's approach and to completely bypass the laborious investigation of the unitary group of Melin.

Let us announce briefly the results of the present paper. We work on a general nilpotent Lie group

$$\mathscr{G} = \mathscr{G}_1\oplus\mathscr{G}_2\oplus\cdots\oplus\mathscr{G}_r,$$

where $r \geqslant 1$ and $\mathcal{G}^{(k)} = \oplus_{j=k}^{r} \mathcal{G}_j$ is the descending central series of $\mathcal{G}$. The composition theorem (Theorem 5.3) holds for symbols satisfying the estimates

$$|D^\alpha a(\xi)| \leqslant C_\alpha \mathbf{m}(\xi) g(\xi)^{-\alpha}, \tag{1.2}$$

where $\mathbf{m}$ is a weight and

$$g(\xi)^\alpha = g_1(\xi)^{\alpha_1} \cdots g_r(\xi)^{\alpha_r},$$

where

$$g_k(\xi) = \left(1 + \left(\sum_{j=k+1}^{r} ||\xi_j||^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

The class of weights admissible for the calculus is considerably larger than those of the existing calculi.

For the $L^2$-boundedness (Theorem 6.2) it is sufficient that

$$|D^\alpha a(\xi)| \leqslant C_\alpha g(\xi)^{-\alpha}$$

which goes in the direction of the above-mentioned Howe's conjecture. It is very likely that our methods could be used to improve the calculus of Melin and to extend it to general homogeneous groups in a similar way.

## 2. Slowly varying metrics

Let $X$ be an $n$-dimensional euclidean space. Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the scalar product and the corresponding euclidean norm. These are fixed throughout the paper.

*A slowly varying metric* on $X$ is, by definition, a family of norms $\{||\cdot||_x\}_{x \in X}$ such that for some $C \geqslant 1$,

$$\frac{1}{C} ||\cdot||_y \leqslant ||\cdot||_x \leqslant C ||\cdot||_y, \tag{2.1}$$

if $||x - y||_x < 1$ (cf. [4, vol. 1]). The metric is said to be *self-tempered*, if it satisfies

$$\left\{\frac{||z||_x}{||z||_y}\right\}^{\pm 1} \leqslant 1 + ||x - y||_x. \tag{2.2}$$

Of course, (2.2) implies (2.1) with $C = 2$.

We are going to define some slowly varying metrics on $X$ and $X \times X$ suitable for our purposes. Suppose that

$$X = \bigoplus_{k=1}^{r} X_r,$$

and let

$$X_{(k)} = \bigoplus_{j=1}^{k} X_j, \quad X^{(k)} = \bigoplus_{j=k}^{r} X_j.$$

Let

$$g_k(x) = \left( d_k + \left( \sum_{j=k+1}^{r} |x_j|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}},$$

where $d_k \geqslant 1$.

**Lemma 2.3.** *Let $d \geqslant 1$. Then*

$$\left( \frac{d+|x|}{d+|y|} \right)^{\pm\frac{1}{2}} \leqslant 1 + \frac{|x-y|}{(d+|x|)^{\frac{1}{2}}}$$

*for $x, y \in X$.*

**Proof.** In fact,

$$\left( \frac{1+a}{1+b} \right)^{\frac{1}{2}} = \left( \frac{1+b}{1+a} \right)^{\frac{1}{2}} + \frac{a-b}{(1+a)^{\frac{1}{2}}(1+b)^{\frac{1}{2}}} \leqslant 1 + \frac{a-b}{(1+a)^{\frac{1}{2}}}$$

for $a \geqslant b \geqslant 0$. □

From Lemma 2.3, we infer the following estimate:

$$\left( \frac{g_k(x)}{g_k(y)} \right)^{\pm\frac{1}{2}} \leqslant 1 + \frac{\left( \sum_{j=k+1}^{r} |x_j - y_j|^2 \right)^{\frac{1}{2}}}{g_k(x)^{\frac{1}{2}}} \leqslant 1 + \mathbf{g}_x(x-y), \tag{2.4}$$

where

$$\mathbf{g}_x(y) = \left( \sum_{k=1}^{r} \frac{|y_k|^2}{g_k(x)^2} \right)^{\frac{1}{2}}, \quad x, y \in X. \tag{2.5}$$

We shall also apply the abbreviated notation

$$\mathbf{g}_x(y) = \left| \frac{y}{g(x)} \right|,$$

where

$$\frac{z}{g(u)} = \left(\frac{z_1}{g_1(u)}, \frac{z_2}{g_2(u)}, \dots, \frac{z_r}{g_r(u)}\right), \quad z, u \in X. \tag{2.6}$$

**Proposition 2.7.** *The family of norms* (2.5) *is a self-tempered metric since it satisfies*

$$\left\{\frac{\mathbf{g}_x(z)}{\mathbf{g}_x(z)}\right\}^{\pm 1} \leqslant 1 + \mathbf{g}_x(x-y). \tag{2.8}$$

**Proof.** This is an immediate consequence of (2.4). □

We shall refer to any metric of this type as an H-metric on $X$. Note that every H-metríc $\mathbf{g}$ is determined by a vector $\mathbf{d} = (d_1, d_2, \dots, d_r) \in \mathbf{R}^r$.

If $\mathbf{g}$ and $\mathbf{h}$ are two **H**-metrics, then

$$\mathbf{G_x}(\mathbf{y}) = (\mathbf{g} \oplus \mathbf{h})_\mathbf{x}(\mathbf{y}) = \mathbf{g}_{x_1}(y_1) + \mathbf{h}_{x_2}(y_2), \quad \mathbf{x}, \mathbf{y} \in X \times X, \tag{2.9}$$

is also a self- tempered metric on $X \times X$.

A strictly positive function $\mathbf{m}$ on $X$ will be called a weight with respect to a slowly varying metric $\mathbf{g}$ on $X$ if it satisfies the condition

$$\mathbf{m}(x+y) \leqslant C\mathbf{m}(x)(1 + \mathbf{g_x}(\mathbf{y}))^M \tag{2.10}$$

for some $C, M > 0$. The weights form a group under multiplication. If $\mathbf{m}_1$ and $\mathbf{m}_2$ are $\mathbf{g}$-weights, then $\mathbf{m} = \max\{\mathbf{m}_1, \mathbf{m}_2\}$ is also a $\mathbf{g}$-weight. A typical example of a $\mathbf{g}$-weight is

$$\mathbf{m}(x) = 1 + \mathbf{g}_x(x).$$

Observe also that, by Lemma 2.3,

$$\mathbf{m}(x) = (1 + |x|)^s, \quad s \in \mathbf{R},$$

is a weight for any H-metric. Other important examples are

$$Q_j(\mathbf{x}) = \max\left\{\frac{1 + |x_{1j}|}{1 + |x_{2j}|}, \frac{1 + |x_{2j}|}{1 + |x_{1j|}}\right\},$$

where

$$\mathbf{x} = (x_1, x_2) = (x_{11}, x_{12}, \dots, x_{1r} | x_{21}, x_{22}, \dots, x_{2r}) = (x_{sj})_{s=1,2}^{1 \leqslant j \leqslant r}. \tag{2.11}$$

These are $\mathbf{G}$-weights, where $\mathbf{G} = \mathbf{g} \oplus \mathbf{g}$. It is important that $Q(\mathbf{x}) = 1$ on the diagonal in $X \times X$, where $x_1 = x_2$.

Let $\mathbf{m}$ be a weight with respect to a slowly varying metric $\mathbf{g}$ on $X$. For $f \in C^\infty(X)$ let

$$|f|^{\mathbf{m}}_{(k)}(\mathbf{g}) = \sup_{x \in X} \frac{\mathbf{g}_x(D^k f(x))}{\mathbf{m}(x)}$$

and

$$|f|^{\mathbf{m}}_{(k)}(\mathbf{g}) = \sum_{j=0}^{k} |f|^{\mathbf{m}}_{(j)}(\mathbf{g}),$$

where $D$ stands for the Fréchet derivative, and

$$\mathbf{g}_x(D^k f(x)) = \sup_{\mathbf{g}_x(y_j) \leqslant 1} \mathbf{g}_x(D^k f(x)(y_1, y_2, \dots, y_k)).$$

Let

$$\mathbf{S}^{\mathbf{m}}(X, \mathbf{g}) = \{a \in C^\infty(X) : |a|^{\mathbf{m}}_k(\mathbf{g}) < \infty, \text{ all } k \in \mathbf{N}\}.$$

$\mathbf{S}^{\mathbf{m}}(X, \mathbf{g})$ is a Fréchet space with the family of seminorms $|\cdot|^{\mathbf{m}}_k(\mathbf{g})$. Thus, $f \in C^\infty(X)$ belongs to $\mathbf{S}^{\mathbf{m}}(X, \mathbf{g})$ if and only if it satisfies the estimates

$$|D^\alpha f(x)| \leqslant C_\alpha \mathbf{m}(x) g(x)^{-\alpha},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbf{N}^r$, and

$$g(x)^\alpha = g_1(x)^{\alpha_1} g_2(x)^{\alpha_2} \cdots g_r(x)^{\alpha_r}.$$

Let also

$$S^{\mathbf{m}}_K(X, \mathbf{g}) = \{f \in S^{\mathbf{m}}(X, \mathbf{g}) : \operatorname{supp} f \subset K \subset X\}.$$

Apart from the Fréchet topology in the spaces $S^{\mathbf{m}}$ it is convenient to introduce a *weak topology* of the $C^\infty$-convergence on Fréchet bounded subsets. By the Ascoli theorem, this is equivalent to the pointwise convergence of bounded sequences in $S^{\mathbf{m}}$. Following Manchon [5], we call a mapping $T : S^{\mathbf{m}_1} \to S^{\mathbf{m}_2}$ *double-continuous*, if it is both Fréchet continuous and weakly continuous. The following lemma implies that $C^\infty_c(X)$ is weakly dense in $S^{\mathbf{m}}(X, \mathbf{g})$.

**Lemma 2.12.** *Let* $\mathbf{g}$ *be an* $H$*-metric on* $X$*. There exists a countable subset* $U$ *of* $X$ *and a family* $\varphi_u$ *of* $C^\infty_c$*-functions on* $X$ *supported in the balls*

$$\Omega_u = \{x \in X : \mathbf{g}_u(x - u) < 1\}$$

*such that* $\varphi_u \in S^{\mathbf{1}}(X, \mathbf{g})$ *uniformly in* $u \in U$ (*and* $\mathbf{g}$), *and*

$$\sum_{u \in U} \varphi_u(x) = 1, \quad x \in X.$$

**Proof.** For every $1\leqslant j\leqslant r$, let $\Gamma_j$ be a discrete subgroup of $X_j$ and $\psi_j$ a $[0,1]$-valued $C_c^\infty$-function supported in the ball $\{x_j\in X_j : |x_j|<r^{-1/2}\}$ such that

$$\sum_{w_j\in\Gamma_j}\psi(x_j-w_j)=1,\quad x_j\in X_j.$$

Let $\Gamma=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_r$ and

$$\psi(x)=\prod_{j=1}^{r}\psi_j(x_j),\quad x\in X.$$

Then

$$\sum_{w\in\Gamma}\psi(x-w)=1,\quad x\in X.$$

For $x\in X$ let $x^\#=\frac{x}{g(x)}$. The mapping $x\to x^\#$ is injective since $g_k$ does not depend on the variables $x_j$ for $1\leqslant j\leqslant k$. Let

$$U=\{\mathbf{u}\in X:\ \mathbf{u}^\#\in\Gamma\}$$

and

$$\varphi_u(x)=\psi\left(\frac{x-u}{g(u)}\right),\quad u\in U.$$

$U$ is no longer a subgroup. It is not hard to see though that the functions $\varphi_u$, where $u\in U$, have all the required properties. $\square$

**Lemma 2.13.** *Let* $\mathbf{g}$ *be an* $H$*-metric on* $X$*. Let*

$$d(x,y)=1+\mathbf{g}_x(x-y),\quad x,y\in X.$$

*Then*

$$d(x,y)\leqslant d(y,x)^2,\quad d(x,y)\leqslant d(x,z)d(z,y)$$

*for* $x,y,z\in X$.

**Proof.** By (2.8),

$$d(x,y)=1+\mathbf{g}_x(x-y)\leqslant 1+\mathbf{g}_y(x-y)(1+\mathbf{g}_y(x-y))\leqslant d(y,x)^2,$$

which proves the first inequality. The other one is proved in a similar way:

$$d(x,y) = 1 + \mathbf{g}_x(x-y) \leqslant 1 + \mathbf{g}_x(x-z) + \mathbf{g}_x(z-y)$$
$$\leqslant d(x,z) + \mathbf{g}_z(z-y)(1+\mathbf{g}_x(x-z)) = d(x,z)d(z,y). \qquad \square$$

**Corollary 2.14.** *There exists a constant* $C_0$ *such that for every H-metric* $\mathbf{g}$

$$\sum_{u \in U} d(u,x)^{-n-1} \leqslant C_0, \quad \sum_{u \in U} d(x,u)^{-2n-1} \leqslant C_0$$

*uniformly in* $x \in X$.

**Proof.** By definition of $U$ and $\mathbf{g}$,

$$\sum_{u \in U} d(u,x)^{-N} = \sum_{u^{\#} \in \Gamma} \left(1 + \left|u^{\#} - \frac{x}{g(u)}\right|\right)^{-N},$$

which implies the first estimate. The other one follows by Lemma 2.13. $\square$

For the general theory of slowly varying metrics and its applications to the theory of pseudodifferential calculus the reader is referred to Hörmander [2, vols. I and III].

## 3. The Melin operator U

Let $\mathscr{G}$ be a nilpotent Lie algebra with a fixed scalar product. The dual vector space $\mathscr{G}^{\star}$ will be identified with $\mathscr{G}$ by means of the scalar product. We shall also regard $\mathscr{G}$ as a Lie group with the Campbell–Hausdorff multiplication

$$x_1 \circ x_2 = x_1 + x_2 + r(x_1, x_2),$$

where

$$r(x_1,x_2) = \tfrac{1}{2}[x_1,x_2] + \tfrac{1}{12}([x_1,[x_1,x_2]] + [x_2,[x_2,x_1]])$$
$$+ \tfrac{1}{48}([x_2,[x_1,[x_2,x_1]]] - [x_1,[x_2,[x_1,x_2]]]) + \cdots$$

is the (finite) sum of terms of order at least 2 in the Campbell–Hausdorff series for $\mathscr{G}$. Let

$$\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \cdots \oplus \mathscr{G}_r,$$

where $r \geqslant 1$ and $\mathscr{G}^{(k)}$ is the desending central series of $\mathscr{G}$. Note that

$$|xy| \leqslant C|y|^{r-1}, \quad |x| \leqslant 1 \leqslant |y|. \tag{3.1}$$

For a function $f \in C_c^\infty(\mathscr{G} \times \mathscr{G})$ let

$$\mathbf{U}f(\mathbf{y}) = \int\int_{\mathscr{G}\times\mathscr{G}} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} f^\vee(\mathbf{x}) e^{i\langle r(\mathbf{x}),\tilde{\mathbf{y}}\rangle}\, d\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathscr{G} \times \mathscr{G}$, and $\tilde{\mathbf{y}} = \frac{y_1+y_2}{2}$. We shall refer to $\mathbf{U}$ as the *Melin operator* on $\mathscr{G}$. The motivation behind this definition is that

$$\widehat{f \bigstar g}(y) = U(\hat{f} \otimes \hat{g})(y, y), \quad y \in \mathscr{G}. \tag{3.2}$$

**Example 3.3.** Let $\mathscr{G} = \mathbf{R}^2 \times \mathbf{R}$ be the Heisenberg Lie algebra with the commutator

$$[(x,t),(y,s)] = (0, \{x,y\}),$$

where $\{x,y\} = x_1 y_2 - x_2 y_1$. Then

$$(x,t)\circ(y,s) = (x+y, t+s+\tfrac{1}{2}\{x,y\})$$

and, for $F \in C_c^\infty(\mathscr{G} \times \mathscr{G})$ and $\lambda + \mu > 0$,

$$\begin{aligned}\mathbf{U}F(\xi,\lambda|\eta,\mu) &= \int\int e^{-i(x\xi+y\eta)} F(x^\vee,\lambda|y^\vee,\mu) e^{i\frac{\lambda+\mu}{2}\{x,y\}}\, dx\, dy \\ &= \frac{1}{4\pi^2}\int\int F\left(\xi + \sqrt{\frac{\lambda+\mu}{2}}u, \lambda|\eta + \sqrt{\frac{\lambda+\mu}{2}}v, \mu\right) e^{-i\{u,v\}}\, du\, dv.\end{aligned}$$

Therefore, for $f, g \in C_c^\infty(\mathscr{G})$ and $\lambda > 0$,

$$\widehat{f \bigstar g}(\xi,\lambda) = \frac{1}{4\pi^2}\int\int \hat{f}(\xi + \sqrt{\lambda}u, \lambda)\hat{g}(\xi + \sqrt{\lambda}v, \lambda) e^{i\{v,u\}}\, du\, dv.$$

Note that for $\lambda = 1$ we obtain the Weyl formula for the symbol of the composition of two pseudodifferential operators

$$a\#b(\xi) = \frac{1}{4\pi^2}\int\int a(\xi+u)b(\xi+v)e^{i\{v,u\}}\, du\, dv, \quad \xi \in \mathbf{R}^2.$$

We return to the general theory. Let $\mathscr{G}' = \mathscr{G}_{(r-1)}$. The commutator

$$\mathscr{G}' \times \mathscr{G}' \ni (x_1, x_2) \to [x_1, x_2]' \ni \mathscr{G}',$$

where $'$ stands for the orthogonal projection onto $\mathscr{G}'$, makes $\mathscr{G}'$ into a Lie algebra isomorphic to $\mathscr{G}/\mathscr{G}_r$ with $x \to x'$ playing the role of the canonical quotient homomorphism. The group multiplication in $\mathscr{G}'$ is

$$x_1 \circ' x_2 = x_1 + x_2 + r(x_1, x_2)'.$$

For $\mathbf{y} = (y_1, y_2) \in \mathscr{G}' \times \mathscr{G}',\ \lambda = (\lambda_1, \lambda_2) \in \mathscr{G}_r \times \mathscr{G}_r$, and $\tilde{\lambda} = \frac{\lambda_1 + \lambda_2}{2}$,

$$\begin{aligned}\mathbf{U}f(\mathbf{y}, \lambda) &= \int\int_{\mathscr{G}' \times \mathscr{G}'} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} e^{-i\langle t, \lambda\rangle} f^\vee(\mathbf{x}, t) e^{-i\langle r'(\mathbf{x}), \tilde{\mathbf{y}}\rangle} e^{-i\langle r(\mathbf{x}), \tilde{\lambda}\rangle}\, d\mathbf{x}\, dt \\ &= \int\int_{\mathscr{G}' \times \mathscr{G}'} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} f(\mathbf{x}^\vee, \lambda) e^{-i\langle r'(\mathbf{x}), \tilde{\mathbf{y}}\rangle} e^{-i\langle r(\mathbf{x}), \tilde{\lambda}\rangle}\, d\mathbf{x} \\ &= \int\int_{\mathscr{G}' \times \mathscr{G}'} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} \mathbf{U}'(f(\cdot, \lambda))^\vee(\mathbf{y}) e^{-i\langle r(\mathbf{x}), \tilde{\lambda}\rangle}\, d\mathbf{x},\end{aligned}$$

where $\mathbf{U}'$ is the Melin operator for $\mathscr{G}'$. Thus,

$$\mathbf{U}f(\mathbf{y}, \lambda) = P_\lambda \circ \mathbf{U}' f(\cdot, \lambda)(\mathbf{y}), \tag{3.4}$$

where $P_\lambda$ is an integral operator on $C_c^\infty(\mathscr{G}')$ defined by

$$P_\lambda f(\mathbf{y}) = \int\int_{\mathscr{G}' \times \mathscr{G}'} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} f^\vee(\mathbf{x}) e^{-i\langle r(\mathbf{x}), \tilde{\lambda}\rangle}\, d\mathbf{x}. \tag{3.5}$$

As explained in the Introduction, formulas (3.4) and (3.5) are of utmost importance for our argument.

Let $\mathbf{g}$ be an H-metric on $\mathscr{G}$ and $\mathbf{G} = \mathbf{g} \oplus \mathbf{g}$. In what follows, we shall employ the following *standard partition of unity* on $\mathscr{G} \times \mathscr{G}$ for $\mathbf{G}$:

$$\Phi_\mathbf{u}(\mathbf{x}) = \varphi_{u_1}(x_1)\varphi_{u_2}(x_2), \quad \mathbf{x} = (x_1, x_2) \in \mathscr{G} \times \mathscr{G},$$

where $\mathbf{u} = (u_1, u_2) \in U \times U$ (see Lemma 2.12). Note that the functions $\Phi_\mathbf{u}$ are supported in $\Omega_\mathbf{u} = \Omega_{u_1} \times \Omega_{u_2}$ and belong uniformly to $S^\mathbf{1}(\mathscr{G} \times \mathscr{G}, \mathbf{G})$.

## 4. The estimate for $P_\lambda$

Let $\mathscr{G}' = \mathscr{G}_{(r-1)}$. Let $\lambda \in \mathscr{G}_r \times \mathscr{G}_r$. Let $\mathbf{g}^1, \mathbf{g}^2$ be H-metrics on $\mathscr{G}'$ such that

$$g_j^k(y) \geqslant (1 + |\lambda_k|)^{1/2}, \quad y \in \mathscr{G}', \tag{4.1}$$

for $1 \leqslant j \leqslant r-1$ and $k = 1, 2$. Let

$$\mathbf{G} = \mathbf{g}^1 \oplus \mathbf{g}^2.$$

Let $\{\Phi_{\mathbf{u}}\}_{\mathbf{u}\in U\times U}$ be the standard partition of unity of $\mathscr{G}'\times\mathscr{G}'$ for $\mathbf{G}$ with the corresponding covering $\Omega_{\mathbf{u}}$. Moreover, let

$$d(\mathbf{u},\mathbf{y}) = 1 + \mathbf{G}_{\mathbf{u}}(\mathbf{u}-\mathbf{y})$$

and $q(\lambda) = \mathbf{Q}_r(\lambda)$. Let $\mathbf{m}$ be a $\mathbf{G}$-weight.

**Lemma 4.2.** *For every $N$, there exist $C$ and $k$ such that*

$$|P_\lambda f(\mathbf{y})| \leqslant C|f|_k^{q(\lambda)^N}(\mathbf{G})d(\mathbf{u},\mathbf{y})^{-N}$$

*uniformly in $\lambda\in\mathscr{G}_r\times\mathscr{G}_r$ and $\mathbf{u}\in\mathscr{G}'\times\mathscr{G}'$, if $f\in S^{\mathbf{1}}(\mathscr{G}'\times\mathscr{G}',\mathbf{G}_\lambda)$ is supported in $\Omega_{\mathbf{u}}$.*

**Proof.** We may assume that $\tilde{\lambda} = \frac{\lambda_1+\lambda_2}{2} \neq 0$ since otherwise $P_\lambda = I$, and there is nothing to prove.

Let $f\in C_c^\infty(\mathscr{G}'\times\mathscr{G}')$ be supported in $\Omega_{\mathbf{u}}$. There exist $C$ and $k$ such that

$$\begin{aligned} |P_\lambda f(\mathbf{y})| &\leqslant \int\int_{\mathscr{G}'\times\mathscr{G}'} |f^\vee(\mathbf{x})|\,d\mathbf{x} = \|f\|_{A(\mathscr{G}'\times\mathscr{G}')} \\ &= \|f_\lambda\|_{A(\mathscr{G}'\times\mathscr{G}')} \leqslant C|f|_k^{\mathbf{1}}(\mathbf{G}_\lambda), \end{aligned} \tag{4.3}$$

where $f_\lambda(\mathbf{y}) = f(g^1(u_1)^{-1}y_1, g^2(u_2)^{-1}y_2)$ and $\|\cdot\|_{A(\mathscr{G}'\times\mathscr{G}')}$ stands for the Fourier algebra norm. The last inequality is achieved by the Sobolev Lemma.

With the notation of (2.11) applied to $\mathbf{x} = (x_1,x_2) = (x_{sj})_{s=1,2}^{1\leqslant j\leqslant r-1}$ and $\mathbf{y} = (y_1,y_2) = (y_{sj})_{s=1,2}^{1\leqslant j\leqslant r-1}$,

$$\begin{aligned} \frac{i(y_{sj}-u_{sj})}{g^s(u_s)}P_\lambda f(\mathbf{y}) &= P_\lambda\left(\frac{i(y_{sj}-u_{sj})}{g^s(u_s)}f\right)(\mathbf{y}) + P_\lambda\left(\frac{|\tilde{\lambda}|}{g^s(u_s)}r'(D)f\right)(\mathbf{y}) \\ &= P_\lambda f_\lambda'(\mathbf{y}) + P_\lambda f_\lambda''(\mathbf{y}), \end{aligned} \tag{4.4}$$

where

$$r'(x) = \left\langle \frac{\tilde{\lambda}}{|\tilde{\lambda}|}, D_{sj}r(x)\right\rangle$$

is a polynomial vanishing at 0 and depending continuously on a parameter $\mu = \frac{\tilde{\lambda}}{|\lambda|}$ ranging over a compact set. Thus, the mappings

$$S^{\mathbf{1}}_{\Omega_{\mathbf{u}}}(\mathscr{G}'\times\mathscr{G}',\mathbf{G}) \ni f \to f_\lambda' \in S^{\mathbf{1}}_{\Omega_{\mathbf{u}}}(\mathscr{G}'\times\mathscr{G}',\mathbf{G}),$$

$$S^{\mathbf{1}}_{\Omega_{\mathbf{u}}}(\mathscr{G}'\times\mathscr{G}',\mathbf{G}) \ni f \to f_\lambda'' \in S^{\mathbf{1}}_{\Omega_{\mathbf{u}}}(\mathscr{G}'\times\mathscr{G}',\mathbf{G})$$

are uniformly continuous in $\lambda$ so, by induction using (4.4) and (4.3), we get our estimate. □

**Proposition 4.5.** *If N is sufficiently large, then for every $\lambda$ there exists a unique double-continuous extension of $P_\lambda$ to a mapping*

$$P_\lambda : S^{\mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G}) \to S^{q(\lambda)^N \mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G}).$$

*If, moreover,* $f \in S^{\mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathscr{G})$, *then*

$$P_\lambda f \in S^{q(\lambda)^N \mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G})$$

*uniformly in* $\lambda$.

**Proof.** By Lemma 4.2,

$$|P_\lambda(\Phi_\mathbf{u} f)(\mathbf{y})| \leqslant C_N |\Phi_\mathbf{u} f|_k^{q(\lambda)^N}(\mathbf{G}) d(u, y)^{-N}$$

and

$$\mathbf{m}(\mathbf{y})^{-1} |P_\lambda(\Phi_\mathbf{u} f)(\mathbf{y})| \leqslant C\mathbf{m}(\mathbf{u})^{-1} d(\mathbf{y}, \mathbf{u}))^M |P_\lambda(\Phi_\mathbf{u} f_\lambda)(\mathbf{y})|$$
$$\leqslant C_1 |f|^{q(\lambda)^N \mathbf{m}} d(\mathbf{u}, \mathbf{y})^{-N+2M}$$

so that

$$\sum_{\mathbf{u} \in U \times U} |P_\lambda(\Phi_\mathbf{u} f)(\mathbf{y})| \leqslant C_1 q(\lambda)^N \mathbf{m}(\mathbf{y}) |f|_k^{q(\lambda)^N \mathbf{m}} \sum_{\mathbf{u} \in U \times U} d(\mathbf{u}, \mathbf{y})^{2M-N},$$

uniformly in $\lambda$. This estimate is valid for $f$ in a bounded subset of $S^{\mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G})$ without any restriction on the support, which implies that for every $\mathbf{y} \in \mathscr{G}'$,

$$f \to \sum_\mathbf{u} P_\lambda(\Phi_\mathbf{u} f)(\mathbf{y})$$

defines a weakly continuous linear form on $S^\mathbf{m}(\mathscr{G}' \times \mathscr{G}', \mathbf{G})$. Consequently, $P_\lambda$ admits a (unique) weakly continuous extension to the whole of $S^\mathbf{m}(\mathscr{G}' \times \mathscr{G}', \mathscr{G})$, and

$$|P_\lambda(f)(\mathbf{y})| = \left| \sum_{\mathbf{u} \in U \times U} P_\lambda(\Phi_\mathbf{u} f)(\mathbf{y}) \right|$$
$$\leqslant C_2 |f|_k^{\mathbf{m}_\lambda} q(\lambda)^N \mathbf{m}_\lambda(\mathbf{y}).$$

The estimates for the derivatives of $P_\lambda f$ follow from the fact that $P_\lambda$ commutes with translations, and hence with differentiations. $\square$

**Remark 4.6.** The estimates of this section are uniform in $\lambda, \mathbf{G}, \mathbf{m}$, and $f$ if $\mathbf{G}$ satisfies (4.1), $\mathbf{m}$ is a $\mathbf{G}$-weight satisfying (2.10) uniformly, and $f \in S^\mathbf{m}(\mathscr{G}', \mathbf{G})$ uniformly.

## 5. Continuity of U

We fix a metric $\mathbf{g}$ with $\mathbf{d} = (1, 1, \ldots, 1)$. Let $\mathbf{G} = \mathbf{g} \oplus \mathbf{g}$. Recall that the Melin operator $\mathbf{U}$ has been defined for $f \in C_c^\infty(\mathscr{G} \times \mathscr{G})$.

**Proposition 5.1.** *Let* $\mathbf{m}$ *be a* $\mathbf{G}$*-weight. For* $N \in \mathbf{N}$ *sufficiently large, there exists a double-continuous extension of the Melin operator to*

$$\mathbf{U} : S^{\mathbf{m}}(\mathscr{G} \times \mathscr{G}, \mathbf{G}) \to S^{Q^N \mathbf{m}}(\mathscr{G} \times \mathscr{G}, \mathbf{G}).$$

**Proof.** Suppose that $\mathscr{G} = \mathscr{G}_{(r)}$ and proceed by induction. If $r = 1, \mathscr{G}$ is abelian and $\mathbf{U} = I$ so the assertion is obvious. Assume that our theorem is true for $\mathscr{G}' = \mathscr{G}_{r-1}$ and $\mathbf{U} = \mathbf{U}'$. For $\lambda \in \mathscr{G}_r$ and $f \in S^{\mathbf{m}}(\mathscr{G} \times \mathscr{G}, \mathbf{G})$, let $f_\lambda(\mathbf{y}) = f(\mathbf{y}, \lambda), (\mathbf{g}_{\lambda_s})_{\mathbf{y}}(\mathbf{z}) = \mathbf{g}_{(\mathbf{y}, \lambda_s)}(\mathbf{z}, 0), \mathbf{G}_\lambda = \mathbf{g}_{\lambda_1} \oplus \mathbf{g}_{\lambda_2}$, and $\mathbf{m}_\lambda(\mathbf{y}) = Q(\mathbf{y}, 0)^{N'} \mathbf{m}(\mathbf{y}, \lambda)$.

Now $\lambda, \mathbf{G}_\lambda, \mathbf{m}_\lambda$, and $f_\lambda$ are as in Remark 4.6, so by the induction hypothesis $\mathbf{U}' f_\lambda \in S^{Q^{N'} \mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G}_\lambda)$ uniformly in $\lambda$. Now Proposition 4.5, where we replace $\mathbf{m}_\lambda$ with $Q^N \mathbf{m}_\lambda$, yields

$$P_\lambda \mathbf{U}' f_\lambda \in S^{q(\lambda)^{N''} Q^{N'} \mathbf{m}_\lambda}(\mathscr{G}' \times \mathscr{G}', \mathbf{G}_\lambda)$$

uniformly in $\lambda$. The same holds true for the derivatives $(\frac{\partial}{\partial \lambda})^j P_\lambda \mathbf{U}' f_\lambda$ which is checked directly. Since

$$q^{N''}(\lambda) Q^{N'}(\mathbf{y}) \leqslant Q(\mathbf{y}, \lambda)^N,$$

where $N = \max\{N', N''\}$ we get by (3.4) the desired estimate: for every $k_1 \in \mathbf{N}$, there exists $k_2 \in \mathbf{N}$ such that

$$|\mathbf{U}f|_{k_1}^{Q^N \mathbf{m}}(\mathbf{G}) \leqslant C |f|_{k_2}^{\mathbf{m}}(\mathbf{G}), \quad f \in S^{\mathbf{m}}(\mathscr{G} \times \mathscr{G}, \mathbf{G}). \qquad \square$$

Let

$$\Phi_{u,v}(\mathbf{x}) = \varphi_u(x_1) \varphi_v(x_2)$$

be the standard partition of unity on $\mathscr{G} \times \mathscr{G}$ for $\mathbf{G}$.

**Corollary 5.2.** *Let* $f \in S^{\mathbf{1}}(\mathscr{G} \times \mathscr{G}, \mathbf{G})$. *Let*

$$f_{u,v}(y) = \mathbf{U}(\Phi_{u,v} f)(y, y).$$

*Then, for every sufficiently large* $N$, *there exists a norm* $|\cdot|^{\mathbf{1}}$ *in* $S^{\mathbf{1}}(\mathscr{G}, \mathbf{g})$ *such that for every* $u, v \in U$

$$\|f_{u,v}\|_{A(\mathscr{G})} \leqslant |f|^{\mathbf{1}} d(u, v)^{-N}.$$

**Proof.** If

$$\mathbf{m}_{u,v}(\mathbf{y}) = d(u, y_1)^{-N} d(v, y_2)^{-4N},$$

then, of course,

$$\Phi_{u,v}f \in S^{\mathbf{m}_{u,v}}(\mathscr{G} \times \mathscr{G}, \mathbf{G})$$

uniformly so, by Proposition 5.1,

$$\mathbf{U}(\Phi_{u,v}f) \in S^{Q^M \mathbf{m}_{u,v}}(\mathscr{G} \times \mathscr{G}, \mathbf{G})$$

uniformly in $(u, v) \in U \times U$. By using Lemma 2.13 and $Q(y, y) = 1$, we get

$$\begin{aligned} |D_y^\alpha f_{u,v}(y)| &\leqslant |f|_k^{\mathbf{1}} \mathbf{m}_{u,v}(y, y) = |f|_k^{\mathbf{1}} d(u, y)^{-N} d(y, v)^{-2N} \\ &\leqslant |f|_k^{\mathbf{1}} d(u, v)^{-N} d(y, v)^{-N}, \quad |\alpha| \leqslant k. \end{aligned}$$

For $F_{u,v}(y) = f_{u,v}(g(v)y)$ (see (2.6)) we have $||f_{u,v}||_{A(\mathscr{G})} = ||F_{u,v}||_{A(\mathscr{G})}$ and

$$\begin{aligned} |D_y^\alpha F_{u,v}(y)| &\leqslant |f|_k^{\mathbf{1}} \frac{g(v)^\alpha}{g(g(v)y)^\alpha} d(u, v)^{-N} (1 + |v^\# - y|)^{-N}, \\ &\leqslant |f|_k^{\mathbf{1}} d(u, v)^{-N} (1 + |v^\# - y|)^{-N+|\alpha|}, \quad |\alpha| \leqslant k, \end{aligned}$$

since, by (2.4),

$$\frac{g_j(v)}{g_j(g(v)y)} \leqslant \frac{g_j(v)}{g_j(g(v)v)} (1 + d(g(v)v, g(v)y)) \leqslant 1 + |v^\# - y|.$$

If $N$ and $k$ are large enough, our assertion follows by the Sobolev inequality. □

**Theorem 5.3.** *Let* $\mathbf{m}_1, \mathbf{m}_2$ be $\mathbf{g}$-*weights on* $\mathscr{G}$. *Then*

$$G_c^\infty(\mathscr{G}) \times C_c^\infty(\mathscr{G}) \ni (a, b) \to (a^\vee \bigstar b^\vee)^\wedge \in \mathscr{S}(\mathscr{G})$$

*extends uniquely to a double-continuous mapping*

$$S^{\mathbf{m}_1}(\mathscr{G}, \mathbf{g}) \times S^{\mathbf{m}_2}(\mathscr{G}, \mathbf{g}) \to S^{\mathbf{m}_1 \mathbf{m}_2}(\mathscr{G}, \mathbf{g}).$$

**Proof.** By (3.2), it is sufficient to apply Proposition 5.1 with $\mathbf{G} = \mathbf{g} \oplus \mathbf{g}$ and $\mathbf{m} = \mathbf{m}_1 \otimes \mathbf{m}_2$. □

## 6. $L^2$-boundedness

Our $L^2$-boundedness result relies on the following Cotlar's Lemma. For the proof see, e.g., Hörmander [2, vol. III].

**Lemma 6.1.** *Let $A_k$ be bounded linear operators on a Hilbert space. If*

$$\sum_{k=1}^{\infty} ||A_k^{\star} A_j||^{1/2} + ||A_j A_k^{\star}||^{1/2} \leqslant M,$$

*then the series $\sum_{k=1}^{\infty} A_k$ is strongly convergent to a bounded operator $A$ whose norm does not exceed $M$.*

**Theorem 6.2.** *Let $a \in S^{\mathbf{1}}(\mathscr{G}, \mathbf{g})$. The linear operator $f \to Af = f \bigstar a^{\vee}$ defined initially on the dense subspace $C_c^{\infty}(\mathscr{G})$ of $L^2(\mathscr{G})$ extends to a bounded mapping of $L^2(\mathscr{G})$. There exists a norm $|\cdot|^{\mathbf{1}}$ in $S^{\mathbf{1}}(\mathscr{G}, \mathbf{g})$ such that*

$$||Af||_{L^2(\mathscr{G})} \leqslant |a|^{\mathbf{1}} ||f||_{L^2(\mathscr{G})}, \quad f \in C_c^{\infty}(\mathscr{G}).$$

**Proof.** Let

$$A_v f = f \bigstar (\varphi_v a)^{\vee}, \quad f \in L^2(\mathscr{G}).$$

Since $\varphi_v \in C_c^{\infty}(\mathscr{G})$, the operators $A_v$ are bounded. Moreover, by (3.2),

$$A_u^{\star} A_v f(y) = (\bar{a} \otimes a)_{u,v}^{\vee} \bigstar f, \quad A_u A_v^{\star} f(y) = (a \otimes \bar{a})_{u,v}^{\vee} \bigstar f,$$

so that, by Corollary 5.2,

$$||A_u^{\star} A_v|| + ||A_u A_v^{\star}|| \leqslant |f|^{\mathbf{1}} d(u,v)^{-N},$$

where $N$ can be taken as large, as we wish, and $|\cdot|^{\mathbf{1}}$ is a norm in $S^{\mathbf{1}}(\mathscr{G}, \mathbf{g})$ depending only on $N$.

On the other hand,

$$a = \sum_u \varphi_u a,$$

where the series is weakly convergent in $S^{\mathbf{1}}(\mathscr{G}, \mathbf{g})$ so that

$$Af = \sum_u A_u f, \quad f \in C_c^{\infty}(\mathscr{G}).$$

Thus, the sequence of operators $A_u$ satisfies the hypothesis of Cotlar's Lemma, and therefore the series $\sum_u A_u$ is strongly convergent to the bounded extension of our operator $A$ whose norm is bounded by $C_0 |a|^{\mathbf{1}}$ (see Corollary 2.14). □

As a corollary we are now going to give a boundedness theorem for certain "variable-coefficient" pseudodifferential operators (cf. Stein [7]).

Let $k$ be a tempered distribution on $\mathcal{G} \times \mathcal{G}$ whose partial Fourier transform

$$a(x, \xi) = \int_{\mathcal{G}} k(x, y) e^{-i\langle y, \xi \rangle} \, dy$$

is a smooth function on $\mathcal{G} \times \mathcal{G}^\star$ satisfying the estimates

$$|\partial^\alpha D^\beta a(x, \xi)| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{-\frac{\beta}{2}},$$

where $\partial^\alpha$ stand for left-invariant derivatives on $\mathcal{G}$ with respect to a certain basis of the Lie algebra. Then $k$ is locally integrable on $\mathcal{G} \times \mathcal{G} \backslash \{0\}$, for every $x \in \mathcal{G}$, $y \to k(x, y)$ is smooth on $\mathcal{G} \backslash \{0\}$, and for every $m \in \mathbf{N}$

$$|k(x, y)| \leqslant C_m |y|^{-m}, \quad y \neq 0. \tag{6.3}$$

The kernel $k$ defines a linear operator

$$T_k f(x) = \int k(x, y^{-1}x) f(y) \, dy = [f \star k(x, \cdot)](x)$$

which maps $C_c^\infty(\mathcal{G})$ into $L^1_{\mathrm{loc}}(\mathcal{G})$.

**Corollary 6.4.** *The mapping $T_k$ satisfies*

$$\int_{|x| \leqslant 1} |T_k f(zx)|^2 \, dx \leqslant C \int \frac{|f(zx)|^2 \, dx}{(1 + |x|)^{n+1}} \tag{6.5}$$

*for $f \in C_c^\infty$ and $z \in \mathcal{G}$. Thus, it extends to a bounded operator from $L^2(\mathcal{G})$ to $L^2(\mathcal{G})$.*

**Proof.** The argument that follows is an adaptation of that of Stein [7]. First let us remark that once (6.5) has been proven, the boundedness of $T_k$ follows by integrating both sides with respect to $z$.

Let us assume for the moment that $k$ is compactly supported with respect to the $x$-variable. For $\lambda \in \mathcal{G}^\star$, let

$$a_\lambda(\xi) = \hat{k}(\lambda, \xi) = \int a(x, \xi) e^{-i\langle \lambda, x \rangle} \, dx.$$

Then

$$|D^\alpha a_\lambda(\xi)| \leqslant C_\alpha (1 + |\lambda|)^{-(n+1)} (1 + |\xi|)^{-\frac{\alpha}{2}},$$

which can be easily seen by integration by parts and using the compactness of the support of $a$ with respect to the $x$-variable. The compactness of the support allows to use the abelian derivatives rather than the left-invariant ones. Thus, by Theorem 6.2, the norms of the operators $T_\lambda f = f \star a_\lambda^\vee$ are bounded by

$C(1+|\lambda|)^{-(n+1)}$ and so

$$T_k f(x) = \int T_\lambda f(x) e^{i\langle \lambda, x\rangle}\, d\lambda$$

is also bounded.

We return to the general case. Let $\eta : \mathscr{G} \to [0,1]$ be a smooth cut-off function equal to 1 for $|x| \leqslant \frac{3}{2}$ and vanishing for $|x| \geqslant 2$. Let

$$f = \eta f + (1-\eta) f = f_1 + f_2 \in C_c^\infty(\mathscr{G}).$$

Then, by the first part of the proof,

$$\int_{|x|\leqslant 1} |T_k f_1(x)|^2\, dx = \int_{|x|\leqslant 1} |T_{\eta k} f_1(x)|^2\, dx$$
$$\leqslant C_1 \|f_1\|_2^2 = C_1 \int_{|x|\leqslant 2} |f(x)|^2\, dx \leqslant 3^{n+1} C_1 \int \frac{|f(x)|^2\, dx}{(1+|x|)^{n+1}}, \tag{6.6}$$

and, by (6.3) and (3.1),

$$|T_k f_2(x)| \leqslant C_2 \int \frac{|f_2(y)|\, dy}{(1+|y^{-1}x|)^{(r-1)(n+1)}},$$
$$\leqslant C_3 \int \frac{|f(y)|\, dy}{(1+|y|)^{n+1}} \leqslant C_4 \left( \int \frac{|f(y)|^2\, dy}{(1+|y|)^{n+1}} \right)^{1/2}.$$

By integration,

$$\int_{|x|\leqslant 1} |T_k f_2(x)|^2\, dx \leqslant C_5 \int \frac{|f(y)|^2\, dy}{(1+|y|)^{n+1}}, \tag{6.7}$$

so that, by putting (6.6) and (6.7) together, we get (6.5) for $z = 0$.

Finally, by applying the particular case of (6.5) to $k_z(x,y) = k(zx,y)$ which satisfies the estimates of $k$ uniformly in $z$, we get

$$\int_{|x|\leqslant 1} |T_k f(zx)|^2\, dx = \int_{|x|\leqslant 1} |T_{k_z} f_z(x)|^2\, dx$$
$$\leqslant C \int \frac{|f(zx)|^2\, dx}{(1+|x|)^{n+1}},$$

which is exactly (6.5). □

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