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Strongly Closed Subgraphs in a Regular Thick Near Polygon

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In this paper we show that a regular thick near polygon has a tower of regular thick near sub-polygons as strongly closed subgraphs if the diameter d is greater than the numerical girth g.

1. INTRODUCTION

Brouwer and Wilbrink [3] studied a regular thick near polygon of the numerical girth $g = 4$ and showed the existence of a tower of regular thick near sub-polygons.

On the other hand we gave a constructing method of strongly closed subgraphs in a distance-regular graph of arbitrary numerical girth [6].

The purpose of this paper is to apply this constructing method to regular thick near polygons of arbitrary numerical girth and to show the existence of a tower of regular thick near sub-polygons as strongly closed subgraphs if the diameter d is larger than the numerical girth g.

First we recall our notation and terminology.

All graphs in this paper are undirected finite simple graphs. Let Γ be a connected graph with usual distance ∂_Γ. We identify Γ with the set of vertices. The *diameter* of Γ, denoted by d_Γ, is the maximal distance of two vertices in Γ. Let $u \in \Gamma$. We denote by $\Gamma_j(u)$ the set of vertices which are at distance j from u.

Let $x, y \in \Gamma$ with $\partial_\Gamma(x, y) = i$. Define

$$\begin{aligned} C(x, y) &:= \Gamma_{i-1}(x) \cap \Gamma_1(y), \\ A(x, y) &:= \Gamma_i(x) \cap \Gamma_1(y) \\ \text{and} \qquad B(x, y) &:= \Gamma_{i+1}(x) \cap \Gamma_1(y). \end{aligned}$$

We say c_i *exists* if $c_i = |C(x, y)|$ does not depend on the choice of x and y under the condition $\partial_\Gamma(x, y) = i$. Similarly, we say a_i *exists,* or b_i *exists*.

A connected graph Γ with the diameter d_Γ is said to be *distance-regular* if c_i, a_i and b_{i-1} exist for all $1 \le i \le d_\Gamma$.

The reader is referred to [1, 2] for more detailed descriptions of distance-regular graphs.

Let Γ be a connected graph of the diameter $d_\Gamma = d \ge 2$.

For any $x, y \in \Gamma$ and $\emptyset \ne \Delta \subseteq \Gamma$, we define

$$\Delta^\perp := \{z \in \Gamma \mid \partial_\Gamma(x, z) \le 1 \text{ for any } x \in \Delta\}$$

and

$$S(x, y) := \{y\} \cup C(x, y) \cup A(x, y) = \{y\}^\perp - B(x, y).$$

We identify Δ with the induced subgraph on it. A subgraph Δ is called a *clique* (resp. *coclique*) if any two vertices on it are adjacent (resp. non-adjacent).

For $v \in \Delta$, Δ is said to be *strongly closed with respect to* v if $S(v, v') \subseteq \Delta$ for any $v' \in \Delta$. Δ is called *strongly closed* if it is strongly closed with respect to v for all $v \in \Delta$.

Singular lines of Γ are the sets of the form $\{x, y\}^{\perp\perp}$ where (x, y) is an edge in Γ. In particular, a singular line of Γ is always a clique.

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Let $(NP)_j$ be the following condition:

$(NP)_j$: If $x \in \Gamma$ and L is a singular line with $\partial_\Gamma(x, L) := \min\{\partial_\Gamma(x, z) \mid z \in L\} = j$, then there is a unique vertex $y \in L$ such that $\partial_\Gamma(x, y) = j$.

We write $(NP)_{<m}$ holds if $(NP)_i$ holds for any $1 \le i < m$.

Let m be an integer with $1 \le m \le d$.

Γ is said to be *of order* $(s, t; t_2, \ldots, t_m)$ if the following conditions hold.

(1) All singular lines have size $s + 1$ and all vertices lie on $t + 1$ singular lines.
(2) $(NP)_{<m}$ holds.
(3) For any $1 \le i \le m$ and $x, y \in \Gamma$ with $\partial_\Gamma(x, y) = i$, there are exactly $t_i + 1$ singular lines containing y at distance $i - 1$ from x, where $t_1 := 0$.

A graph Γ of order $(s, t; t_2, \ldots, t_d)$ with the diameter $d \ge 2$ is called (the collinearity graph of) *a regular near polygon.* A regular near polygon is called a *regular near* $2d$*-gon* if $t_d = t$, a *regular near* $(2d + 1)$*-gon* otherwise. A regular near polygon is also called a *regular thick near polygon* if $s > 1$.

It is known that regular near polygons are distance-regular (see Section 3).

More detailed descriptions of a regular near polygon will be found in [1, Section III.3] and [2, Section 6.4].

The main results of this paper are the following.

THEOREM 1.1. *Let r and m be positive integers with $r + 1 \le m$. Let Γ be a graph of order $(s, t; t_2, \ldots, t_{m+r})$ with $0 = t_1 = \cdots = t_r < t_{r+1}$. Suppose $s > 1$. Then $t_{r+1} < \cdots < t_{m-1} < t_m$. Moreover, for any integer q with $r + 1 \le q \le m$ and any pair of vertices (u, v) at distance q, there exists a regular near $2q$-gon of order $(s, t_q; t_2, \ldots, t_q)$ containing (u, v) as a strongly closed subgraph in Γ.*

As a direct consequence of our theorem we have the following.

COROLLARY 1.2. *Let Γ be a regular thick near polygon of order $(s, t; t_2, \ldots, t_d)$ with $0 = t_1 = \cdots = t_r < t_{r+1}$. If $2r + 1 \le d$, then $t_{r+1} < \cdots < t_{d-r}$ and for any integer q with $r + 1 \le q \le d - r$ there exists a regular near $2q$-gon of order $(s, t_q; t_2, \ldots, t_q)$ as a strongly closed subgraph in Γ.*

Our results are generalizations of the result of Brouwer and Wilbrink [3] and an application of the result of [6].

In Section 2, we recall the method and results introduced in the previous paper [6]. In Section 3, we collect several basic properties and show that regular near polygons are distance-regular. We prove our main theorem in Section 4.

2. STRONGLY CLOSED SUBGRAPHS

In this section, we recall a constructing method of strongly closed subgraphs and the results obtained in the previous paper [6]. For the proofs and more detailed descriptions the reader is referred to [6].

Let Γ be a distance-regular graph of the diameter $d_\Gamma = d \ge 2$. Fix an integer q with $1 \le q < d$.

A quadruple (w, x, y, z) of vertices is called a *root of size* q if

$$\partial_\Gamma(w, x) = \partial_\Gamma(y, z) = q, \quad \partial_\Gamma(w, y) \le 1, \quad \partial_\Gamma(x, z) \le 1,$$
$$\partial_\Gamma(w, z) \le q \quad \text{and} \quad \partial_\Gamma(x, y) \le q.$$

A triple (x, y, z) of vertices with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = q$ is called a *conron of size q* if there exist three sequences of vertices

$$(x_0, x_1, \ldots, x_m = x), \qquad (y_0, y_1, \ldots, y_m = y) \qquad \text{and} \qquad (z_0, z_1, \ldots, z_m = z)$$

such that $\partial_\Gamma(x_0, y_0) \leq 1$, $(x_{i-1}, z_{i-1}, x_i, z_i)$ and $(y_{i-1}, z_{i-1}, y_i, z_i)$ are roots of size q for all $1 \leq i \leq m$.

The conditions $(SS)_q$, $(CR)_q$ and $(SC)_q$ are defined as follow:

$(SS)_q$: $S(x, z) = S(y, z)$ for any triple of vertices (x, y, z) with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = q$ and $\partial_\Gamma(x, y) \leq 1$.

$(CR)_q$: $S(x, z) = S(y, z)$ for any conron (x, y, z) of size q.

$(SC)_q$: For any given pair of vertices at distance q, there exists a strongly closed subgraph of the diameter q containing them.

We also write $(SS)_{<t}$ holds if $(SS)_i$ holds for any $1 \leq i < t$.

DEFINITION 2.1. Let Γ be a distance-regular graph and q be a fixed integer with $b_{q-1} > b_q$. Assume $(CR)_q$ holds. Let $u, v \in \Gamma$ with $\partial_\Gamma(u, v) = q$. For any $x, y \in \Gamma_q(u)$ define the relation $x \approx y$ iff (x, y, u) is a conron of size q. Then this is an equivalence relation on $\Gamma_q(u)$. (See [6, Lemma 2.2(2)].) Let $\Psi(u, v)$ be the equivalence class containing v. Define

$$\Delta(u, v) := \{x \in \Gamma \mid \partial_\Gamma(u, x) + \partial_\Gamma(x, z) = q \text{ for some } z \in \Psi(u, v)\}$$

the subgraph induced on all vertices lying on shortest paths between u and vertices in $\Psi(u, v)$.

PROPOSITION 2.2 [6, Theorem 1.1]. *Let Γ be a distance-regular graph and q be a fixed integer with $b_{q-1} > b_q$. Suppose the following conditions hold.*

(i) $(SS)_{<q}$ *holds,*
(ii) $(CR)_q$ *holds and* $\Delta(w, x) = \Delta(y, z)$ *if* (w, x, y, z) *is a root of size* q.

Then for any pair of vertices (u, v) in Γ at distance q, $\Delta(u, v)$ is a strongly closed subgraph of the diameter q which is $(c_q + a_q)$-regular. In particular, $(SC)_q$ holds.

A *circuit* of length m is a sequence of distinct vertices $(x_0, x_1, \cdots, x_{m-1})$ such that (x_{i-1}, x_i) is an edge of Γ for all $1 \leq i \leq m$, where $x_m = x_0$. A circuit of length m is called *reduced* if $m \geq 4$ and any proper subset of it does not form a circuit. A shortest reduced circuit is called a *minimal circuit.* The *numerical girth* of Γ, denoted by g, is the length of a minimal circuit.

PROPOSITION 2.3 [6, Proposition 3.1(2)]. *Let q be a positive integer. Let Γ be a distance-regular graph with the numerical girth $g = 2r + 2$, the diameter $d \geq q + r$. If the following conditions* (a) *and* (b) *hold, then $(CR)_q$ holds.*

(a) *Let $u, v, p, p' \in \Gamma$ with $\partial_\Gamma(u, p) = \partial_\Gamma(v, p) = q$, $\partial_\Gamma(u, v) \leq 1$ and $\partial_\Gamma(p, p') = r$. Then $\partial_\Gamma(u, p') = q + r$ implies $\partial_\Gamma(v, p') = q + r$.*
(b) *Let (w, x, y, z) be a root of size q with $x \neq z$ and $(x = x_0, x_1, \ldots, x_r, z_r, \ldots, z_0 = z)$ be a minimal circuit. Then $\partial_\Gamma(w, x_r) = q + r$ implies $\partial_\Gamma(y, z_r) = q + r$.*

LEMMA 2.4 [6, Lemmas 2.4 and 2.6]. *Let Γ be a distance-regular graph with $b_{q-1} > b_q$ and $(CR)_q$ holds. Then we have the following.*

(1) *If (w, x, y, z) is a root of size q, then $\Psi(y, z) \subseteq \Delta(w, x)$.*
(2) *If $(SS)_{<q}$ holds, then for any pair of vertices (u, v) at distance q, $\Delta(u, v)$ is strongly closed with respect to u.*

LEMMA 2.5 [6, Lemma 4.4]. *Let Γ be a distance-regular graph of the diameter d_Γ, and h be an integer with $h < d_\Gamma$. Assume $c_{h+1} > 1, b_{h-1} > b_h$ and $(SC)_h$ holds. If there exist a vertex u and a path $(x_0, \ldots, x_h)$ of length h such that $\partial_\Gamma(x_0, x_h) = \partial_\Gamma(u, x_i) = h$ for all $0 \leq i \leq h$, then $a_h < a_{h+1}$.*

REMARK. For the results in this section Γ need not be a distance-regular graph. Suppose Γ is a graph such that c_i, a_i and b_i exist for all $0 \leq i \leq q$. Then the results are proved by the same manner as in [6].

Let Δ be a strongly closed subgraph of the diameter q in Γ. Then c_i and a_i of Δ exist for all $1 \leq i \leq q$ which are the same as those of Γ. Moreover, if Δ is a regular graph of valency k_Δ, then b_i of Δ exists with $b_i = k_\Delta - c_i - a_i$ for all $0 \leq i \leq q - 1$, and hence it is distance-regular.

3. SOME BASIC PROPERTIES

In this section we collect some basic properties and prove the following result.

PROPOSITION 3.1. *Let Γ be a graph of order $(s, t; t_2, \ldots, t_m)$. Then $(SS)_{<m}$ holds. Moreover, c_i, a_{i-1} and b_{i-1} exist for all $1 \leq i \leq m$ which satisfy*

$$c_i = t_i + 1, \qquad a_{i-1} = (t_{i-1} + 1)(s - 1) \qquad \text{and} \qquad b_{i-1} = s(t - t_{i-1}),$$

where $t_0 = -1$ and $t_1 = 0$.

In particular, regular near polygons are distance-regular.

Throughout this section Γ denotes a graph of the diameter $d_\Gamma = d \geq 2$.

LEMMA 3.2. *Suppose $(NP)_h$ holds. Then $(SS)_h$ holds.*

PROOF. Let (x, y, z) be a triple of vertices with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = h$ and $\partial_\Gamma(x, y) \leq 1$. Suppose there exists $w \in S(y, z) - S(x, z)$ to derive a contradiction. Then we have $\partial_\Gamma(x, w) = h + 1, \partial_\Gamma(x, y) = 1$ and $\partial_\Gamma(y, w) = h$. As $(NP)_h$ holds, there exists $v \in \{z, w\}^{\perp\perp} \cap \Gamma_{h-1}(y)$. Then $\partial_\Gamma(x, v) = h$ from the triangle inequality on (x, y, w, v). This shows $\{v, z\} \subseteq \{z, w\}^{\perp\perp} \cap \Gamma_h(x)$ contradicting our assumption. Hence $S(y, z) \subseteq S(x, z)$. By symmetry we have $S(x, z) = S(y, z)$. □

LEMMA 3.3. *If $(SS)_{<h}$ holds, then the following hold.*

(1) $C(u, x)$ is a coclique for any $u, x \in \Gamma$ with $\partial_\Gamma(u, x) = i \leq h$.
(2) Let $1 \leq m < h$ and (u, v, p, p') be a quadruple of vertices with $\partial_\Gamma(u, p) = \partial_\Gamma(v, p) = m, \partial_\Gamma(u, v) \leq 1$ and $\partial_\Gamma(p, p') = h - m$. Then $\partial_\Gamma(u, p') = h$ implies $\partial_\Gamma(v, p') = h$.

PROOF. (1) We prove the assertion by induction on i. The case $i = 1$ is clear. Let $2 \leq i \leq h$. Suppose there exists an edge (y, z) in $C(u, x)$. Let $v \in C(y, u) \subseteq C(x, u)$. From our inductive assumption $C(v, x)$ is a coclique and thus $\partial_\Gamma(v, z) = i - 1$. Then $\partial_\Gamma(u, z) = \partial_\Gamma(v, z) = i - 1$ and $x \in S(v, z) - S(u, z)$ contradicting our assumption.

(2) Let $(p = p_m, p_{m+1}, \dots, p_h = p')$ be a shortest path connecting them. Assume $\partial_\Gamma(u, p') = h$. Then we have $\partial_\Gamma(u, p_i) = i$ for all $m \le i \le h$. Since $(SS)_m$ holds, we have $S(u, p_m) = S(v, p_m)$. This implies $p_{m+1} \in B(u, p_m) = B(v, p_m)$ and $\partial_\Gamma(u, p_{m+1}) = \partial_\Gamma(v, p_{m+1}) = m + 1$. Inductively, we have $p_i \in B(u, p_{i-1}) = B(v, p_{i-1})$ and $\partial_\Gamma(v, p_i) = i$ for all $m + 1 \le i \le h$. The desired result is proved. □

Next we show the following well-known result.

LEMMA 3.4. *Let $2 \le h \le d$. Suppose a_1 and c_i exist for all $1 \le i \le h$. Then the following conditions are equivalent:*

(i) $(NP)_{<h}$ holds.

(ii) For any $1 \le i \le h$ and any pair of vertices u and x at distance i, we have $C(u, x)$ is a coclique and

$$\bigcup_{z \in C(u,x)} A(z, x) \subseteq A(u, x). \qquad (*)$$

Moreover if $i \ne h$, then the equality holds.

(iii) There exists no induced subgraph $K_{2,1,1}$ and a_i exists with $a_i = c_i a_1$ for all $1 \le i < h$.

PROOF. (i) ⇒ (ii): The first assertion follows from Lemmas 3.2 and 3.3. Assume $i < h$. Take any $y \in A(u, x)$. Then there exists $z \in \{x, y\}^{\perp\perp} \cap \Gamma_{i-1}(u)$ as $(NP)_i$ holds. Hence y is in the left-hand side of $(*)$.

(ii) ⇒ (iii): Γ has no induced subgraph $K_{2,1,1}$ since $C(u, u')$ is a coclique for any u and u' at distance 2. This implies that the left-hand side of $(*)$ is a disjoint union and

$$|A(u, x)| = \left| \bigcup_{z \in C(u,x)} A(z, x) \right| = c_i a_1$$

for any $u, x \in \Gamma$ with$\partial_\Gamma(u, x) = i < h$. Thus the desired result follows.

(iii) ⇒ (ii): We prove the assertion by induction on i. The case $i = 1$ is clear. If there exists an edge (y, z) in $C(u, x)$, then $z \in A(u, y)$ and there exists $w \in C(u, y)$ such that $z \in A(w, y)$ from our inductive hypothesis. Then (w, y, z, x) forms $K_{2,1,1}$ which contradicts our assumption. Hence $C(u, x)$ is a coclique and the left-hand side of $(*)$ is included in $A(u, x)$. Comparing the sizes of both sides we have the assertion.

(ii) ⇒ (i): Let $u \in \Gamma$ and L be a singular line of Γ such that $\partial_\Gamma(u, L) = i < h$. If there exist distinct vertices x and x' in L such that $\partial_\Gamma(u, x) = \partial_\Gamma(u, x') = i$, then

$$x' \in A(u, x) = \bigcup_{z \in C(u,x)} A(z, x).$$

Thus there exists $z \in C(u, x)$ such that $x' \in A(z, x)$. Then we have $z \in \{x, x'\}^{\perp\perp} = L$ which contradicts $\partial_\Gamma(u, L) = i$. □

PROOF OF PROPOSITION 3.1. The first assertion is a direct consequence of Lemma 3.2.

Since all singular lines have size $s + 1$ and all vertices lie on $t + 1$ singular lines, c_1, a_1, a_0 and b_0 exist such that $c_1 = 1$, $a_1 = s - 1$, $a_0 = 0$ and $b_0 = s(t + 1)$. For any integer i with $1 \le i \le m$ and any vertices $x, y \in \Gamma$ at distance i, there are exactly $t_i + 1$ singular lines containing y at distance $i - 1$ from x, and each singular line has unique vertex at distance $i - 1$ from x. It follows that c_i exists with $c_i = t_i + 1$. Then Lemma 3.4 shows that a_i and b_i exist such that $a_i = c_i a_1 = (t_i + 1)(s - 1)$ and $b_i = b_0 - c_i - a_i = s(t - t_i)$ for all $1 \le i \le m - 1$.

If $m = d$, then b_m exists with $b_m = 0$ and hence a_m exists with $a_m = b_0 - c_m$. Hence Γ is distance-regular. The proposition is proved. □

From a basic property of graphs we have the following corollary.

COROLLARY 3.5. *For a graph of order* $(s, t; t_2, \dots, t_m)$*, we have* $0 \le t_2 \le \cdots \le t_m$.

The rest of this section we prove the following result.

LEMMA 3.6. *Let* q *be a positive integer and* Γ *be a graph of the diameter* $d_\Gamma > q$ *such that* $B(x, y) \ne \emptyset$ *for any* $x, y \in \Gamma$ *with* $\partial_\Gamma(x, y) = i \le q$*. Suppose* $(SC)_q$ *holds. Then* $(SS)_{\le q}$ *holds.*

PROOF. Let (x, y, z) be a triple of vertices with $\partial_\Gamma(x, y) \le 1$ and $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = h \le q$. Suppose there exists $w \in S(y, z) - S(x, z)$ to derive a contradiction. Then $\partial_\Gamma(x, w) = h+1$, $\partial_\Gamma(x, y) = 1$ and $\partial_\Gamma(y, w) = h$. Let $w_h := w$ and take $w_i \in B(x, w_{i-1}) \subseteq B(y, w_{i-1})$ for $h + 1 \le i \le q$. Then $\partial_\Gamma(x, w_q) = q + 1$ and $\partial_\Gamma(y, w_q) = q$. Since $(SC)_q$ holds, there exists a strongly closed subgraph Δ of the diameter q containing (y, w_q). Then $w_h \in \Delta$ as it is on a shortest path between y and w_q. Thus $z \in S(y, w_h) \subseteq \Delta$ and $x \in S(z, y) \subseteq \Delta$. We have $q + 1 = \partial_\Gamma(x, w_q) \le d_\Delta = q$, which is a contradiction. The lemma is proved. □

REMARK. Γ has no induced subgraph $K_{2,1,1}$ iff $(SS)_1$ holds. More information about the relations among $(SS)_h$, $(CR)_i$ and $(SC)_j$, the reader is referred to [6, 7].

4. PROOF OF THE THEOREM

In this section we prove our main theorem. First we prove the following result.

PROPOSITION 4.1. *Let* Γ *be a graph such that* c_i*,* a_i *and* b_i *exist for all* $i \le q$ *with* $a_1 > 0$ *and* $b_{q-1} > b_q$*. Suppose* $(CR)_q$ *and* $(SS)_{<q}$ *hold. Then* $\Delta(w, x) = \Delta(y, z)$ *for any root* (w, x, y, z) *of size* q*. In particular,* $(SC)_q$ *holds.*

PROOF. The second assertion follows from Proposition 2.2 and the first assertion.

Let (w, x, y, z) be a root of size q. Suppose $\Delta(y, z) \not\subseteq \Delta(w, x)$ to derive a contradiction. We take a vertex $p \in \Delta(y, z) - \Delta(w, x)$ that has the maximal distance from y. Let $m := \partial_\Gamma(y, p) = \max\{\partial_\Gamma(y, v) \mid v \in \Delta(y, z) - \Delta(w, x)\}$. There exists $z' \in \Psi(y, z)$ such that p is on a shortest path between y and z'. We have $p \notin \Psi(y, z)$ from Lemma 2.4(1) and thus $z' \ne p$. Take $p' \in C(z', p) \subseteq B(y, p)$. Since p' is on a shortest path between y and z', we find that $p' \in \Delta(y, z)$ and hence $p' \in \Delta(w, x)$ from the maximality of m. We have $\partial_\Gamma(w, p) = \partial_\Gamma(w, p') + 1$, otherwise $p \in S(w, p') \subseteq \Delta(w, x)$ from Lemma 2.4(2). The triangle inequality on (w, y, p, p') implies $\partial_\Gamma(w, p') = \partial_\Gamma(y, p) = m$, $\partial_\Gamma(w, y) = 1$ and $\partial_\Gamma(w, p) = \partial_\Gamma(y, p') = m + 1$. We can take $v \in A(p, p')$ as $a_1 > 0$. Then $\partial_\Gamma(w, v) = \partial_\Gamma(y, v) = m + 1$ from Lemma 3.3(1). Lemma 2.4(2) implies $v \in A(y, p') \subseteq \Delta(y, z)$ and thus $v \in \Delta(w, x)$ from the maximality of m. Hence we have $p \in A(w, v) \subseteq \Delta(w, x)$ from Lemma 2.4(2). This is a contradiction. Therefore $\Delta(y, z) \subseteq \Delta(w, x)$.

By symmetry, we have $\Delta(w, x) = \Delta(y, z)$. The proposition is proved. □

Next we prove the following result.

PROPOSITION 4.2. *Let* r *and* q *be positive integers with* $r + 1 \le q$*. Let* Γ *be a graph with the numerical girth* $g = 2r + 2$*, the diameter* $d_\Gamma \ge q + r$ *such that* c_i*,* a_i *and* b_i *exist for all* $i \le q$ *with* $a_1 > 0$ *and* $b_{q-1} > b_q$*. Suppose* $(NP)_q$ *and* $(SS)_{<q+r}$ *hold. Then* $(CR)_q$ *and* $(SC)_q$ *hold.*

To show this we prove the conditions (a) and (b) of Proposition 2.3 hold.

LEMMA 4.3. *Let Γ be a graph as in Proposition 4.2 satisfying $(NP)_q$ and $(SS)_{<q+r}$. Let (x, x', z, z') be a quadruple of vertices with $\partial_\Gamma(x, z) = \partial_\Gamma(x', z') = 1$, $\partial_\Gamma(x, x') = \partial_\Gamma(z, z') = r$ and $\partial_\Gamma(x, z') = \partial_\Gamma(z, x') = r + 1$. Then the following hold.*

(1) Let $p \in \Gamma_q(x) \cap \Gamma_q(z)$. Then $\partial_\Gamma(p, x') = q + r$ implies $\partial_\Gamma(p, z') = q + r$.
(2) Let $u \in \Gamma_q(x) \cap \Gamma_{q-1}(z)$ and $v \in \Gamma_{q-1}(x) \cap \Gamma_q(z)$ with $\partial_\Gamma(u, v) = 1$. Then $\partial_\Gamma(u, x') = q + r$ implies $\partial_\Gamma(v, z') = q + r$.
(3) Let $w, y \in \Gamma$ such that (w, x, y, z) is a root of size q. Then $\partial_\Gamma(w, x') = q + r$ implies $\partial_\Gamma(y, z') = q + r$.

PROOF. Note that $C(\alpha, \beta)$ is a coclique if $\partial_\Gamma(\alpha, \beta) \le q + r$ from Lemma 3.3(1).

(1) Since $(NP)_q$ holds, there exists $p' \in A(x, z) \cap \Gamma_{q-1}(p)$. Then we have $\partial_\Gamma(x', p') = \partial_\Gamma(z', p') = r + 1$. Applying Lemma 3.3(2) to (x', z', p', p) we have the assertion.
(2) We have $\partial_\Gamma(v, x') = q + r - 1$ from the triangle inequality on (v, u, x, x'). Let $p \in A(u, v)$. Then we have $\partial_\Gamma(x, p) = \partial_\Gamma(z, p) = q$ and $\partial_\Gamma(p, x') = q + r$. Hence $\partial_\Gamma(p, z') = q + r$ from (1). Thus $\partial_\Gamma(u, z') = q + r - 1$ from the triangle inequality on (u, p, z, z'). Therefore $\partial_\Gamma(v, z') = q + r$ since $C(z', p)$ is a coclique.
(3) Since (w, x, y, z) is a root of size q, we have $\{\partial_\Gamma(w, z), \partial_\Gamma(x, y)\} \subseteq \{q - 1, q\}$. If $\partial_\Gamma(w, z) = \partial_\Gamma(x, y) = q - 1$, then the assertion follows from (2).
If $\partial_\Gamma(w, z) = q$, then $\partial_\Gamma(w, z') = q + r$ from (1). Applying Lemma 3.3(2) to (w, y, z, z') we have $\partial_\Gamma(y, z') = q + r$.
If $\partial_\Gamma(x, y) = q$, then $\partial_\Gamma(y, x') = q + r$ by applying Lemma 3.3(2) to (w, y, x, x'). Therefore $\partial_\Gamma(y, z') = q + r$ from (1).
In each case we have $\partial_\Gamma(y, z') = q + r$. The lemma is proved. □

PROOF OF PROPOSITION 4.2. Conditions (a) and (b) of Proposition 2.3 hold from Lemmas 3.3(2) and 4.3(3). Hence $(CR)_q$ holds. Therefore $(SC)_q$ holds from Proposition 4.1. □

LEMMA 4.4. *Let Γ be a graph of order $(s, t; t_2, \dots, t_h)$ with $s > 1$. If $t_{h-1} < t_h$, then there exist a vertex u and a path $(x_0, \dots, x_h)$ of length h in Γ such that $\partial_\Gamma(x_0, x_h) = \partial_\Gamma(u, x_i) = h$ for all $0 \le i \le h$.*

PROOF. Fix a vertex u in Γ. First we claim that $A(u, w) \cap B(v, w) \ne \emptyset$ for any $v, w \in \Gamma_h(u)$ with $\partial_\Gamma(v, w) = i < h$. Suppose $A(u, w) \cap B(v, w) = \emptyset$. Then

$$A(u, w) \subseteq C(v, w) \cup A(v, w).$$

The right-hand side is a disjoint union of $(t_i + 1)$ cliques of size s and the left-hand side contains a disjoint union of $(t_h + 1)$ cliques of size $s - 1$ from Lemma 3.4. This contradicts $t_i \le t_{h-1} < t_h$. Hence our claim is proved.

Take $x_0 \in \Gamma_h(u)$. Inductively we can take $x_i \in A(u, x_{i-1}) \cap B(x_0, x_{i-1})$ for all $1 \le i \le h$ from our claim. The lemma is proved. □

PROOF OF THEOREM 1.1. Proposition 3.1 shows that c_i, a_{i-1} and b_{i-1} exist for all $i \le m + r$ such that

$$c_i = t_i + 1, \qquad a_{i-1} = (t_{i-1} + 1)(s - 1) \qquad \text{and} \qquad b_{i-1} = s(t - t_{i-1}).$$

In particular, $(SS)_{<m+r}$ holds and Γ has the numerical girth $g = 2r + 2$.

We prove $t_{h-1} < t_h$ and $(SC)_h$ holds for all $r + 1 \le h \le m$ by induction on h.

From our assumption we have $t_r < t_{r+1}$ and hence $b_r > b_{r+1}$. Hence $(SC)_{r+1}$ holds from Proposition 4.2.

Let $r+1 \leq h < m$. Suppose $t_{h-1} < t_h$ and $(SC)_h$ holds. Then $a_h < a_{h+1}$ from Lemmas 4.4 and 2.5. Thus $t_h < t_{h+1}$ and $(SC)_{h+1}$ holds from Proposition 4.2. Note that c_i and a_i of a strongly closed subgraph are the same as those of Γ. The theorem is proved. □

REMARK. A regular near $2d$-gon of order $(s, t; t_2, \ldots, t_d)$ is called a *generalized $2d$-gon of order* (s, t) if $t_1 = \cdots = t_{d-1} = 0$ and $t_d = t$.

Feit and Higman showed that a generalized $2d$-gon has $d \in \{2, 3, 4, 6\}$, unless it is an ordinary polygon (see [4] or [2, Theorem 6.5.1]).

Let r and m be positive integers with $r+1 \leq m$. Let Γ be a graph of order $(s, t; t_2, \ldots, t_{m+r})$ with $s > 1$ and $0 = t_1 = \cdots = t_r < t_{r+1}$. Theorem 1.1 shows that a graph Γ has a generalized $2(r+1)$-gon of order (s, t_{r+1}) as a strongly closed subgraph. Hence we have $r \in \{1, 2, 3, 5\}$ from the result of Feit and Higman. This result was first proved in [5].

Here we conjecture the following.

CONJECTURE. Let Γ be a regular thick near polygon of the diameter d and the numerical girth $g \geq 6$. Then $d < g$.

Suppose Γ is a regular thick near polygon of order $(s, t; t_2, \ldots, t_d)$ with the numerical girth $g = 2r + 2 \geq 6$. Suppose $2r + 2 \leq d$. Then Corollary 1.2 shows that $0 = t_1 = \cdots = t_r < t_{r+1} < \cdots < t_{d-r}$ and there exists a tower of regular near sub-polygons

$$\Delta^{r+1} \subset \Delta^{r+2} \subset \cdots \subset \Delta^{d-r}$$

where Δ^q is a regular near $2q$-gon of order $(s, t_q; t_2, \ldots, t_q)$. In particular, $r \in \{2, 3, 5\}$.

To prove our conjecture it is enough to show that there does not exist a regular thick near $2(r+2)$-gon of order $(s, t; t_2, \ldots, t_{r+2})$ with $r \in \{2, 3, 5\}$ and $0 = t_1 = \cdots = t_r < t_{r+1} < t_{r+2}$ which satisfying the condition $(SC)_{r+1}$.

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