On the stability of the additive Cauchy functional equation in random normed spaces

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Abstract

Some stability results for the functional equations of Cauchy and Jensen in probabilistic setting are proved by using the fixed point method.

Keywords: Generalized stability; Functional equation; Random normed space; Fixed point

Introduction


It is worth noting that almost all proofs in this topic used the direct method: the exact solution of the functional equation is explicitly constructed as a limit of a (Hyers) sequence, starting from the given approximate solution $f$ [1,6,12–14]. In 2003, Radu [20] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative. This method has recently been used by many authors (see, e.g., [4,5, 7,19]).

The first result on the stability of Cauchy equation in the setting of fuzzy normed spaces has been given in [18]. By using the fixed point method, in this short note we provide a probabilistic counterpart of the generalized stability result for the Cauchy equation.

1. Preliminaries

We recall some useful notions and results. First we evoke the fixed point alternative of Diaz and Margolis, to which we will refer to as

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Lemma 1.1. (Cf. [15,20].) Let \((X, d)\) be a complete generalized metric space and \(J : X \to X\) be a strictly contractive mapping, that is,
\[
d(Jx, Jy) \leq Ld(x, y) \quad (x, y \in X),
\]
for some \(L < 1\). Then, for each fixed element \(x \in X\), either
\[
d(J^n x, J^{n+1} x) = +\infty, \quad \forall n \geq 0,
\]
or
\[
d(J^n x, J^{n+1} x) < +\infty, \quad \forall n \geq n_0,
\]
for some natural number \(n_0\). Moreover, if the second alternative holds then:

(i) the sequence \((J^n x)\) is convergent to a fixed point \(y^*\) of \(J\);

(ii) \(y^*\) is the unique fixed point of \(J\) in the set \(Y := \{y \in X, \ d(J^{n_0} x, y) < +\infty\}\) and \(d(y, y^*) \leq \frac{1}{1-L}d(y, Jy)\) \((x, y \in Y)\).

A function \(F: \mathbb{R} \to [0, 1]\) is called a distribution function if it is nondecreasing and left-continuous, with \(\sup_{t \in \mathbb{R}} F(t) = 1\) and \(\inf_{t \in \mathbb{R}} F(t) = 0\). The class of all distribution functions \(F\) with \(F(0) = 0\) is denoted by \(D_+\). For any \(a \geq 0\), \(\varepsilon_a\) is the element of \(D_+\) defined by
\[
\varepsilon_a(t) = \begin{cases} 
0 & \text{if } t \leq a, \\
1 & \text{if } t > a.
\end{cases}
\]

The notion of random normed space goes back to Sherstnev (see [9,23] or [22]).

Definition 1.2. Let \(X\) be a real vector space, \(F\) be a mapping from \(X\) into \(D_+\) (for any \(x \in X\), \(F(x)\) is denoted by \(F_x\)) and \(T\) be a \(t\)-norm. The triple \((X, F, T)\) is called a random normed space (briefly RN-space) iff the following conditions are satisfied:

\((RN1)\) \(F_\theta = \varepsilon_0\) iff \(x = \theta\), the null vector;

\((RN2)\) \(F_{\alpha x} = F_x \left(\frac{t}{|\alpha|}\right)\) for all \(\alpha \in \mathbb{R}, \alpha \neq 0\) and \(x \in X\);

\((RN3)\) \(F_{x+y}(t_1 + t_2) \geq F_x(t_1) + F_y(t_2)\), for all \(x, y \in X\) and \(t_1, t_2 > 0\).

Every normed space \((X, \| \cdot \|)\) defines a random normed space \((X, F, T_M)\) where
\[
F_u(t) = \frac{1}{t + \|u\|}, \quad \forall t > 0,
\]
and \(T_M\) is the minimum \(t\)-norm. This space is called the induced random normed space.

If the \(t\)-norm \(T\) is such that \(\sup_{0 < a < 1} T(a, a) = 1\), then every RN-space \((X, F, T)\) is a metrizable linear topological space with the topology \(\tau\) (called the \(F\)-topology or the \((\varepsilon, \lambda)\)-topology) induced by the base of neighborhoods of \(\theta\)
\[
\{U(\varepsilon, \lambda) \mid \varepsilon > 0, \ \lambda \in (0, 1)\},
\]
where
\[
U(\varepsilon, \lambda) = \{x \in X \mid F_x(\varepsilon) > 1 - \lambda\}.
\]
A sequence \(\{x_n\}\) in an RN-space \((X, F, T)\) converges to \(x \in X\) in the topology \(\tau\) (we denote \(\lim_{n \to \infty} x_n = x\)) if \(\lim_{n \to \infty} F_{x_n-x}(t) = 1, \forall t > 0\). \(\{x_n\}\) is called a Cauchy sequence if \(\lim_{m,n \to \infty} F_{x_m-x_n}(t) = 1, \forall t > 0\). The RN-space \((X, F, T)\) is said to be complete if every Cauchy sequence in \(X\) is convergent. Notice that if \(T = T_M\), then \((X, F, T)\) is locally convex.

2. The main result

Let \(X\) be a linear space, \((Y, F, T_M)\) be a complete random normed space and \(G\) be a mapping from \(X \times \mathbb{R}\) into \([0,1]\), such that \(G(x, \cdot) \in D_+\) for all \(x\). Consider the set \(E := \{g : X \to Y, \ g(0) = 0\}\) and the mapping \(d_G\) defined on \(E \times E\) by
\[
d_{G}(g, h) = \inf\{a \in R_+, \quad F_{g(x)-h(x)}(at) \geq G(x, t) \text{ for all } x \in X \text{ and } t > 0\}
\]
where, as usual, \(\inf \emptyset = +\infty\).

In the proof of the main result we need the following lemma (cf. [10,16]), whose proof is given for the sake of convenience.

**Lemma 2.1.** \(d_{G}\) is a complete generalized metric on \(E\).

**Proof.** It is immediate that \(d_{G}\) is symmetric and \(d_{G}(f, f) = 0\) for all \(f \in E\). If \(d_{G}(f, g) = 0\), then for every fixed \(x\) and \(t\) we have \(F_{f(x)-g(x)}(t) \geq G(x, \frac{1}{a})\) for all \(a > 0\). Therefore, \(F_{f(x)-g(x)}(t) = 1\) for all \(x\) and \(t\), which implies \(f = g\).

Next, if \(d_{G}(f, g) = a < \infty\) and \(d_{G}(g, h) = b < \infty\), then \(F_{f(x)-g(x)}(at) \geq G(x, t)\) and \(F_{g(x)-h(x)}(bt) \geq G(x, t)\) for all \(x\) and \(t\), which shows that \(d_{G}(f, h) \leq a + b\), so that \(d_{G}(f, h) = d_{G}(f, g) + d_{G}(g, h)\).

Suppose that \(\{g_{n}\}\) is \(d_{G}\)-Cauchy. We fix \(x\) in \(X\) and denote \(G(x, t)\) by \(H(t)\). Let \(\varepsilon > 0\) and \(\lambda \in (0, 1)\) be given and let \(t > 0\) be such that \(H(t) > 1 - \lambda\). For \(a < \frac{1}{\lambda}\), we choose \(n_{0}\) such that \(d_{G}(g_{n}, g_{m}) < a\) for all \(n \geq n_{0}\). Then

\[
F_{g_{n}(x)-g_{m}(x)}(\varepsilon) \geq F_{g_{n}(x)-g_{m}(x)}(at) \geq H(t) \geq 1 - \lambda, \quad \forall n \geq n_{0},
\]
hence \(\{g_{n}(x)\}\) is Cauchy. Since \((Y, F, T_{M})\) is complete, there exists a mapping \(g : X \rightarrow Y\) with \(g(0) = 0\), such that \(\{g_{n}(x)\}\) converges to \(g(x)\).

Let \(a, \delta > 0\) be given. Then there exists \(n_{0}\) such that \(F_{g_{n}(x)-g_{n+1}(x)}(at) \geq H(t)\) for all \(n > n_{0}\), all \(m \geq 1\) and each \(t\). Fix \(n > n_{0}\) and \(t > 0\). Since

\[
F_{g_{n}(x)-g(x)}((a + \delta)t) \geq \min\{F_{g_{n}(x)-g_{n+1}(x)}(at), F_{g_{n+1}(x)-g(x)}(\delta t)\} \geq \min\{H(t), F_{g_{n+1}(x)-g(x)}(\delta t)\},
\]
by letting \(m \rightarrow \infty\) we obtain \(F_{g_{n}(x)-g(x)}((a + \delta)t) \geq \min\{H(t), 1\} = H(t)\). Therefore \(d_{G}(g_{n}, g) \leq a + \delta, \quad \forall n \geq n_{0}\), so that \(\{g_{n}\}\) is \(d_{G}\)-convergent. \(\square\)

**Theorem 2.2.** Let \(X\) be a real linear space, let \(f\) be a mapping from \(X\) into a complete random normed space \((Y, F, T_{M})\) with \(f(0) = 0\) and let \(\Phi : X^{2} \rightarrow D_{+}\) be a symmetric mapping with the property

\[
\exists \alpha \in (0, 2): \quad \Phi(2x, 2y)(at) \geq \Phi(x, y)(t), \quad \forall x, y \in X, \quad \forall t > 0.
\]

(2.1)

If

\[
F_{f(x+y)-f(x)-f(y)}(at) \geq \Phi(x, y)(t), \quad \forall x, y \in X,
\]

(2.2)

then there is a unique additive mapping \(g : X \rightarrow Y\) such that

\[
F_{g(x)-f(x)}(t) \geq \Phi(x, x)((2 - \alpha)t), \quad \forall x \in X, \quad \forall t > 0.
\]

(2.3)

Moreover,

\[
g(x) = \lim_{n \rightarrow \infty} \frac{f(2^{n}x)}{2^{n}}.
\]

(2.4)

**Proof.** By setting \(y = x\) in (2.2), we immediately see that \(F_{2f(x)-f(2x)}(at) \geq \Phi(x, x)(t)\) for all \(x\), whence

\[
F_{f(x)-f(2x)}(at) \geq \Phi(x, x)(2t), \quad \forall x \in X, \quad \forall t > 0.
\]

Let \(G(x, t) := \Phi(x, x)(2t)\). Consider the set \(E := \{g : X \rightarrow Y, \quad g(0) = 0\}\) together with the mapping \(d_{G}\) defined on \(E \times E\) by

\[
d_{G}(g, h) = \inf\{a \in R_+, \quad F_{g(x)-h(x)}(at) \geq G(x, t) \text{ for all } x \in X \text{ and } t > 0\}.
\]

By Lemma 1.2, \((E, d_{G})\) is a complete generalized metric space.

Now, let us consider the linear mapping

\[
J : E \rightarrow E, \quad Jg(x) := \frac{1}{2}g(2x).
\]

It is easy to see that \(J\) is a strictly contractive self-mapping of \(E\) with the Lipschitz constant \(\frac{2}{3}\).
Indeed, let \( g, h \) in \( E \) be given such that \( d_G(g, h) < \varepsilon \). Then

\[
F_{g(x) - h(x)}(\varepsilon t) \geq G(x, t), \quad \forall x \in X, \forall t > 0,
\]

whence

\[
F_{Jg(x) - Jh(x)}\left(\frac{\alpha}{2} \varepsilon t\right) = F_{g(2x) - h(2x)}(\alpha \varepsilon t) \geq G(2x, \alpha t)
\]

for all \( x \) and \( t \). Since \( G(2x, \alpha t) \geq G(x, t) \) for all \( x \) and \( t \), then \( F_{Jg(x) - Jh(x)}\left(\frac{\alpha}{2} \varepsilon t\right) \geq G(x, t) \), that is, \( d_G(g, h) < \varepsilon \Rightarrow d_G(Jg, Jh) \leq \frac{\alpha}{2} \varepsilon \). This means that

\[
d_G(Jg, Jh) \leq \frac{\alpha}{2} d_G(g, h) \quad \text{for all} \ g, h \ \text{in} \ E.
\]

Next, from \( F_{f(x) - 2^{-1}f(2x)}(t) \geq G(x, t) \) it follows that \( d_G(f, Jf) \leq 1 \).

Using the fixed point alternative we deduce the existence of a fixed point of \( J \), that is, the existence of a mapping \( g : X \to Y \) such that

\[
g(2x) = 2g(x), \quad \forall x \in X.
\]

Also, \( d_G(f, g) \leq \frac{1}{1 - \varepsilon} d(f, Jf) \) implies the inequality \( d_G(f, g) \leq \frac{1}{1 - \varepsilon} \), from which it immediately follows

\[
F_{g(x) - f(x)}\left(\frac{\alpha - 2}{\alpha} t\right) \geq G(x, t) \quad \text{for all} \ t > 0 \ \text{and} \ x \in X \ \text{(recall that} \ G \ \text{is left continuous in the second variable)}.
\]

This means that

\[
F_{g(x) - f(x)}(t) \geq G\left(x, \frac{2 - \alpha}{2} t\right), \quad \forall x \in X, \forall t > 0,
\]

whence we obtain the estimation

\[
F_{g(x) - f(x)}(t) \geq \Phi(x, x)\left(2 - \alpha\right) t, \quad \forall x \in X, \forall t > 0.
\]

Since for any \( x \) and \( t \), \( d_G(u, v) < \varepsilon \Rightarrow F_{u(x) - v(x)}(t) \geq G\left(x, \frac{t}{2}\right) \), from \( (J^n f, g) \to 0 \), it follows

\[
\lim_{n \to \infty} F_{2^n x}(2^n t) = g(x), \quad \forall x \in X.
\]

The additivity of \( g \) can be proven in the standard way. In fact, since \( T_M \) is continuous, then \( z \to F_z \) is continuous (cf. [22, Chapter 12]). Therefore, for almost all \( t \),

\[
F_{g(x+y) - g(x) - g(y)}(t) = \lim_{n \to \infty} F_{f\left(\frac{2^n(x+y)}{2}\right) - f\left(\frac{2^n x}{2}\right) - f\left(\frac{2^n y}{2}\right)}(t) = \lim_{n \to \infty} F_{f(2^n(x+y)) - f(2^n x) - f(2^n y)}(2^n t)
\]

\[
\geq \lim_{n \to \infty} \Phi(x, y)\left(\frac{2}{\alpha}\right) \frac{n}{t} = 1,
\]

so that \( F_{g(x+y) - g(x) - g(y)}(t) = 1 \) for all \( t > 0 \), which implies \( g(x+y) - g(x) - g(y) = 0 \).

The uniqueness of \( g \) follows from the fact that \( g \) is the unique fixed point of \( J \) with the following property: there is \( C \in (0, \infty) \) such that \( F_{g(x) - f(x)}(Ct) \geq G(x, t) \) for all \( x \in X \) and \( t > 0 \).  \( \square \)

\textbf{Remark 2.3.} Except for obvious modifications, the above method can be used to prove the following complementary result:

\textbf{Theorem 2.4.} Let \( X \) be a real linear space, let \( f \) be a mapping from \( X \) into a complete random normed space \( (Y, F, T_M) \) with \( f(0) = 0 \) and let \( \Phi : X^2 \to D_+ \) be a symmetric mapping with the property

\[
\exists \alpha \in (0, 2): \quad \Phi(x, y)(t) \geq \Phi(2x, 2y)(\alpha t), \quad \forall x, y \in X, \forall t > 0.
\]

(2.5)

If the control condition (2.2) holds, then there is a unique additive mapping \( g : X \to Y \) such that

\[
F_{g(x) - f(x)}(t) \geq \Phi\left(\frac{x}{2}, \frac{x}{2}\right)\left(\frac{2 - \alpha}{2} t\right), \quad \forall x \in X, \forall t > 0.
\]

(2.6)
Moreover,
\[ g(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right). \]

Example 2.5. Let \( X \) and \( Y \) be normed spaces and \((X, F, T_M)\) be the induced random normed space. If
\[ \Phi(x, y)(t) := \frac{t}{t + \varphi(x, y)}, \quad \forall t > 0, \]
then the condition (2.1) holds if and only if \( \varphi(2x, 2y) \leq \alpha \varphi(x, y) \) for all \( x, y \) in \( X \), while (2.5) is equivalent to \( \varphi(x, y) \leq \alpha \varphi(2x, 2y) \). We note that \( \varphi(x, y) = \|x\|^p + \|y\|^p \) verifies the first condition for \( p < 1 \) and the second one in case \( p > 1 \). Since (2.2) reduces to
\[ \|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \forall x, y \in X, \]
our theorems slightly extend the results of Aoki [2] and Gajda [8] (see also [5]).

Remark 2.6. In the same way we can prove the following stability result for Jensen equation in random normed spaces (compare with [17, Theorem 2.1]).

Theorem 2.7. Let \( X \) be a real linear space, let \( f \) be a mapping from \( X \) into a complete random normed space \((Y, F, T_M)\) with \( f(0) = 0 \) and let \( \Phi : X^2 \to D_+ \) be a symmetric mapping with the following property:
\[ \exists \alpha \in (0, 2): \quad \Phi(2x, 2y)(\alpha t) \geq \Phi(x, y)(t), \quad \forall x, y \in X, \forall t > 0. \]
If
\[ F_2 f \left( \frac{x+y}{2} \right) - f(x) - f(y)(t) \geq \Phi(x, y)(t) \]
for all \( x, y \in X \) and \( t > 0 \), then there is a unique additive mapping \( g : X \to Y \) such that
\[ F_g(x) - f(x) \geq \Phi(2x, 0)((2 - \alpha)t), \quad \forall x \in X, \forall t > 0. \]

Remark 2.8. In our approach, the \( t \)-norm \( T_M \) has been used in proving the triangle inequality for \( d_{G} \). On the other hand, the problem of replacing \( T_M \) by a weaker \( t \)-norm is related to a more difficult problem in the theory of fixed points in probabilistic metric spaces (see [9, Chapter 3]).

References