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Universal traversal sequences with backtracking

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Abstract

In this paper we introduce a new notion of traversal sequences that we call *exploration sequences*. Exploration sequences share many properties with the traversal sequences defined in Aleliunas et al. (Proceedings on the 20th Annual Symposium of Foundations of Computer Science, 1979, pp. 218–223), but they also exhibit some new properties. In particular, they have an ability to backtrack, and their random properties are robust under choice of the probability distribution on labels.

Further, we present simple constructions of polynomial-length universal exploration sequences for some previously studied classes of graphs (e.g., 2-regular graphs, cliques, expanders), and we also present universal exploration sequences for trees. These constructions do not obey previously known lower bounds on the length of universal traversal sequences; thus, they highlight another difference between exploration and traversal sequences.

1. Introduction

The s–t-connectivity problem has drawn a lot of attention in computer science. One direction of study is aimed at investigating the space complexity of this problem. It is known that the directed version of this problem is complete for non-deterministic log-space, and its undirected version is complete for symmetric log-space (SL), which is defined to be the class of problems log-space reducible to undirected s–t-connectivity. It is conjectured that symmetric log-space is equal to deterministic log-space ($SL = L$.) The main evidence supporting this conjecture is (a) the existence of short universal traversal sequences, which were introduced by Cook and studied in [A,AKLLR], and (b) the success of derandomization techniques which, applied to the random walk algorithm, have shown that SL is in $DSPACE(\log^{4/3} n)$ [NSW,SZ,ATWZ].

A traversal sequence is a sequence which guides a walk in a graph. Aleliunas et al. shows in [AKLLR] that with high probability a sequence of length $O(d^2n^3 \log n)$ chosen uniformly at random guides a walk in any undirected d-regular connected graph in such a way that all vertices are visited. A sequence with this property is called a *universal traversal sequence* (UTS). If there is a universal traversal sequence

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for 3-regular undirected graphs constructible in log-space, then the s–t-connectivity problem is in log-space, hence, $SL = L$ [AKLLR].

Explicit universal traversal sequences for 2-regular graphs were constructed in [BBKLW,B,I,K]; for cliques in [KPS], and for expanders in [IIW]. The sequences in [B,BBKLW,KPS] are of length $n^{O(\log n)}$, thus they are not log-space constructible. On the other hand, the sequences constructed in [HW,I,K] are of polynomial length, and they are log-space constructible. However, the construction for expanders in [HW] requires the graph to be labeled "*consistently*". (We define what consistent labeling means in the next section.) Hoory and Wigderson [HW] does not address the question of finding such a labeling in small space, although it is known that this problem can be reduced to finding a perfect matching in a bipartite graph.

The only known explicit constructions of UTSs of subexponential length for 3-regular graphs are based on pseudo–random generators for randomized log-space, [BNS,N,INW]. The pseudo-random generator presented in [BNS] yields a UTS of length $2^{2^{O(\sqrt{n})}}$, and the pseudo-random generators presented in [INW,N] yield UTSs of length $n^{O(\log n)}$.

We introduce a new type of traversal sequences that we call *exploration sequences*. We show that exploration sequences have the same useful properties as traversal sequences, and moreover, they also exhibit a new property—the ability to backtrack. Using that new ability we construct *universal exploration sequences* (UXS) for some classes of graphs. These UXSs are shorter than the corresponding lower-bounds on the length of UTSs for cycles and cliques given by [BBKLW,AAR,BRT,T,BT,D].

Under our definition it can be shown that a random exploration sequence of length $O(d^2n^3 \log n)$ is a UXS for d-regular graphs with high probability. Thus, constructions of UTSs of length $n^{O(\log n)}$ for 3-regular graphs that are based on pseudo-random generators apply to UXSs, too. Also the results of [ATWZ,NSW] can be formulated in terms of exploration sequences, as well as other results on random traversal sequences. In particular, the strengthening of the upper-bound on the length of non-uniform UTSs of [CRRST,KLNS] apply to UXSs, too. Beside that, random exploration sequences exhibit robustness under the choice of probability distribution on labels.

Our definition is motivated by the following. If we are told that we reach a vertex v in a graph following a traversal sequence t, it is impossible in principle to determine where the walk started (or even to tell what vertex was visited immediately previously to v). This ambiguity does not occur if the labeling of the graph is consistent as required for the construction of [HW]. However, in neither of these cases is there a traversal sequence that depends only on t and that brings us back from v to the starting vertex. In contrast, with exploration sequences we can have such a sequence, we can *backtrack*. This gives us a lot of power.

2. Definitions, basic properties, and explicit constructions

Let G be an undirected graph. A *labeling* l of G is a map assigning to every vertex v and every edge e incident to v a label $l(e,v) \in \{0, \dots, deg(v) - 1\}$, so that for every two distinct edges e, e' incident to v, $l(e,v) \neq l(e',v)$. (A labeling would be *consistent* if for every vertex v, and every two distinct edges $\{v,u\}, \{v,w\}$, $l(\{v,u\},u) \neq l(\{v,w\},w)$.)

In our new notion, during a walk on G guided by an exploration sequence t, digits of t will be interpreted as labels *relative* to the edge by which we got to the vertex. In contrast, in definition of traversal sequences of [AKLLR], digits of t are interpreted directly as edge labels. More formally:

Let l be a labeling of G. Let $e \in E(G)$ and $v \in e$. We say that $i \in \mathbb{N}$ *takes us* from the pair (e, v) to a pair (e', v'), if $e' = \{v, v'\}$ and $l(e', v) = l(e, v) + i \bmod deg(v)$. We denote that by $(e, v) \rightarrow^{i} (e', v')$. (Intuitively, pair (e, v) corresponds to being at v's end of edge e.) The symbol Λ denotes the empty move, i.e., $(e, v) \rightarrow^{\Lambda} (e, v)$. For $t \in (\mathbb{N} \cup \{\Lambda\})^*$, we extend the notation recursively: if $t = \varepsilon$, then $(e, v) \rightarrow^{t} (e, v)$, and if $t = t'i$, for $t' \in (\mathbb{N} \cup \{\Lambda\})^*$ and $i \in \mathbb{N} \cup \{\Lambda\}$, then $(e, v) \rightarrow^{t} (e'', v'')$, given that $(e, v) \rightarrow^{t'} (e', v')$ and $(e', v') \rightarrow^{i} (e'', v'')$. We call sequence t an *exploration sequence*. (If t contains symbol Λ, then it is a *general exploration sequence*, otherwise it is *simple*.) We say that *t visits a vertex u starting at* (e, v) if there is a prefix t' of t and an edge e_u incident to u, such that $(e, v) \rightarrow^{t'} (e_u, u)$.

Let $\mathcal{G}$ be a class of connected undirected graphs. We say that an exploration sequence t is a *universal exploration sequence for* $\mathcal{G}$, if for every $G \in \mathcal{G}$ and every labeling of G, t visits every vertex in G starting at any pair (e, v), where $e \in E(G)$ and $v \in e$. Some of our constructions deal with d-regular graphs but note that many of our theorems and constructions do not require regularity.

For comparison, let us restate the notion of traversal sequences that are defined in [AKLLR] in our terms. Traversal sequences require $l(e', v) = i$ rather than $l(e', v) = l(e, v) + i \bmod deg(v)$, when choosing where to go next according to i. (There is a minor technical difference between the original definition and this definition since we go from one edge–vertex pair to another edge–vertex pair, whereas in [AKLLR] you go from one vertex to another vertex.)

Let $d > 1$ be an integer, and $t \in \{0, \ldots, d-1\}^*$, where $t = t_1 t_2 \cdots t_k$. We define a *backtrack* $\bar{t}$ of t to be the sequence $\overline{t_k t_{k-1}} \cdots \overline{t_1}$, where $\overline{t_i} = (d - t_i) \bmod d$. We state the following simple theorem.

Theorem 1 (Backtracking theorem). *Let $d > 1$ be an integer. Let $t \in \{0, \ldots, d-1\}^*$. Then there is an exploration sequence $t' = 0\bar{t}0$ such that for any d-regular graph G, and any labeling of G the following holds. If $e \in E(G)$, $v \in e$ then $(e, v) \rightarrow^{tt'} (e, v)$.*

The proof by induction on the length of t is rather simple, so we leave it as an exercise. We defined backtracking only for simple exploration sequences but it can be easily extended to general exploration sequences.

The following theorem states that (analogous to the situation with random traversal sequences) a random exploration sequence of sufficient length is a UXS.

Theorem 2. *Let $d > 1$ be an integer, and let $n > 1$ be an integer. With high probability, a sequence chosen uniformly at random from $\{0, \ldots, d-1\}^{O(d^2 n^3 \log n)}$ is a universal exploration sequence for d-regular graphs on n vertices.*

This theorem follows from the fact that with high probability, a general exploration sequence of length $O(d^2 n^3 \log n)$ chosen uniformly at random is a UXS for d-regular graphs on n vertices. This fact can be proven in exactly the same way as the equivalent theorem for traversal sequences (see [AKLLR]), since regardless of whether we are walking using the traversal sequences or the exploration sequences, the next edge is chosen uniformly at random among the incident edges. In the next section we give a full proof of a related statement. (Results of [CRRST,KLNS] can be used to improve the upper-bound on the length of universal exploration sequences to $O(dn^3 \log n)$ for $3 \leqslant d \leqslant n/2 - 1$, and $O(n^3 \log n)$ for $n/2 \leqslant d$, respectively.)

If we could construct a UXS of polynomial length for 3-regular graphs in log-space, we could conclude $SL = L$, since s–t-connectivity on 3-regular undirected graphs is a complete problem for SL and any UXS can be used for s–t-connectivity testing [AKLLR]. (There is a log-space algorithm that transforms any undirected graph into a 3-regular graph while preserving connectivity.) We do not know how to construct a UXS for 3-regular graphs in log-space. However, we present several log-space constructions of UXSs for some other classes of graphs.

Let us first note that the constructions of universal traversal sequences for cliques and expanders of [KPS,HW] can be "lifted-up", and reformulated in terms of exploration sequences using the backtracking ability. This reformulation is possible, although technically complicated, so we do not present it and we rather present different constructions.

Proposition 3. *1^n is a UXS for cycles of length n.*

The following construction for cliques corresponds to visiting all neighbors of the starting vertex in the order given by a labeling of the clique.

Proposition 4. *$(10)^n$ is a UXS for cliques of size n.*

Next, we give a construction of UXS for constant degree expanders. Indeed, the constructed sequence is a UXS for any class of d-regular graphs of bounded maximum distance of any two vertices (the *diameter*). The construction employs a full backtracking up to the diameter of the graph.

We follow [HW] in the definition of expanders. Let $d \geqslant 3$ be an integer. A d-regular graph $G = (V, E)$ on n vertices is a *c-expander* if for every $U \subset V$, the number of edges between U and $V - U$ is at least $c|U|(n - |U|)/n$. It is called an expander if it is c-expander for some $c > 0$. It is straightforward to verify that any d-regular c-expander has diameter at most $c' \log n$ for some constant c' independent of n.

Define recursively: $s_d^0 = 0$, and for $h \geqslant 1$, define $s_d^h = 1(s_d^{h-1}1)^{d-1}$, and $t_d^h = (1s_d^{h-1})^d$.

Theorem 5. *t_d^h is a UXS for d-regular graphs with maximum distance (diameter) at most h.*

Corollary 6. *$t_d^{O(\log n)}$ is a UXS for d-regular expanders.*

As in [HW], we hide the expansion factor of the expander in $O(\log n)$. The length of t_d^k is at most $2d^k$, hence, the UXS for d-regular expanders is of length $n^{O(\log d)}$. The UTS for expanders, that is presented in [HW], is of length $n^{O(d \log d)}$.

We prove Theorem 5 for the special case of $d = 3$. For $d > 3$, the proof is analogous. The proof uses the following two lemmas. The first lemma says that s_3^h brings us always back to the starting edge but in the opposite direction. The second lemma says that s_3^h visits all vertices in a graph G that are reachable from the starting point by a path of length at most h without crossing the starting edge. Then, the proof of Theorem 5 combines these two lemmas.

Lemma 7. *Let G be a 3-regular graph. For any integer $h \geqslant 0$, and any edge $e = \{v, u\}$ of G, $(e, v) \rightarrow^{s_3^h} (e, u)$.*

Proof. The proof is by induction on h. For $h = 0$, the lemma is trivial. Assume $h > 0$. $s_3^h = 1s_3^{h-1}1s_3^{h-1}1$ and G is 3-regular, so by the induction hypothesis we get $(e, v) \rightarrow^1 (e_1, u_1) \rightarrow^{s_3^{h-1}} (e_1, v) \rightarrow^1 (e_2, u_2) \rightarrow^{s_3^{h-1}} (e_2, v) \rightarrow^1 (e, u)$, where $e_1 = \{v, u_1\}$, and $e_2 = \{v, u_2\}$ for some vertices u_1 and u_2. $\square$

Lemma 8. *Let G be a 3-regular graph. Let $e = \{u, v\}$ be an edge of G and $h_0 \geqslant 0$ be an integer. Let $B^{h_0}(e, v)$ be the set of vertices of G that are reachable from v by a path of length at most h_0 without using edge e. Then for any $h \geqslant h_0$, s_3^h visits all vertices of $B^{h_0}(e, v)$ starting at (e, v).*

Proof. We prove the lemma by induction on h_0. For $h_0 = 0$, the lemma is trivial, so consider $h_0 > 0$. Let $h \geqslant h_0$ be arbitrary. Let $e_1 = \{v, u_1\}$, and $e_2 = \{v, u_2\}$, so that $(e, v) \rightarrow^1 (e_1, u_1)$ and $(e_1, v) \rightarrow^1 (e_2, u_2)$. Clearly, $(e_2, v) \rightarrow^1 (e, u)$. By the previous lemma, $(e, v) \rightarrow^1 (e_1, u_1) \rightarrow^{s_3^{h-1}} (e_1, v) \rightarrow^1 (e_2, u_2) \rightarrow^{s_3^{h-1}} (e_2, v) \rightarrow^1 (e, u)$. By the induction hypothesis, the first s_3^{h-1} visits all vertices in $B^{h_0-1}(e_1, u_1)$, and the other s_3^{h-1} visits all vertices in $B^{h_0-1}(e_2, u_2)$. Since $B^{h_0}(e, v) \subseteq B^{h_0-1}(e_1, u_1) \cup B^{h_0-1}(e_2, u_2) \cup \{v\}$, the lemma follows. $\square$

Proof of Theorem 5. Let G be any 3-regular graph with diameter at most h, and let $e = \{u, v\} \in E(G)$. Further, let $e_1 = \{v, u_1\}$, and $e_2 = \{v, u_2\}$, so that $(e, v) \rightarrow^1 (e_1, u_1)$ and $(e_1, v) \rightarrow^1 (e_2, u_2)$. Similarly to the previous lemma, the traversal by t_3^h goes as follows: $(e, v) \rightarrow^1 (e_1, u_1) \rightarrow^{s_3^{h-1}} (e_1, v) \rightarrow^1 (e_2, u_2) \rightarrow^{s_3^{h-1}} (e_2, v) \rightarrow^1 (e, u) \rightarrow^{s_3^{h-1}} (e, v)$. Since every vertex in G is at distance at most h from v, the claim follows by the previous lemma. $\square$

The universal exploration sequence for trees presented in the next proposition employs depth first search of trees (as in the standard Euler tour technique for graph algorithms.) There is no known explicit polynomial-length UTS for trees.

Proposition 9. 1^{2n} *is a UXS for trees of size n.*

Propositions 3 and 4 in conjunction with known super-linear lower-bounds on the length of universal traversal sequences for cycles and cliques suggest that exploration sequences are in some sense more powerful than traversal sequences.[2]

The universal exploration sequences constructed in this section are sequences from $\{0, 1\}^*$. Theorems of the following section show that we can restrict the alphabet of exploration sequences for traversal of any graph.

3. Unbalanced probability distributions on labels

In this section we show that in contrast to traversal sequences, properties of random exploration sequences do not change dramatically with a change of the probability distribution on labels. In particular, for the randomized s–t-connectivity algorithm one can use a source of randomness that does not give a uniform probability distribution on labels at every step.

Let us first show that traversal sequences are sensitive to the choice of the probability distribution on labels. Consider a labeled graph G in Fig. 1. It is a 3-regular undirected graph on $2n$ vertices.

Consider a random walk on G, where at every step of the walk the edge labeled by 0 is chosen with probability $1/2$, the edge labeled by 1 is chosen with probability $1/4$, and the edge labeled by 2 is chosen with probability $1/4$. (This is the distribution one

[2] Recently, H. Karloff, M. Koucký, and O. Reingold have shown that any universal traversal sequence t for 3-regular graphs on nd vertices can be converted into a universal exploration sequence t' for d-regular graphs on n vertices such that $|t'| \leqslant |t|$. A similar conversion also works for UTSs for d'-regular graphs, for $d' \geqslant 3$.

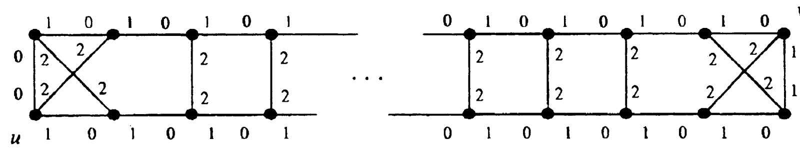

Fig. 1.

obtains via a naïve scheme to use a two-sided coin to generate a random number from $\{0,1,2\}$. I.e., using two independent uniformly distributed random bits r_i and r_{i+1} to obtain a number from $\{0,1,2\}$ by formula $(2r_i + r_{i+1}) \bmod 3$.) It is straightforward to verify that the stationary distribution of the random walk is not uniform. Moreover, the probability of being at vertex v is exponentially small. (If p_i denotes the probability of being at the top ith vertex (or the bottom one, the graph is symmetric), then $p_2 = 2p_1/3$, $p_n = 2p_{n-1}/3$, and for $1 < i < n$, $p_i = p_{i-1}/2$.) Hence, the expected recurrence time of v is exponential in n. This implies that a random walk of polynomial length starting from u that uses the above defined probability distribution visits v with exponentially small probability.

Thus, the standard randomized log-space algorithm for s–t-connectivity, that performs a polynomial-length random walk on a graph, will produce incorrect answers using the biased distribution on labels.

The following theorem asserts that this is not the case if we perform the random walk using relative labeling as in an exploration sequence.

Theorem 10. *Let $d \geqslant 2$ be an integer. Let $\mathscr{D}$ be a probability distribution on $\{0,\ldots,d-1\}$, such that there exist $0 \leqslant r < q < d$, for which G.C.D.$(q-r,d) = 1$, and r and q occur with positive probabilities under $\mathscr{D}$. Then there is a constant $c > 0$ such that for any integer $n > 1$, for every connected d-regular graph G on n vertices, and every labeling of G, the following holds.*

If $e \in E(G)$ and $v \in e$, then with probability at least $1/2$, a random walk of length cd^2n^3 starting at (e,v), where at every step of the walk the next relative label is chosen according to $\mathscr{D}$, visits all vertices of G.

Constant c depends on $\mathscr{D}$. In particular, we can set $c = 3/\min\{Pr_{x\in\mathscr{D}}[x = r], Pr_{x\in\mathscr{D}}[x = q]\}$.

To establish the theorem we need the following lemma which ensures that all vertices are reachable under $\mathscr{D}$.

Lemma 11. *Let $d \geqslant 2$, and let integers $0 \leqslant r < q < d$ satisfy that G.C.D.$(q-r,d) = 1$. Let G be any connected d-regular graph, and let l be its labeling. Then $\forall e, e' \in E(G)$, $\forall v \in e$, and $\forall v' \in e'$, there exists $t \in \{r,q\}^{\leqslant 2|E(G)|}$ such that $(e,v) \to^t (e',v')$.*

Proof. Let r, q, and G be given. Fix a pair (e,v). We want to show that any other pair (e',v') is reachable by a sequence consisting of only r's and q's. (Note, if it is reachable then it is also reachable by such an exploration sequence of length at most $2|E(G)|$.) For $u \in V(G)$, define the in-degree and out-degree of u as follows:

$$d_+(u) = |\{\{u,w\} \in E(G);\ \exists t \in \{r,q\}^*;\ (e,v) \to^t (\{u,w\},u)\}|$$

and

$$d_-(u) = |\{\{u,w\} \in E(G);\ \exists t \in \{r,q\}^*;\ (e,v) \to^t (\{u,w\},w)\}|.$$

We claim that for any $u \in V(G)$, $d_+(u) \leqslant d_-(u)$. Moreover, $d_+(u) < d_-(u)$, unless $d_+(u) = d_-(u) = d$. We prove this claim. Fix $u \in V(G)$, and define the sets of incoming and outgoing edge labels:

$$L_+(u) = \{l(\{u,w\},u); \{u,w\} \in E(G);\ \exists t \in \{r,q\}^*;\ (e,v) \to^t (\{u,w\},u)\}$$

and

$$L_-(u) = \{l(\{u,w\},u); \{u,w\} \in E(G);\ \exists t \in \{r,q\}^*;\ (e,v) \to^t (\{u,w\},w)\}.$$

Observe, $d_+(u) = |L_+(u)|$ and $d_-(u) = |L_-(u)|$. Clearly, $\forall x \in L_+(u),\ x + r \in L_-(u)$, where the arithmetic is done modulo d. Hence, $|L_+(u)| \leqslant |L_-(u)|$, which implies that $d_+(u) \leqslant d_-(u)$. For the second part of the claim: $|L_+(u)| = |L_-(u)| \Rightarrow \forall x \in L_+(u),\ \exists y \in L_+(u)$ such that $x + q = y + r \Rightarrow \forall x \in L_+(u),\ \exists y \in L_+(u)$ such that $y = x + q - r \Rightarrow \exists a \in \{0, \dots, d-1\}$ such that $\{a + (q-r)b;\ b \in \{0, \dots, d-1\}\} \subseteq L_+(u)$, where all the arithmetic is done modulo d. If $G.C.D.(q-r, d) = 1$, then $\{a + (q-r)b;\ b \in \{0, \dots, d-1\}\} = \{0, \dots, d-1\}$. Hence, $d_+(u) = d_-(u)$ implies that $L_+(u) = \{0, \dots, d-1\}$ and $d_+(u) = d_-(u) = d$, which establishes the claim.

Clearly, $\sum_{u \in V} d_+(u) = \sum_{u \in V} d_-(u)$. Hence, if there were a vertex w such that $d_+(w) < d$, then there would have to be a vertex w' such that $d_+(w') > d_-(w')$, which would be a contradiction. Thus, for all $v' \in V(G)$, $d_+(v') = d$, which means that all pairs (e', v') are reachable from (e, v). $\square$

Note, the key property in the previous lemma is that for every $u \in V(G)$, $G.C.D.(q - r, deg\, u) = 1$. Hence, if $q = r + 1$ then the lemma holds for any graph, not only for d-regular graphs.

Proof of Theorem 10. Let $\mathscr{D}, n, G, e, v$ be given. Instead of considering a traversal of G by a random simple exploration sequence, we consider a traversal of G by a random general exploration sequence, i.e., we allow also Λ to be chosen. We define a probability distribution $\mathscr{D}_\varepsilon$ on $\{\Lambda, 0, \dots, d-1\}$ as follows: $Pr_{x \in \mathscr{D}_\varepsilon}[x = \Lambda] = \varepsilon$, and for every $i \in \{0, \dots, d-1\}$, $Pr_{x \in \mathscr{D}_\varepsilon}[x = i] = (1-\varepsilon) Pr_{x \in \mathscr{D}}[x = i]$, where we set $\varepsilon = 1/3$.

Let us consider a random walk on G during which distribution $\mathscr{D}_\varepsilon$ is used for choosing a label at every step of the walk. This random walk corresponds to a Markov chain with the set of states $S = \{(e, v);\ e \in E(G),\ v \in e\}$. (We assume general knowledge of Markov chains. For background see [IM].) The previous lemma implies that this Markov chain is irreducible, and the non-zero probability of Λ implies that the chain is aperiodic. Hence, it has a stationary distribution. It is easy to verify that the uniform distribution is stationary. Thus for every $a \in S$, the expected recurrence time $T_{a,a} = |S| = 2|E(G)|$.

The previous lemma implies that there is $t \in \{r, q\}^*$ of length m that is less than or equal to $2|E(G)| \cdot |V(G)|$, such that t visits all vertices in G starting at (e, v). Let $t = t_1 t_2 \cdots t_m$, where $t_1, t_2, \dots, t_m \in \{q, r\}$. Let $a_0, a_1, \dots, a_m \in S$ be such that $(e, v) = a_0 \to^{t_1} a_1 \to^{t_2} a_2 \cdots \to^{t_m} a_m$. We want to upper bound the expected time T of a walk that visits $a_1, a_2, \dots, a_m$ in this order starting from a_0 and ending at a_m. (Some a_i's may be the same.) We use the following lemma.

Lemma 12. *Let a and b be states of a finite state, irreducible, aperiodic Markov chain M. Let $p_{a,b} > 0$ be the transition probability of going from a to b. Then the expected time $T_{a,b}$ of the first visit to b starting from a satisfies: $T_{a,b} \leqslant T_{a,a}/p_{a,b}$, where $T_{a,a}$ is the recurrence time of a.*

We give a proof of this lemma in the appendix.

Let $T_{a_i,a_{i+1}}$ denote the expected time of the first visit to state a_{i+1} starting from a_i. Then by linearity of expectation $T = \sum_{i=0}^{m-1} T_{a_i,a_{i+1}}$. By Lemma 12, for all $i \in \{0, \ldots, m-1\}$, $T_{a_i,a_{i+1}} \leqslant T_{a_i,a_i}/p_{a_i,a_{i+1}}$, where $T_{a_i,a_i} = 2|E(G)|$. Let $\gamma_\varepsilon = \min\{Pr_{x\in\mathscr{D}_\varepsilon}[x = r], Pr_{x\in\mathscr{D}_\varepsilon}[x = q]\}$. Clearly, $p_{a_i,a_{i+1}} \geqslant \gamma_\varepsilon$. Hence, $T \leqslant 2\gamma_\varepsilon^{-1}|E(G)|m \leqslant \gamma_\varepsilon^{-1}d^2n^3$.

Set $c = 2\gamma_\varepsilon^{-1}$. By the Markov inequality, with probability at least 1/2, a random walk starting at (e, v) of length $l = cd^2n^3$ visits all vertices of G. This is a random walk according to $\mathscr{D}_\varepsilon$. Clearly, with at least the same probability, a random walk of length l according to $\mathscr{D}$ starting at (e, v) visits all vertices of G. (By removing all empty moves the event that all vertices of G are visited happens only earlier. We can also argue formally: $1/2 \leqslant Pr_{t\in\mathscr{D}_\varepsilon^l}[t$ visits all vertices of $G] = \sum_{i=0}^{l}\binom{l}{i}\varepsilon^i(1-\varepsilon)^{l-i}Pr_{t\in\mathscr{D}^{l-i}}[t$ visits all vertices of $G]$. The claim follows by noting that for any $j \leqslant l$, $Pr_{t\in\mathscr{D}^j}[t$ visits all vertices of $G] \leqslant Pr_{t\in\mathscr{D}^l}[t$ visits all vertices of $G]$.) □

By the same argument as in [AKLLR], Theorem 10 implies that for $l = 2cd^3n^4 \log n$, an exploration sequence that is chosen randomly from $\{0, \ldots, d-1\}^l$ according to distribution $\mathscr{D}^l$ is a UXS for d-regular graphs, with high probability: a random walk of length $(cd^2n^3) \times (2dn \log n)$ according to distribution $\mathscr{D}$ fails to visit all vertices in a given labeled graph G starting from a given point with probability at most $2^{-2dn \log n}$. There are less than $n^{dn} = 2^{dn \log n}$ different d-regular graphs on n vertices including all possible labelings, starting edge and vertex. Thus, the probability that a sequence of length l chosen randomly according to $\mathscr{D}^l$ fails to visit all vertices in some graph is at most $2^{-dn \log n}$.

The next theorem is an extension of Theorem 10 to all graphs. The proof of this theorem is similar to the proof of Theorem 10; hence the proof is omitted. (This theorem could be further generalized for r and $r + 1$ to have different probabilities.)

Theorem 13. *Let $r \geqslant 0$ be an integer. Then there is a constant $c > 0$ such that for any integer $n > 1$, for every connected graph G on n vertices, and every labeling of G, the following holds.*

If $e \in E(G)$ and $v \in e$, then with probability at least 1/2, a random walk of length cn^5 starting at (e, v), where at every step of the walk the next relative label is chosen uniformly at random from $\{r, r+1\}$, visits all vertices of G.

As a consequence we obtain the next claim that states that in search for UXS we can restrict our attention to exploration sequences consisting of only two labels, for instance, labels 1 and 2.

Corollary 14. *Let $r \geqslant 0$ and $n > 1$ be integers. With high probability, a sequence chosen uniformly at random from $\{r, r+1\}^{O(n^7 \log n)}$ is a universal exploration sequence for graphs on n vertices.*

4. Conclusion

We have shown that properly defined traversal sequences may have surprising computational power, in particular, they can backtrack. Can they compute something more? In particular, can we convert any s–t-connectivity algorithm into a universal exploration sequence, e.g., the one of [ATWZ] running in space $O(\log^{4/3} n)$? (Such a possibility could allow us to further optimize the existing connectivity algorithms.)

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Appendix

Lemma A.1. *Let a and b be states of a finite state, irreducible, aperiodic Markov chain M. Let* $p_{a,b} > 0$ *be the transition probability of going from a to b. Then the expected time* $T_{a,b}$ *of the first visit to b starting from a satisfies*: $T_{a,b} \leqslant T_{a,a}/p_{a,b}$, *where* $T_{a,a}$ *is the recurrence time of a.*

Proof. The expected time of the first visit to b starting from a is bounded from above by the expected time between crossings of the transition from a to b, as any crossing of the transition from a to b goes through a. Let S be the set of the states of M and for any $c, d \in S$, let $p_{c,d}$ denote the transition probability of going from c to d. Let M' be the Markov chain that corresponds to transitions between states of M. Thus, M' has states $S' = \{(c,d); c, d \in S \& p_{c,d} > 0\}$. For $(c,d), (e,f) \in S'$, the transition probability $p_{(c,d),(e,f)}$ of going from (c,d) to (e,f) is equal to $p_{e,f}$ if $d = e$ and 0 otherwise.

It is easy to verify that since M is aperiodic and irreducible, so is M'. Let π be the stationary distribution of M. For every $(c,d) \in S'$, define $\pi'(c,d) = \pi(c)p_{c,d}$. We claim that π' is the stationary distribution of M'. By definition, $\pi(d) = \sum_{c \in S} \pi(c)p_{c,d}$. For any $(c,d) \in S'$, consider:

$$\sum_{(e,f) \in S'} \pi'(e,f)p_{(e,f),(c,d)} = \sum_{e \in S} \pi'(e,c)p_{(e,c),(c,d)} = p_{c,d} \sum_{e \in S} \pi'(e,c)$$
$$= p_{c,d} \sum_{e \in S} \pi(e)p_{e,c} = p_{c,d}\pi(c) = \pi'(c,d).$$

Hence, π' is the stationary distribution of M'. Therefore, $T_{a,b} \leqslant 1/\pi'(a,b) = 1/\pi(a)p_{a,b} = T_{a,a}/p_{a,b}$. $\square$

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