Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Journal of Algebra 319 (2008) 2761–2771

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Conditions for generic initial ideals to be almost reverse lexicographic ☆

Young Hyun Cho [a], Jung Pil Park [b,*]

[a] *Department of Mathematical Sciences and Research Institute for Mathematics, Seoul National University, Seoul 151-747, South Korea*

[b] *National Institute for Mathematical Sciences (NIMS), Daejeon 305-340, South Korea*

Received 16 April 2007

Available online 5 February 2008

Communicated by Steven Dale Cutkosky

Abstract

Let I be a homogeneous Artinian ideal in a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k of characteristic 0. We study an equivalent condition for the generic initial ideal $\text{gin}(I)$ with respect to reverse lexicographic order to be almost reverse lexicographic. As a result, we show that the Moreno-Socias conjecture implies the Fröberg conjecture and the Pardue conjecture. And for the case $\operatorname{codim} I \leqslant 3$, we show that R/I has the strong Lefschetz property if and only if $\text{gin}(I)$ is almost reverse lexicographic. Finally we give a positive partial answer to the Moreno-Socias conjecture, and hence to the Pardue conjecture.

Keywords: Generic algebra; Generic initial ideal; Hilbert function; Hilbert series; Lefschetz properties; Strong Stanley property

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k. Throughout this paper, we assume that k is a field of characteristic 0, and use only the reverse lexicographic order as a multiplicative term order on the set of the monomials in R. A monomial ideal I in R is said to be *almost*

☆ The first author is partially supported by BK21, and the second author is supported by National Institute for Mathematical Sciences (NIMS).

* Corresponding author.
E-mail addresses: youngcho@math.snu.ac.kr (Y.H. Cho), jppark@nims.re.kr (J.P. Park).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.01.014

reverse lexicographic if I contains every monomial M which is bigger than a minimal generator of I having the same degree as M. One of the conjectures associated with the almost reverse lexicographic ideal is the Moreno-Socias conjecture [12].

Conjecture 1.1 *(Moreno-Socias). If I is a homogeneous ideal generated by generic forms in R, then the generic initial ideal* $\operatorname{gin}(I)$ *of I is almost reverse lexicographic.*

In this paper, we study the relation between Lefschetz properties of standard graded Artinian k-algebras R/I and the condition for $\operatorname{gin}(I)$ to be almost reverse lexicographic (Theorem 2.8). Let $A = R/I = \bigoplus_{i=0}^{t} A_i$ be a standard graded Artinian k-algebra with $t = \max\{i \mid A_i \neq 0\}$. Then we say that $A = R/I$ has the weak (resp. strong) Lefschetz property if there exists a linear form L such that the multiplication $\times L^i : (R/I)_d \to (R/I)_{d+i}$ is either injective or surjective for each d and $i = 1$ (resp. for all i). In that case the linear form L is called a weak (resp. strong) Lefschetz element on A. And $A = R/I$ is said to have the strong Stanley property if there exists a linear form $L \in R$ such that the multiplication map $\times L^{t-2i} : A_i \to A_{t-i}$ is bijective for each $i = 0, 1, \ldots, [t/2]$. Note that if such a linear form L exists, then it is a generic element in R_1, that is, the set of such linear forms in R_1 forms a Zariski open set of $\mathbb{P}^{n-1}$. Note that A has the strong Stanley property if and only if it has the strong Lefschetz property and its Hilbert function is symmetric. We will abuse the notation and refer to the strong Lefschetz or Stanley properties for I rather than for $A = R/I$.

Stanley [14] and Watanabe [15] independently showed that any monomial complete intersection Artinian ideal has the strong Stanley property. Using this, Watanabe showed that any generic complete intersection Artinian ideal has the strong Stanley property (see [15, Remark 3.9]). And there are some results on the strong Lefschetz property [8,10,11]. But the question whether a homogeneous complete intersection Artinian ideal I has the strong Lefschetz (or Stanley) property is still open for the case $\operatorname{codim} I \geqslant 3$. One of the longstanding conjectures on generic algebras is the Fröberg conjecture [6].

Conjecture 1.2 *(Fröberg). If I is a homogeneous ideal generated by generic forms $F_1, \ldots, F_r$ in R of degrees* $\deg F_i = d_i$*, then the Hilbert series $S_{R/I}(z)$ of R/I is given by*

$$S_{R/I}(z) = \left| \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1 - z)^n} \right|,$$

where for a power series $\sum a_i z^i$ with integer coefficients we let $|\sum a_i z^i| = \sum b_i z^i$ with $b_i = a_i$ if $a_j > 0$ for all $0 \leqslant j \leqslant i$, and $b_i = 0$ if $a_j \leqslant 0$ for some $j \leqslant i$.

Let I be a homogeneous Artinian ideal in R which has the strong Lefschetz property. For a degree d form $F \in R$ we have the following exact sequence

$$0 \to \big((I : F)/I\big)(-d) \to (R/I)(-d) \xrightarrow{\times F} R/I \to R/I + (F) \to 0.$$

If F is generic, then the Hilbert function of $R/I + (F)$ is given by

$$H\big(R/I + (F), t\big) = \max\big\{H(R/I, t) - H(R/I, t - d), 0\big\}.$$

Hence the Hilbert series of $R/I + (F)$ is given by

$$S_{R/I+(F)}(z) = \left|\left(1 - z^d\right) S_{R/I}(z)\right|.$$

Hence the study for the strong Lefschetz (or Stanley) property of standard graded k-algebras defined by generic forms is closely related with the study for the Fröberg conjecture. And, as shown by Pardue in [13], the Moreno-Socias conjecture implies the Fröberg conjecture: i.e., if the Moreno-Socias conjecture is true for any number r of generic forms, then the Fröberg conjecture is also true for any r. We will give another proof of this at Corollary 2.4.

Another conjecture on generic algebras is the Pardue conjecture [13], which is also closely related with the study for the Lefschetz properties.

Conjecture 1.3 *(Pardue). Let k be an infinite field. If I is an ideal in $R = k[x_1, \ldots, x_n]$ generated by generic forms $F_1, \ldots, F_n$, then x_{n-i} is a weak Lefschetz element on $R/\mathrm{gin}(I) + (x_n, \ldots, x_{n-i+1})$ for each $0 \leqslant i \leqslant n-1$.*

We will show that the Moreno-Socias conjecture implies the Pardue conjecture (see Corollary 2.5), and give a partial answer for the Pardue conjecture (see Corollary 2.13).

Recently, there are some achievements for the Moreno-Socias conjecture. For codimension 2 case, Moreno-Socias [12], Aguirre et al. [1] proved the Moreno-Socias conjecture is true. And Cimpoeas [4] showed that every complete intersection Artinian ideal I satisfying the strong Stanley property has the almost reverse lexicographic $\mathrm{gin}(I)$ for codimension 3 case.

In this paper, we give an equivalent condition for $\mathrm{gin}(I)$ to be almost reverse lexicographic in the view point of the minimal system of generators of $\mathrm{gin}(I)$ (Lemma 2.1 and Theorem 2.8). And we generalize the result of Cimpoeas: For any homogeneous Artinian ideal I of $S = k[x_1, x_2, x_3]$, the ideal I has the strong Lefschetz property if and only if $\mathrm{gin}(I)$ is almost reverse lexicographic (Proposition 2.7). Then we show that $\mathrm{gin}(I)$ is almost reverse lexicographic for a monomial complete intersection ideal $I = (x_1^{d_1}, \ldots, x_n^{d_n})$ in $R = k[x_1, \ldots, x_n]$ if $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for $i \geqslant 4$ (Lemma 2.10). At last we show that the Moreno-Socias conjecture is true if I is a complete intersection Artinian ideal of R which is generated by generic forms of degrees d_i satisfying the same condition (Corollary 2.12). As a recent work Harima and Wachi [9] show similar results with ours independently.

Suppose that I is a homogeneous Artinian ideal of $R = k[x_1, \ldots, x_n]$. In the paper [3], Cho et al. showed that the minimal system of generators of $\mathrm{gin}(I)$ is completely determined by the positive integer f_1 and functions $f_i : \mathbb{Z}_{\geqslant 0}^{i-1} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ defined as follows:

$$\begin{aligned} f_1 &= \min\left\{t \;\middle|\; x_1^t \in \mathrm{gin}(I)\right\} \quad \text{and} \\ f_i(\alpha_1, \ldots, \alpha_{i-1}) &= \min\left\{t \;\middle|\; x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^t \in \mathrm{gin}(I)\right\}, \end{aligned} \tag{1.1}$$

for each $2 \leqslant i \leqslant n$.

Proposition 1.4. *(See [3].) Let I be a homogeneous Artinian ideal in $R = k[x_1, \ldots, x_n]$. Suppose that $f_1, \ldots, f_n$ are defined for $\mathrm{gin}(I)$ as in (1.1). Then the minimal system of generators $\mathcal{G}(\mathrm{gin}(I))$ of $\mathrm{gin}(I)$ is*

$$\mathcal{G}\big(\mathrm{gin}(I)\big) = \{x_1^{f_1}\} \cup \left\{ x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{f_i(\alpha_1,\ldots,\alpha_{i-1})} \;\middle|\; \begin{array}{l} 2 \leqslant i \leqslant n, \\ 0 \leqslant \alpha_1 < f_1, \textit{ and} \\ 0 \leqslant \alpha_j < f_j(\alpha_1,\ldots,\alpha_{j-1}) \\ \textit{for each } 2 \leqslant j \leqslant i \end{array} \right\}.$$

For each $1 \leqslant i \leqslant n-1$, let the set J_i be defined as

$$J_i = \left\{ (\alpha_1,\ldots,\alpha_i) \;\middle|\; \begin{array}{l} 0 \leqslant \alpha_1 < f_1 \text{ and} \\ 0 \leqslant \alpha_j < f_j(\alpha_1,\ldots,\alpha_{j-1}) \\ \text{for each } 2 \leqslant j \leqslant i \end{array} \right\}. \tag{1.2}$$

We mainly use the sets J_i to prove the main theorem and its corollaries. Hence we need to investigate the sets J_i closely. In what follows, we use the following notations for simplicity. For $\alpha = (\alpha_1,\ldots,\alpha_i) \in \mathbb{Z}^i_{\geqslant 0}$, we denote $\sum_{j=0}^{i} \alpha_j$ by $|\alpha|$. And for $\beta = (\beta_1,\ldots,\beta_i) \in \mathbb{Z}^i_{\geqslant 0}$, we say that $\beta > \alpha$ if $x_1^{\beta_1} \cdots x_i^{\beta_i} > x_1^{\alpha_1} \cdots x_i^{\alpha_i}$.

Lemma 1.5. *Let I be a homogeneous Artinian ideal in $R = k[x_1,\ldots,x_n]$. For each $1 \leqslant i \leqslant n-1$, let J_i be the set defined for* $\mathrm{gin}(I)$ *as in* (1.2).

(1) *An element $(\alpha_1,\ldots,\alpha_i) \in \mathbb{Z}^i_{\geqslant 0}$ belongs to J_i if and only if the monomial $x_1^{\alpha_1} \cdots x_i^{\alpha_i}$ is not contained in* $\mathrm{gin}(I)$.
(2) *If $\alpha = (\alpha_1,\ldots,\alpha_i) \in J_i$, then the element $(0,\ldots,0,|\alpha|) \in \mathbb{Z}^i_{\geqslant 0}$ belongs to J_i. Furthermore, we have $f_{i+1}(\alpha_1,\ldots,\alpha_i) \leqslant f_{i+1}(0,\ldots,0,|\alpha|)$.*
(3) *For two elements $(0,\ldots,0,a)$, $(0,\ldots,0,b)$ of J_i, if $a \leqslant b$, then we have*

$$a + f_{i+1}(0,\ldots,0,a) \geqslant b + f_{i+1}(0,\ldots,0,b).$$

Proof. (1) The assertion follows easily from Proposition 1.4 and the definition of J_i.

(2) If $(0,\ldots,0,|\alpha|)$ is not an element of J_i, then the monomial $x_i^{|\alpha|}$ belongs to $\mathrm{gin}(I)$ as shown in (1). But this implies that the monomial $x_1^{\alpha_1} \cdots x_i^{\alpha_i}$ is also contained in $\mathrm{gin}(I)$, since $\mathrm{gin}(I)$ is strongly stable. This contradicts $(\alpha_1,\ldots,\alpha_i) \in J_i$. And the last assertion follows from strongly stableness of $\mathrm{gin}(I)$ and the definition of f_{i+1}.

(3) Set $\mu = f_{i+1}(0,\ldots,0,a)$ and $t = \min\{b-a, \mu\}$. By the definition of f_{i+1}, we have $x_i^a x_{i+1}^{\mu} \in \mathrm{gin}(I)$. And since $\mathrm{gin}(I)$ is strongly stable, this implies that $x_i^{a+t} x_{i+1}^{\mu-t} \in \mathrm{gin}(I)$. If $\mu \leqslant b-a$, then $x_i^{a+t} x_{i+1}^{\mu-t} = x_i^{a+\mu} \in \mathrm{gin}(I)$, and hence x_i^b is also contained in $\mathrm{gin}(I)$. But, this contradicts $(0,\ldots,0,b) \in J_i$. Hence $\mu > b-a$ and $x_i^b x_{i+1}^{\mu-t} = x_i^{a+t} x_{i+1}^{\mu-t} \in \mathrm{gin}(I)$. This shows that

$$f_{i+1}(0,\ldots,0,b) \leqslant \mu - t = f_{i+1}(0,\ldots,0,a) - (b-a). \qquad \square$$

In the paper [2], Ahn et al. gave the following tool detecting whether I has the strong Lefschetz (or Stanley) property, from the view point of the minimal system of generators of $\mathrm{gin}(I)$. This tool gives us the chance to prove the main theorems easily.

Proposition 1.6. *(See [2].) Let I be a homogeneous Artinian ideal in $R = k[x_1,\ldots,x_n]$ with $t = \max\{i \mid (R/I)_i \neq 0\}$.*

(1) *I has the strong Lefschetz property if and only if we have*

$$f_n(0,\dots,0,|\alpha|+1)+1 \leqslant f_n(\alpha),$$

for any $\alpha=(\alpha_1,\dots,\alpha_{n-1})\in J_{n-1}$.

(2) *I has the strong Stanley property if and only if we have*

$$f_n(\alpha)=t-2|\alpha|+1,$$

for any $\alpha=(\alpha_1,\dots,\alpha_{n-1})\in J_{n-1}$.

2. Main results

Let I be a homogeneous Artinian ideal in $R=k[x_1,\dots,x_n]$. Unless otherwise stated, we assume that the integer f_1, the functions f_i and the sets J_i are defined for $\operatorname{gin}(I)$ as in (1.1) and (1.2), respectively.

The following lemma gives an equivalent condition for $\operatorname{gin}(I)$ to be almost reverse lexicographic.

Lemma 2.1. *Let I be a homogeneous Artinian ideal in $R=k[x_1,\dots,x_n]$. Then* $\operatorname{gin}(I)$ *is almost reverse lexicographic if and only if for any $1\leqslant i\leqslant n-1$ the following conditions are satisfied:*

(1) *$f_{i+1}(0,\dots,0,|\alpha|+1)+1\leqslant f_{i+1}(\alpha)$ for any $\alpha=(\alpha_1,\dots,\alpha_i)\in J_i$, and*
(2) *$f_{i+1}(\beta)\leqslant f_{i+1}(\alpha)$ for any $\alpha=(\alpha_1,\dots,\alpha_i)$, $\beta=(\beta_1,\dots,\beta_i)\in J_i$ with $|\alpha|=|\beta|$ and $\alpha<\beta$.*

Proof. ($\Rightarrow$) Fix i. Note that $f_{i+1}(\alpha)>0$, otherwise $x_1^{\alpha_1}\cdots x_i^{\alpha_i}\in\operatorname{gin}(I)$ by the definition of f_{i+1}, and this contradicts to $\alpha\in J_i$ as shown in Lemma 1.5(1). Hence we have

$$x_i^{|\alpha|+1}x_{i+1}^{f_{i+1}(\alpha)-1} > x_1^{\alpha_1}\cdots x_i^{\alpha_i}x_{i+1}^{f_{i+1}(\alpha)}.$$

This shows that the monomial $x_i^{|\alpha|+1}x_{i+1}^{f_{i+1}(\alpha)-1}$ is contained in $\operatorname{gin}(I)$, since $\operatorname{gin}(I)$ is almost reverse lexicographic. So the assertion (1) follows by the definition of f_{i+1}. And the assertion (2) follows easily by the hypothesis and the definition of f_{i+1}.

($\Leftarrow$) Let $1\leqslant i\leqslant n-1$, and let $M=x_1^{\beta_1}\cdots x_{i-1}^{\beta_{i-1}}x_i^b$, $N=x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^a$ be monomials in R having the same degree. Suppose that $M>N$ and N is a minimal generator of $\operatorname{gin}(I)$. If we set $\alpha=(\alpha_1,\dots,\alpha_{i-1})$, $\beta=(\beta_1,\dots,\beta_{i-1})$, then we have $\alpha\in J_{i-1}$, $a=f_i(\alpha)$ and $b\leqslant a$. We have to show that M belongs to $\operatorname{gin}(I)$. We may assume $\beta\in J_{i-1}$ by Lemma 1.5(1). Hence we have to show that $f_i(\beta)\leqslant b$.

If $b=a$, then $|\alpha|=|\beta|$ and $x_1^{\beta_1}\cdots x_{i-1}^{\beta_{i-1}}>x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}$. By the condition (2), we have

$$f_i(\beta)\leqslant f_i(\alpha)=a=b.$$

If $b<a$, then $|\beta|>|\alpha|$. Hence we have

$$\begin{aligned}|\beta| + f_i(\beta) &\leqslant |\beta| + f_i\bigl(0, \ldots, 0, |\beta|\bigr)\\ &\leqslant |\alpha| + 1 + f_i\bigl(0, \ldots, 0, |\alpha| + 1\bigr)\\ &\leqslant |\alpha| + f_i(\alpha) = |\alpha| + a = |\beta| + b,\end{aligned}$$

where the first and second inequalities follow from Lemma 1.5(2) and (3), respectively, and the third one follows from the condition (1). This shows that $f_i(\beta) \leqslant b$. □

Corollary 2.2. *For every homogeneous Artinian ideal K in the polynomial ring $S = k[x_1, x_2]$, $\operatorname{gin}(K)$ is almost reverse lexicographic.*

Proof. If $\alpha \in J_1$, then $x_1^{\alpha} x_2^{f_2(\alpha)} \in \operatorname{gin}(K)$ and $f_2(\alpha) > 0$. Since $\operatorname{gin}(K)$ is strongly stable, we have $x_1^{\alpha+1} x_2^{f_2(\alpha)-1} \in \operatorname{gin}(K)$. Hence $f_2(\alpha + 1) \leqslant f_2(\alpha) - 1$. This shows that two conditions (1) and (2) in Lemma 2.1 are satisfied. □

Note that we must verify that the two conditions (1) and (2) in Lemma 2.1 are satisfied for every i from 1 to $n - 1$, in order to show that $\operatorname{gin}(I)$ is almost reverse lexicographic. Using Corollary 2.2, we can reduce the range of i to check, furthermore we can rephrase Lemma 2.1 as follows.

Proposition 2.3. *Let I be a homogeneous Artinian ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$. Then $\operatorname{gin}(I)$ is almost reverse lexicographic if and only if for each $2 \leqslant i \leqslant n - 1$ the following conditions are satisfied*:

(1) *There exist generic linear forms $L_1, \ldots, L_{n-i-1}$ in R such that the ring $R/I + (L_1, \ldots, L_{n-i-1})$ has the strong Lefschetz property.*
(2) *For any two elements $\alpha = (\alpha_1, \ldots, \alpha_i)$ and $\beta = (\beta_1, \ldots, \beta_i)$ of J_i with $|\alpha| = |\beta|$ and $\alpha < \beta$, we have $f_{i+1}(\beta) \leqslant f_{i+1}(\alpha)$.*

Proof. Let $1 \leqslant i \leqslant n - 1$. For generic linear forms $L_1, \ldots, L_{n-i-1}$ in R, if we set $\bar{I}$ to be the image of I in the ring $R/(L_1, \ldots, L_{n-i-1})$, then $\operatorname{gin}(\bar{I}) = \operatorname{gin}(I)_{x_n \to 0, \ldots, x_{i+2} \to 0}$ by Green (see Corollary 2.15 in the paper [7], and see also Wiebe [16]). If we denote f_j^I and J_j^I the functions and the sets defined for $\operatorname{gin}(I)$ as in (1.1) and (1.2), respectively, then this implies that $f_{j+1}^I = f_{j+1}^{\bar{I}}$ and $J_j^I = J_j^{\bar{I}}$ for any $1 \leqslant j \leqslant i$. Since

$$\frac{R/(L_1, \ldots, L_{n-i-1})}{\bar{I}} = R/I + (L_1, \ldots, L_{n-i-1}),$$

the first condition in Lemma 2.1 implies that $R/I + (L_1, \ldots, L_{n-i-1})$ has the strong Lefschetz property for each $1 \leqslant i \leqslant n-1$. But since every homogeneous Artinian ideal K of codimension 2 has almost reverse lexicographic $\operatorname{gin}(K)$, it is enough to check only for i from 2 to $n - 1$. □

As shown in Introduction, Proposition 2.3 implies

Corollary 2.4. *The Moreno-Socias conjecture implies the Fröberg conjecture, that is, if the Moreno-Socias conjecture is true for any number r of generic forms in a polynomial ring $R = k[x_1, \ldots, x_n]$, then the Fröberg conjecture is also true for any r.*

As shown in the proof of Proposition 2.3, if $\mathrm{gin}(I)$ is almost reverse lexicographic, then x_{n-i} is a Strong Lefschetz element on $R/\mathrm{gin}(I) + (x_n, \dots, x_{n-i+1})$ for each $0 \leqslant i \leqslant n-1$. Hence we have

Corollary 2.5. *The Moreno-Socias conjecture implies the Pardue conjecture.*

The following example shows the second condition in Proposition 2.3 cannot be omitted.

Example 2.6. Consider the following strongly stable monomial ideal

$$I = \begin{pmatrix} x^2, xy^2, y^4, y^3z, xyz^2, y^2z^2, xz^3, yz^3, z^4, \\ y^3w, xyzw, \underline{xz^2w^2, y^2zw^3}, yz^2w^3, z^3w^3, \\ xyw^4, y^2w^4, xzw^4, yzw^4, z^2w^4 \\ xw^5, yw^5, zw^5, w^6 \end{pmatrix} \subset S = k[x,y,z,w].$$

Note that both S/I and $S/I + (L)$ have the strong Lefschetz property by Proposition 1.6. Although xz^2w^2 is a minimal generator of I and $y^2zw^2 > xz^2w^2$, y^2zw^2 does not belong to I. Hence I is not almost reverse lexicographic.

But if we restrict our interests to the case that I is a homogeneous Artinian ideal in $S = k[x_1, x_2, x_3]$, then we can show that the second condition in Proposition 2.3 is superfluous, that is, $\mathrm{gin}(I)$ is almost reverse lexicographic if and only if S/I has the strong Lefschetz property.

Proposition 2.7. *Let I be a homogeneous Artinian ideal of $S = k[x_1, x_2, x_3]$. Then S/I has the strong Lefschetz property if and only if* $\mathrm{gin}(I)$ *is almost reverse lexicographic.*

Proof. ($\Rightarrow$) It suffices to show that the second condition in Proposition 2.3 is fulfilled. Let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in J_2$. If $|\alpha| = |\beta|$ and $\alpha < \beta$, then $\alpha_2 \geqslant \beta_2$ and $\beta_1 = \alpha_1 + (\alpha_2 - \beta_2)$. Since the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{f_3(\alpha)}$ is contained in $\mathrm{gin}(I)$, and since $\mathrm{gin}(I)$ is strongly stable, we have

$$x_1^{\beta_1} x_2^{\beta_2} x_3^{f_3(\alpha)} = x_1^{\alpha_1 + (\alpha_2 - \beta_2)} x_2^{\alpha_2 - (\alpha_2 - \beta_2)} x_3^{f_3(\alpha)} \in \mathrm{gin}(I).$$

This shows that $f_3(\beta) \leqslant f_3(\alpha)$ by the definition of f_3.

($\Leftarrow$) This follows from Proposition 2.3. □

Putting Propositions 2.3 and 2.7 together, we obtain the main result of this paper.

Theorem 2.8. *Let I be a homogeneous Artinian ideal in the polynomial ring $R = k[x_1, \dots, x_n]$. Then* $\mathrm{gin}(I)$ *is almost reverse lexicographic if and only if the following conditions are satisfied*:

(1) *For any $0 \leqslant i \leqslant n-3$, there exist generic linear forms $L_1, \dots, L_i$ in R such that the ring $R/I + (L_1, \dots, L_i)$ has the strong Lefschetz property.*
(2) *For each $3 \leqslant i \leqslant n-1$, if $\alpha = (\alpha_1, \dots, \alpha_i)$, $\beta = (\beta_1, \dots, \beta_i)$ are two elements of J_i with $|\alpha| = |\beta|$ and $\alpha < \beta$, then we have $f_{i+1}(\beta) \leqslant f_{i+1}(\alpha)$.*

In Example 2.6, we showed that the first condition in Proposition 2.3 is not enough to ensure that $\mathrm{gin}(I)$ is almost reverse lexicographic, if the number of variables of the ring R is greater than or equal to 4. But for any $0 \leqslant i \leqslant n-3$, if there exist generic linear forms $L_1, \ldots, L_i$ such that $I + (L_1, \ldots, L_i)$ has the strong Stanley property, then $\operatorname{gin}(I)$ is almost reverse lexicographic as shown in the following corollary.

Corollary 2.9. *Let I be a homogeneous Artinian ideal in $R = k[x_1, \ldots, x_n]$. For each $0 \leqslant i \leqslant n-3$, if there exist generic linear forms $L_1, \ldots, L_i$ in R such that $I + (L_1, \ldots, L_i)$ has the strong Stanley property, then* $\operatorname{gin}(I)$ *is almost reverse lexicographic.*

Proof. It is enough to show that the second condition in Theorem 2.8 is satisfied for each $3 \leqslant i \leqslant n-1$. Fix i and let $\alpha = (\alpha_1, \ldots, \alpha_i)$ and $\beta = (\beta_1, \ldots, \beta_i)$ be two elements of J_i with $|\alpha| = |\beta|$ and $\beta > \alpha$. By the assumption, there exist generic linear forms $L_1, \ldots, L_{n-i-1}$ in R such that $I + (L_1, \ldots, L_{n-i-1})$ has the strong Stanley property. From the reason described in the proof of Proposition 2.3, we have

$$f_{i+1}(\beta) = t - 2|\beta| + 1 = t - 2|\alpha| + 1 = f_{i+1}(\alpha),$$

where $t = \max\{j \mid (R/I + (L_1, \ldots, L_{n-i-1}))_j \neq 0\}$. Hence the assertion follows. □

As a result, we will show that the Moreno-Socias conjecture is true for the case that K is a complete intersection Artinian ideal generated by generic forms $F_1, \ldots, F_n$ in R of degrees d_i with $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for each $i \geqslant 4$. To do so, we will show first that if $I = (x_1^{d_1}, \ldots, x_n^{d_n})$ under the same condition on the d_i, then $\operatorname{gin}(I)$ is almost reverse lexicographic. We use the result of Stanley and Watanabe: every monomial complete intersection Artinian ideal has the strong Stanley property (see [14] or [15, Corollary 3.5]).

Lemma 2.10. *Let $I = (x_1^{d_1}, \ldots, x_n^{d_n})$. If $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for each $i \geqslant 4$, then* $\operatorname{gin}(I)$ *is almost reverse lexicographic.*

Proof. By Corollary 2.9, it suffices to show that for each $0 \leqslant i \leqslant n-3$, there exist generic linear forms $L_1, \ldots, L_i$ such that $R/I + (L_1, \ldots, L_i)$ has the strong Stanley property. Hence it is enough to show that $R/I + (L_1, \ldots, L_i)$ is isomorphic to $k[x_1, \ldots, x_{n-i}]/(x_1^{d_1}, \ldots, x_{n-i}^{d_{n-i}})$ for each $0 \leqslant i \leqslant n-3$. We will show this by induction on i. At first, for simplicity, we denote by $\mathcal{M}_j$ the set of monomials $\{x_1^{d_1}, \ldots, x_{n-j}^{d_{n-j}}\}$ for each $0 \leqslant j \leqslant n-3$.

For the case $i = 0$, the claim is true by the hypothesis. Assume that the claim is true for the case $i = s < n-3$. Then there exist generic linear forms $L_1, \ldots, L_s$ such that $R/I + (L_1, \ldots, L_s)$ is isomorphic to $S/(\mathcal{M}_s)$, where $S = k[x_1, \ldots, x_{n-s}]$. Choose a generic linear form $L_{s+1} \in R$ such that the ideal generated by $\mathcal{M}_{s+1} \cup \{L_1, \ldots, L_s, L_{s+1}\}$ is a complete intersection Artinian ideal in R. If we set L' to be the image of L_{s+1} in $S/(\mathcal{M}_s)$, then we have $R/I + (L_1, \ldots, L_{s+1}) = S/(\mathcal{M}_s) + (L')$. Let K be the ideal generated by $\mathcal{M}_{s-1} \cup \{L''\}$ in S, where L'' is a generic form in S such that the image of L'' in $S/(\mathcal{M}_s)$ is L'. Note that K is a complete intersection Artinian ideal in S, and that the Castelnuovo–Mumford regularity of the ideal K is

$$\operatorname{reg} K = 1 + \sum_{j=1}^{n-s-1} d_j - (n-s-1).$$

Since $s < n-3$, we have $d_{n-s} \geqslant \operatorname{reg} K$. This implies that $x_{n-s}^{d_{n-s}} = 0$ in S/K. Hence we have

$$\begin{aligned} R/I + (L_1, \ldots, L_s, L_{s+1}) &= S/(\mathcal{M}_s) + (L') = S/K + \left(x_{n-s}^{d_{n-s}}\right) = S/K \\ &= k[x_1, \ldots, x_{n-s-1}, x_{n-s}]/(\mathcal{M}_{s-1}) + (L'') \\ &= k[x_1, \ldots, x_{n-s-1}]/(\mathcal{M}_{s-1}), \end{aligned}$$

the last equation follows since L'' is generic. So we are done. □

We need to know another way to compute generic initial ideals. The following is introduced in Eisenbud [5]:

By a monomial of $\bigwedge^d R_i$ we mean an element of the form $N = n_1 \wedge \cdots \wedge n_d$, where the n_j are degree i monomials of R, and we denote the support of N by $\operatorname{supp}(N) = \{n_1, \ldots, n_d\}$. We define a term in $\bigwedge^d R_i$ to be a product $p \cdot N$, where $p \in k$ and N is a monomial in $\bigwedge^d R_i$. We will say that $p \cdot N = p \cdot n_1 \wedge \cdots \wedge n_d$ is a normal expression if the n_j are ordered so that $n_1 > \cdots > n_d$.

We order the monomials of $\bigwedge^d R_i$ by ordering their normal expressions lexicographically: if $N = n_1 \wedge \cdots \wedge n_d$ and $N' = n'_1 \wedge \cdots \wedge n'_d$ are normal expressions, then $N > N'$ if and only if $n_j > n'_j$ for the smallest j with $n_j \neq n'_j$. We extend the order to terms, and define the initial term of an element $F \in \bigwedge^d R_i$ to be the greatest term with respect to the given order.

If K is a homogeneous ideal of R, then there is a Zariski open set $U \subset \mathrm{GL}(n, k)$ such that for each $i \geqslant 0$, $\bigwedge^d \operatorname{gin}(K)_i$ is spanned by the greatest monomial of $\bigwedge^d R_i$ that appears in any $\bigwedge^d (gK_i)$ with $g \in U$, where $d = \dim_k K_i$, that is, if $N = n_1 \wedge \cdots \wedge n_d$ is the greatest monomial that appears in any $\bigwedge^d (gK_i)$ with $g \in U$, then $\operatorname{gin}(K)_i = kn_1 \oplus \cdots \oplus kn_d$ (see Theorem 15.18 in Eisenbud [5] for the details).

Hence, in order to compute $\operatorname{gin}(K)_i$, choose the basis $F_1, \ldots, F_d$ of gK_i with $g \in U$, where $d = \dim_k K$. Let $N_1, \ldots, N_s$ be the monomial basis of $\bigwedge^d R_i$ with $N_1 > \cdots > N_s$ with respect to the given order. If we write $F_1 \wedge \cdots \wedge F_d = \sum p_j \cdot N_j$ with $p_j \in k$, then the greatest monomial of $\bigwedge^d (gK_i)$ is the first monomial N_j such that $p_j \neq 0$. Note that each p_j is given as a polynomial expression in the coefficients of the F_i. Hence each p_j is given as a polynomial expression in the coefficients of a minimal system of generators of gK. And note that if $m_1, \ldots, m_t$ are the monomials of degree i contained in $\operatorname{gin}(K)_{\leqslant i-1}$, the ideal generated by the elements of $\operatorname{gin}(K)$ of degrees $\leqslant i - 1$, then $p_j = 0$ for any j with $\{m_1, \ldots, m_t\} \nsubseteq \operatorname{supp}(N_j)$. Hence if $M_1, \ldots, M_u$ is the monomial basis of $\bigwedge^d R_i$ such that $\{m_1, \ldots, m_t\} \subset \operatorname{supp}(M_j)$ and $M_1 > \cdots > M_u$, then we can write $F_1 \wedge \cdots \wedge F_d = \sum p_j \cdot M_j$, where each $p_j \in k$ is a polynomial expression in the coefficients of a minimal system of generators of gK. Note the greatest monomial of $\bigwedge^d (gK_i)$ is the first monomial M_j such that $p_j \neq 0$, and that the monomials in $\operatorname{supp}(M_j)$ except $m_1, \ldots, m_t$ are the minimal generators of $\operatorname{gin}(K)$ having degree i. The following theorem will give a positive answer to the Moreno-Socias conjecture in our case.

Theorem 2.11. *Let I be a homogeneous ideal in $R = k[x_1, \ldots, x_n]$ such that* $\operatorname{gin}(I)$ *is almost reverse lexicographic. Suppose that K is a homogeneous ideal generated by generic forms in R. If $H(R/I, d) = H(R/K, d)$ for all d, then* $\operatorname{gin}(K)$ *is also almost reverse lexicographic.*

Proof. We will prove this by showing that $\operatorname{gin}(I)_i = \operatorname{gin}(K)_i$ for every $i \geqslant 0$. It is clear for the case $i = 0$.

Suppose that $\operatorname{gin}(I)_j = \operatorname{gin}(K)_j$ for $j = 0, \ldots, i$. Let $d = \dim_k I_{i+1} = \dim_k K_{i+1}$. Without loss of generality, we may assume that we make a general choice of coordinates for K and I, i.e. $\operatorname{gin}(K) = \operatorname{in}(K)$ and $\operatorname{gin}(I) = \operatorname{in}(I)$. Let $m_1, \ldots, m_t$ be the monomials of degree $i + 1$ in

$\operatorname{gin}(I)_{\leqslant i} = \operatorname{gin}(K)_{\leqslant i}$. And let $n_{t+1}, \ldots, n_d$ be the minimal generators of $\operatorname{gin}(I)$ of degree $i+1$ with $n_{t+1} > \cdots > n_d$. Then we know that the monomial $N = m_1 \wedge \cdots \wedge m_t \wedge n_{t+1} \wedge \cdots \wedge n_d$ is the greatest monomial appearing in $\bigwedge^d I_{i+1}$. Let $M_1, \ldots, M_u$ be the monomial basis of $\bigwedge^d R_{i+1}$ such that $\{m_1, \ldots, m_t\} \subset \operatorname{supp}(M_j)$ and $M_1 > \cdots > M_u$. If $F_1, \ldots, F_d$ is a basis of K_{i+1}, then $F_1 \wedge \cdots \wedge F_d$ can be written as $F_1 \wedge \cdots \wedge F_d = \sum p_j \cdot M_j$ with $p_j \in k$.

Now we will show that $N = M_1$. Let $M_1 = m_1 \wedge \cdots \wedge m_t \wedge l_{t+1} \wedge \cdots \wedge l_d$ for some monomials $l_{t+1}, \ldots, l_d$ in R_{i+1} with $l_{t+1} > \cdots > l_d$. It suffices to show that $n_{t+1} \wedge \cdots \wedge n_d \geqslant l_{t+1} \wedge \cdots \wedge l_d$ as monomials in $\bigwedge^{d-t} R_{i+1}$ by the choice of M_1. But this is clear because $\operatorname{gin}(I)$ is almost reverse lexicographic and $n_{t+1}, \ldots, n_d$ are minimal generators of $\operatorname{gin}(I)$ having degrees $i+1$.

This shows that p_1, the coefficient of M_1, is a nonzero polynomial expression in the coefficients of a minimal system of generators of K. Since K is generated by generic forms and k is a field of characteristic 0, this implies that $p_1 \neq 0$. Hence $\operatorname{gin}(I)_{i+1}$ and $\operatorname{gin}(K)_{i+1}$ have the same monomial basis $\{m_1, \ldots, m_t, n_{t+1}, \ldots, n_d\}$.

Since k is a field of characteristic 0 and $\operatorname{gin}(K)$ is finitely generated, for each step i, we can have only finitely many closed subsets in a projective space which are defined by polynomials having the variables as the coefficients of a minimal system of generators of K. Since K is generated by generic forms, we have $\operatorname{gin}(I)_i = \operatorname{gin}(K)_i$ for all $i \geqslant 0$. □

Corollary 2.12. *Let K be a generic complete intersection Artinian ideal generated by generic forms $G_1, \ldots, G_n$ in $R = k[x_1, \ldots, x_n]$ with* $\deg G_i = d_i$*, if $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for each $i \geqslant 4$, then* $\operatorname{gin}(K)$ *is almost reverse lexicographic. In particular, every generic complete intersection Artinian ideal of codimension* 3 *has almost reverse lexicographic generic initial ideal, and hence has the strong Stanley property.*

Proof. Let $I = (x_1^{d_1}, \ldots, x_n^{d_n})$. Then $H(R/I, d) = H(R/K, d)$ for all d. By Theorem 2.11, we are done. □

This gives a positive partial answer for the Pardue conjecture, as shown in 2.5.

Corollary 2.13. *Let K be a generic complete intersection Artinian ideal generated by generic forms $F_1, \ldots, F_n$ in R of degrees d_i with $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for each $i \geqslant 4$. Then x_{n-i} is a strong Lefschetz elements on $R/\operatorname{gin}(I) + (x_n, \ldots, x_{n-i+1})$ for each* $0 \leqslant i \leqslant n-1$.

References

[1] E. Aguirre, A.S. Jarrah, R. Laubenbacher, J.A. Ortiz-Navarro, R. Torrez, Generic ideals and Moreno-Socias conjecture, arXiv:math.AC/0104047.
[2] J. Ahn, Y.H. Cho, J.P. Park, Generic initial ideals of Artinian ideals having Lefschetz properties or the strong Stanley property, J. Algebra 318 (2) (2007) 589–606.
[3] H.M. Cho, Y.H. Cho, J.P. Park, Generic initial ideals of arithmetically Cohen–Macaulay projective subschemes, Comm. Algebra 35 (7) (2007) 2281–2297.
[4] M. Cimpoeas, Generic initial ideal for complete intersections of embedding dimension three with strong Lefschetz property, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 50(98) (1) (2007) 33–66.
[5] D. Eisenbud, Commutative Algebra. With a View Toward Algebraic Geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995.
[6] R. Fröberg, An inequality for Hilbert series of graded algebras, Math. Scand. 56 (1985) 117–144.
[7] M. Green, Generic initial ideals, in: Six Lectures on Commutative Algebra, in: Progr. Math., vol. 166, Birkhäuser, 1988, pp. 119–186.
[8] T. Harima, J.C. Migliore, U. Nagel, J. Watanabe, The weak and strong Lefschetz properties for Artinian K-algebras, J. Algebra 262 (1) (2003) 99–126.

[9] T. Harima, A. Wachi, Generic initial ideals, graded Betti numbers and k-Lefschetz properties, arXiv:0707.2247v2.
[10] J. Herzog, D. Popescu, The strong Lefschetz property and simple extensions, arXiv:math.AC/0506537.
[11] J.C. Migliore, R.M. Miró-Roig, Ideals of general forms and the ubiquity of the Weak Lefschetz property, J. Pure Appl. Algebra 182 (1) (2003) 79–107.
[12] G. Moreno-Socias, Degrevlex Grobner bases of generic complete intersections, J. Pure Appl. Algebra 180 (3) (2003) 263–283.
[13] K. Pardue, Generic Polynomials, unpublished preprint, 1999.
[14] R.P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods 1 (2) (1980) 168–184.
[15] J. Watanabe, The Dilworth number of Artinian rings and finite posets with rank function, in: Commutative Algebra and Combinatorics, in: Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 303–312.
[16] A. Wiebe, The Lefschetz property for componentwise linear ideals and Gotzmann ideals, Comm. Algebra 32 (2004) 4601–4611.