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Cyclic sum formula for multiple L-values

Gaku Kawashima [a], Tatsushi Tanaka [b,*], Noriko Wakabayashi [c]

[a] Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan
[b] Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto-city 603-8555, Japan
[c] Faculty of Engineering, Kyushu Sangyo University, 3-1 Matsukadai 2-chome, Higashi-ku, Fukuoka, 813-8503, Japan

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ABSTRACT

The cyclic sum formula for multiple L-values, which can be viewed as a generalization of the cyclic sum formula for multiple zeta values proved by Hoffman and Ohno (or Ohno and Wakabayashi), is shown. An algebraic formulation of the cyclic sum formula is also presented.

1. Introduction/main theorem

The multiple L-values (MLV's for short) are studied for example in [1–3,8] as a generalization of Euler–Zagier's multiple zeta values. For $r \geqslant 1$, we denote the set of rth roots of unity by μ_r. The MLV is defined, for $l \geqslant 1$, $k_1, \dots, k_l \geqslant 1$ and $\lambda_1, \dots, \lambda_l \in \mu_r$ with $(k_1, \lambda_1) \neq (1, 1)$, by the convergent series

$$L(k_1, \dots, k_l; \lambda_1, \dots, \lambda_l) = \sum_{m_1 > \dots > m_l > 0} \frac{\lambda_1^{m_1 - m_2} \cdots \lambda_{l-1}^{m_{l-1} - m_l} \lambda_l^{m_l}}{m_1^{k_1} \cdots m_l^{k_l}},$$

* Corresponding author.
E-mail addresses: kawashima@math.nagoya-u.ac.jp (G. Kawashima), t.tanaka@cc.kyoto-su.ac.jp (T. Tanaka), noriko@ip.kyusan-u.ac.jp (N. Wakabayashi).

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which is known as an MLV of shuffle-type in [1]. Using same parameters, we define the 'non-strict' analogue of the MLV, which we call the multiple L-star value (MLSV for short) here, by the convergent series

$$L^\star(k_1,\dots,k_l;\lambda_1,\dots,\lambda_l)=\sum_{m_1\geqslant\cdots\geqslant m_l>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_{l-1}^{m_{l-1}-m_l}\lambda_l^{m_l}}{m_1^{k_1}\cdots m_l^{k_l}}.$$

We call the number $k_1+\cdots+k_l$ weight and the number l depth. When $r=1$ (and $k_1>1$), MLV and MLSV are reduced to the following multiple zeta value (MZV for short) and the multiple zeta-star value (MZSV for short), respectively:

$$\zeta(k_1,\dots,k_l)=\sum_{m_1>\cdots>m_l>0}\frac{1}{m_1^{k_1}\cdots m_l^{k_l}},\qquad \zeta^\star(k_1,\dots,k_l)=\sum_{m_1\geqslant\cdots\geqslant m_l>0}\frac{1}{m_1^{k_1}\cdots m_l^{k_l}}.$$

The $\mathbb{Q}$-vector space generated by MLSV's coincides with the $\mathbb{Q}$-vector space generated by MLV's because of the simple linear transformation between them. Deligne, Goncharov and Zhao proved in [2,8] that the upper bound of the dimension of the $\mathbb{Q}$-vector space generated by MLV's (or MLSV's) for fixed r and weight k is $d_k[r]$ which is given by

$$\sum_{k\geqslant 0}d_k[r]t^k=\begin{cases}\frac{1}{1-t^2-t^3}, & r=1,\\ \frac{1}{1-t-t^2}, & r=2,\\ \frac{1}{1-(\frac{\varphi(r)}{2}+v)t+(v-1)t^2}, & r\geqslant 3,\end{cases}$$

where v denotes the number of prime factors of r and φ Euler's totient function. This suggests that there are several relations among MLV's. To find and prove concrete relations among MLV's or MLSV's is one of the crucial problems to know the algebraic structure of the $\mathbb{Q}$-vector space generated by MLV's.

In this paper, we establish a family of $\mathbb{Q}$-linear relations for MLV's and MLSV's, namely the cyclic sum formula. This result can be regarded as a kind of generalization of the cyclic sum formula for MZV's proved in Hoffman and Ohno [4] or that for MZSV's proved in Ohno and Wakabayashi [5] (see [7] also), which is stated as follows.

Theorem 1.1. *Let $r\geqslant 1$. For $k_1,\dots,k_l\geqslant 1$, $\lambda_1,\dots,\lambda_l\in\mu_r$ with $k_q\neq 1$ for some $1\leqslant q\leqslant l$ or $\lambda_i\neq\lambda_j$ for some i,j $(i\neq j)$, we have*:

(i)
$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}L(k_j-i+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1},i;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_{j-1},1)$$
$$=\sum_{j=1}^{l}L(k_j+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1};\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_{j-1})$$
$$-\sum_{j=1}^{l}(1-\delta_{\lambda_j,1})L(1,k_{j+1},\dots,k_l,k_1,\dots,k_j;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_{j-1},1)$$
$$+\sum_{j=1}^{l}(1-\delta_{\lambda_j,1})L(1,k_{j+1},\dots,k_l,k_1,\dots,k_j;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_j).$$

(ii)
$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1} L^\star(k_j-i+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1},i;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_{j-1},1)$$
$$= (k_1+\cdots+k_l)\zeta(k_1+\cdots+k_l+1)$$
$$-\sum_{j=1}^{l}(1-\delta_{\lambda_j,1})L^\star(1,k_{j+1},\dots,k_l,k_1,\dots,k_j;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_{j-1},1)$$
$$+\sum_{j=1}^{l}(1-\delta_{\lambda_j,1})L^\star(1,k_{j+1},\dots,k_l,k_1,\dots,k_j;\lambda_j,\dots,\lambda_l,\lambda_1,\dots,\lambda_j),$$

where $\delta_{a,b}$ is Kronecker's delta.

2. Proof

We begin with the proof of the relation (i) of our main theorem. We fix $r>0$. For $k_1,\dots,k_l \geqslant 1$, $k_l \geqslant 0$, $\lambda_1,\dots,\lambda_{l+1}\in\mu_r$, we define two infinite series by

$$S(k_1,\dots,k_l;\lambda_1,\dots,\lambda_l) = \sum_{m_1>\cdots>m_l>0} \frac{\lambda_1^{m_1-m_2}\cdots\lambda_{l-1}^{m_{l-1}-m_l}\lambda_l^{m_l}}{(m_1-m_l)m_1^{k_1}\cdots m_l^{k_l}},$$
$$T(k_1,\dots,k_l;\lambda_1,\dots,\lambda_{l+1}) = \sum_{m_1>\cdots>m_{l+1}\geqslant 0} \frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1}\cdots m_l^{k_l}}.$$

Then the following lemma holds.

Lemma 2.1. *For $k_1,\dots,k_l\geqslant 1$, $k_{l+1}\geqslant 0$, $\lambda_1,\dots,\lambda_{l+1}\in\mu_r$, we have*:

(i) *The series $T(k_1,\dots,k_l;\lambda_1,\dots,\lambda_{l+1})$ converges if $k_1\geqslant 2$ or $\lambda_1\lambda_{l+1}\neq\lambda_l$.*
(ii) *If the series $T(k_1,\dots,k_l;\lambda_1,\dots,\lambda_{l+1})$ converges, we have the identity*

$$S(k_1,\dots,k_l,0;\lambda_1,\dots,\lambda_{l+1}) = T(k_1,\dots,k_l;\lambda_1,\dots,\lambda_{l+1}) - L(k_1+1,k_2,\dots,k_l;\lambda_1,\dots,\lambda_l).$$

(iii) *If the series $S(k_1+1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1})$ converges, we have the identity*

$$S(k_1+1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1}) = S(k_1,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_{l+1})$$
$$- L(k_1+1,k_2,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_{l+1}).$$

(iv) *If the series $S(1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1})$ converges, we have the identity*

$$S(1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1}) = T(k_2,\dots,k_l,k_{l+1}+1;\lambda_2,\dots,\lambda_l,\lambda_1\lambda_{l+1},\lambda_{l+1})$$
$$-(1-\delta_{\lambda_1,1})L(1,k_2,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_{l+1})$$
$$+(1-\delta_{\lambda_1,1})L(1,k_2,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_l,\lambda_1\lambda_{l+1}).$$

Proof. We firstly prove (i). If $k_1 \geqslant 2$, we have

$$\begin{aligned}
\left|T(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1})\right| &\leqslant T(2,\underbrace{1,\ldots,1}_{l-1};\underbrace{1,\ldots,1}_{l+1}) \\
&= \sum_{m_1>\cdots>m_{l+1}\geqslant 0} \frac{1}{(m_1-m_{l+1})m_1^2 m_2\cdots m_l} \\
&\leqslant \sum_{\substack{m_1>\cdots>m_l>0\\ m_1\geqslant j>0}} \frac{1}{j\cdot m_1^2 m_2\cdots m_l} \\
&= \sum_{q=1}^{l-1}\zeta(2,\underbrace{1,\ldots,1}_{q-1},2,\underbrace{1,\ldots,1}_{l-q-1}) + l\zeta(2,\underbrace{1,\ldots,1}_{l}) + \zeta(3,\underbrace{1,\ldots,1}_{l-1}),
\end{aligned}$$

and hence the series $T(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1})$ converges absolutely.

Next we show the sum

$$T_M(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1}) = \sum_{M\geqslant m_1>\cdots>m_{l+1}\geqslant 0} \frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1}\cdots m_l^{k_l}}$$

converges as $M\to\infty$ when $k_1=1$, $\lambda_1\lambda_{l+1}\neq\lambda_l$. Let

$$\Lambda(m_2,\ldots,m_{l+1}) = \lambda_1^{-m_2}\lambda_2^{m_2-m_3}\cdots\lambda_{l-1}^{m_{l-1}-m_l}\lambda_l^{m_l+m_{l+1}}\lambda_{l+1}^{-m_{l+1}}.$$

We write m_1-m_{l+1} by m_{l+1} to obtain

$$T_M(1,k_2,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1}) = \sum_{\substack{M\geqslant m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l}} \frac{(\frac{\lambda_1\lambda_{l+1}}{\lambda_l})^{m_1}\Lambda(m_2,\ldots,m_{l+1})}{m_1 m_2^{k_2}\cdots m_l^{k_l} m_{l+1}}.$$

We put

$$a(m) = \sum_{i=0}^{m}\left(\frac{\lambda_1\lambda_{l+1}}{\lambda_l}\right)^i.$$

By the assumption, the sum $a(m)$ is bounded and we have

$$\begin{aligned}
&T_M(1,k_2,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1}) \\
&\quad= \sum_{\substack{M\geqslant m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l}} \frac{(a(m_1)-a(m_1-1))\Lambda(m_2,\ldots,m_{l+1})}{m_1 m_2^{k_2}\cdots m_l^{k_l} m_{l+1}} \\
&\quad= \sum_{\substack{M\geqslant m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l}} \frac{a(m_1)\Lambda(m_2,\ldots,m_{l+1})}{m_1 m_2^{k_2}\cdots m_l^{k_l} m_{l+1}} - \sum_{\substack{M>m_1\geqslant m_2>\cdots>m_l>0\\ m_1+1\geqslant m_{l+1}>m_1+1-m_l}} \frac{a(m_1)\Lambda(m_2,\ldots,m_{l+1})}{(m_1+1) m_2^{k_2}\cdots m_l^{k_l} m_{l+1}}.
\end{aligned}$$

First sum is decomposed into

$$\sum_{\substack{M\geqslant m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l}} = \underset{\text{(a)}}{\sum_{\substack{M=m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l}}} + \underset{\text{(b)}}{\sum_{\substack{M>m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}=m_1-m_l+1}}} + \underset{\text{(c)}}{\sum_{\substack{M>m_1>\cdots>m_l>0\\ m_1\geqslant m_{l+1}>m_1-m_l+1}}}$$

and the second one into

$$\sum_{\substack{M>m_1\geqslant m_2>\cdots>m_l>0\\ m_1+1\geqslant m_{l+1}>m_1+1-m_l}} = \underset{\text{(d)}}{\sum_{\substack{M>m_1=m_2>\cdots>m_l>0\\ m_1+1\geqslant m_{l+1}>m_1+1-m_l}}} + \underset{\text{(e)}}{\sum_{\substack{M>m_1>m_2>\cdots>m_l>0\\ m_1+1=m_{l+1}>m_1+1-m_l}}} + \underset{\text{(f)}}{\sum_{\substack{M>m_1>m_2>\cdots>m_l>0\\ m_1+1>m_{l+1}>m_1+1-m_l}}}.$$

Each of (a), (b), (d) and (e) converges. For example, the convergence of (b) is shown as follows

$$\begin{aligned}
&\sum_{m_1>\cdots>m_l>0}\left|\frac{a(m_1)\Lambda(m_2,\ldots,m_l,m_1+1-m_l)}{m_1m_2^{k_2}\cdots m_l^{k_l}(m_1+1-m_l)}\right| \\
&\quad\leqslant \sum_{m_1>\cdots>m_l>0}\frac{K}{m_1m_2^{k_2}\cdots m_l^{k_l}(m_1+1-m_l)} \quad (\exists K>0) \\
&\quad\leqslant \sum_{m_1>\cdots>m_l>0}\frac{K}{m_1m_2^{k_2}\cdots m_{l-1}^{k_{l-1}}m_l(m_1+1-m_l)} \\
&\quad= \sum_{m_1>\cdots>m_l>0}\frac{K}{m_1(m_1+1)m_2^{k_2}\cdots m_{l-1}^{k_{l-1}}}\left(\frac{1}{m_l}+\frac{1}{m_1+1-m_l}\right) \\
&\quad= \left(\sum_{m_1>\cdots>m_l>0} + \sum_{\substack{m_1>\cdots>m_l>0\\ m_1+1>m_l>m_1+1-m_{l-1}}}\right)\frac{K}{m_1(m_1+1)m_2^{k_2}\cdots m_{l-1}^{k_{l-1}}m_l} < \infty.
\end{aligned}$$

We also find that (c) + (f) converges and hence the series $T(1,k_2,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1})$ converges if $\lambda_1\lambda_{l+1}\neq\lambda_l$.

The property (ii) is shown by an easy calculation as follows

$$\begin{aligned}
S(k_1,\ldots,k_l,0;\lambda_1,\ldots,\lambda_{l+1}) &= \sum_{m_1>\cdots>m_{l+1}>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1}\cdots m_l^{k_l}} \\
&= \left(\sum_{m_1>\cdots>m_{l+1}\geqslant 0} - \sum_{\substack{m_1>\cdots>m_{l+1}>0\\ (m_{l+1}=0)}}\right)\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1}\cdots m_l^{k_l}} \\
&= T(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1}) - L(k_1+1,k_2,\ldots,k_l;\lambda_1,\ldots,\lambda_l).
\end{aligned}$$

Next, using the partial fraction expansion

$$\frac{1}{(m_1-m_{l+1})m_1} = \frac{1}{m_{l+1}}\left(\frac{1}{m_1-m_{l+1}}-\frac{1}{m_1}\right), \tag{1}$$

we have

$$\begin{aligned}
&S(k_1+1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1})\\
&\quad=\sum_{m_1>\dots>m_{l+1}>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1+1}m_2^{k_2}\cdots m_{l+1}^{k_{l+1}}}\\
&\quad=\sum_{m_1>\dots>m_{l+1}>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{m_1^{k_1}\cdots m_l^{k_l}m_{l+1}^{k_{l+1}+1}}\left(\frac{1}{m_1-m_{l+1}}-\frac{1}{m_1}\right)\\
&\quad=S(k_1,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_{l+1})-L(k_1+1,k_2,\dots,k_l,k_{l+1}+1;\lambda_1,\dots,\lambda_{l+1}).
\end{aligned}$$

Thus the property (iii) holds.

Lastly, the proof of (iv) goes as follows. Using the partial fraction expansion (1), we have

$$\begin{aligned}
&S(1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1})\\
&\quad=\sum_{m_2>\dots>m_{l+1}>0}\frac{\lambda_1^{-m_2}\lambda_2^{m_2-m_3}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{m_2^{k_2}\cdots m_l^{k_l}m_{l+1}^{k_{l+1}+1}}\sum_{m_1=m_2+1}^{\infty}\lambda_1^{m_1}\left(\frac{1}{m_1-m_{l+1}}-\frac{1}{m_1}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{m_1=m_2+1}^{\infty}\lambda_1^{m_1}\left(\frac{1}{m_1-m_{l+1}}-\frac{1}{m_1}\right)&=\sum_{m_1=m_2+1}^{\infty}\left\{\left(\frac{\lambda_1^{m_1}}{m_1-m_{l+1}}-\frac{\lambda_1^{m_1+m_{l+1}}}{m_1}\right)+\left(\frac{\lambda_1^{m_1+m_{l+1}}}{m_1}-\frac{\lambda_1^{m_1}}{m_1}\right)\right\}\\
&=\sum_{j=0}^{m_{l+1}-1}\frac{\lambda_1^{m_2+m_{l+1}-j}}{m_2-j}+\left(\lambda_1^{m_{l+1}}-1\right)\sum_{m_1=m_2+1}^{\infty}\frac{\lambda_1^{m_1}}{m_1},
\end{aligned}$$

we have

$$\begin{aligned}
S(1,k_2,\dots,k_{l+1};\lambda_1,\dots,\lambda_{l+1})=&\sum_{m_2>\dots>m_{l+1}>j\geqslant 0}\frac{\lambda_2^{m_2-m_3}\cdots\lambda_l^{m_l-m_{l+1}}(\lambda_1\lambda_{l+1})^{m_{l+1}-j}\lambda_{l+1}^{j}}{(m_2-j)m_2^{k_2}\cdots m_l^{k_l}m_{l+1}^{k_{l+1}+1}}\\
&-(1-\delta_{\lambda_1,1})\sum_{m_1>\dots>m_{l+1}>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{m_1m_2^{k_2}\cdots m_l^{k_l}m_{l+1}^{k_{l+1}+1}}\\
&+(1-\delta_{\lambda_1,1})\sum_{m_1>\dots>m_{l+1}>0}\frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}(\lambda_1\lambda_{l+1})^{m_{l+1}}}{m_1m_2^{k_2}\cdots m_l^{k_l}m_{l+1}^{k_{l+1}+1}}.
\end{aligned}$$

Therefore we conclude (iv). □

We have the following corollary.

Corollary 2.2. *For any $k_1,\dots,k_l\geqslant 1$, $\lambda_1,\dots,\lambda_{l+1}\in\mu_r$ with some $k_q\neq 1$ or $\lambda_1\lambda_{l+1}\neq\lambda_l$, we have*

$$\begin{aligned}
&T(k_1,\dots,k_l;\lambda_1,\dots,\lambda_{l+1})-T(k_2,\dots,k_l,k_1;\lambda_2,\dots,\lambda_l,\lambda_1\lambda_{l+1},\lambda_{l+1})\\
&\quad=-\sum_{i=1}^{k_1-1}L(k_1-i+1,k_2,\dots,k_l,i;\lambda_1,\dots,\lambda_{l+1})+L(k_1+1,k_2,\dots,k_l;\lambda_1,\dots,\lambda_l)
\end{aligned}$$

$$- (1-\delta_{\lambda_1,1})L(1,k_2,\ldots,k_l,k_1;\lambda_1,\ldots,\lambda_{l+1})$$
$$+ (1-\delta_{\lambda_1,1})L(1,k_2,\ldots,k_l,k_1;\lambda_1,\ldots,\lambda_l,\lambda_1\lambda_{l+1}).$$

Proof. According to (iii) in Lemma 2.1, we have

$$S(k_1,\ldots,k_l,0;\lambda_1,\ldots,\lambda_{l+1}) = S(1,k_2,\ldots,k_l,k_1-1;\lambda_1,\ldots,\lambda_{l+1}) - \sum_{i=1}^{k_1-1} L(k_1-i+1,k_2,\ldots,k_l,i;\lambda_1,\ldots,\lambda_{l+1}).$$

Combining this identity with (ii) and (iv) in Lemma 2.1, we conclude the corollary. □

Proof of main theorem (i). We establish the relation (i) of our main theorem by adding up the identity of Corollary 2.2 for all cyclically equivalent indices when $\lambda_{l+1}=1$. (When $\lambda_{l+1}=1$, applying Corollary 2.2 repeatedly, we find that the series $T(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_l,1)$ converges if $k_q \neq 1$ for some $1 \leqslant q \leqslant l$ or $\lambda_i \neq \lambda_j$ for some i, j $(i \neq j)$, and hence Corollary 2.2 holds under this condition.) □

When $\lambda_{l+1} \neq 1$, we have the identity

$$\begin{aligned}
&\sum_{m=0}^{r-1}\sum_{j=1}^{l}\sum_{i=1}^{k_j-1} L\big(k_j-i+1,k_{j+1},\ldots,k_l,k_1,\ldots,k_{j-1},i;\\
&\qquad \lambda_j\lambda_{l+1}^m,\ldots,\lambda_l\lambda_{l+1}^m,\lambda_1\lambda_{l+1}^{m+1},\ldots,\lambda_{j-1}\lambda_{l+1}^{m+1},\lambda_{l+1}\big)\\
&\quad = \sum_{m=0}^{r-1}\sum_{j=1}^{l} L\big(k_j+1,k_{j+1},\ldots,k_l,k_1,\ldots,k_{j-1};\lambda_j\lambda_{l+1}^m,\ldots,\lambda_l\lambda_{l+1}^m,\lambda_1\lambda_{l+1}^{m+1},\ldots,\lambda_{j-1}\lambda_{l+1}^{m+1}\big)\\
&\qquad - \sum_{m=0}^{r-1}\sum_{j=1}^{l}(1-\delta_{\lambda_j\lambda_{l+1}^m,1}) L\big(1,k_{j+1},\ldots,k_l,k_1,\ldots,k_j;\\
&\qquad\quad \lambda_j\lambda_{l+1}^m,\ldots,\lambda_l\lambda_{l+1}^m,\lambda_1\lambda_{l+1}^{m+1},\ldots,\lambda_{j-1}\lambda_{l+1}^{m+1},\lambda_{l+1}\big)\\
&\qquad + \sum_{m=0}^{r-1}\sum_{j=1}^{l}(1-\delta_{\lambda_j\lambda_{l+1}^m,1}) L\big(1,k_{j+1},\ldots,k_l,k_1,\ldots,k_j;\lambda_j\lambda_{l+1}^m,\ldots,\lambda_l\lambda_{l+1}^m,\lambda_1\lambda_{l+1}^{m+1},\ldots,\lambda_j\lambda_{l+1}^{m+1}\big)
\end{aligned}$$

by Lemma 2.1 because $\lambda_{l+1}^r = 1$.

The relation (ii) of our theorem can be obtained by considering the series

$$C(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1}) = \sum_{\substack{m_1 \geqslant \cdots \geqslant m_{l+1} \geqslant 1\\ m_1 \neq m_{l+1}}} \frac{\lambda_1^{m_1-m_2}\cdots\lambda_l^{m_l-m_{l+1}}\lambda_{l+1}^{m_{l+1}}}{(m_1-m_{l+1})m_1^{k_1}\cdots m_l^{k_l}}$$

instead of the series $T(k_1,\ldots,k_l;\lambda_1,\ldots,\lambda_{l+1})$. The proof goes similar to the above (also see [5]). Instead of the proof, we introduce another way to prove the relation (ii) of the main theorem in the next section.

3. Algebraic formulation

Arakawa and Kaneko introduced the algebraic setup of MLV's by using the non-commutative algebra $\mathcal{A} = \mathcal{A}_r := \mathbb{Q}\langle x, y_\lambda \mid \lambda \in \mu_r\rangle$ in [1]. At first, we express the relation (i) of our main theorem, which has already shown in the previous section, by using the language of $\mathcal{A}$. Then we formulate and prove the relation (ii) of our main theorem.

We define two subalgebras of $\mathcal{A}$ by

$$\mathcal{A} \supset \mathcal{A}^1 := \mathbb{Q} + \sum_{\lambda\in\mu_r} \mathcal{A}y_\lambda \supset \mathcal{A}^0 := \mathbb{Q} + \sum_{\lambda\in\mu_r} x\mathcal{A}y_\lambda + \sum_{\substack{\lambda,\nu\in\mu_r\\ \nu\neq 1}} y_\nu\mathcal{A}y_\lambda.$$

Let $z_{k,\lambda} = x^{k-1}y_\lambda$ $(k \geqslant 1,\ \lambda \in \mu_r)$. We give the $\mathbb{Q}$-linear map $\mathcal{L} : \mathcal{A}^0 \to \mathbb{C}$ by $\mathcal{L}(1) = 1$ and

$$\mathcal{L}(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}) = L(k_1, \ldots, k_l; \lambda_1, \ldots, \lambda_l).$$

Under these notations, finding a linear relation among MLV's corresponds to find an element in $\ker \mathcal{L}$.

For $n \geqslant 1$, we denote the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes(n+1)}$ by $\diamond$, which is given by

$$a \diamond (w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \otimes \cdots \otimes w_n \otimes aw_{n+1},$$
$$(w_1 \otimes \cdots \otimes w_{n+1}) \diamond b = w_1 b \otimes w_2 \otimes \cdots \otimes w_{n+1}$$

for any $a, b, w_1, \ldots, w_{n+1} \in \mathcal{A}$. We notice that the action $\diamond$ gives a two-sided $\mathcal{A}$-module structure on $\mathcal{A}^{\otimes(n+1)}$. For $n \geqslant 1$, let $\mathcal{C}_n : \mathcal{A} \to \mathcal{A}^{\otimes(n+1)}$ be the $\mathbb{Q}$-linear map defined by

$$\mathcal{C}_n(x) = x \otimes (x + y_1)^{\otimes(n-1)} \otimes y_1,$$
$$\mathcal{C}_n(y_\lambda) = -x \otimes (x + y_1)^{\otimes(n-1)} \otimes y_\lambda + y_\lambda \otimes (x + y_1)^{\otimes(n-1)} \otimes y_1 - y_\lambda \otimes (x + y_1)^{\otimes(n-1)} \otimes y_\lambda$$

and

$$\mathcal{C}_n\big(ww'\big) = \mathcal{C}_n(w) \diamond w' + w \diamond \mathcal{C}_n\big(w'\big)$$

for any $w, w' \in \mathcal{A}$. Because of the properties

$$a \diamond (b \diamond w) = ab \diamond w, \qquad (w \diamond a) \diamond b = w \diamond ab$$

for any $a, b, w \in \mathcal{A}$, the map $\mathcal{C}_n$ is well defined. We define the map $M_n : \mathcal{A}^{\otimes(n+1)} \to \mathcal{A}$ by

$$M_n(w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \cdots w_{n+1}$$

and $\rho_n = M_n\mathcal{C}_n$. Let $\check{\mathcal{A}}^1$ be the subalgebra of $\mathcal{A}^1$ generated by words 1 and $z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}$ satisfying $k_q \neq 1$ for some $1 \leqslant q \leqslant l$ or $\lambda_i \neq \lambda_j$ for some i, j $(i \neq j)$. Then we have the following proposition.

Proposition 3.1. *For any $n \geqslant 1$, we have $\rho_n(\check{\mathcal{A}}^1) \subset \ker \mathcal{L}$.*

The relation (i) of our main theorem is just the case of $n = 1$ in Proposition 3.1, which is stated as follows.

Proposition 3.2. $\rho_1(\check{\mathcal{A}}^1) \subset \ker \mathcal{L}$.

Proposition 3.2 is shown first and then we show Proposition 3.1 by reducing it to Proposition 3.2.

Before we prove Proposition 3.2, we firstly show a lemma.

Lemma 3.3. *For cyclically equivalent words $w, w' \in \mathcal{A}$, we have $\rho_1(w) = \rho_1(w')$.*

Proof. Let $u_1, \dots, u_l \in \{x, y_\lambda \mid \lambda \in \mu_r\}$, $\varepsilon(u) = 1$ or λ according to $u = x$ or y_λ, and $\nu(u) = 0$ or 1 according to $u = x$ or y_λ. Since

$$\mathcal{C}_1(u) = (-1)^{\nu(u)} x \otimes (x + y_1)^{\otimes(n-1)} \otimes y_{\varepsilon(u)} + \nu(u)\big(y_{\varepsilon(u)} \otimes (x + y_1)^{\otimes(n-1)} \otimes y_1 - y_{\varepsilon(u)} \otimes (x + y_1)^{\otimes(n-1)} \otimes y_{\varepsilon(u)}\big)$$

for $u \in \{x, y_\lambda \mid \lambda \in \mu_r\}$, we have

$$\begin{aligned}\mathcal{C}_1(u_1 \cdots u_l) &= \sum_{j=1}^{l} u_1 \cdots u_{j-1} \diamond \mathcal{C}_1(u_j) \diamond u_{j+1} \cdots u_l \\ &= \sum_{j=1}^{l} \big\{ x u_{j+1} \cdots u_l \otimes u_1 \cdots u_{j-1} y_{\varepsilon(u_j)} + \nu(u_j)(y_{\varepsilon(u_j)} u_{j+1} \cdots u_l \otimes u_1 \cdots u_{j-1} y_1 \\ &\quad - y_{\varepsilon(u_j)} u_{j+1} \cdots u_l \otimes u_1 \cdots u_{j-1} y_{\varepsilon(u_j)})\big\},\end{aligned}$$

where we assume $u_1 \cdots u_{j-1} = 1$ if $j = 1$ and $u_{j+1} \cdots u_l = 1$ if $j = l$. Therefore we have

$$\begin{aligned}\rho_1(u_1 \cdots u_l) = \sum_{j=1}^{l} \big\{ & x u_{j+1} \cdots u_l u_1 \cdots u_{j-1} y_{\varepsilon(u_j)} + \nu(u_j)(y_{\varepsilon(u_j)} u_{j+1} \cdots u_l u_1 \cdots u_{j-1} y_1 \\ & - y_{\varepsilon(u_j)} u_{j+1} \cdots u_l u_1 \cdots u_{j-1} y_{\varepsilon(u_j)})\big\}.\end{aligned}$$

Since the right-hand side does not change under the cyclic permutation of $\{u_1, \dots, u_l\}$, we conclude the lemma. □

Now we prove Proposition 3.2.

Proof of Proposition 3.2. It is enough to show that

$$\rho_1(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}) \in \ker \mathcal{L},$$

where $k_1, \dots, k_l \geqslant 1$, $\lambda_1, \dots, \lambda_l \in \mu_r$ with $k_q \neq 1$ for some $1 \leqslant q \leqslant l$ or $\lambda_i \neq \lambda_j$ for some i, j $(i \neq j)$. By the definition of $\mathcal{C}_1$, we have

$$\begin{aligned}\mathcal{C}_1(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}) = & \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} \otimes z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{i,1} \\ & - \sum_{j=1}^{l} x \cdot z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} \otimes z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j}\end{aligned}$$

$$+\sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} \otimes z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{k_j,1}$$

$$-\sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} \otimes z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j}.$$

Applying M_1 to this, we obtain

$$\begin{aligned}\rho_1(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}) = {} & \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{i,1} \\ & - \sum_{j=1}^{l} z_{k_{j+1}+1,\lambda_{j+1}} z_{k_{j+2},\lambda_{j+2}} \cdots z_{k_l,\lambda_l} z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j} \\ & + \sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{k_j,1} \\ & - \sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j}.\end{aligned}$$

This is an element in $\ker \mathcal{L}$ if $k_q \neq 1$ for some $1 \leqslant q \leqslant l$ or $\lambda_i \neq \lambda_j$ for some i, j $(i \neq j)$ because of the relation (i) of the main theorem, which is proved in the previous section. □

Proof of Proposition 3.1. Actually, we have the identity

$$\rho_n\big((x + y_1)w\big) = \rho_{n+1}(w) \tag{2}$$

for $n \geqslant 1$, $w \in \mathcal{A}$ because of $\mathcal{C}_n(x + y_1) = 0$. Let $\check{\mathcal{A}}^1_{(d)}$ denote the degree-d homogenous part of $\check{\mathcal{A}}^1$. For $n, d \geqslant 1$, we let $\mathrm{CSF}^n_d[r] = \langle \rho_n(w) \mid w \in \check{\mathcal{A}}^1_{(d)} \rangle_{\mathbb{Q}}$. Because of the identity (2), we find the filtration structure

$$\mathrm{CSF}^{n+1}_d[r] \subset \mathrm{CSF}^n_{d+1}[r]$$

for any $n, d \geqslant 1$. Thanks to this filtration structure and Proposition 3.2, we conclude Proposition 3.1. □

We proceed to prove the relation (ii) of our main theorem. Let $\bar{\mathcal{L}} : \mathcal{A}^0 \to \mathbb{C}$ be the $\mathbb{Q}$-linear map defined by $\bar{\mathcal{L}}(1) = 1$ and

$$\bar{\mathcal{L}}(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l}) = L^{\star}(k_1, \ldots, k_l; \lambda_1, \ldots, \lambda_l).$$

We let γ be the automorphism on $\mathcal{A}$ characterized by $\gamma(x) = x$, $\gamma(y_\lambda) = x + y_\lambda$. It is known that the identity

$$\bar{\mathcal{L}} = \mathcal{L}d \tag{3}$$

holds, where d stands for the $\mathbb{Q}$-linear map given by $d(wy_\lambda) = \gamma(w)y_\lambda$ $(w \in \mathcal{A})$.

For $n \geqslant 1$, we define the $\mathbb{Q}$-linear map $\bar{\mathcal{C}}_n : \mathcal{A} \to \mathcal{A}^{\otimes(n+1)}$ by

$$\bar{\mathcal{C}}_n(x) = x \otimes y_1^{\otimes n},$$
$$\bar{\mathcal{C}}_n(y_\lambda) = -x \otimes y_1^{\otimes(n-1)} \otimes y_\lambda + (y_\lambda - x) \otimes y_1^{\otimes n} - (y_\lambda - x) \otimes y_1^{\otimes(n-1)} \otimes y_\lambda$$

and

$$\bar{\mathcal{C}}_n\bigl(ww'\bigr) = \bar{\mathcal{C}}_n(w) \diamond \gamma^{-1}\bigl(w'\bigr) + \gamma^{-1}(w) \diamond \bar{\mathcal{C}}_n\bigl(w'\bigr)$$

for any $w, w' \in \mathcal{A}$, where γ^{-1} is the inverse map of γ (i.e., $\gamma^{-1} \in \mathrm{Aut}(\mathcal{A})$ is characterized by $\gamma^{-1}(x) = x$, $\gamma^{-1}(y_\lambda) = y_\lambda - x$). Let $\bar{\rho}_n = M_n\bar{\mathcal{C}}_n$. Then we have the following proposition.

Proposition 3.4. *For any $n \geqslant 1$, we have $\bar{\rho}_n(\check{\mathcal{A}}^1) \subset \ker \bar{\mathcal{L}}$.*

To prove Proposition 3.4, we need the following lemma.

Lemma 3.5. *For any $n \geqslant 1$, we have $\rho_n = d\bar{\rho}_n$ on $\mathcal{A}$.*

Proof. It suffices to show

$$\rho_n(w) = d\bar{\rho}_n(w) \tag{4}$$

for $w = z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l} x^q$ $(q \geqslant 1,\ l \geqslant 0,\ z_{k,\lambda} = x^{k-1} y_\lambda)$. By the definition of $\mathcal{C}_n$ and $\bar{\mathcal{C}}_n$, we calculate

$$\begin{aligned}
\mathcal{C}_n(w) = {} & \sum_{p=1}^{q} x^{q-p+1} \otimes (x+y_1)^{\otimes(n-1)} \otimes z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l} z_{p,1} \\
& + \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} x^q \otimes (x+y_1)^{\otimes(n-1)} \otimes z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{i,1} \\
& - \sum_{j=1}^{l} x \cdot z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} x^q \otimes (x+y_1)^{\otimes(n-1)} \otimes z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j} \\
& + \sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} x^q \otimes (x+y_1)^{\otimes(n-1)} \otimes z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} z_{k_j,1} \\
& - \sum_{j=1}^{l} z_{1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} x^q \otimes (x+y_1)^{\otimes(n-1)} \otimes z_{k_1,\lambda_1} \cdots z_{k_j,\lambda_j},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{C}}_n(w) = {} & \sum_{p=1}^{q} \gamma^{-1}\bigl(x^{q-p+1}\bigr) \otimes y_1^{\otimes(n-1)} \otimes \gamma^{-1}\bigl(z_{k_1,\lambda_1} \cdots z_{k_l,\lambda_l} x^{p-1}\bigr) y_1 \\
& + \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \gamma^{-1}\bigl(z_{k_j-i+1,\lambda_j} z_{k_{j+1},\lambda_{j+1}} \cdots z_{k_l,\lambda_l} x^q\bigr) \otimes y_1^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1,\lambda_1} \cdots z_{k_{j-1},\lambda_{j-1}} x_{i-1}) y_1
\end{aligned}$$

$$-\sum_{j=1}^{l}\gamma^{-1}\bigl(x\cdot z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}x^q\bigr)\otimes y_1^{\otimes(n-1)}\otimes\gamma^{-1}\bigl(z_{k_1,\lambda_1}\cdots z_{k_{j-1},\lambda_{j-1}}x^{k_j-1}\bigr)y_{\lambda_j}$$
$$+\sum_{j=1}^{l}\gamma^{-1}\bigl(z_{1,\lambda_j}z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}x^q\bigr)\otimes y_1^{\otimes(n-1)}\otimes\gamma^{-1}\bigl(z_{k_1,\lambda_1}\cdots z_{k_{j-1},\lambda_{j-1}}x^{k_j-1}\bigr)y_1$$
$$-\sum_{j=1}^{l}\gamma^{-1}\bigl(z_{1,\lambda_j}z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}x^q\bigr)\otimes y_1^{\otimes(n-1)}\otimes\gamma^{-1}\bigl(z_{k_1,\lambda_1}\cdots z_{k_{j-1},\lambda_{j-1}}x^{k_j-1}\bigr)y_{\lambda_j}.$$

Then we find concrete expressions of $\rho_n(w)$ and $\bar{\rho}_n(w)$. According to the definition of the map d, we conclude (4). □

Let d^{-1} be the $\mathbb{Q}$-linear map defined by $d^{-1}(wy_\lambda)=\gamma^{-1}(w)y_\lambda$ $(w\in\mathcal{A})$. We easily find that $dd^{-1}=d^{-1}d=\mathrm{id}$.

Proof of Proposition 3.4. We have

$$\bar{\mathcal{L}}\bar{\rho}_n\bigl(\check{\mathcal{A}}^1\bigr)=\bar{\mathcal{L}}d^{-1}\rho_n\bigl(\check{\mathcal{A}}^1\bigr)=\mathcal{L}\rho_n\bigl(\check{\mathcal{A}}^1\bigr)=\{0\},$$

because of Lemma 3.5, the identity (3) and Proposition 3.1. Thus we conclude Proposition 3.4. □

Proof of main theorem (ii). We find

$$\begin{aligned}\bar{\mathcal{C}}_1\bigl(\gamma(z_{k_1,\lambda_1}\cdots z_{k_l,\lambda_l})\bigr)=&\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}z_{k_j-i+1,\lambda_j}z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}\otimes z_{k_1,\lambda_1}\cdots z_{k_{j-1},\lambda_{j-1}}z_{i,1}\\&-\sum_{j=1}^{l}z_{1,\lambda_j}z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}\otimes z_{k_1,\lambda_1}\cdots z_{k_{j-1},\lambda_{j-1}}z_{k_j,1}\\&+\sum_{j=1}^{l}z_{1,\lambda_j}z_{k_{j+1},\lambda_{j+1}}\cdots z_{k_l,\lambda_l}\otimes z_{k_1,\lambda_1}\cdots z_{k_j,\lambda_j}\end{aligned}$$

and

$$\bar{\mathcal{C}}_1\bigl(x^k\bigr)=\sum_{i=1}^{k}x^{k-i+1}\otimes z_{i,1}.$$

Thanks to Proposition 3.4, we obtain

$$\bar{\mathcal{L}}\bar{\rho}_1\bigl(\gamma(z_{k_1,\lambda_1}\cdots z_{k_l,\lambda_l})-x^{k_1+\cdots+k_l}\bigr)=0,$$

which is the relation (ii) of our main theorem. □

Remark 3.6. As in the case of ρ_n, we have the identity

$$\bar{\rho}_n\bigl((x+y_1)w\bigr)=\bar{\rho}_{n+1}(w) \tag{5}$$

for $n \geqslant 1$, $w \in \mathcal{A}$ because of $\bar{\mathcal{C}}_n(x+y_1)=0$. For $n, d \geqslant 1$, we let

$$\overline{\mathrm{CSF}}^n_d[r] = \langle \bar{\rho}_n(w) \mid w \in \check{\mathcal{A}}^1_{(d)} \rangle_{\mathbb{Q}}.$$

Because of the identity (5), we find the filtration structure

$$\overline{\mathrm{CSF}}^{n+1}_d[r] \subset \overline{\mathrm{CSF}}^n_{d+1}[r]$$

for any $n, d \geqslant 1$.

Remark 3.7. The dimension of $\mathrm{CSF}^n_d[1]$ is discussed in [7,6], which is given by

$$\dim_{\mathbb{Q}} \mathrm{CSF}^n_d[1] = -2 + \frac{1}{d+n-1} \sum_{m \mid d+n-1} \varphi\left(\frac{d+n-1}{m}\right)\left(2^m - L^{(n-1)}_m\right),$$

where the sequence $\{L^{(n)}_m\}$ is the Lucas n-step sequence defined by

$$L^{(n)}_m = \begin{cases} 0, & n=0, \\ 2^m - 1, & n>0,\ m=1,\dots,n, \\ L^{(n)}_{m-1} + \dots + L^{(n)}_{m-n}, & n>0,\ m \geqslant n+1. \end{cases}$$

Similar combinatorial observation to [6] shows that the dimension of $\mathrm{CSF}^n_d[r]$ is given by

$$\dim_{\mathbb{Q}} \mathrm{CSF}^n_d[r] = -r-1 + \frac{1}{d+n-1} \sum_{m \mid d+n-1} \varphi\left(\frac{d+n-1}{m}\right)\left\{(r+1)^m - L^{(n-1)}_m[r]\right\},$$

where the sequence $\{L^{(n)}_m[r]\}$ is defined by

$$L^{(n)}_m[r] = \begin{cases} 0, & n=0, \\ (r+1)^m - 1, & n>0,\ m=1,\dots,n, \\ r(L^{(n)}_{m-1}[r] + \dots + L^{(n)}_{m-n}[r]), & n>0,\ m \geqslant n+1. \end{cases}$$

Remark 3.8. Let us compare our cyclic sum formula with the derivation relation proved by Arakawa and Kaneko [1] in a simple case. The derivation relation for multiple L-values is stated algebraically as follows. Let $\partial_n : \mathcal{A} \to \mathcal{A}$ be the derivation (i.e. $\mathbb{Q}$-linear map with Leibniz rule) characterized by

$$\partial_n(x) = x(x+y_1)^{n-1}y_1,$$
$$\partial_n(y_\lambda) = -x(x+y_1)^{n-1}y_\lambda + y_\lambda(x+y_1)^{n-1}y_1 - y_\lambda(x+y_1)^{n-1}y_\lambda.$$

Then, the derivation relation states that

$$\partial_n\left(\mathcal{A}^0\right) \subset \ker \mathcal{L}$$

for any integer $n \geqslant 1$.

Denote the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes(n+1)}$ by $\cdot$, which is given by

$$a\cdot(w_1\otimes\cdots\otimes w_{n+1})=aw_1\otimes w_2\otimes\cdots\otimes w_{n+1},$$
$$(w_1\otimes\cdots\otimes w_{n+1})\cdot b=w_1\otimes\cdots\otimes w_n\otimes w_{n+1}b$$

for any $a,b,w_1,\dots,w_{n+1}\in\mathcal{A}$. Like the definition of the operator ρ_n, the operator ∂_n is also regarded as a composition $M_n\mathcal{D}_n$, where $\mathcal{D}_n:\mathcal{A}\to\mathcal{A}^{\otimes(n+1)}$ be the $\mathbb{Q}$-linear map defined by

$$\mathcal{D}_n(x)=x\otimes(x+y_1)^{\otimes(n-1)}\otimes y_1,$$
$$\mathcal{D}_n(y_\lambda)=-x\otimes(x+y_1)^{\otimes(n-1)}\otimes y_\lambda+y_\lambda\otimes(x+y_1)^{\otimes(n-1)}\otimes y_1-y_\lambda\otimes(x+y_1)^{\otimes(n-1)}\otimes y_\lambda$$

and

$$\mathcal{D}_n\big(ww'\big)=\mathcal{D}_n(w)\cdot w'+w\cdot\mathcal{D}_n\big(w'\big)$$

for any $w,w'\in\mathcal{A}$.

Applying the operator ∂_1 to $z_{k,\lambda}=x^{k-1}y_\lambda\in\mathcal{A}^0$, we obtain a kind of sum formula for MLV's of depth 2:

$$\sum_{i=1}^{k-1}L(i+1,k-i;1,\lambda)+L(k,1;\lambda,1)-L(k,1;\lambda,\lambda)=L(k+1;\lambda),$$

while applying the operator ρ_1 to $z_{k,\lambda}\in\mathcal{A}^0$, we obtain another sum formula for MLV's of depth 2:

$$\sum_{i=1}^{k-1}L(i+1,k-i;1,\lambda)+(1-\delta_{\lambda,1})\big(L(1,k;\lambda,1)-L(1,k;\lambda,\lambda)\big)=L(k+1;\lambda).$$

These two formulas are different in general. But they coincide and state the sum formula for MZV's of depth 2 if $r=1$ or $\lambda=1$.

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