Extremal Polynomials Associated with a System of Curves in the Complex Plane*

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1. Introduction

The nth Tchebycheff polynomial $T_n(z)$ associated with a compact set E in the plane is that monic nth degree polynomial whose maximum absolute value on E is an small as possible. If we write

$$M_n = \max | T_n(\zeta)|, \qquad \zeta \in E,$$

a classical result of Szegö (*15*) asserts that

$$\lim_{n\to\infty} M_n^{1/n} = C(E)$$

where $C(E)$ denotes the logarithmic capacity of E. [Earlier Fekete (*4*) had established the existence of the limit, identifying it with what he called the "transfinite diameter" of E.]

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All asymptotic formulas have refinements. In this case we consider the behavior of the ratio $M_n/C(E)^n$. Faber (*3*), using certain polynomials now named after him, was able to show that if E is a single sufficiently smooth (he actually assumed analytic) Jordan curve then this ratio tends to 1 as $n \to \infty$. In addition he determined the asymptotic behavior of the Tchebycheff polynomials on and outside the curve.

If E consists not of one curve but a union of two or more curves or curvilinear arcs, very little seems to be known. The case of E a union of two intervals on the real line was studied in detail by Akhiezer (*1*) [see also (*2*), pp. 280–289], not only for the Tchebycheff but also the weighted Tchebycheff polynomials (which we shall define presently), and he determined the asymptotic behavior of the ratio $M_n/C(E)^n$ in terms of elliptic functions. An interesting phenomenon was that the limit did not exist; in almost all cases the set of limit points of the ratio filled out an interval.

The purpose of this paper is to study this and related questions for the general case of E a finite union of (sufficiently smooth) Jordan curves, or curves and arcs.

Let ρ be a nonnegative function on E, and set

$$M_{n,\rho} = \min_{a_1,\ldots,a_n} \sup |(\zeta^n + a_1\zeta^{n-1} + \cdots + a_n)|\, \rho(\zeta), \qquad \zeta \in E, \tag{1.1}$$

so that $M_{n,1}$ is what we denoted above by M_n. The extremal polynomial, for which the minimum is assumed, is called the weighted Tchebycheff polynomial and we denote it by $T_{n,\rho}$. We shall determine, at least when E consists exclusively of closed curves, the asymptotic behavior of $M_{n,\rho}$ and $T_{n,\rho}$ (on and outside E) for large values of n, and again encounter the phenomenon of not having convergence. We shall see that the set of limit points of the ratio $M_{n,\rho}/C(E)^n$ is generally an interval, whose end points can be expressed explicitly in terms of the weight function ρ and the critical values of Green's function, with pole at infinity, for the exterior of E. In case E contains arcs there are difficulties which we have not been able to overcome, although in the very special case when E is a union of real intervals we can determine the limiting behavior of $M_{n,\rho}$ (but not $T_{n,\rho}$) and so recover the results of Akhiezer.

We also consider the weighted orthogonal polynomials. If

$$m_{n,\rho} = \min_{a_1,\ldots,a_n} \int_E |\, \zeta^n + a_1\zeta^{n-1} + \cdots + a_n\,|^2\, \rho(\zeta)\, |\, d\zeta\,|$$

(here $|d\zeta|$ denotes the differential of arc length on E) then the extremal polynomial $P_{n,\rho}$ is the nth monic orthogonal polynomial associated with the weight function ρ on E. For results on the limiting behavior of $m_{n,\rho}$ and $P_{n,\rho}$ when E is a single closed curve we refer the reader to the survey article of Suetin (*14*). We shall be concerned with the case of several curves and arcs. Surprisingly perhaps, the orthogonal polynomials turn out to be simpler than the Tchebycheff, as will be seen at several points in our study. In particular the case of arcs presents no insurmountable difficulty and even in that case we can determine the asymptotic behavior of $P_{n,\rho}$ as well as $m_{n,\rho}$. [It is possible that our results are new even for the case of one curvilinear arc. Of course for an interval everything is classical. See for example (*17*), Chapter XII.]

A related extremal quantity is $\lambda_{n,\rho}$, the smallest eigenvalue of the $(n+1) \times (n+1)$ moment matrix

$$(c_{i,j}) = \left(\int_E \zeta^i \bar{\zeta}^j \rho(\zeta) \mid d\zeta \mid\right), \qquad i, j = 0, \ldots, n.$$

These matrices are of considerable interest. If E is a subset of the unit circle $c_{i,j}$ depends only on the difference $i - j$ and we have a Toeplitz matrix, whose eigenvalues have been extensively studied in recent years. If E is a subset of the real line $c_{i,j}$ depends only on the sum $i + j$ and we have a Hankel matrix. It turns out that a thorough understanding of the behavior of the orthogonal polynomials associated with ρ allows one to get results on the asymptotic behavior of $\lambda_{n,\rho}$. See for example (*21*), where for a large class of orthogonal polynomials and corresponding moment matrices (much more general than we shall be considering here) the limits of $m_{n,\rho}^{1/n}$ and $\lambda_{n,\rho}^{1/n}$ were found, and (*20*), where from the known precise behavior of polynomials orthogonal on an interval an asymptotic formula for the corresponding $\lambda_{n,\rho}$ was found. Thus from our work on the orthogonal polynomials associated with a general E and ρ it will not be hard to determine the asymptotic behavior of the corresponding $\lambda_{n,\rho}$. It should be mentioned that for all cases we treat here $\lambda_{n,\rho}$ tends exponentionally to zero as $n \to \infty$. But in most cases treated in the past [for example in the original work (*8*) in this area] λ_n tends to zero like a negative power of n, and so they are not covered by our treatment. It would be interesting to see whether some refinement of the method presented here could be developed that would handle those cases as well.

There would not be much point to all this if the extension from one to several curves were automatic. But it is not, and the reason it is not can be expressed in one work: multiple-valuedness. The exterior of a single curve is simply connected and so admits only single valued analytic functions; but the exterior of a disconnected set is not simply connected and functions which arise naturally turn out to be multiple-valued. To make the problem clearer we shall present now an outline of the derivation of Faber's results. We shall see why things work smoothly in the case of one Jordan curve, and get an idea of what else has to be done in the more general case.

2. Tchebycheff Polynomials for a Single Curve

There are two parts. First we find a lower bound for all

$$\max |Q(\zeta)|, \qquad \zeta \in E$$

(Q monic of degree n) in case E is a Jordan curve, and then show that this lower bound is practically achieved if n is large. For the first, since we know very well how Q behaves at infinity (it is asymptotically z^n), what we want to do is compare the behavior at infinity with the maximum absolute value on E. The maximum modulus theorem immediately comes to mind (applied to the region Ω exterior to E) but of course since Q has a pole of order n at infinity rather than being regular there, we cannot apply the theorem directly.

We shall first divide Q by a function analytic in Ω except for a pole of order n at infinity, nowhere zero, and having boundary values on E of absolute value one. In fact if $\Phi(z)$ is the function that maps Ω conformally on the exterior of the unit circle, with infinity corresponding to infinity in such a manner that

$$\lim_{z\to\infty} \Phi(z)/z > 0,$$

then $\Phi(z)^n$ is such a function. Hence by the maximum modulus theorem

$$\begin{aligned} \max_{\zeta\in E} |Q(\zeta)| &= \max_{\zeta\in E} |Q(\zeta)\,\Phi(\zeta)^{-n}| \\ &\geqslant |Q\Phi^{-n}(\infty)| = (\lim_{z\to\infty} z/\Phi(z))^n \end{aligned} \tag{2.1}$$

The limit inside the parentheses is just the capacity $C(E)$. [As far as we are concerned this may be taken as the definition of $C(E)$.] Hence we have the inequality

$$\max_{\zeta\in E} |Q(\zeta)| \geqslant C(E)^n \tag{2.2}$$

valid for all Q.

In order to come close to achieving equality in (2.2) or equivalently in (2.1) it is clear that we must find a Q such that $Q\Phi^{-n}$ is close to constant throughout Ω. Thus we want to find a Q which is as close as possible to a constant times Φ^n in Ω. Now if Φ^n were a polynomial it would be equal to

$$\frac{1}{2\pi i}\int_C \frac{\Phi^n(\zeta)}{\zeta - z}\, d\zeta, \tag{2.3}$$

where C is a Jordan curve, described once in the positive direction, containing E and z in its interior. The fact is that in any case (2.3) is a polynomial, the Faber polynomial $F_n(z)$, and that $C(E)^n F_n(z)$ is a Q with the desired property.

To see first that F_n is a polynomial of degree n observe that F_n is an entire function, since it is clearly analytic inside C and C may be made arbitrarily large. Now we may write

$$F_n(z) = \Phi^n(z) + \frac{1}{2\pi i}\int_C \frac{\Phi^n(\zeta)}{\zeta - z}\, d\zeta \tag{2.4}$$

for z exterior to C. Since the integral tends to zero as $z \to \infty$ we see that

$$F_n(z) \sim C(E)^{-n} z^n \qquad z \to \infty$$

and so F_n is a polynomial of degree n with leading coefficient $C(E)^{-n}$.

If we can show that the integral in (2.4) is very small for $z \in \Omega$ we shall have accomplished our purpose [since then $Q = C(E)^n F_n$ will have the desired properties]. This is slightly technical and in general involves smoothness properties of E. If E is analytic though (the case treated by Faber), it is quite easy. For then Φ extends analytically a little into the interior of E where, since it has absolute value greater than one outside E, it has absolute value smaller than one. The curve C in (2.4) may thus be taken to be a level curve

$$E_r : |\Phi(z)| = r$$

with $r < 1$ and we immediately get the estimate

$$F_n(z) = \Phi^n(z) + O(r^{-n}) \qquad z \in \bar{\Omega}$$

where the constant in the "O" depends only on the length of E_r and the distance from E_r to E. Hence $Q = C(E)^n F_n$ is a monic polynomial of degree n satisfying

$$\max |Q(\zeta)| \leqslant C(E)^n (1 + O(r^n)).$$

This inequality for our special Q, together with the general inequality (2.2), gives

$$M_n = C(E)^n (1 + O(r^n)) \tag{2.5}$$

in the analytic case. This of course is considerably stronger than the asserted

$$M_n \sim C(E)^n. \tag{2.6}$$

Now let us see how the asymptotic behavior of $T_n(z)$ may be determined from that of the constants M_n. The functions

$$C(E)^{-n} \Phi(z)^{-n} T_n(z)$$

are all analytic in Ω, one at infinity, and have maximum absolute value on E, and so in Ω, tending to 1 as $n \to \infty$. Every sequence of n's has a subsequence for which the corresponding functions converge uniformly on each closed subset of Ω. The limit function is 1 at infinity and has maximum absolute value at most 1 in Ω; the limit function is therefore the constant 1. It follows that

$$T_n(z) \sim C(E)^n \Phi(z)^n \tag{2.7}$$

uniformly on each closed subset of Ω.

To show that this holds uniformly on $\bar{\Omega}$ itself is considerably more subtle. The asymptotic formula (2.6) itself is not enough to deduce this, but something sharper such as (2.5) is needed. We shall present here not Faber's argument, which depended too much for our purposes on the fact that E was a single curve, but a different one which we shall be able to use later. It depends on an analogue for general curves of a theorem of S. Bernstein for the circle which, since we shall be referring to it later, we state formally. First a little notation. A function of a real variable belongs to class $C^{\alpha+}$ (α a non-negative integer) if its α'th derivative satisfies a Lipschitz condition with some positive exponent. A curve belongs to class $C^{\alpha+}$ if it is rectifiable and its coordinates are $C^{\alpha+}$ functions of arc length.

Lemma 2.1. *Let E be a simple closed curve of class C^{1+}. Then there is a constant A such that for any* nth *degree polynomial Q we have*

$$\max_E |Q'(\zeta)| \leqslant An \max_E |Q(\zeta)|.$$

This is implicitly contained in the work of Szegö (*16*). We give the proof here since it is simple, elegant, and uses ideas already introduced. If we write M for the maximum absolute value of Q on E then just as before

$$|Q(z)\,\Phi(z)^{-n}| \leqslant M, \qquad z \in \Omega.$$

In particular on the level curve

$$E_{1+\delta} : |\Phi(z)| = 1 + \delta$$

we have

$$|Q(z)| \leqslant M(1+\delta)^n,$$

and this inequality persists throughout the interior of $E_{1+\delta}$. Hence by an elementary inequality

$$|Q'(\zeta)| \leqslant M(1+\delta)^n \{\mathrm{dist}(\zeta, E_{1+\delta})\}^{-1}$$

for all ζ interior to $E_{1+\delta}$, and so in particular for $\zeta \in E$. Now it is known (*19*) that for a curve of class C^{1+} the mapping function has derivative bounded away from zero. Thus the distance from E to $E_{1+\delta}$ is at least a constant times δ, so that

$$\max_E |Q'(\zeta)| \leqslant M(1+\delta)^n A\delta^{-1}.$$

The lemma follows upon setting $\delta = n^{-1}$.

Now let us see how a sharp asymptotic formula for M_n such as (2.5) implies that (2.7) holds throughout $\bar{\Omega}$. Let us write

$$U_n(z) = C(E)^{-n}\, T_n(z)\, \Phi(z)^{-n}$$

so that $U_n(\infty) = 1$ and

$$\max_E |U_n(\zeta)| = 1 + O(r^n). \tag{2.8}$$

Since $U_n(\infty) = 1$ we have

$$\frac{1}{2\pi i}\int_E U_n(\zeta)\,\Phi(\zeta)^{-1}\,\Phi'(\zeta)\,d\zeta = 1. \tag{2.9}$$

[Note that the function $\Phi'(z)$ extends continuously to E (*19*).]

Now on E,

$$\frac{1}{2\pi i}\,\Phi(\zeta)^{-1}\,\Phi'(\zeta)\,d\zeta = \psi(\zeta)\mid d\zeta\mid,$$

where ψ is positive and continous and satisfies

$$\int_E \psi(\zeta)\mid d\zeta\mid = 1. \tag{2.10}$$

In fact since $|\Phi(\zeta)| = 1$, the identity

$$\Phi^{-1}\Phi' d\zeta = d\log\Phi$$

makes it clear ψ is real and nonzero, and (2.10) implies it must be positive. Thus if we take real parts in (2.9) we get

$$\int_E \mathscr{R}U_n(\zeta)\,\psi(\zeta)\mid d\zeta\mid = 1.$$

Now the idea is this: $|U_n|$ is never much more than 1. If U_n differed from 1 by a substantial amount at some point $\zeta \in E$ then by the lemma it would differ from 1 by a substantial amount on a little arc around ζ, and so, by elementary geometry, $\mathscr{R}U_n$ would be substantially less than 1 on that arc. Since $\mathscr{R}U_n$ could only be a little more than 1 on the rest of E, this would give

$$\int_E \mathscr{R}U_n(\zeta)\,\psi(\zeta)\mid d\zeta\mid < \int_E \psi(\zeta)\mid d\zeta\mid,$$

contradicting the fact that both integrals equal 1.

More precisely now, it follows from the lemma and the boundedness of Φ', that

$$|U_n'(\zeta)| \leqslant An$$

for all $\zeta \in E$ and all n. Thus if

$$|1 - U_n(\zeta_0)| \geqslant \delta$$

then

$$|1 - U_n(\zeta)| \geqslant \delta/2, \qquad \widehat{\zeta\zeta_0} \leqslant 1/2An.$$

Now by elementary geometry this inequality and (2.8) imply

$$\mathscr{R}U_n(\zeta) \leqslant 1 - \delta^2/4, \qquad \widehat{\zeta\zeta_0} \leqslant 1/2An$$

for large n. We find, therefore,

$$\int_E \{\mathscr{R} U_n(\zeta) - 1\} \psi(\zeta) \mid d\zeta \mid \leqslant O(r^n) - \frac{\delta^2}{4} \int_{\Omega_{\zeta_0} \leqslant 1/2An} \psi(\zeta) \mid d\zeta \mid,$$

which will be negative for large n since $\psi(\zeta)$ is bounded below.

This contradiction shows that for any $\delta > 0$ we must have

$$| 1 - U_n(\zeta)| < \delta, \qquad \zeta \in E$$

for sufficiently large n. Thus U_n converges uniformly to 1 on E and so also on $\bar{\Omega}$. Notice that for this argument we did not use the full strength of (2.5). The estimate

$$M_n = C(E)^n (1 + o(n^{-1}))$$

is exactly what was needed.

3. Introduction (continued)

Now let us see what happens if E consists of several mutually exterior curves. The most obvious difficulty is that Ω, the exterior of E (i.e., the component of the complement of E which contains infinity), is not conformally equivalent to the exterior of a circle. There is, though, a completely standard analogue of $\Phi(z)$ in this case.

In fact let $g(z)$ denote Green's function for Ω with pole at infinity, so that g is harmonic in Ω, of the form

$$\log | z | + \text{harmonic function}$$

near $z = \infty$ and

$$\lim_{z \to \zeta} g(z) = 0, \qquad \zeta \in E.$$

Then if $\tilde{g}$ is the harmonic conjugate of g we have in the case of a single curve

$$\Phi(z) = \exp[g(z) + i\tilde{g}(z)]$$

(except possibly for a constant factor of absolute value one due to the presence of an arbitrary constant in $\tilde{g}$). In the more general case we simply define Φ by this formula. The problem is that Φ is multiple-valued.

However, $|\Phi|$ is single-valued and so one can still use the maximum modulus theorem. Thus, for any monic polynomial Q,

$$\max_E |Q(\zeta)| = \max_E |Q(\zeta)\,\Phi(\zeta)^{-n}| \geqslant \{\lim_{z\to\infty} |z|/|\Phi(z)|\}^n.$$

The limit in brackets is (or may be defined as) the capacity $C(E)$, i.e., we have

$$g(z) = \log|z| - \log C(E) + o(1) \qquad z \to \infty. \tag{3.1}$$

Consequently, we still have the lower bound

$$M_n \geqslant C(E)^n. \tag{3.2}$$

But now we cannot hope to come close to achieving equality. For we could achieve equality only if $Q\Phi^{-n}$ is constant, and we could come close to achieving equality only if $Q\Phi^{-n}$ is close to constant. But since Q is single-valued and Φ^n multiple-valued, this cannot be expected to occur.

It is also seen that the argument of §2, using Faber polynomials, breaks down. The Faber polynomials are still defined by (2.3), which makes perfectly good sense since Φ^n is single-valued near infinity. But now in the representation (2.4) we cannot bring the path of integration all the way down to E because of the multiple-valuedness of Φ^n, and this is what we had to do in order to get a good estimate for F_n.

All this leads us to expect that there is a larger lower bound than (3.2), due to the fact that the functions we are dealing with have a certain type of multiple valued behavior. This is exactly the case, and finding the correct lower bound, which is trivial in the case of a single curve, is the extra problem to be handled.

To set things up precisely, we must discuss our multiple-valued functions in some detail. We suppose that E consists of mutually exterior Jordan curves $E_1, \ldots, E_p$. Let F be a multiple-valued meromorphic function in Ω for which $|F(z)|$ is single-valued and assume F has only finitely many zero and poles, i.e., finitely many points where $|F| = 0$ or ∞. Any two function elements of F defined in the same disc differ by a constant factor of absolute value one, so F'/F is single-valued. It follows that for any curve (or, more generally, chain) C

$$\mathop{\Delta}_C \log F = \int_C \frac{F'(z)}{F(z)}\,dz \tag{3.3}$$

and so depends only on the homology class of C rather than its homotopy class as would be the case for an arbitrary multiple-valued function. Since $\log|F|$ is single-valued we need only concern ourselves with $\Delta \arg F$.

Let us therefore take Jordan curves E_k' $(k = 1,..., p)$ slightly exterior to the E_k (we may take the E_k' to be the level curves $|\Phi(z)| = 1 + \delta$ for appropriate small δ) and define

$$\gamma_k = \gamma_k(F) = \frac{1}{2\pi} \underset{E_k'}{\Delta} \arg F \qquad k = 1,... p,$$

where the curves are described counterclockwise. The γ_k are not independent but must satisfy

$$\sum_{k=1}^{p} \gamma_k \equiv 0 \qquad (\text{mod } 1). \tag{3.4}$$

In fact (3.3) shows that this sum is the number of poles of F minus the number of zeros.

Suppose we have two such functions, F_1 and F_2. Then F_1/F_2 is single-valued if and only if

$$\gamma_k(F_1) \equiv \gamma_k(F_2) \quad (\text{mod } 1) \qquad k = 1,..., p.$$

(Strictly speaking F_1/F_2 consists of many single-valued functions. Continuation of any function element of the quotient leads to a single-valued function.) We shall say simply that F_1 and F_2 belong to the same class and identify a class with a p-tuple

$$\Gamma = (\gamma_1, ..., \gamma_p)$$

of reals mod 1 satisfying (3.4). It is easy to see that every such p-tuple is the class of some function.

The family of all classes will be denoted by $\mathscr{T}_p{}^0$, the $\mathscr{T}_p$ denoting the p-torus and the superscript 0 referring to the fact that we are dealing with that subset of $\mathscr{T}_p$ for which (3.4) is satisfied. The class of F will be denoted by $\Gamma(F)$ and classes will be written additively, so that $\Gamma(F_1F_2) = \Gamma(F_1) + \Gamma(F_2)$.

Now let us see how to improve (3.2). Given a monic nth degree polynomial Q, the function $Q\Phi^{-n}$ is analytic in Ω, belongs to the class

$$\Gamma_n = -n\Gamma(\Phi),$$

has absolute value $C(E)^n$ at infinity, and absolute value at most

$$\max_E |Q|$$

in Ω.

We now ask the following question. Among all F analytic in Ω, of class Γ, and satisfying $|F(\infty)| = 1$, how small can

$$\sup_\Omega |F(z)| \tag{3.5}$$

be? In case Γ is the class of single valued functions the answer is of course 1, but for other Γ it is larger than one. If we denote by $\mu(\Gamma)$ the minimum of (3.5) (it is actually attained) we see that

$$\max_E |Q| \geqslant C(E)^n \mu(\Gamma_n).$$

This is the correct substitute for (3.2), and in fact we shall be able to prove under quite general conditions that

$$M_n \sim C(E)^n \mu(\Gamma_n).$$

The idea is to introduce as substitutes for Faber polynomials the polynomials

$$\frac{1}{2\pi i} \int_C \Phi^n(\zeta) F_n(\zeta) \frac{d\zeta}{\zeta - z} \tag{3.6}$$

where $F_n \in \Gamma_n$ is the function minimizing (3.5). The integrand is now single-valued, the contour can be brought down to E, and the estimation goes through.

This explains, by the way, the apparently erratic behavior of the sequence $M_n/C(E)^n$: As n varies, Γ_n wanders through $\mathscr{T}_p{}^0$, generally even being dense in $\mathscr{T}_p{}^0$, so that $\mu(\Gamma_n)$ cannot be expected to converge.

Let us now give a brief description of the organization of the paper. In the next section, which is also preparatory, harmonic measures and the functions of Green and Neumann associated with Ω are discussed. Section 5 is devoted to a study of the minimum problem just described (generalized by the introduction of a weight function). In particular the extremal quantities $\mu(\Gamma)$ are more or less determined. We say more or less since the exact determination depends on solving a certain system of equations involving harmonic measures. However, the range of $\mu(\Gamma)$ as Γ runs through all classes can be determined explicitly.

Most of the ideas of this section are not new. We mention for example the work of Garabedian (*5*), Rudin (*11*), and Tamarkin and Havison (*18*) concerning related extremal problems for single valued functions.

In §6 the L_2 analogue of all this, which will be needed for the study of orthogonal polynomials, will be taken up. The situation will in some ways be simpler. In particular, the determination of the analogue of the extremal quantity $\mu(\Gamma)$ will always be reducible to the Jacobi inversion problem for abelian integrals. The next section concerns the Szegö kernel function, which is intimately connected with the extremal problem of §6, and needed for the study of the eigenvalues of the moment matrices.

With this background prepared, in the next three sections we discuss respectively the Tchebycheff polynomials, orthogonal polynomials and moment matrices associated with systems E of Jordan curves and weight functions ρ on E. Asymptotic formulas are obtained for both the extremal quantities $M_{n,\rho}$, $m_{n,\rho}$, $\lambda_{n,\rho}$ and the corresponding extremal polynomials. The following three sections are concerned with the case when one or more of the components of E is a Jordan arc rather than a closed curve. As mentioned earlier, only partial results are obtained in the Tchebycheff case.

Finally in §14 we consider two cases where everything can be worked out somewhat explicitly, namely when E is a union of intervals on the real line and when E has two components.

4. The Basic Potential-Theoretic Functions

In this section we assume that E consists of Jordan curves of class C^{1+}; as before Ω is the component of the complement of E containing infinity.

Green's function $g(z, z_0)$ for Ω with pole at $z_0 \in \Omega$ is determined by the properties

(a) $g(z, z_0)$ is harmonic in $\Omega - \{z_0\}$,

(b) $g(z, z_0) - \log | z - z_0 |^{-1}$ is harmonic near z_0,

(c) $\lim_{z \to \zeta} g(z, z_0) = 0$ for all $\zeta \in E$.

In case $z_0 = \infty$ property (b) is replaced by

(b′) $g(z, \infty) - \log | z |$ is harmonic near ∞.

It is well known that $g(z, z_0) = g(z_0, z)$ (so that in particular $g(z, z_0)$ is a harmonic function of z_0 in $\Omega - \{z\}$) and that

$$h(z) = \frac{1}{2\pi} \int_E f(\zeta) \frac{\partial}{\partial n_\zeta} g(\zeta, z) \mid d\zeta \mid \tag{4.1}$$

solves the Dirichlet problem for Ω with boundary function f. This means that for any continuous function f defined on E, $h(z)$ is harmonic in Ω and has limit $f(\zeta)$ at each point $\zeta \in E$. The n_ζ appearing in the integral denotes the unit normal at $\zeta \in E$ directed into Ω. If we take for f the characteristic function of E_k we obtain

$$\omega_k(z) = \frac{1}{2\pi} \int_{E_k} \frac{\partial}{\partial n_\zeta} g(\zeta, z) \mid d\zeta \mid, \tag{4.2}$$

the harmonic measure of E_k. This has limit one on E_k and zero on the rest of E.

All these harmonic functions have multiple-valued harmonic conjugates, which we denote by putting a tilde over the letter denoting the function. A basic fact is

$$\underset{z \in E_k}{\Delta} \tilde{g}(z, z_0) = 2\pi\omega_k(z_0) \tag{4.3}$$

which is an easy consequence of (4.2):

$$\begin{aligned} 2\pi\omega_k(z_0) &= \int_{E_k} \frac{\partial}{\partial n_\zeta} g(\zeta, z_0) \mid d\zeta \mid \\ &= \int_{E_k} \frac{\partial}{\partial t_\zeta} \tilde{g}(\zeta, z_0) \mid d\zeta \mid = \underset{E_k}{\Delta} \tilde{g}(\zeta, z_0). \end{aligned}$$

Here t_ζ is the tangent vector obtained by rotating n_ζ counterclockwise through an angle of $\pi/2$.

It follows from this and the symmetry of Green's function that the function $h(z)$ given by (4.1) satisfies

$$\underset{E_k'}{\Delta} \tilde{h}(z) = \int_E f(\zeta) \frac{\partial \omega_k(\zeta)}{\partial n_\zeta} \mid d\zeta \mid = \int_E f(\zeta)\, d\tilde{\omega}_k(\zeta). \tag{4.4}$$

where E_k' is any curve lying in Ω and homologous to E_k.

We shall denote by $\Omega_k(z)$, $G(z, z_0)$ the multiple-valued functions obtained by adding to $\omega_k(z)$, $g(z, z_0)$ respectively their conjugates. Thus

$$\Omega_k(z) = \omega_k(z) + i\tilde{\omega}_k(z),$$

$$G(z, z_0) = g(z, z_0) + i\tilde{g}(z, z_0).$$

The derivatives of these multiple-valued functions are clearly single-valued.

The function

$$\Phi(z, z_0) = e^{G(z,z_0)}$$

is a multiple-valued function of the sort discussed in the preceding section. It has no zeros and a simple pole at z_0, and satisfies

$$|\Phi(\zeta, z_0)| = 1, \qquad \zeta \in E.$$

By (4.3) its class is

$$\Gamma(\Phi(z, z_0)) = (\omega_1(z_0),..., \omega_p(z_0)).$$

As things stand $\tilde{g}(z, z_0)$ is only determined up to an arbitrary additive constant and so $\Phi(z, z_0)$ up to a multiplicative constant of absolute value one. At times we shall determined these constants to suit our convenience.

In what follows we shall, in order to simplify notation, write $g(z)$ and $\Phi(z)$ for $g(z, \infty)$ and $\Phi(z, \infty)$, respectively.

Neumann's function is slightly less familiar and more complicated. Given $z_1, z_2 \in \Omega$ the Neumann function $N(z, z_1, z_2)$ has the following characteristic properties:

(a) $N(z, z_1, z_2) - \log \left| \dfrac{z - z_2}{z - z_1} \right|$ is harmonic in Ω and continuous on $\bar{\Omega}$,

(b) $\dfrac{\partial N}{\partial n_\zeta} = 0 \quad$ for $\zeta \in E$.

The condition (b) is equivalent to $\tilde{N}$ being constant on each E_k. It is clear that $N(z, z_1, z_2)$ is only determined up to an arbitrary additive constant. Let us see how to express N in terms of Green's functions and harmonic measures.

If we make a cut e between z_1 and z_2 then there is defined in $\bar{\Omega} - e$ a harmonic

$$a(z) = \arg \frac{z - z_2}{z - z_1}.$$

The function

$$\log \left| \frac{z - z_2}{z - z_1} \right| - \frac{1}{2\pi} \int_E a(\zeta) \frac{\partial}{\partial n_\zeta} \tilde{g}(z, \zeta) \, | d\zeta |$$

has harmonic conjugate which is constant on each E_k and it has the right behavior at z_1 and z_2. But it is not the Neumann function since, because of the $\tilde{g}(z, \zeta)$ appearing in the integral, it is not single-valued. We shall show though that for appropriate choice of constants λ_k $(k = 1,\ldots, p)$ we have

$$N(z, z_1, z_2) = \log \left| \frac{z - z_2}{z - z_1} \right| - \frac{1}{2\pi} \int_E a(\zeta) \frac{\partial}{\partial n_\zeta} \tilde{g}(z, \zeta) \mid d\zeta \mid + \sum_{k=1}^{p} \lambda_k \tilde{\omega}_k(z). \tag{4.5}$$

In fact the right side still has conjugate which is constant on each E_k and all we have to do is determine the λ_k so that the right side is single-valued.

If we write

$$P_{jk} = \underset{E_j}{\Delta} \tilde{\omega}_k(z)$$

and refer to (4.3) we see that the single-valuedness of the right side of (4.5) is equivalent to

$$\sum_{k=1}^{p} \lambda_k P_{jk} = \int_E a(\zeta) \frac{\partial \omega_j(\zeta)}{\partial n_\zeta} \mid d\zeta \mid, \qquad j = 1,\ldots, p.$$

This can be written in a more convenient form if we use Green's theorem. Applying that theorem to the harmonic functions $a(z)$ and $\omega_k(z)$ in $\Omega - e$ gives

$$\int_{E+e} a(\zeta) \frac{\partial \omega_j(\zeta)}{\partial n_\zeta} \mid d\zeta \mid = \int_{E+e} \omega_j(\zeta) \frac{\partial a(\zeta)}{\partial n_\zeta} \mid d\zeta \mid$$

where e is to be described twice, once in each direction. Now

$$\int_E \omega_j \frac{\partial a}{\partial n} \mid d\zeta \mid = \int_{E_j} \frac{\partial}{\partial n_\zeta} \arg \frac{\zeta - z_2}{\zeta - z_1} \mid d\zeta \mid$$

$$= \int_{E_j} \frac{\partial}{\partial t_\zeta} \log \left| \frac{\zeta - z_2}{\zeta - z_1} \right| \mid d\zeta \mid = \underset{E_j}{\Delta} \log \left| \frac{\zeta - z_2}{\zeta - z_1} \right|$$

and this is clearly zero. As for

$$\int_e \omega_j(\zeta) \frac{\partial a(\zeta)}{\partial n_\zeta} \mid d\zeta \mid,$$

this is also zero since the values of $a(\zeta)$ on opposite sides of e differ by 2π and so the two normal derivatives cancel. Finally, again using the fact that the two values of $a(\zeta)$ on e differ by 2π, we find

$$\int_e a(\zeta) \frac{\partial \omega_j(\zeta)}{\partial n_\zeta} \mid d\zeta \mid = \int_e a(\zeta) \frac{\partial \tilde{\omega}_j(\zeta)}{\partial t_j} \mid d\zeta \mid$$
$$= 2\pi\{\tilde{\omega}_j(z_1) - \tilde{\omega}_j(z_2)\}.$$

Consequently the condition on the λ_k may be written

$$\sum_{k=1}^{p} \lambda_k P_{jk} = 2\pi\{\tilde{\omega}_j(z_2) - \tilde{\omega}_j(z_1)\}, \qquad j = 1,\ldots, p. \tag{4.6}$$

Since N is determined only up to an arbitrary additive constant we may (and do) also require

$$\sum_{k=1}^{p} \lambda_k = 0.$$

To complete the construction we show that the system of equations

$$\sum_{k=1}^{p} \lambda_k P_{jk} = \alpha_j\,, \qquad j = 1,\ldots, p,$$
$$\sum_{k=1}^{p} \lambda_k = 0$$

always has a unique solution provided $\Sigma\alpha_j = 0$. (This is certainly the case in (4.6) since $\Sigma\omega_j(z) = 1$.) Since

$$\sum_{j=1}^{p} P_{jk} = \underset{E}{\Delta}\, \tilde{\omega}_k(z) = 0$$

(E being homologous to zero in Ω), all that we have to show is that the only solution to the homogeneous system

$$\sum_{k=1}^{p} \lambda_k P_{jk} = 0,$$
$$\sum_{k=1}^{p} \lambda_k = 0$$

is identically zero. If these equations are satisfied then the function

$$\omega(z) = \sum_{k=1}^{p} \lambda_k \omega_k(z)$$

satisfies

$$\underset{E_j}{\Delta}\,\tilde{\omega}(z) = 0, \qquad j = 1,\dots,p.$$

Consequently

$$\int_E \omega(\zeta)\frac{\partial\omega(\zeta)}{\partial n_\zeta}\,|\,d\zeta\,| = \sum_{k=1}^{p}\lambda_k\,\underset{E_k}{\Delta}\,\tilde{\omega}(\zeta) = 0.$$

But since the integral is just the Dirichlet integral of ω this implies that ω is constant. Since $\Sigma\lambda_k = 0$ the constant is zero and so all $\lambda_k = 0$.

Notice that even with the λ_k determined there is still an arbitrary additive constant involved in the construction of $N(z, z_1, z_2)$. This is because of the conjugate harmonic functions appearing on the right side of (4.5). How this constant is determined is largely immaterial, as long as it is done in such a way that $N(z, z_1, z_2)$ varies continuously with the parameters z_1, z_2. For example we can choose an arbitrary point in Ω and determine all conjugate functions on the right side of (4.5) by requiring them to vanish at this point.

Green's function becomes trivial when the parameter approaches E. This is not so with Neumann's function and in fact we must also concern ourselves with the case when z_1 or z_2 lies on E. In the second case, for example, the appropriate substitute for property (a) of $N(z, z_1, z_2)$ is

(a′) $N(z, z_1, z_2) - \log\dfrac{|\,z - z_2\,|^2}{|\,z - z_1\,|}$ is harmonic in Ω and continuous on $\bar{\Omega}$.

One reason for the change is that now

$$a(z) = \arg\frac{(z - z_2)^2}{z - z_1},$$

which is continuous on $\bar{\Omega} - e$, extends continuously to E. In fact with only this modification everything proceeds as before and the formulas (4.5) (with $|\,z - z_2\,|$ replaced by $|\,z - z_2\,|^2$) and (4.6) continue to hold.

We define the apparently multiple-valued function

$$\Psi(z, z_1, z_2) = \exp\{N(z, z_1, z_2) + i\tilde{N}(z, z_1, z_2)\}.$$

This function has a simple pole at z_1 and a simple zero at z_2 (a double pole if $z_1 \in E$ and a double zero if $z_2 \in E$). Each continuous determination of its argument is constant on each E_k. Therefore the class of $\Psi(z, z_1, z_2)$ is

$$\Gamma(\Psi(z, z_1, z_2)) = (0,\dots,0),$$

and so Ψ is actually single-valued (except for an arbitrary constant factor of absolute value one).

The reason we went through so much detail in the construction of Neumann's function is that continuity properties of the various functions, with their parameters varying, will be very important for us. In the case of Neumann's function these properties are most simply derived from the representation we have obtained.

Recall that we said a function of a real variable belongs to class $C^{\alpha+}$ (α a nonnegative integer) if it belongs to C^{α} and its α'th derivative satisfies a Lipschitz condition with some positive exponent. We shall say that a family of functions are uniformly of class $C^{\alpha+}$ if the derivatives up to order α are all uniformly bounded and the Lipschitz norms, with some fixed positive exponent, of the αth derivatives are uniformly bounded. This definition is extended in the obvious way to functions defined on rectifiable curves. The various continuity results we shall need are summarized in the following lemma. In addition the continuity up to the boundary of the various conjugate functions, which may have been a matter of some concern to the reader during the preceding discussion, are established.

Lemma 4.1. *Suppose* $E \in C^{\alpha+}$ (i.e., *each* $E_k \in C^{\alpha+}$) *with* $\alpha \geqslant 1$. *Then the following hold.*

(1) *$G'(z, z_0)$ [and so also $G(z, z_0)$ and $\Phi(z, z_0)$] extends continuously to E; $G'(\zeta, z_0)$ is of class $C^{\alpha-1+}$ on E and $G'(\zeta, z_0)$ bounded away from zero uniformly for z_0 in any closed subset of Ω.*

(2) *$\Psi(z, z_1, z_2)$ extends continuously to E; $\Psi(\zeta, z_1, z_2)$ is of class $C^{\alpha+}$ on E and*

$$\Psi(\zeta, z_1, z_2)\,|(\zeta - z_1)/(\zeta - z_2)|^2$$

bounded and bounded away from zero uniformly for $z_1, z_2 \in \bar{\Omega}$.

(3) *The functions*

$$\Phi(\zeta, z_0)^{\pm 1}\,\Psi(\zeta, z_1, z_0), \qquad \zeta \in E$$

are of class $C^{\alpha+}$ uniformly for $z_0 \in \Omega$ and z_1 in any closed subset of Ω.

(4) *If $f(\zeta) \in C^{\beta+}$ ($\beta \leqslant \alpha$) then the harmonic function $h(z)$ which solves the Dirichlet problem in Ω with boundary function f has conjugate $\tilde{h}(z)$ which extends continuously to E and belongs to $C^{\beta+}$ there. Moreover a family of f's uniformly of class $C^{\beta+}$ give rise to $\tilde{h}(\zeta)$ uniformly of class $C^{\beta+}$.*

Proof. We first take the case when a particular E_k, say E_1, is the unit circle and establish the continuity properties of the various functions for $\zeta \in E_1$.

Let $h(z, z_0)$ be the harmonic function in Ω whose boundary values on E are

$$\log \left| \frac{1 - \bar{z}_0 \zeta}{\zeta - z_0} \right|. \tag{4.7}$$

Then

$$g(z, z_0) = \log \left| \frac{1 - \bar{z}_0 z}{z - z_0} \right| - h(z, z_0).$$

Now for $| z_0 | \geqslant 1 + \delta$ the functions (4.7) are uniformly bounded on E, and so by the maximum principle for harmonic functions the $h(z, z_0)$ are uniformly bounded in Ω. Hence for some constant M we have

$$g(z, z_0) \leqslant M, \qquad | z | \leqslant 1 + \delta/2$$

as long as $| z_0 | \geqslant 1 + \delta$. Now by the reflection principle g can be extended harmonically to the annulus

$$(1 + \delta/2)^{-1} < | z | < 1 + \delta/2$$

by defining

$$g(z, z_0) = -g(\bar{z}^{-1}, z_0), \qquad (1 + \delta/2)^{-1} < | z | \leqslant 1,$$

and we still have $| g(z, z_0)| \leqslant M$. But then each partial derivative of $g(z, z_0)$ has a uniform bound on each closed subset of this annulus (as follows, for example, from the Poisson integral representation) and so the same is true for $\tilde{g}(z, z_0)$ which is, after all, just defined as a line integral involving the first partial derivatives of g. This establishes the first part of (1), at least on E_1, for an arbitrary α. For the second part, what we must show is that

$$\partial \tilde{g}(\zeta, z_0)/\partial t_\zeta$$

is bounded away from zero. This derivative is the same as

$$\partial g(\zeta, z_0)/\partial n_\zeta \, .$$

Now there is a constant $m > 0$ such that

$$g(z, z_0) \geqslant m, \qquad | z | = 1 + \delta/2,$$

for all z_0 satisfying $|z_0| \geqslant 1+\delta$; for otherwise we could find some such z, z_0 for which $g(z, z_0) = 0$. Thus, by the maximum principle again,

$$g(z, z_0) \geqslant \frac{m}{\log(1+\delta/2)} \log|z| \qquad 1 \leqslant |z| \leqslant 1+\delta/2$$

and so

$$\partial g(\zeta, z_0)/\partial n_\zeta \geqslant m/\log(1+\delta/2).$$

Next we consider the fourth statement of the lemma. If E consists only of the unit circle E_1 then the Plemelj formulas [(*9*), §17] show that

$$h(z) + i\tilde{h}(z) = \frac{1}{2\pi i}\int_{E_1} f(\zeta)\frac{z+\zeta}{z-\zeta}\frac{d\zeta}{\zeta}, \qquad z \in \Omega$$

and that $\tilde{h}$ extends continuously to E_1 and is given there by a principal value integral:

$$\tilde{h}(\zeta_1) = -\frac{1}{2\pi}\int f(\zeta)\frac{\zeta_1+\zeta}{\zeta_1-\zeta}\frac{d\zeta}{\zeta}.$$

The asserted behavior of $\tilde{h}$ is therefore a consequence of a well-known theorem of Privaloff [(*9*), §§19, 20].

If there are other E_k, let h_1 be the solution of the Dirichlet problem, with boundary function f, for the entire exterior of E_1. We know that $\tilde{h}_1$ has the right behavior on E_1. But $h - h_1$ is zero on E_1 and so extends by reflection to an annulus around E_1. On any closed subset of the annulus each partial derivative of $h - h_1$, and so also of $\tilde{h} - \tilde{h}_1$, is given by a bound which depend only on $\max |f(\zeta)|$. It follows therefore that $\tilde{h}$ is also of class $C^{\beta+}$ on E_1, and uniformly so if the f are uniformly of class $C^{\beta+}$.

To establish (2) and (3) we shall slightly modify (4.5) in our case of E_1 the unit circle. In fact if we set

$$a_1(z) = \arg\frac{(z-z_2)(1-\bar{z}_2 z)}{(z-z_1)(1-\bar{z}_1 z)}$$

then the formula

$$N(z, z_1, z_2) = \log\left|\frac{(z-z_2)(1-\bar{z}_2 z)}{(z-z_1)(1-\bar{z}_1 z)}\right| - \frac{1}{2\pi}\int_E a_1(\zeta)\frac{\partial}{\partial n_\zeta}\tilde{g}(z,\zeta)\,|d\zeta| + \sum_{k=1}^{p}\lambda_k\tilde{\omega}_k(z) \qquad (4.8)$$

holds, where the λ_k are the same as in (4.6). To see this, note that the right side here differs from the right side of (4.5) by

$$\log\left|\frac{1-\bar{z}_2 z}{1-\bar{z}_1 z}\right| - \frac{1}{2\pi}\int_E \arg\frac{1-\bar{z}_2\zeta}{1-\bar{z}_1\zeta}\frac{\partial}{\partial n_\zeta}\tilde{g}(z,\zeta)\mid d\zeta\mid. \tag{4.9}$$

But since $\arg(1-\bar{z}_2 z)(1-\bar{z}_1 z)^{-1}$ is harmonic in Ω we have

$$\arg\frac{1-\bar{z}_2 z}{1-\bar{z}_1 z} = \frac{1}{2\pi}\int_E \arg\frac{1-\bar{z}_2\zeta}{1-\bar{z}_1\zeta}\frac{\partial}{\partial n_\zeta} g(z,\zeta)\mid d\zeta\mid,$$

and if we take the conjugate harmonic function of both sides we see that (4.9) is zero.

The identity (4.8) holds even when z_1 or z_2 belongs to E, by the same argument. Now let us look at continuity behavior. From what has already been done we deduce that the integral in (4.8) is uniformly of class $C^{\alpha+}$ on E_1, and that each $\tilde{\omega}_k \in C^{\alpha+}$. By the uniform boundedness of the right side of (4.6) we see that the constants λ_k are also uniformly bounded. Thus if we apply part (4) once more we see that

$$N(\zeta, z_1, z_2) + i\tilde{N}(\zeta, z_1, z_2) - \log\frac{(\zeta - z_2)(1-\bar{z}_2\zeta)}{(\zeta - z_1)(1-\bar{z}_1\zeta)}$$

are uniformly of class $C^{\alpha+}$ on E_1. Taking exponentials,

$$\left\{\Psi(\zeta, z_1, z_2)\frac{(\zeta - z_1)(1-\bar{z}_1\zeta)}{(\zeta - z_2)(1-\bar{z}_2\zeta)}\right\}^{\pm 1} \in C^{\alpha+} \tag{4.10}$$

uniformly. Thus statement (2) of the lemma follows.

It follows from our discussions of $g(\zeta, z_0)$ above that

$$\Phi(\zeta, z_0)\frac{\zeta - z_0}{1-\bar{z}_0\zeta}$$

and its reciprocal are uniformly of class $C^{\alpha+}$ on E_1 for all $z_0 \in \Omega$. Combining this with (4.10) gives the third assertion on E_1.

It remains to reduce the general case to the case where E_1 is the unit circle, and this is done by conformal mapping. Thus, let $z = \varphi(s)$ map the exterior of the unit circle in the s-plane conformally on the exterior of E_1. Then the region Ω in the z-plane corresponds to a region in the s-plane, the curves E_k corresponding to curves E'_k, and now E_1' is the unit circle. Since (*19*) φ' extends continuously to a $C^{\alpha-1+}$ function on E_1' with

$$\varphi'(\sigma) > 0, \qquad \sigma \in E_1',$$

and since all the functions under discussion correspond under a conformal transformation, the assertions in the general case follow from the special case.

5. The Extremum Problem for Uniform Norms

We take up here the extremum problem relevant to the study of the Tchebycheff polynomials. We assume throughout that $E \in C^{1+}$ and that we have defined on E a function $\rho(\zeta)$. For the present ρ is assumed positive and of class C^{0+}, a condition that will be relaxed later.

For $0 < p < \infty$ a (single-valued) analytic function F in Ω is said to belong to $H_p(\Omega)$ if $|F|^p$ has a harmonic majorant; $H_\infty(\Omega)$ consists of the bounded analytic functions. Many of the properties of $H_p(\Omega)$ (for example the existence of nontangential limits a.e. on E) follow by conformal mapping from the corresponding results for the disc. If one uses as norm the L_p norm of the boundary function then for $1 \leqslant p \leqslant \infty$, $H_p(\Omega)$ becomes a Banach space. The reader is referred to (*11*) for details.

We now study the following problem.

$(1, \rho)$ For all $F \in H_\infty(\Omega)$ satisfying $F(\infty) = 1$, determine the greatest lower bound of

$$\text{ess sup}\, |F(\zeta)|\, \rho(\zeta), \qquad \zeta \in E.$$

Here of course $F(\zeta)$ denote the (a.e. defined) nontangential limit of $F(z)$ as $z \to \zeta$. We shall show that this g.l.b., which we call $\mu(\rho)$, is attained and determine both it and the extremal function in terms of Green's and Neumann's functions.

Copying the method of (*11*) we introduce the following dual problem:

$(2, \rho)$ For all $f \in H_1^0(\Omega)$ (this means $f \in H_1(\Omega)$ and vanishes at $z = \infty$) satisfying

$$\lim_{z\to\infty} zf(z) > 0, \qquad \frac{1}{2\pi}\int_E |f(\zeta)|\, \rho(\zeta)^{-1}\, |d\zeta| = 1 \tag{5.1}$$

determine the least upper bound of

$$\lim_{z\to\infty} zf(z).$$

This l.u.b. we call $\mu'(\rho)$ for the moment although it will transpire that it is the same as $\mu(\rho)$.

Since

$$\int_E f(\zeta)\, d\zeta = 2\pi i \lim_{z\to\infty} zf(z), \qquad f \in H_1{}^0(\Omega),$$

we see that

$$f \to \lim_{z\to\infty} zf(z)$$

is a continuous linear functional on $H_1{}^0(\Omega)$ and that, if the norm

$$\frac{1}{2\pi}\int_E |f(\zeta)|\, \rho(\zeta)^{-1}\, |\, d\zeta\, |$$

is used on the space, then $\mu'(\rho)$ is exactly the norm of this functional. Hence by the Hahn–Banach theorem and the fact that the dual space of L_1 is L_∞, we know that there is a function $u \in L_\infty(E)$ with the following properties:

$$\begin{aligned} &\max |\, u(\zeta)| = \mu'(\rho), && \zeta \in E, \\ &\lim_{z\to\infty} zf(z) = \frac{1}{2\pi}\int_E f(\zeta)\, u(\zeta)\, \rho(\zeta)^{-1}\, |\, d\zeta\, |, && f \in H_1{}^0(\Omega). \end{aligned} \tag{5.2}$$

In particular we have

$$\int_E f(\zeta)\, u(\zeta)\, \rho(\zeta)^{-1} \frac{|\, d\zeta\, |}{d\zeta}\, d\zeta = 0 \qquad \text{if} \quad \lim_{z\to\infty} zf(z) = 0$$

and this implies that there is an $F_0 \in H_\infty(\Omega)$ such that

$$F_0(\zeta) = iu(\zeta)\, \rho(\zeta)^{-1} \frac{|\, d\zeta\, |}{d\zeta} \qquad \text{a.e. on } E.$$

[See (*11*), Corollary 3.6. F_0 is given by the formula

$$F_0(z) = \frac{1}{2\pi}\int_E u(\zeta)\, \rho(\zeta)^{-1} \frac{|\, d\zeta\, |}{\zeta - z} + \text{const.}]$$

We now rewrite (5.2) as

$$\begin{aligned} \text{ess sup}\, |\, F_0(\zeta)|\, \rho(\zeta) &= \mu'(\rho), && \zeta \in E, \\ 2\pi i \lim_{z\to\infty} zf(z) &= \int_E f(\zeta)\, F_0(\zeta)\, d\zeta, && f \in H_1{}^0(\Omega). \end{aligned} \tag{5.3}$$

Note that the last identity applied to $f(z) = z^{-1}$ gives

$$F_0(\infty) = 1.$$

This function F_0 is exactly the extremal function we were looking for. For any $F \in H_\infty(\Omega)$ satisfying $F(\infty) = 1$ and any $f \in H_1{}^0(\Omega)$ satisfying (5.1) we have

$$\begin{aligned}\lim_{z\to\infty} zf(z) &= \frac{1}{2\pi i}\int_E f(\zeta)F(\zeta)\,d\zeta = \frac{1}{2\pi}\int_E i^{-1}f(\zeta)F(\zeta)\frac{d\zeta}{|d\zeta|}\,|d\zeta| \\ &\leqslant \frac{1}{2\pi}\int_E |f(\zeta)|\,\rho(\zeta)^{-1}\,|F(\zeta)|\,\rho(\zeta)\,|d\zeta| \leqslant \operatorname*{ess\,sup}_E |F(\zeta)|\,\rho(\zeta).\end{aligned} \tag{5.4}$$

This gives the inequality $\mu'(\rho) \leqslant \mu(\rho)$. But then (5.3) shows that $\mu'(\rho) = \mu(\rho)$ and that F_0 is extremal.

It is not hard to see that there is an extremal function $f_0 \in H_1{}^0(\Omega)$ with

$$\lim_{z\to\infty} zf_0(z) = \frac{\mu(\rho)}{2\pi}, \qquad \int_E |f_0(\zeta)|\,\rho(\zeta)^{-1}\,|d\zeta| = 1.$$

In fact if we take a sequence of f_n satisfying (5.1) such that the quantities

$$\lim_{z\to\infty} zf_n(z)$$

approach $\mu(\rho)$ as $n \to \infty$, then there is a subsequence converging uniformly on closed subsets of Ω to an f_0 which is necessarily extremal for problem $(2, \rho)$.

The following lemma states properties of the extremals which, as we shall see, characterize them.

Lemma 5.1. *The extremal functions F_0, f_0 satisfy*

$$|F_0(\zeta)|\,\rho(\zeta) = \mu(\rho), \qquad F_0(\zeta)f_0(\zeta)/G'(\zeta) \geqslant 0 \tag{5.5}$$

almost everywhere on E.

Proof. We have equality in (5.4) for $F = F_0$, $f = f_0$. But the only way equality can occur is for

$$|F_0(\zeta)|\,\rho(\zeta) = \mu(\rho), \qquad i^{-1}f_0(\zeta)F_0(\zeta)\frac{d\zeta}{|d\zeta|} \geqslant 0 \tag{5.6}$$

to hold almost everywhere.

Let us see why the last inequality is equivalent to the inequality in (5.5). Since $g(\zeta) = 0$,

$$G'(\zeta) = \frac{d}{d\zeta}\{g(\zeta) + i\tilde{g}(\zeta)\} = i\frac{d\tilde{g}}{|d\zeta|}\frac{|d\zeta|}{d\zeta}.$$

Since $d\tilde{g}/|d\zeta| \geqslant 0$ we obtain

$$iG'(\zeta)^{-1}\frac{|d\zeta|}{d\zeta} \geqslant 0 \tag{5.7}$$

which shows the equivalence of the inequalities in (5.5) and (5.6).

The next lemma will be used to show that F_0 and f_0 can be smoothly continued to the boundary.

Lemma 5.2. *Suppose F and G belong respectively to $H_\infty(\Omega)$ and $H_1(\Omega)$, and that almost everywhere on some arc of E we have*

$$F(\zeta)\,G(\zeta) \geqslant 0, \qquad |F(\zeta)| = \sigma(\zeta)$$

where $\sigma(\zeta)$ is a nonzero function of class C^{0+}. Then both F and G extend continuously to C^{0+} functions on the arc and any zero of G on this arc is of even multiplicity. (This means, for a zero ζ_0, that there is an even integer m such that $G(\zeta)(\zeta - \zeta_0)^{-m}$ is bounded and bounded away from zero in a neighborhood of ζ_0.)

Proof. Note first that since the situation is entirely a local one, we may assume by applying a conformal mapping that Ω is the unit disc. Then the case $\sigma \equiv 1$ is contained in Lemmas 4.4 and 4.5 of (*11*). (We won't reproduce the proof here.) In this case F and G are actually analytically continuable across the arc. To reduce the general case to this, we assume σ is positive and defined everywhere on the circumference and let $\Sigma(z)$ be a nonzero analytic function in the disc with boundary values of absolute value $\sigma(\zeta)$. Specifically, if $h(z)$ is the Poisson integral of $\log \sigma$ then

$$\Sigma(z) = \exp\{h(z) + i\tilde{h}(z)\}.$$

By Lemma 4.1(4) we see that $\Sigma(z)^{\pm 1}$ extend to C^{0+} functions on the circumference. The pair of functions

$$\Sigma^{-1}F, \qquad \Sigma G$$

are now in the special case with $\sigma = 1$ and the result follows.

It follows from Lemma 5.1 and 5.2 together with the fact [Lemma 4.1(1)] that G' is continuous and nonzero on and near E, that F_0 and f_0 extend continuously to E, F_0 of course having no zeros on E and f_0 having finitely many zeros all of even multiplicity.

We can now show that Lemma 5.1 characterizes two of the three extremal objects.

Lemma 5.3. *Suppose we have $F_1 \in H_\infty(\Omega)$ with $F_1(\infty) = 1$, $f_1 \in H_1{}^0(\Omega)$ satisfying (5.1), and a constant μ_1 such that*

$$|F_1 \mid \rho = \mu_1 , \qquad F_1 f_1 / G' \geqslant 0$$

almost everywhere on E. Then $F_1 = F_0$ and $\mu_1 = \mu(\rho)$. Moreover, f_1 is extremal.

Proof. Our assumptions imply that we have equality in (5.4) when $F = F_1$, $f = f_1$. This implies that F_1 and f_1 are extremal and that $\mu_1 = \mu(\rho)$. Thus also $| F_0 | = | F_1 |$ almost everywhere on E.

Next, since we obtain equality in (5.4) if $f = f_0$ and $F = F_1$, we must have $F_1 f_0 / G' \geqslant 0$ almost everywhere on E. Thus F_0 and F_1 have almost everywhere on E the same absolute value and argument and are consequently equal.

One may well ask about the uniqueness of f_0 . It follows from what we have already done that any two such extremal f's have non-negative quotient on E, and so if there is one nonzero in $\bar{\Omega}$ it is the unique extremal function. This will be the case, as we shall see in a moment, for the case $p = 1$ of one boundary curve. But if $p > 1$ there may be an f_0 with a zero somewhere and this always leads to other extremal functions.

Lemmas 5.1 and 5.3 show the equivalence of our extremal problems with a pair of boundary problems which we now proceed to solve in terms of Green's functions and Neumann functions.

First observe that, by (5.7),

$$- \underset{E}{\Delta} \arg G'(\zeta) = \underset{E}{\Delta} \arg \frac{d\zeta}{| d\zeta |} .$$

But this is just the total change of the inclination of the tangent vector to E and so is equal to $2\pi p$. It follows that $G'(z)$ has a total of p zeros in Ω, counting the one at infinity. Call the other zeros

$$z_1^*, \ldots, z_{p-1}^* ,$$

where each point is repeated according to its multiplicity.

It follows from what we have shown, and the second statement of Lemma 5.1, that $F_0 f_0$ also has a total of p zeros in $\bar{\Omega}$ counting the one f_0 has at infinity; here any zero on E, necessarily of even multiplicity by Lemma 5.2, is counted with half its multiplicity. Denote the zeros of $F_0 f_0$ other than the one at infinity by

$$z_1, \ldots, z_{p-1} \tag{5.8}$$

where each zero in Ω is repeated according to its multiplicity and each zero on E half its multiplicity.

The function

$$U(z) = F_0(z) f_0(z)/G'(z) \prod_{j=1}^{p-1} \Psi(z, z_j^*, z_j)$$

has neither zeros nor poles in $\bar{\Omega}$, extends continuously and nonzero to E, and on each component E_k of E has constant argument. (Recall that the characteristic property $\partial N/\partial n_\zeta = 0$ of N implies that $\tilde{N}$ is constant on each E_k.) This implies U is constant, as is seen by an argument already used: If $\arg U(\zeta) = \lambda_k$ on E_k then the Dirichlet integral of $\arg U$ in Ω equals

$$\int_E \arg U(\zeta) \frac{\partial \arg U(\zeta)}{\partial n_\zeta} \mid d\zeta \mid = \sum_k \lambda_k \underset{E_k}{\Delta} \log \mid U(\zeta) \mid = 0.$$

Thus $\arg U$, and so also U, is constant.

We have

$$F_0(z) f_0(z) = UG'(z) \prod_{j=1}^{p-1} \Psi(z, z_j^*, z_j). \tag{5.9}$$

If we make the normalizations $\tilde{N}(\infty, z^*, z_j) = 0$ as we may, then we find that $U > 0$. But then the second statement of Lemma 5.1 tells us that on each E_k

$$\arg \prod_{j=1}^{p-1} \Psi(\zeta, z_j^*, z_j)$$

is not only a constant, but that it is of the form $2\pi m_k$ for some integer m_k. Now the constants λ_k appearing in the formula (4.5) for $N(z, z_1, z_2)$ are just the values of $\tilde{N}(\zeta, z_1, z_2)$ on E_k. These constants satisfy the equation (4.6). It follows that our points $z_1, \ldots, z_{p-1}$ must satisfy

$$\sum_{j=1}^{p-1} \{\tilde{\omega}_k(z_j) - \tilde{\omega}_k(z_j^*)\} = \sum_{l=1}^{p} m_l P_{kl}, \qquad k = 1, \ldots, p. \tag{5.10}$$

Conversely if this is satisfied for certain integers $m_1, \ldots, m_p$ then (5.9) gives us a solution to the second half of (5.5).

In order to take account of the first half of (5.5) we introduce the important function $R(z)$ which is a multiple-valued function with single-valued absolute value, having no zeros or poles in Ω, and having boundary values satisfying

$$| R(\zeta)| = \rho(\zeta).$$

We define R as follows: If $h(z)$ is the harmonic function with boundary values $\log \rho(\zeta)$ then

$$R(z) = \exp[h(z) + i\tilde{h}(z)].$$

By Lemma 4.1(4), R extends continuously to a C^{0+} function on E. Notice that the description we gave of R determines it up to a constant factor of absolute value one; for the quotient of two such has no zeros or poles and has constant absolute value E, and this implies (even for multiple-valued functions) that it must be constant. We shall normalize R so that $R(\infty) > 0$. We have by (4.1) the formula

$$R(\infty) = \exp\left\{\frac{1}{2\pi}\int_E \log \rho(\zeta)\,\frac{\partial g(\zeta)}{\partial n_\zeta}\,| d\zeta |\right\}. \tag{5.11}$$

The list (5.8) includes the zeros of both F_0 and f_0. Suppose $z_1, \ldots, z_q$ are the zeros of F_0, the remainder those of f_0. If we normalize the functions $\Phi(z, z_j)$ to be positive at ∞ (which only involves defining $\tilde{g}(\infty, z_j) = 0$) we find that the multiple-valued function

$$\mu(\rho)/F_0(z) \prod_{j=1}^{q} \Phi(z, z_j)$$

is analytic and nonzero in Ω, continuous up to E, and has there absolute value $\rho(\zeta)$. It is also positive at infinity. Consequently it must be $R(z)$ and so we have

$$F_0(z) = \mu(\rho)\, R(z)^{-1} \prod_{j=1}^{q} \Phi(z, z_j)^{-1}. \tag{5.12}$$

Since $F_0(\infty) = 1$, this gives for the extremal quantity $\mu(\rho)$ the formula

$$\mu(\rho) = R(\infty) \exp\left\{\sum_{j=1}^{q} g(z_j)\right\}. \tag{5.13}$$

We have used here the symmetry of Green's function to obtain

$$|\Phi(\infty, z_j)| = \exp g(\infty, z_j) = \exp g(z_j, \infty),$$

and our convention not to display the parameter value ∞.

The fact that the rigth side of (5.12) is single-valued imposes another condition on $z_1, \ldots, z_q$. In fact we have [see (4.3) and (4.4)]

$$\gamma_k(\Phi(z, z_j)) = \frac{1}{2\pi} \underset{E_k}{\Delta} \arg \Phi(z, z_j) = \omega_k(z_j)$$

$$\gamma_k(R) = \frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} |d\zeta|, \tag{5.14}$$

and these give the system of equations

$$\sum_{j=1}^{q} \omega_k(z_j) \equiv -\frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} |d\zeta| \pmod 1, \qquad k = 1, \ldots, p. \tag{5.15}$$

Conversely if these equations are satisfied then (5.12) defines a single-valued function. Thus we have shown that the equations (5.10) and (5.15) may always be solved for $z_1, \ldots, z_p$ [the number q appearing in (5.15) being unique] and that the extremals are given, in terms of these points, by the formula (5.9), (5.12), and (5.13).

We shall now extend all that we have done to the case of multiple-valued functions. In order to do this we introduce for each class Γ (see §3) a canonical function of that class, having no zero or poles in Ω, by which we may divide an arbitrary function of the class to obtain a single-valued function. Specifically let

$$\Gamma = (\gamma_1, \ldots, \gamma_p)$$

where, recall, the γ_k are only determined mod 1. If however we make specific choices for γ_k, requiring of course that $\Sigma \gamma_k = 0$, and solve the system

$$\sum_{k=1}^{p} \lambda_k P_{jk} = \gamma_j, \qquad j = 1, \ldots, p,$$

$$\sum_{k=1}^{p} \lambda_k = 0,$$

as we know can be done uniquely, then the function

$$V_\Gamma(z) = \exp\left\{\sum_{k=1}^{p} \lambda_k[\omega_k(z) + i\tilde{\omega}_k(z)]\right\} \tag{5.16}$$

has the required properties.

Now by $H_p(\Omega, \Gamma)$ we mean all multiple-valued functions F of class Γ in Ω for which $|F|^p$ has a harmonic majorant; $H_\infty(\Omega, \Gamma)$ means $|F|$ is bounded. Since V_Γ is everywhere analytic and nonzero and continuous up to E, all questions relating to $H_p(\Omega, \Gamma)$ may be reduced to $H_p(\Omega)$ by simply dividing by $V_\Gamma(z)$. Now we consider the following extremum problems:

$(1, \rho, \Gamma)$ For all $F \in H_\infty(\Omega, \Gamma)$ satisfying $|F(\infty)| = 1$ determine

$$\mu(\rho, \Gamma) = \inf_F \operatorname{ess\,sup}_E |F(\zeta)|\, \rho(\zeta),$$

$(2, \rho, \Gamma)$ For all $f \in H_1{}^0(\Omega, -\Gamma)$ satisfying

$$\frac{1}{2\pi}\int_E |f(\zeta)|\, \rho(\zeta)^{-1}\, |d\zeta| = 1$$

determine

$$\mu'(\rho, \Gamma) = \sup_f \lim_{z\to\infty} |zf(z)|.$$

Notice the change from Γ to $-\Gamma$ in problem $(2, \rho, \Gamma)$. Since $F \to FV$ maps $H_\infty(\Omega)$ onto $H_\infty(\Omega, \Gamma)$ it is easy to see that the extremal function F_Γ for problem $(1, \rho, \Gamma)$, determined up to a constant factor of absolute value 1, is given by

$$F_\Gamma(z) = V_\Gamma(z) F_0(z)/V_\Gamma(\infty) \tag{5.17}$$

where $F_0(z)$ is the extremal function for problem $(1, \sigma)$ (class $\Gamma = 0$) with weight function

$$\sigma(\zeta) = \rho(\zeta)\, |V_\Gamma(\zeta)|.$$

Also since $f \to f/V$ maps $H_1{}^0(\Omega)$ onto $H_1{}^0(\Omega, -\Gamma)$ we find that the (an) extremal function f_Γ for problem $(2, \rho, \Gamma)$ is given by

$$f_\Gamma(z) = f_0(z)/V_\Gamma(z)$$

where $f_0(z)$ is extremal for problem $(2, \sigma)$ with the same weight σ.

Hence $\mu'(\rho, \Gamma) = \mu(\rho, \Gamma)$ and the equations characterizing our extremals (up to constant factors of absolute value one) are

$$|F_\Gamma(\zeta)|\,\rho(\zeta) = \mu(\rho, \Gamma), \qquad F_\Gamma(\zeta) f_\Gamma(\zeta)/G'(\zeta) \geqslant 0$$

just as before. The only change in the solution, then, occurs at the very last step. (Notice that $F_\Gamma f_\Gamma$ is single-valued.) Now $F_0(z)$ as given by (5.12) must belong to class Γ, and so we must substitute for (5.15) the system

$$\sum_{j=1}^{q} \omega_k(z_j) \equiv -\gamma_k - \frac{1}{2\pi}\int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} |\, d\zeta \,| \quad (\mathrm{mod}\ 1), \qquad k = 1,..., p. \tag{5.18}$$

Finally let us see how all the results obtained can be extended to the case where we assume merely that ρ is a nonnegative function on E satisfying

$$\int_E |\log \rho(\zeta)|\, |\, d\zeta \,| < \infty. \tag{5.19}$$

It turns out that the only problem is stating the problem right, and once this is done the solution drops out of what went before. We have to define our analytic function spaces taking careful account now of the possibly singular behavior of ρ on E. This can be done in various ways and for our purpose the simplest is the following: $H_p(\Omega, \rho, \Gamma)$ consists of those functions everywhere analytic in Ω, of class Γ, and for which $|F(z)^p R(z)|$ has a harmonic majorant (or, in the case $p = \infty$, $|F(z) R(z)|$ is bounded); here $R(z)$ is what it was before, the condition (5.19) being all that is necessary for its existence. The function $R(z)$ has nontangential limit of absolute value $\rho(\zeta)$ almost everywhere on E. This is classical in the case of the unit disc and one way the statement follows in general by is mapping the universal covering space of Ω (the disc) conformally onto Ω. Thus the absolute value of any $F \in H_p(\Omega, \rho, \Gamma)$ has nontangential limit almost everywhere on E; we denote this limit by $|F(\zeta)|$.

The problems can now be formulated as follows:

$(1, \rho, \Gamma)$ For all $F \in H_\infty(\Omega, \rho, \Gamma)$ satisfying $|F(\infty)| = 1$, determine

$$\mu(\rho, \Gamma) = \inf_F \operatorname*{ess\,sup}_E |F(\zeta)|\,\rho(\zeta).$$

$(2, \rho, \Gamma)$ For all $f \in H_1{}^0(\Omega, \rho^{-1}, -\Gamma)$ satisfying

$$\frac{1}{2\pi}\int_E |f(\zeta)|\,\rho(\zeta)^{-1}\,|\, d\zeta \,| = 1$$

determine

$$\mu'(\rho, \Gamma) = \sup_f \lim_{z\to\infty} | zf(z)|.$$

The answers are now exactly as they were before, as can be seen as follows. The mapping $F \to F/R$ sends $H_\infty(\Omega, \Gamma + \Gamma(R))$ onto $H_\infty(\Omega, \rho, \Gamma)$. Hence the extremal function $F_{\rho,\Gamma}$ for our problem $(1, \rho, \Gamma)$ is given by

$$F_{\rho,\Gamma}(z) = R(\infty) F_0(z)/R(z) \tag{5.20}$$

where F_0 is the solution to problem $(1, \Gamma + \Gamma(R))$ with weight function equal to 1. Similarly our extremal function $f_{\rho,\Gamma}$ for $(2, \rho, \Gamma)$ is given by

$$f_{\rho,\Gamma}(z) = f_0(z)\, R(z)$$

where f_0 is extremal for problem $(2, \Gamma + \Gamma(R))$ with weight function 1. Thus the extremal functions and constants are given by exactly the same formulas as before.

We now sum up what we have done so far.

Theorem 5.4. *The extremal constants for problems $(1, \rho, \Gamma)$ and $(2, \rho, \Gamma)$ are given by*

$$\mu(\rho, \Gamma) = \mu'(\rho, \Gamma) = R(\infty) \exp\left\{\sum_{j=1}^{q} g(z_j)\right\}$$

and the extremal functions by

$$F_{\rho,\Gamma}(z) = \mu(\rho, \Gamma)\, R(z)^{-1} \prod_{j=1}^{q} \Phi(z, z_j)^{-1}$$

$$F_{\rho,\Gamma}(z) f_{\rho,\Gamma}(z) = UG'(z) \prod_{j=1}^{p-1} \Psi(z, z_j^*, z_j) \qquad (U > 0).$$

The points $z_1, \ldots, z_{p-1}$ are obtained as solutions of

$$\sum_{j=1}^{p-1} \{\tilde{\omega}_k(z_j) - \tilde{\omega}_k(z_j^*)\} = \sum_{l=1}^{p} m_l P_{kl}\,, \qquad k = 1, \ldots, p$$

$$\sum_{j=1}^{q} \omega_k(z_j) \equiv -\gamma_k - \frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} | d\zeta | \quad (\text{mod } 1), \qquad k = 1, \ldots, p$$

(where the m_l may be arbitrary integers) and any solution of the system gives rise to extremal functions by the above formulas. Moreover the number $q \leqslant p - 1$ and points $z_1, \ldots, z_q$ are uniquely determined.

One might well ask how explicit this solution is and the answer is "not as explicit as we would like". If $p = 1$ then of course there are no points z_j or z_j^*, there is only the trivial class $\Gamma = 0$, and we have

$$\mu(\rho) = R(\infty), \qquad F_\rho(z) = R(\infty)/R(z), \qquad f_\rho(z) = G'(z)\, R(z).$$

If $p = 2$ it is also possible to write down the solution, in terms, however, of certain mapping functions. The details will be given in §14. For $p > 2$ we know of no analogous explicit solution.

It is however possible to determine the range of $\mu(\rho, \Gamma)$ as Γ runs through all classes. Clearly $\mu(\rho, \Gamma) \geqslant R(\infty)$ and this is achieved for $\Gamma = -\Gamma(R)$. There is a simple heuristic argument for guessing the maximum of $\mu(\rho, \Gamma)$. We know from the form of $F_{\rho,\Gamma}$ that for a fixed Γ,

$$\log\{\mu(\rho, \Gamma)/R(\infty)\}$$

is the minimum, over all $z_1, \ldots, z_p \in \bar{\Omega}$ such that

$$\prod_{j=1}^{p} \Phi(z, z_j)^{-1} \in \Gamma,$$

of

$$\sum_{j=1}^{p} g(z_j).$$

Thus we first minimize this for all $(z_1, \ldots, z_p)$ corresponding to a certain class, and then maximize over all classes. This leads to the conjecture that the "minimax" occurs when

$$\operatorname{grad}_{z_k} \sum_{j=1}^{p} g(z_j) = 0$$

for each k, i.e., that the points z_j are just the zeros z_j^* of $G'(z)$. This argument, though leading to a correct guess, seems difficult to carry out. A different, although less transparent, method will be used.

First, though, we want to examine the continuity behavior of μ as a function of ρ and Γ. On the set $\mathscr{T}_p{}^0$ of classes we have of course the topology of the p-torus $\mathscr{T}_p$. As for the weight functions we use the metric

$$\operatorname{dist}(\rho_1, \rho_2) = \int_E |\log \rho_1(\zeta) - \log \rho_2(\zeta)|\, |d\zeta|.$$

Lemma 5.5. *$\mu(\rho, \Gamma)$ is jointly continuous.*

Proof. We have seen [see (5.20)] that

$$\mu(\rho, \Gamma) = R(\infty)\, \mu(1, \Gamma + \Gamma(R)).$$

Since $R(\infty)$ and $\Gamma(R)$ are continuous in ρ [see (5.11) and (5.14)] we have reduced the question to the case where $\rho = 1$. We now reduce it the other way using the functions $V_\Gamma(z)$ given by (5.16). In fact we have by (5.17)

$$\mu(1, \Gamma) = |\, V_\Gamma(\infty)|^{-1}\, \mu(|\, V_\Gamma\,|, 0).$$

Now as Γ varies continuously the coefficients λ_k in (5.16) vary continuously and so the V_Γ vary continuously in the metric for uniform convergence. Thus $\mu(1, \Gamma)$ is continuous in Γ.

We now give the range of μ.

Theorem 5.6. *The range of $\mu(\rho, \Gamma)$ as Γ runs through $\mathscr{T}_p^{\,0}$ is the closed interval*

$$\left[R(\infty), R(\infty) \exp\left\{\sum_{j=1}^{p-1} g(z_j^*)\right\}\right].$$

Proof. Since the space of pairs ρ, Γ is connected it follows from Lemma 5.5 that the range of μ is an interval. We know the minimum of μ is $R(\infty)$ and it suffices to show its maximum is as claimed.

For any class Γ we have

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_E |f_{\rho,\Gamma}(\zeta)|\, \rho(\zeta)^{-1}\, |\, d\zeta\,| \\ &= \frac{1}{2\pi} \int_E |f_{\rho,\Gamma}(\zeta)\, R(\zeta)^{-1}\, G'(\zeta)^{-1} \prod_{j=1}^{p-1} \Phi(\zeta, z_j^*)^{-1}\,|\,|\, dG(\zeta)| \end{aligned}$$

since $|\,\Phi(\zeta, z_j^*)| = 1$ for $\zeta \in E$. This can also be written

$$1 = \frac{1}{2\pi} \int_E \left| f_{\rho,\Gamma}(\zeta)\, R(\zeta)^{-1}\, G'(\zeta)^{-1} \prod_{j=1}^{p-1} \Phi(\zeta, z_j^*)^{-1} \right| \frac{\partial g(\zeta)}{\partial n_\zeta}\, |\, d\zeta\,|. \quad (5.21)$$

Now the function

$$\left| f_{\rho,\Gamma}(z)\, R(z)^{-1}\, G'(z)^{-1} \prod_{j=1}^{p-1} \Phi(z, z_j^*)^{-1} \right|$$

is subharmonic in Ω and continuous up to E [the zeros of the product cancelling the poles of $G'(z)^{-1}$], and so its value at ∞ is at most the value at ∞ of the harmonic function with the same boundary values. This latter value is given by the right side of (5.21) and so is equal to 1. This gives us the inequality

$$\mu(\rho, \Gamma)/R(\infty) \prod_{j=1}^{p-1} |\Phi(\infty, z_j^*)| \leqslant 1$$

or

$$\mu(\rho, \Gamma) \leqslant R(\infty) \exp\left\{\sum_{j=1}^{p-1} g(z_j^*)\right\}.$$

This value is actually achieved by $\mu(\rho, \Gamma)$ for

$$\Gamma = -\sum_{j=1}^{p-1} \Gamma(\Phi(z, z_j^*)) - \Gamma(R),$$

for then the equations in Theorem 5.4 determining the z_j are satisfied by the z_j^*.

The product of exponential Neumann functions appearing in the representation of $F_{\rho,\Gamma} f_{\rho,\Gamma}$ may be given another form more pleasant than that obtained by using (4.5). Recall the notation $\Omega_k = \omega_k + i\tilde{\omega}_k$.

Theorem 5.7. *We have*

$$F_{\rho,\Gamma} f_{\rho,\Gamma} = \mu(\rho, \Gamma)\, G' \left\{1 + \sum_{k=1}^{p-1} a_k \frac{d\Omega_k}{dG}\right\},$$

where a_k are the unique real constants satisfying the system

$$G'(z_j) + \sum_{k=1}^{p-1} a_k \Omega_k'(z_j) = 0, \qquad j = 1,\ldots, p-1. \tag{5.22}$$

Proof. Let us form the Riemann surface S of genus $p - 1$ which is the double of Ω. Thus S is formed by joining Ω and a copy along their common boundary E and giving the result an appropriate analytic structure.

The functions ω_k, $g(z)$ extend to all of S by reflecting across E, since they vanish on E. Any $p - 1$ of the differentials $d\Omega_k$ form a basis for the everywhere analytic differentials on S (recall $\sum_{k=1}^{p} d\Omega_k = 0$) and dG is an abelian differential with poles at ∞ and $\overline{\infty}$ (the point of the copy of Ω corresponding to $\infty \in \Omega$) and zeros at the z_j^* and $\overline{z_j^*}$. It follows that the most general meromorphic function on S whose only poles are among the z_j^* and $\overline{z_j^*}$ is of the form

$$a + \sum_{k=1}^{p-1} a_k \frac{d\Omega_k}{dG}.$$

We have in mind the function

$$H(z) = F_{\rho,\Gamma}(z) f_{\rho,\Gamma}(z)/\mu(\rho, \Gamma)\, G'(z).$$

The poles of this function in Ω are among the z_j^* ; it is nonnegative on E and so extends by reflection to the rest of S where its only poles are among the $\overline{z_j^*}$. Since $H(\infty) = 1$ and dG has a pole at ∞ we see that $a = 1$. Further, since H vanishes at both z_j and $\bar{z}_j$ we obtain

$$1 + \sum_{k=1}^{p-1} a_k \frac{d\Omega_k}{dG}(z_j) = 0, \qquad j = 1,\dots, p-1,$$

$$1 + \sum a_k \frac{d\Omega_k}{dG}(\bar{z}_j) = 0, \qquad j = 1,\dots, p-1.$$

Now since $d\Omega_k/dG$ is real on E [in fact

$$\frac{d\Omega_k}{dG} = \frac{\partial \omega_k}{\partial n_\zeta} \Big/ \frac{\partial g}{\partial n_\zeta}$$

on E] its extension to all of S takes conjugate values at conjugate points. Thus if we write

$$a_k = a_k' + i a_k'' \qquad (a_k', a_k'' \text{ real}),$$

then the two systems may be written equivalently as

$$\begin{aligned} 1 + \sum a_k' \frac{d\Omega_k}{dG}(z_j) &= 0, \\ \sum a_k'' \frac{d\Omega_k}{dG}(z_j) &= 0. \end{aligned} \tag{5.23}$$

Now the second system implies that all $a_k'' = 0$. For if we write

$$H_1(z) = \sum a_k'' \frac{d\Omega_k}{dG},$$

then H_1 vanishes at all the z_j and, since the a''_k are real, also at the $\bar{z}_j$. Thus the differential

$$\sum a''_k \, d\Omega_k$$

has at least $2(p - 1)$ zeros. But on a surface of genus $p - 1$ each everywhere analytic differential, not identically zero, has exactly $2(p - 2)$ zeros. [See for example (*12*), Theorem 10-11.] This proves each $a''_k = 0$, so the a_k are real and the representation is as claimed.

The immediately preceding argument also gives uniqueness. For if we had two real solutions of (5.22) their difference would be a real solution of (5.23) and so be identically zero.

There are slight modifications of both the statement and proof of the theorem in case some z_j lie on E or have multiplicity larger than one. These modifications are straightforward and left to the reader.

6. The Extremum Problem for L_2 Norms

We assume again that $E \in C^{1+}$ and, temporarily, that $\rho > 0$ and belongs to C^{0+}. The next problem we consider is the following.

(3, ρ) For all $F \in H_2(\Omega)$ satisfying $F(\infty) = 1$, determine

$$\nu(\rho) = \inf_F \int_E |F(\zeta)|^2 \rho(\zeta) \, |d\zeta|.$$

As in the preceding section, once we have this solved it will be easy to give extensions to the cases of multiple-valued functions and more general weights.

$H_2(\Omega)$ is a Hilbert space with inner product

$$(F_1, F_2) = \int_E F_1(\zeta) \overline{F_2(\zeta)} \rho(\zeta) \, |d\zeta|.$$

The map $F \to F(\infty)$ is a continuous linear functional in $H_2(\Omega)$, by the identity

$$F(\infty) = \frac{1}{2\pi} \int_E F(\zeta) \frac{\partial g}{\partial n_\zeta} \, |d\zeta|$$

and Schwarz's inequality. Hence it is of the form

$$F \to (F, K)$$

for some appropriate $K \in H_2(\Omega)$. (K is the Szegö kernel function at infinity. More about this in the next section.) Since the functional has norm $\nu(\rho)^{-1/2}$,

$$K(\infty) = \int_E |K(\zeta)|^2 \rho(\zeta) |d\zeta| = \nu(\rho)^{-1}.$$

Thus if we set

$$F_0(z) = \nu(\rho) K(z)$$

then we have

$$F_0(\infty) = 1, \qquad \int_E |F_0(\zeta)|^2 \rho(\zeta) |d\zeta| = \nu(\rho),$$

so that F_0 is extremal, and

$$F(\infty) = \nu(\rho)^{-1} \int_E F(\zeta) \overline{F_0(\zeta)} \rho(\zeta) |d\zeta|, \qquad F \in H_2(\Omega).$$

This last statement implies, just as for the analogue in §5, that

$$i\overline{F_0(\zeta)} \rho(\zeta) \frac{|d\zeta|}{d\zeta} = f_0(\zeta) \tag{6.1}$$

where $f_0(z) \in H_2{}^0(\Omega)$ and

$$\lim_{z\to\infty} zf(z) = \nu(\rho)/2\pi.$$

Theorem 6.1. *The functions $F_0 \in H_2(\Omega)$ and $f_0 \in H_2{}^0(\Omega)$ satisfy*

$$F_0(\infty) = 1, \qquad \lim_{z\to\infty} zf_0(z) = \nu(\rho)/2\pi \tag{6.2}$$

$$|f_0(\zeta)| = |F_0(\zeta)| \rho(\zeta), \qquad F_0(\zeta) f_0(\zeta)/G'(\zeta) \geqslant 0 \qquad \text{a.e. } on\ E. \tag{6.3}$$

These statements characterize F_0, f_0, $\nu(\rho)$ in the sense that if we have $F_1 \in H_2(\Omega)$, $f_1 \in H_2{}^0(\Omega)$ with

$$F_1(\infty) = 1, \qquad \lim_{z\to\infty} zf_1(z) = \nu_1/2\pi > 0,$$

$$|f_1(\zeta)| = |F_1(\zeta)| \rho(\zeta), \qquad F_1(\zeta) f_1(\zeta)/G'(\zeta) \geqslant 0 \qquad \text{a.e. } on\ E \tag{6.4}$$

then $F_1 = F_0$, $f_1 = f_0$, $\nu_1 = \nu(\rho)$.

Proof. The four properties of F_0, f_0, $\nu(\rho)$ follow from what just preceded since, by (5.6), the statement (6.1) is equivalent to the pair (6.3).

It remains to prove the second assertion. For any $F \in H_2(\Omega)$ we have

$$F(\infty) = i^{-1}\nu_1^{-1} \int_E F(\zeta) f_1(\zeta)\, d\zeta$$

$$= \nu_1^{-1} \int_E F(\zeta) \overline{F_1(\zeta)}\, \rho(\zeta)\, d\zeta$$

and so if $F(\infty) > 0$,

$$F(\infty)^2 \leqslant \nu_1^{-2} \int_E |F(\zeta)|^2 \rho(\zeta)\, d\zeta \cdot \int_E |F_1(\zeta)|^2 \rho(\zeta)\, d\zeta. \tag{6.5}$$

Thus we have

$$\nu(\rho) \geqslant \nu_1^2 \Big/ \int_E |F(\zeta)|^2 \rho(\zeta)\, d\zeta.$$

However if we take $F = F_1$ we get equality in (6.5) so that

$$\nu(\rho) \leqslant \nu_1^2 \Big/ \int_E |F_1(\zeta)|^2 \rho(\zeta)\, d\zeta.$$

Thus F_1 is also an extremal function for problem $(3, \rho)$. But on any Hilbert space the norm of a nonzero linear functional is assumed for a unique point of the unit sphere, and this means that the extremal function F_0 is unique. Thus $F_1 = F_0$ and so also

$$\int_E |F_1(\zeta)|^2 \rho(\zeta)\, d\zeta = \int_E |F_0(\zeta)|^2 \rho(\zeta)\, d\zeta = \nu(\rho)$$

which gives $\nu_1 = \nu(\rho)$. That $f_1 = f_0$ follows from the uniqueness of F_0, since (6.3) and (6.4) then give

$$|f_1(\zeta)| = |f_0(\zeta)|, \qquad \arg f_1(\zeta) = \arg f_0(\zeta) \qquad \text{a.e. on } E.$$

Having reduced the extremum problem to a pair of boundary problems, we find the solution by essentially the same method as in §5. First we show by using Lemma 5.2 [modified to the case where the two functions belong to $H_2(\Omega)$] that F_0 and f_0 extend to C^{0+} functions on E. In fact the second statement of (6.3) implies that the function

$$F_0(z) f_0(z)/G'(z)$$

so extends, and so $|F_0(\zeta) f_0(\zeta)| \in C^{0+}$. But since we already know from the first statement of (6.3) that

$$|f_0(\zeta)/F_0(\zeta)| \in C^{0+},$$

we deduce

$$|F_0(\zeta)| \in C^{0+}, \qquad |f_0(\zeta)| \in C^{0+};$$

then by Lemma 5.2

$$F_0(\zeta) \in C^{0+}, \qquad f_0(\zeta) \in C^{0+}$$

and any zero of F_0 or f_0 on E is of even multiplicity.

The representation of the product $F_0 f_0$ is given by the formula (5.9) exactly as before, the zeros z_j $(j = 1,\dots, p-1)$ of the product satisfying (5.10). To get the representation of F_0, we see that its absolute value is determined by (5.9) and the first half of (6.3). In fact

$$|F_0(\zeta)|^2 = U\rho(\zeta)^{-1} \left| G'(\zeta) \prod_{j=1}^{p-1} \Psi(\zeta, z_j^*, z_j) \right|.$$

This shows incidentally that any z_j on E is a zero of F_0, and that if a particular point appears m times among the $z_j \in E$ then it is a zero of F_0 of multiplicity m [recall Lemma 4.1(2)]. Consequently the function

$$H(z) = U^{-1}R(z)\, F_0(z)^2/G'(z) \prod_{j=1}^{p-1} \Psi(z, z_j^*, z_j)$$

extends continuously and nonzero to E and has there absolute value one. It is therefore expressible in terms of exponential Green's functions Φ with parameter values the zeros and poles of H.

Now since the zeros z_j^* of $G'(z)$ are cancelled by the poles of the product of the exponential Neumann function (and we know there is a pole at ∞ because $G'(\infty) = 0$) we need only concern ourselves with the z_j which are zeros of the product but may or may not be a zero of F_0. In fact any z_j which is a zero of F_0 gives rise to a zero of H and any z_j not a zero of F_0 gives rise to a pole of H. Thus if we set

$$\epsilon_j = \begin{cases} +1 & \text{if } F_0(z_j) = 0, \\ -1 & \text{if } f_0(z_j) = 0, \end{cases}$$

we have

$$H(z) = \Phi(z) \prod_{j=1}^{p-1} \Phi(z, z_j)^{\epsilon_j},$$

and so (recall $\Phi'/\Phi = G'$)

$$F_0(z)^2 = UR(z)^{-1}\,\Phi'(z) \prod_{j=1}^{p-1} \{\Phi(z, z_j)^{\epsilon_j}\,\Psi(z, z_j^*, z_j)\},$$

$$f_0(z)^2 = UR(z)\,G'(z)\,\Phi(z)^{-1} \prod_{j=1}^{p-1} \{\Phi(z, z_j)^{-\epsilon_j}\,\Psi(z, z_j^*, z_j)\}.$$

If the same point is a zero of both F_0 and f_0, it appears twice among the z_j, say, as z_{j_1} and z_{j_2}. One can set $\epsilon_{j_1} = +1$, $\epsilon_{j_2} = -1$ quite arbitrarily since in the product of the Φ^{ϵ_j} the two terms cancel.

One can now substitute $z = \infty$ (or let $z \to \infty$) in the formulas for F_0 and f_0 and use (6.2) to determine the two constants U and $\nu(\rho)$. We obtain [recall the definition (3.1) of $C(E)$]

$$U = R(\infty)\,C(E) \prod_{j=1}^{p-1} |\,\Phi(z_j)^{-\epsilon_j}\,\Psi(\infty, z_j^*, z_j)^{-1}\,|$$

$$\nu(\rho) = 2\pi R(\infty)\,C(E) \prod_{j=1}^{p-1} |\,\Phi(z_j)^{-\epsilon_j}\,|.$$

We are not through because the full set of conditions on the z_j must still be found. Given z_j satisfying (5.10), the functions $F_0(z)$, $f_0(z)$ given by the above formulas satisfy (6.3), but they will generally not be single-valued. The condition that the formulas determine single-valued functions is that the variations of the argument of the right sides around each E_k be integral multiples of 4π. Thus using our formulas (4.3) and (5.14) giving the variations of the arguments of $\Phi(z, z_0)$ and $R(z)$, and the fact [recall (5.7)]

$$\begin{aligned}\underset{E_k}{\Delta} \arg \Phi'(z) &= \underset{E_k}{\Delta}\, \tilde{g}(\zeta) + \underset{E_k}{\Delta} \arg G'(\zeta)\\ &= 2\pi\omega_k(\infty) - \underset{E_k}{\Delta} \arg \frac{d\zeta}{|\,d\zeta\,|} = 2\pi\omega_k(\infty) - 2\pi,\end{aligned}$$

we obtain the system

$$\sum_{j=1}^{p-1} \epsilon_j \omega_k(z_j) \equiv \frac{1}{2\pi} \int_E \log \rho(\zeta)\, \frac{\partial \omega_k}{\partial n_\zeta}\, |\,d\zeta\,| - \omega_k(\infty) + 1 \qquad (\text{mod } 2).$$

Conversely if we have z_j satisfying this system for some choice of $\epsilon_j = \pm 1$, and the system (5.10), then our formulas determine F_0, f_0, $\nu(\rho)$.

The passage to the more general problem:

$(3, \rho, \Gamma)$ For all $F \in H_2(\Omega, \rho, \Gamma)$ satisfying $|F(\infty)| = 1$, determine

$$\nu(\rho, \Gamma) = \inf_F \int_E |F(\zeta)|^2 \rho(\zeta)\, d\zeta,$$

with the weaker assumption (5.19) on ρ, proceeds just as in §5. There is no need to go through the details. Everything is summarized in the following theorem.

Theorem 6.2. *The extremal value for problem* $(3, \rho, \Gamma)$ *is given by*

$$\nu(\rho, \Gamma) = 2\pi R(\infty)\, C(E) \exp\left\{- \sum_{j=1}^{p-1} \epsilon_j g(z_j)\right\}$$

and the extremal functions by

$$F_{\rho,\Gamma}(z)^2 = UR(z)^{-1}\, \Phi'(z) \prod_{j=1}^{p-1} \{\Phi(z, z_j)^{\epsilon_j}\, \Psi(z, z_j^*, z_j)\}$$

where

$$U = R(\infty)\, C(E) \exp\left\{- \sum_{j=1}^{p-1} \epsilon_j g(z_j) - \sum_{j=1}^{p-1} N(\infty, z_j^*, z_j)\right\}.$$

The constants $\epsilon_j = \pm 1$ *and points* z_j *are uniquely determined by the systems*

$$\sum_{j=1}^{p-1} \{\tilde{\omega}_k(z_j) - \tilde{\omega}_k(z_j^*)\} = \sum_{l=1}^{p} m_l P_{kl}\,, \qquad k = 1,\dots, p,$$

$$\sum_{j=1}^{p-1} \epsilon_j \omega_k(z_j) \equiv \frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} |\, d\zeta\,| - \omega_k(\infty) + 1 + 2\gamma_k \pmod 2,$$
$$k = 1,\dots, p.$$

The exact analogues of Theorem 5.7 and Lemma 5.5 also hold.

Theorem 6.3. *In the representation for* $F_0(z)^2$ *the product*

$$U \prod_{j=1}^{p-1} \Psi(z, z_j^*, z_j)$$

is also given by

$$\frac{\nu(\rho, \Gamma)}{2\pi} \left\{1 + \sum_{k=1}^{p-1} a_k \frac{d\Omega_k}{dG}\right\},$$

where a_k are the unique real constants satisfying the system

$$G'(z_j) + \sum_{k=1}^{p-1} a_k \Omega_k'(z_j) = 0, \qquad j = 1, \ldots, p-1.$$

Lemma 6.4. *$\nu(\rho, \Gamma)$ is jointly continuous.*

Neither of the above assertions need be proved again in the present context. However we shall give the details for the analogue of Theorem 5.6.

Theorem 6.5. *The range of $\nu(\rho, \Gamma)$ as Γ runs through $\mathscr{T}_p^{\,0}$ is the closed interval with endpoints*

$$2\pi R(\infty)\, C(E) \exp\{\pm \sum g(z_j^*)\}.$$

Proof. We have to show that $\nu(\rho, \Gamma)$ always lies in this interval and that the two end points are attained for appropriate classes Γ.

The function

$$\prod_{j=1}^{p-1} | \Psi(z, z_j^*, z_j)\, \Phi(z, z_j)\, \Phi(z, z_j^*)^{-1} |$$

is subharmonic in Ω and continuous up to E. Therefore

$$\prod_{j=1}^{p-1} | \Psi(\infty, z_j^*, z)\, \Phi(z_j)\, \Phi(z_j^*)^{-1} |$$

$$\leqslant \frac{1}{2\pi} \int_E \prod_{j=1}^{p-1} | \Psi(\zeta, z_j^*, z_j)| \frac{\partial g}{\partial n_\zeta} | d\zeta |$$

$$= \frac{U^{-1}}{2\pi} \int_E | F_{\rho,\Gamma}(\zeta)|^2 \rho(\zeta) \, | d\zeta | = \nu(\rho, \Gamma)/2\pi U.$$

If we use the expressions for U and $\nu(\rho, \Gamma)$ given in Theorem 6.2 we see that the right side is just

$$\prod_{j=1}^{p-1} | \Psi(\infty, z_j^*, z_j)|$$

and so we obtain

$$\prod_{j=1}^{p-1} | \Phi(z_j)| \leqslant \prod_{j=1}^{p-1} | \Phi(z_j^*)|.$$

The two inequalities for $\nu(\rho, \Gamma)$ follow immediately from this and the form of $\nu(\rho, \Gamma)$.

To show that $\nu(\rho, \Gamma)$ assumes the two end-points, we have, clearly, to find the two classes Γ which give rise to the solutions

$$z_j = z_j^*, \qquad \epsilon_j = +1, \qquad j = 1,\dots,p-1,$$
$$z_j = z_j^*, \qquad \epsilon_j = -1, \qquad j = 1,\dots,p-1,$$

respectively. Thus the two extreme Γ are given by

$$\gamma_k \equiv -\frac{1}{4\pi}\int_E \log \rho(\zeta)\frac{\partial\omega_k}{\partial n_\zeta}\,|\,d\zeta\,|\pm\frac{1}{2}\sum_{j=1}^{p-1}\omega_k(z_j^*)+\frac{1}{2}\,\omega_k(\infty)-\frac{1}{2} \qquad (\text{mod } 1)$$
$$k = 1 \dots p.$$

Finally we would like to say a few words about the systems of equations, appearing in the statement of Theorem 6.2, which determine the z_j and ϵ_j. They appear similar to the corresponding system, in Theorem 5.4, for the z_j and q. There are however a couple of differences. We already know that the present system always has a unique solution whereas the other may not. A further difference is that the solution of the present system may be reduced to a classical problem, namely the Jacobi inversion problem for abelian integrals.

We consider again the Riemann surface S which is the double of Ω, and the differentials $d\Omega_k$ which span the space of everywhere-analytic differentials on S. Let us determine the periods of the $d\Omega_k$. A homology basis of S consists of any $p-1$ of the E_k together with any $p-1$ of the closed curves E_k', which consists of an arc lying in Ω joining E_k to E_{k+1} (to E_1 if $k = p$) together with its conjugate lying in the copy of Ω. The period of $d\Omega_k$ about E_j is

$$\int_{E_j} d(\omega_k + i\tilde{\omega}_k) = iP_{jk}\,.$$

As for the period about E_j', observe that since $\tilde{\omega}_k$ takes negative values at conjugate points, the period is just the period of $d\omega_k$, that is

$$\mathop{\Delta}_{E_j'} \omega_k\,.$$

Now if neither j nor $j+1$ is k, this is zero because ω_k vanishes on E_j and E_{j+1}; if $j+1 = k$, then ω_k increases from 0 to 1 as we go along the "upper half" of E_j' (the part lying in Ω) and continues to increase from 1 to 2 along the lower half. Thus

$$\mathop{\Delta}_{E'_{k+1}} \omega_k = 2$$

and similarly

$$\mathop{\Delta}_{E'_k} \omega_k = -2.$$

In any case,

$$\int_{E'_j} d\Omega_k \equiv 0 \qquad (\text{mod } 2).$$

Now let us consider the following Jacobi inversion problem [see (*12*), section 10-8] on S: Determine points z'_j on S ($j = 1,\ldots, p-1$) such that for each k

$$\sum_{j=1}^{p-1} \int_{z_j^*}^{z'_j} d\Omega_k \equiv \frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} \mid d\zeta \mid - \omega_k(\infty) + 1 + 2\gamma_k - \sum_{j=1}^{p-1} \omega_k(z_j^*)$$
$$(\text{mod periods}).$$

This system can always be solved for points z'_j on S; it is equivalent to the system of Theorem 6.2 with all $\epsilon_j = +1$, but of course in the latter system we must have all $z_j \in S$. The exact connection between the two is the following. Some of the z'_j lie in Ω, some in its copy. Set

$$z_j = \begin{cases} z'_j & \text{if} \quad z'_j \in \Omega \quad \text{or} \quad z'_j \in E, \\ \overline{z'_j} & \text{otherwise;} \end{cases}$$

$$\epsilon_j = \begin{cases} +1 & \text{if} \quad z'_j \in \Omega \quad \text{or} \quad z'_j \in E, \\ -1 & \text{otherwise.} \end{cases}$$

Then all the z_j lie in Ω or E and the two systems are completely equivalent; it is only necessary to notice that

$$\omega_k(\bar{z}) \equiv -\omega_k(z) \qquad (\text{mod } 2).$$

The Jacobi inversion problem may be solved (more exactly, reduced to an algebraic equation) by the use of theta functions. We refer the reader to (*10*) for details.

7. The Szegö Kernel Function

The kernel function $K(z, z_0)$ associated with a region Ω and weight function ρ on E is characterized by the reproducing property

$$F(z_0) = \int_E F(\zeta) \overline{K(\zeta, z_0)} \rho(\zeta) \mid d\zeta \mid, \qquad F \in H_2(\Omega, \rho). \tag{7.1}$$

This function was introduced by G. Szegö who considered the case of Ω the interior of a circle, or more generally the interior on a Jordan curve, and was shown by him to be expressible as an infinite series involving the polynomials orthonormal with respect to the weight ρ. See (*17*), Chapter XVI.

The existence of the kernel function follows from abstract considerations. As before Ω will be the exterior of $E = E_1 \cup \cdots \cup E_p$. With the inner product

$$\int_E F_1(\zeta) \overline{F_2(\zeta)} \rho(\zeta) \mid d\zeta \mid$$

$H_2(\Omega, \rho)$ is a Hilbert space. The linear functional $F \to F(z_0)$ is continuous since

$$| F(z_0)^2 R(z_0)| \leqslant \frac{1}{2\pi} \int_E | F(\zeta)|^2 \rho(\zeta) \frac{\partial g(\zeta, z_0)}{\partial n_\zeta} \mid d\zeta \mid. \tag{7.2}$$

(The right side of the inequality is the value at z_0 of the smallest harmonic majorant of $| F(z)^2 R(z)|$.) Thus by the Riesz representation theorem there is a unique function

$$K(z, z_0) \in H_2(\Omega, \rho)$$

for which (7.1) holds.

Let us see how to obtain K for the space $H_2(\Omega, \rho, \Gamma)$. We use the same trick as before to reduce this to the single-valued case. Define $V_\Gamma(z)$ as before by (5.16), let the weight function σ be defined by

$$\sigma(\zeta) = e^{2\lambda_k} \rho(\zeta), \qquad \zeta \in E_k$$

and let $K_1(z, z_0)$ be the kernel function associated with the space $H_2(\Omega, \sigma)$. Then for any $F \in H_2(\Omega, \rho, \Gamma)$ we have $FV_\Gamma^{-1} \in H_2(\Omega, \sigma)$ and so

$$F(z_0) V_\Gamma(z_0)^{-1} = \int_E F(\zeta) V_\Gamma(\zeta)^{-1} \overline{K_1(\zeta, z_0)} \sigma(\zeta) \mid d\zeta \mid.$$

Now for $\zeta \in E$ we have

$$\sigma(\zeta) = \exp[2 \sum_{k=1}^{p} \lambda_k \omega_k(\zeta)]\, \rho(\zeta) = V_\Gamma(\zeta)\, \overline{V_\Gamma(\zeta)}\, \rho(\zeta)$$

and so we can write

$$F(z_0) = \int_E F(\zeta)\, \overline{V_\Gamma(\zeta)}\, \overline{K_1(\zeta, z_0)}\, V_\Gamma(z_0)\, \rho(\zeta) \mid d\zeta \mid.$$

It follows that if we set

$$K(\zeta, z) = V_\Gamma(\zeta)\, K_1(\zeta, z)\, \overline{V_\Gamma(z)}$$

then

$$F(z_0) = \int_E F(\zeta)\, \overline{K(\zeta, z_0)}\, \rho(\zeta) \mid d\zeta \mid, \qquad F \in H_2(\Omega, \rho, \Gamma) \tag{7.3}$$

and K is the required kernel function. We shall write $K_{\rho,\Gamma}(z, z_0)$ instead of $K(z, z_0)$ if it is necessary to exhibit its dependence on ρ and Γ.

If we set $F(z) = K(z, z_1)$ in (7.3) we obtain

$$K(z_0 , z_1) = \int K(\zeta, z_1)\, \overline{K(\zeta, z_0)}\, \rho(\zeta) \mid d\zeta \mid \tag{7.4}$$

and so, in particular, we have the identities

$$K(z_0 , z_1) = \overline{K(z_1 , z_0)},$$

$$\int_E \mid K(\zeta, z_0)\mid^2 \rho(\zeta)\, d\zeta = K(z_0 , z_0). \tag{7.5}$$

Because of all the multiple-valuedness it might perhaps be in order to say a few words about the meaning of statements such as (7.3). Of course one could use the functions V_Γ to reduce everything to the single-valued case, but this is needlessly cumbersome. One could reduce everything to the simply-connected case by considering the universal covering surface of Ω, so that the integral in (7.3) is transformed into an integral involving single-valued functions taken over its boundary. This is probably the most elegant, although least elementary, way of dealing with the situation. The simplest way, however, is probably the following. Introduce a system of cuts C so that $\Omega - C$ is simply connected. Each function of $H_2(\Omega, \rho, \Gamma)$ has a representative (generally infinitely many) which is single-valued in $\Omega - C$. Then for each such representative $F(z)$, and for the representative of $K(z, z_0)$ satisfying $K(z_0 , z_0) > 0$, the identity (7.3) and the succeeding ones hold.

It follows immediately from (7.3) using Schwarz's inequality and (7.5) that, for all $F \in H_2(\Omega, \rho, \Gamma)$ satisfying $|F(z_0)| = 1$, we have

$$\int_E |F(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid \geqslant K(z_0, z_0)^{-1},$$

with equality occurring for

$$F(z) = K(z, z_0)/K(z_0, z_0).$$

Thus, in the notation of the last section,

$$K(z, \infty) = \nu(\rho, \Gamma)^{-1} F_{\rho,\Gamma}(z). \tag{7.6}$$

The form of $K(z, z_0)$ for general z_0 would have been easy to derive from this if the problem $(3, \rho, \Gamma)$ had been stated in a conformally invariant manner. But it was not, since clearly $\rho(\zeta) \mid d\zeta \mid$ is not invariant. What we shall do therefore is replace the problem by an equivalent but conformally invariant one, for which the transition from $z_0 = \infty$ to general z_0 will be immediate. Let us define ρ_0 by

$$\rho(\zeta) = \rho_0(\zeta) \frac{\partial g(\zeta, \infty)}{\partial n_\zeta} \tag{7.7}$$

Then $(3, \rho, \Gamma)$ is equivalent to the following:

For all $F \in H_2(\Omega, \rho, \Gamma)$ satisfying $|F(\infty)| = 1$, determine

$$\nu = \inf_F \int_E |F(\zeta)|^2 \rho_0(\zeta) \mid dG(\zeta, \infty)|. \tag{7.8}$$

To express the solution of this problem in terms of ρ_0 rather than ρ we must write down the function, without zeros or poles in Ω, which on E has absolute value $\partial g(\zeta, \infty)/\partial n_\zeta$. This is just

$$\Phi'(z) \prod_{j=1}^{p-1} \Phi(z, z_j^*).$$

Hence if $R_0(z)$ is the function without zeros or poles in Ω which on E has absolute value $\rho_0(\zeta)$ we find

$$R(z) = R_0(z)\, \Phi'(z) \prod_{j=1}^{p-1} \Phi(z, z_j^*),$$

and so the solution to the problem in terms of ρ_0 rather than ρ is given by the formulas [see Theorem 6.2 and recall $C(E)^{-1} = \Phi'(\infty)$]

$$\nu_{\rho,\Gamma} = 2\pi R_0(\infty) \exp\left\{\sum_{j=1}^{p-1} g(z_j^*) - \sum_{j=1}^{p-1} \epsilon_j g(z_j)\right\},$$

$$F_{\rho,\Gamma} = UR_0(z)^{-1} \prod_{j=1}^{p-1} \{\Phi(z, z_j^*)^{-1}\, \Phi(z, z_j)^{\epsilon_j}\, \Psi(z, z_j^*, z_j)\}.$$

From this solution it is clear how to solve the modification of the problem where, in (7.8), the parameter value ∞ is replaced by an arbitrary z_0. From this solution the kernel function is obtained from the analogue of (7.6) with ρ obtained from ρ_0 by (7.7) with ∞ replaced by z_0. In this case the connection between R and R_0 is given by

$$R(z) = R_0(z)\, \Phi(z)^2\, \Phi(z, z_0)^{-2}\, \Phi'(z, z_0) \prod_{j=1}^{p-1} \Phi(z, z_j^*(z_0)),$$

where $z_j^*(z_0)$ are the zeros of $\Phi'(z, z_0)$ other than the double zero at $z = \infty$. (Thus $z_j^*(z_0)$ are exactly the zeros of the differential $dG(z, z_0)$.)

In analogy with the definition of $C(E)$ in terms of the behavior of $\Phi(z)$ at ∞, we set

$$C_{z_0}(E)^{-1} = \lim_{z\to z_0} |(z - z_0)\, \Phi(z, z_0)|.$$

The preceding discussion gives the following result.

Theorem 7.1. *The square of the Szegö kernel function $K(z, z_0)$ is a constant (depending on z_0) times*

$$R(z)^{-1}\, \Phi'(z, z_0)\, \Phi(z)^2\, \Phi(z, z_0)^{-2} \prod_{j=1}^{p-1} \{\Phi(z, z_j)^{\epsilon_j}\, \Psi(z, z_j^*(z_0), z_j)\}$$

where the constant is determined by

$$K(z_0, z_0) = (2\pi)^{-1}\, R(z_0)^{-1}\, C_{z_0}(E)\, \Phi(z_0)^2 \exp\left\{\sum_{j=1}^{p-1} \epsilon_j\, g(z_j, z_0)\right\}.$$

The constants $\epsilon_j = \pm 1$ and points z_j are uniquely determined by the systems

$$\sum_{j=1}^{p-1} \{\tilde{\omega}_k(z_j) - \tilde{\omega}_k(z_j^*(z_0))\} = \sum_{l=1}^{p} m_l P_{kl}, \qquad k = 1,\dots, p,$$

$$\sum_{j=1}^{p-1} \epsilon_j \omega_k(z_j) \equiv \frac{1}{2\pi} \int_E \log \rho(\zeta)\, \frac{\partial \omega_k}{\partial n_\zeta}\, |\, d\zeta\, | + \omega_k(z_0) - 2\omega_k(\infty) + 1 + 2\gamma_k \pmod 2, \qquad k = 1,\dots, p.$$

The last system is equivalent to $K(z, z_0)$, as given by the square root of the expression above, belonging to class Γ; the system is obtained more easily by using this fact directly than by passing back and forth between ρ and ρ_0 in the corresponding system in the statement of Theorem 6.2.

The analogue of Theorem 6.3 is

Theorem 7.2. *In the representation for $K(z, z_0)^2$, the product*

$$\prod_{j=1}^{p-1} \Psi(z, z_j^*(z_0), z_j)$$

is a constant times

$$1 + \sum_{k=1}^{p-1} a_k \frac{d\Omega_k(z)}{dG(z, z_0)},$$

where the a_k are the unique real constants satisfying the system

$$G'(z_j, z_0) + \sum_{k=1}^{p-1} a_k' \Omega_k(z_j) = 0, \qquad j = 1,..., p - 1.$$

We also have the following description of K as a function of its variables. We assume $E \in C^{1+}$.

Theorem 7.3. (1) *$K_{\rho,\Gamma}(z, z_0)$ is jointly continuous in ρ, Γ, and z, $z_0 \in \Omega$.*

(2) *The range of $K_{\rho,\Gamma}(z_0, z_0)$ for fixed ρ and z_0 with variable Γ is the closed interval with endpoints*

$$(2\pi)^{-1} R(z_0)^{-1} C_{z_0}(E) \Phi(z_0)^2 \exp\left\{\pm \sum_{j=1}^{p-1} g(z_j^*(z_0), z_0)\right\}.$$

(3) *If $E \in C^{\alpha+}$ ($\alpha \geqslant 1$) then*

$$R(z)^{1/2} K_{\rho,\Gamma}(z, z_0)$$

as a function of z, extends continuously to E and belongs to class $C^{\alpha-1+}$, uniformly for all Γ and z_0 in any closed subset of Ω.

Proof. The second statement of the theorem follows easily from Theorem 6.5 by conformal mapping. The third follows from the form of K given by Theorem 7.1 upon application of Lemma 4.1.

After all we have done perhaps the simplest way to get the continuity of K is to use the equivalent Jacobi inversion problem. (See the end of §6.) If we take any point $z^\dagger$ then our two systems are equivalent to

$$\sum_{j=1}^{p-1} \int_{z^\dagger}^{z_j'} d\Omega_k \equiv \frac{1}{2\pi} \int_E \log \rho(\zeta) \frac{\partial \omega_k}{\partial n_\zeta} \mid d\zeta \mid + \sum_{j=1}^{p-1} \{ i\tilde{\omega}_k(z_j^*(z_0)) - \Omega_k(z^\dagger)\}$$
$$+ \omega_k(z_0) - 2\omega_k(\infty) + 1 + 2\gamma_k \text{ (mod periods)}, \qquad k = 1,\dots, p,$$

where the z_j' and ϵ_j, z_j are connected by

$$z_j = \begin{cases} z_j' & \text{if } z_j' \in \Omega \text{ or } E, \\ \overline{z_j'} & \text{otherwise;} \end{cases}$$

$$\epsilon_j = \begin{cases} +1 & \text{if } z_j' \in \Omega \text{ or } E, \\ -1 & \text{otherwise.} \end{cases}$$

Now the expressions on the right in this system are clearly continuous in ρ, Γ, and z_0. [The points $z_j^*(z_0)$, being the zeros of a function in which the parameter z_0 appears analytically, are continuous in z_0.] Also

$$(z_j') \to \left(\sum_{j=1}^{p-1} \int_{z^\dagger}^{z_j'} d\Omega_k \right),$$

which maps the $(p - 1)$-fold direct product of S into the p-fold direct sum of the complex numbers mod periods, is an open mapping. Therefore the solution of the system for z_j', which we know to be unique, varies continuously with the right sides. Hence the z_j' are continuous in ρ, Γ, z_0.

If we recall that the extension of $g(z, \zeta)$ to S is given by

$$g(\bar{z}, \zeta) = -g(z, \zeta), \qquad z \in \Omega,$$

we see that

$$\sum_{j=1}^{p-1} \epsilon_j g(z_j, z_0) = \sum_{j=1}^{p-1} g(z_j', z_0).$$

Thus the continuity of $K_{\rho,\Gamma}(z_0, z_0)$ follows from the continuity of $g(z, \zeta)$ in its variables as long as we can show that z_0 is never among the z_j. If it were, however, the corresponding $K_{\rho,\Gamma}(z_0, z_0)$ would be zero or infinite which is clearly not so. The continuity of $C_{z_0}(E)\, \Phi(z_0)^2$, which also appears in the expression for $K(z_0, z_0)$, is obvious for

$z_0 \neq \infty$ and is easily checked even for $z_0 = \infty$, where it has the limiting value $C(E)^{-1}$.

The joint continuity of $K_{\rho,\Gamma}(\zeta, z_0)\,\rho(\zeta)^{1/2}$ with $\zeta \in E$ follows from what we have done if we use parts (1) and (3) of Lemma 4.1, and the continuity of $K_{\rho,\Gamma}(z, z_0)$ follows from this if we use (7.4).

The following corollary shows how estimates on E for $H_2(\Omega, \rho, \Gamma)$ functions imply estimates in Ω.

Corollary 7.4. *Given ρ and a closed subset A of Ω, there is a constant M, independent of Γ, for which we have*

$$\max_A |F(z)|^2 \leqslant M \int_E |F(\zeta)|^2 \rho(\zeta) \, | d\zeta |, \qquad F \in H_2(\Omega, \rho, \Gamma).$$

Proof. We have to check that

$$\max_\Gamma K_{\rho,\Gamma}(z_0, z_0),$$

as given by the theorem, is bounded for $z_0 \in A$. In fact it is continuous for $z_0 \in \Omega$.

The corollary of course could have been proved without using the kernel function. It is for example an immediate consequence of (7.2) and Lemma 4.1.

8. Tchebycheff Polynomials for a System of Jordan Curves

We now carry through in detail what was outlined in §3. The constants $M_{n,\rho}$ are defined by (1.1); the corresponding extremal polynomial, the Tchebycheff polynomial, is denoted by $T_{n,\rho}$. The class Γ_n always denotes $-n\Gamma(\Phi)$. Thus for any monic n'th degree polynomial Q we have

$$\Gamma(Q\Phi^{-n}) = \Gamma_n, \qquad |Q\Phi^{-n}(\infty)| = C(E)^n. \tag{8.1}$$

All weight functions $\rho(\zeta)$ considered in this section will be assumed to be bounded as well as to satisfy (5.19). This implies that $|R(z)|$ is bounded, and so

$$Q(z)\,\Phi(z)^{-n} \in H_\infty(\Omega, \rho, \Gamma_n).$$

It follows therefore from (8.1) that

$$M_{n,\rho} \geqslant C(E)^n \mu(\rho, \Gamma_n),$$

where $\mu(\rho, \Gamma)$ is the extremal quantity in problem $(1, \rho, \Gamma)$ of §5.

What we have to do is produce a polynomial for which the corre-ponding maximum is not much more than $C(E)^n \mu(\rho, \Gamma_n)$, and the idea is to use $C(E)^n$ times the polynomial given by (3.6) with

$$F_n = F_{\rho,\Gamma_n}.$$

This will generally work, although there has to be a little modification in case some of the z_j appearing in the representation for $F_{\rho,\Gamma}$ (see Theorem 5.4) are close to E.

The following technical lemmas allow us to estimate integrals such as (3.6).

Lemma 8.1. *Suppose we have a function f of class $C^{\alpha+m+}$ on a closed interval. Then*

$$f(x) = f(y) + (x-y)f'(y) + \cdots + \frac{(x-y)^{m-1}}{(m-1)!} f^{(m-1)}(y) + (x-y)^m g(x\ y),$$

where $g(x, y)$ is of class $C^{\alpha+}$ on the square. Moreover if we have a family of f uniformly of class $C^{\alpha+m+}$ then the corresponding g are uniformly $C^{\alpha+}$.

Proof. By Taylor's theorem with integral remainder,

$$g(x, y) = \frac{1}{(m-1)!} \int_0^1 t^{m-1} f^{(m)}(x + t(y-x))\, dt.$$

The conclusions of the lemma are immediate.

Lemma 8.2. *Suppose that $E \in C^{\alpha+}$ $(\alpha \geqslant 2)$ and that*

$$F \in H_\infty(\Omega, \Gamma_n)$$

and extends continuously to a function of class $C^{\beta+}$ $(\beta \leqslant \alpha - 2)$ on E. Define the polynomial Q by

$$Q(z) = \frac{1}{2\pi i} \int_C F(\zeta)\, \Phi(\zeta)^n \frac{d\zeta}{\zeta - z}$$

where C is a Jordan curve, described once counterclockwise, containing E and z in its interior. Then uniformly in $\bar{\Omega}$

$$Q(z) = \Phi(z)^n \{F(z) + o(n^{2-\alpha} + n^{-\beta})\},$$

and this holds uniformly for any family of F's, which are uniformly of class $C^{\beta+}$ on E.

Proof. We have for $z \in \Omega$

$$Q(z) = F(z)\,\Phi(z)^n + \frac{1}{2\pi i}\int_E F(\zeta)\,\Phi(\zeta)^n \frac{d\zeta}{\zeta - z},$$

the integral representing a function $f(z)$ analytic in Ω and continuous up to E. It suffices to show that uniformly on E

$$f(\zeta) = o(n^{2-\alpha} + n^{-\beta}), \tag{8.2}$$

and for this the analyticity of $F(z)$ will no longer be necessary, but only the continuity of $F(\zeta)$. We may assume therefore that $F(\zeta)$ is supported on a single component, say E_1, of E. In addition we need prove (8.2) only for $\zeta \in E_1$ for the integral, representing a function analytic outside E_1, assumes its maximum on E_1.

The boundary function $f(\zeta)$ is given by

$$f(\zeta_0) = -\frac{1}{2}F(\zeta_0)\,\Phi(\zeta_0)^n + \frac{1}{2\pi i}\int_{E_1} F(\zeta)\,\Phi(\zeta)^n \frac{d\zeta}{\zeta - \zeta_0},$$

where the integral is a Cauchy principal value. Now we would like to introduce the variable change $s = \Phi(\zeta)$ and say that as ζ runs over E_1, s runs over the unit cricle. This would be true if E consisted only of E_1, but in our case

$$\underset{E_1}{\Delta} \arg \Phi(\zeta) = 2\pi\omega_1(\infty),$$

which is less than 2π if $p > 1$. Accordingly we set

$$s = \Phi(\zeta)^{1/\omega_1(\infty)}$$

so that s does run over the unit circle. However because Φ is multiple-valued, and also involves an arbitrary constant factor of absolute value 1, this is not quite definite. We make it definite by prescribing that the point $\zeta = \zeta_0$ corresponds to $s = 1$. We can then write (note $\Phi(\zeta_0) = 1$)

$$\begin{aligned} f(\zeta_0) &= -\frac{1}{2}F(\zeta_0) + \frac{1}{2\pi i}\int_{|s|=1} s^{n\omega_1(\infty)} F(\zeta(s)) \frac{\zeta'(s)}{\zeta(s)-\zeta(1)}\, ds \\ &= -\frac{1}{2}F(\zeta_0) + \frac{F(\zeta_0)}{2\pi i}\int_{|s|=1} s^{[n\omega_1(\infty)]}\frac{ds}{s-1} \\ &\quad + \frac{1}{2\pi i}\int_{|s|=1} s^{[n\omega_1(\infty)]}\left(s^{\{n\omega_1(\infty)\}}F(\zeta(s))\frac{\zeta'(s)}{\zeta(s)-\zeta(1)} - \frac{F(\zeta_0)}{s-1}\right) ds \end{aligned}$$

where square brackets denote "greatest integer in" and curly brackets "fractional part of."

Now the next to last integral is exactly πi since for $|s_0| > 1$ we have

$$\begin{aligned} 0 &= \int_{|s|=1} s^{[n\omega_1(\infty)]}\frac{ds}{s-s_0} \\ &= \lim_{s_0\to 1}\int_{|s|=1} s^{[n\omega_1(\infty)]}\frac{ds}{s-s_0} \qquad (|s_0| > 1) \\ &= -\pi i + \int_{|s|=1} s^{[n\omega_1(\infty)]}\frac{ds}{s-1}. \end{aligned}$$

Hence

$$f(\zeta_0) = \frac{1}{2\pi i}\int_{|s|=1} s^{[n\omega_1(\infty)]}\left(s^{\{n\omega_1(\infty)\}}F(\zeta(s))\frac{\zeta'(s)}{\zeta(s)-\zeta(1)} - \frac{F(\zeta_0)}{s-1}\right) ds.$$

Thus $f(\zeta_0)$ is just the Fourier coefficient with index

$$-[n\omega_1(\infty)] - 1$$

of the function inside the large parentheses. This function may be written

$$F_1(s)\frac{\left\{\zeta'(s) - \dfrac{\zeta(s)-\zeta(1)}{s-1}\right\}\dfrac{1}{s-1}}{\dfrac{\zeta(s)-\zeta(1)}{s-1}} + \frac{F_1(s)-F_1(1)}{s-1}, \tag{8.3}$$

where we have set

$$F_1(s) = s^{\{n\omega_1(\infty)\}}F(\zeta(s)).$$

The functions $F_1(s)$ are uniformly of class $C^{\beta+}$ of the unit circle; by Lemma 4.1(1) the functions $\zeta'(s)$ are uniformly of class $C^{\alpha-1+}$ and bounded away from zero. Let us apply Lemma 8.1 to $\zeta(s)$ twice, once with $m = 1$ and once with $m = 2$. We find that the first of the two functions in (8.3) is of class $C^{\alpha-2+}$. By well-known estimates of Fourier coefficients [(*22*), p. 45], if this function is multiplied by s^m and integrated over the unit circle the result is $o(m^{2-\alpha})$ as $m \to \infty$. Consequently we can write

$$f(\zeta_0) = \frac{1}{2\pi i}\int_{|s|=1} s^m \frac{F_1(s)-1}{s-1}\,ds + o(n^{2-\alpha}), \tag{8.4}$$

where we have set

$$m = [n\omega_1(\infty)].$$

We now show that this last integral is $o(n^{-\beta})$. Note that the sort of argument just used gives the estimate $o(n^{1-\beta})$. This however can be improved as we shall now see. The function

$$\int_{|s|=1} \frac{F_1(s)}{s-t}\,ds, \qquad |t| > 1$$

is analytic for $|t| > 1$ and, by Privaloff's theorem on singular integrals quoted in §4, extends continuously to a $C^{\beta+}$ function on $|t| = 1$. If we denote by $S_m(t)$ the sum of m terms of the Fourier series of the function then we have for $m < m'$,

$$\begin{aligned} S_{m'}(1) - S_m(1) &= \sum_{k=m}^{m'-1} \frac{1}{2\pi i}\int_{|t|=2} t^k\,dt \int_{|s|=1} \frac{F_1(s)}{s-t}\,ds \\ &= \frac{1}{2\pi i}\int_{|s|=1} F_1(s)\,ds \int_{|t|=2} \frac{t^{m'}-t^m}{(s-t)(t-1)}\,dt \\ &= \int_{|s|=1} \frac{s^m - s^{m'}}{s-1} F_1(s)\,ds \\ &= \int_{|s|=1} \frac{s^m - s^{m'}}{s-1}\{F_1(s)-1\}\,ds. \end{aligned}$$

Now it is known [(*22*), p. 64] that for a function satisfying a Lipschitz condition of order δ the difference between the function and S_m is uniformly $O(m^{-\delta}\log m)$; it follows easily that for a function of class $C^{\beta+}$ this difference is $o(m^{-\beta})$. Hence

$$\int_{|s|=1} \frac{s^m - s^{m'}}{s-1}\{F_1(s)-1\}\,ds = o(m^{-\beta}) \qquad (m < m'). \tag{8.5}$$

Since the function

$$\frac{F_1(s) - 1}{s - 1}$$

belongs to L_1, we know that

$$\int_{|s|=1} s^{m'} \frac{F_1(s) - 1}{s - 1} ds \to 0, \qquad m' \to \infty.$$

Hence letting $m' \to \infty$ in (8.5) we find

$$\int_{|s|=1} s^{m} \frac{F_1(s) - 1}{s - 1} ds = o(m^{-\beta}).$$

If we combine this with (8.4) we obtain

$$f(\zeta_0) = o(n^{2-\alpha} + n^{-\beta})$$

and the lemma is established.

In the special case where $F = 1$ and E consists of a single contour, the lemma is a known asymptotic formula for the Faber polynomials [(*13*), Theorem 2].

Before we state the main result of this section, we should like to point out that the constant $M_{n,\rho}$ is unchanged if ρ is replaced by its upper semicontinuous modification

$$\overline{\lim_{\zeta_0 \to \zeta}}\, \rho(\zeta_0),$$

in which case the "sup" in the expression (1.1) defining $M_{n,\rho}$ may be replaced by "max". There is therefore no loss of generality if we assume ρ is itself upper semicontinuous.

We denote by $F_n(z)$ the extremal function F_{ρ,Γ_n} for the problem $(1, \rho, \Gamma_n)$ found in §5 and given by Theorem 5.4.

Theorem 8.3. *Suppose that $E \in C^{2+}$ and that ρ is bounded above, upper semicontinuous, and satisfies*

$$\int_E \log \rho(\zeta) \mid d\zeta \mid > -\infty.$$

Then as $n \to \infty$, we have

$$M_{n,\rho} \sim C(E)^n \mu(\rho, \Gamma_n)$$

and

$$T_{n,\rho}(z) = C(E)^n \Phi(z)^n \{F_n(z) + o(1)\} \tag{8.6}$$

uniformly in each closed subset of Ω.

If $E \in C^{2p+1+}$ *and* ρ *is nonzero and of class* C^{2p-1+}, *then* (8.6) *holds uniformly in* $\bar{\Omega}$ *for any sequence* $n \to \infty$ *with the property that the zeros of* F_n *are bounded away from* E.

Proof. Let us assume first that $E \in C^{\alpha+}$ ($\alpha \geqslant 2$) and that ρ is nonzero and of class $C^{\alpha-2+}$.

The function F_n is $\mu(\rho, \Gamma_n)$ times

$$R(z)^{-1} \prod_{j=1}^{q} \Phi(z, z_j)^{-1},$$

where the constant q and points z_j depend on n. It follows from Lemma 4.1(4) that $R(z)^{-1}$ extends continuously to a $C^{\alpha-1+}$ function on E. This also holds for each $\Phi(z, z_j)$, by Lemma 4.1(1), but not uniformly in n unless the z_j are bounded away from E. We therefore modify the product by deleting all terms corresponding to z_j within δ of E. We denote the modified product by Π_1 and the deleted terms by Π_2.

But now we have changed the class of the function. To restore it to class Γ_n we must multiply by a function of class

$$\Gamma' = \Gamma\left(\prod\nolimits_2 \Phi(z, z_j)^{-1}\right) = \left(-\sum\nolimits_{j\,2} \omega_k(z_j)\right),$$

which, however, is very smooth on E. Such a function is V_Γ, as given by (5.16). To determine the coefficients λ_k in the expression for $V_{\Gamma'}$ we must specify constants

$$\gamma_k \equiv -\sum\nolimits_2 \omega_k(z_j) \qquad (\text{mod } 1) \tag{8.7}$$

satisfying $\Sigma\gamma_k = 0$. We do this as follows. Since each z_j is very close to (within δ of) E, each $\omega_k(z_j)$ is very close to either 0 or 1. We define γ_k to be the difference between

$$-\sum\nolimits_2 \omega_k(z_j)$$

and its nearest integer. Of course each γ_k is very small for δ small, and by (8.7) we have

$$\sum_{k=1}^{p} \gamma_k \equiv -\sum\nolimits_2 \left(\sum_{k=1}^{p} \omega_k(z_j)\right) = -\sum\nolimits_2 1 \equiv 0 \qquad (\text{mod } 1)$$

This implies $\Sigma\gamma_k = 0$ and so we have chosen an acceptable representative (γ_k) for Γ'. The corresponding λ_k are uniformly bounded (indeed they tend to zero as $\delta \to 0$, uniformly in n). Consequently, by Lemma 4.1(4), $V_{\Gamma'}$ is in $C^{\alpha+}$ uniformly in n.

We now introduce the nth-degree polynomial

$$Q(z) = \frac{1}{2\pi i} \int_C R(\zeta)^{-1} \prod_1 \Phi(z, z_j)^{-1} V_{\Gamma'}(\zeta)\, \Phi(\zeta)^n \frac{d\zeta}{\zeta - z}, \tag{8.8}$$

where C is a Jordan curve described once counterclockwise and containing E and z in its interior. It follows from Lemma 8.2 that

$$Q(z) = R(z)^{-1} \prod_1 \Phi(z, z_j)^{-1} V_{\Gamma'}(z)\, \Phi(z)^n + o(n^{2-\alpha}) \tag{8.9}$$

uniformly for $z \in \bar{\Omega}$.

Suppose now that δ was so chosen that the corresponding λ_k in the expression for $V_{\Gamma'}$ have absolute value at most ϵ/p. Then

$$| V_{\Gamma'}(z)|^{\pm 1} \leqslant e^{\epsilon}, \qquad z \in \bar{\Omega}.$$

Hence we find that

$$\max_E | Q(\zeta)|\, \rho(\zeta) \leqslant e^{\epsilon} + o(n^{2-\alpha}) \leqslant e^{2\epsilon}$$

for sufficiently large n, and that the coefficient of z^n in $Q(z)$ has absolute value at least

$$R(\infty)^{-1}\, C(E)^{-n}\, e^{-\epsilon} \exp\{-\textstyle\sum_1 g(z_j)\} \geqslant e^{-\epsilon} C(E)^{-n}\, \mu_n(\rho, \Gamma)^{-1}.$$

This gives the upper bound

$$M_{n,\rho} \leqslant e^{3\epsilon} C(E)^n\, \mu_n(\rho, \Gamma)$$

and so the first statement of the theorem is established, at least for $\rho(\zeta)$ nonzero and of class C^{1+}.

To remove this restriction, take any $\rho(\zeta)$ satisfying the hypotheses of the theorem. Since ρ is upper semicontinuous we can find a sequence

$$\rho_m \searrow \rho$$

where each ρ_m is nonzero and of class C^{1+}. By the Lebesgue monotone convergence theorem

$$\mathrm{dist}(\rho_m , \rho) = \int_E | \log \rho_m(\zeta) - \log \rho(\zeta)|\, | d\zeta | = \int_E \log \frac{\rho_m(\zeta)}{\rho(\zeta)}\, | d\zeta | \to 0.$$

It follows therefore by Lemma 5.5, and the compactness of $\mathscr{T}_p^0$, that for m sufficiently large we shall have

$$\mu(\rho_m, \Gamma_n) \leqslant \mu(\rho, \Gamma_n) + \epsilon$$

for all n. With this m,

$$M_{n,\rho_m} \leqslant C(E)^n \{\mu(\rho_m, \Gamma_n) + \epsilon\}$$

for large enough n, and so

$$M_{n,\rho} \leqslant M_{n,\rho_m} \leqslant C(E)^n \{\mu(\rho, \Gamma_n) + 2\epsilon\}.$$

This completes the proof the first statement of the theorem.

The second statement is a direct consequence of the first. The functions

$$F_n(z)\, R(z), \qquad T_{n,\rho}(z)\, R(z)/C(E)^n\, \Phi(z)^n$$

are both bounded on E, and so also in Ω, by

$$\mu(\rho, \Gamma_n) + o(1)$$

and they have absolute value $R(\infty)$ at infinity. Now in order to prove (8.6) it suffices to show that any sequence has a subsequence for which the relation holds. Take any subsequence $n \to \infty$ for which Γ_n converges to a class Γ, and these functions converge uniformly in each closed subset of Ω. Call the limits of these functions

$$F(z)\, R(z), \qquad T(z)\, R(z).$$

What we must show is that $T(z) = F(z)$. However we have (again using the continuity of μ)

$$|F(\infty)| = 1, \qquad \sup_{\Omega} |F(z)\, R(z)| \leqslant \mu(\rho, \Gamma)$$

so that F is the extremal function for problem $(1, \rho, \Gamma)$. But $T(z)$ satisfies these same two relations, and so the uniqueness of the extremal function gives $T(z) = F(z)$.

The proof of the last part of the theorem is more complicated and is based on the idea of the proof of Faber's result (2.7) presented in §2. Let us go back to the polynomials Q used in the proof of the first part of the theorem. If δ is smaller than the minimum distance of the z_j to E then the product Π_1 is the full product and $V_{\Gamma'}$ is just the function identically one. Therefore with our assumptions on E and ρ we can write (8.9) as

$$Q(z) = \mu(\rho, \Gamma_n)^{-1} F_n(z) + o(n^{1-2p})$$

and this gives the inequality

$$M_{n,\rho} \leqslant C(E)^n \mu(\rho, \Gamma_n)\{1 + o(n^{1-2p})\}. \tag{8.10}$$

We are going to use the function

$$f_n(z) = f_{\rho,\Gamma_n}(z)$$

which is extremal for problem $(2, \rho, \Gamma_n)$. (See Theorem 5.4.) Since at $z = \infty$ we have

$$\lim_{z\to\infty} (zf_n(z)) = \mu(\rho, \Gamma_n)$$

we find, after choosing an appropriate value for our arbitrary constant of absolute value one,

$$\frac{1}{2\pi i}\int_E C(E)^{-n}\,\Phi(\zeta)^{-n}\,T_{n,\rho}(\zeta) f_n(\zeta)\,d\zeta = \mu(\rho, \Gamma_n).$$

This can also be written

$$\frac{U}{2\pi}\int_E C(E)^{-n}\,\Phi(\zeta)^{-n}\,T_{n,\rho}(\zeta) F_n(\zeta)^{-1} \prod \Psi(\zeta, z_j^*, z_j)\,\frac{\partial g}{\partial n_\zeta}\,|\,d\zeta\,| = \mu(\rho, \Gamma_n).$$

Of course the same identity holds if

$$C(E)^{-n}\,\Phi(\zeta)^{-n}\,T_{n,\rho}(\zeta)$$

is replaced by $F_n(\zeta)$, so that subtracting the two gives

$$\int_E \left\{C(E)^{-n}\,\Phi(\zeta)^{-n}\,\frac{T_{n,\rho}(\zeta)}{F_n(\zeta)} - 1\right\} \prod \Psi(\zeta, z_j^*, z_j)\,\frac{\partial g}{\partial n_\zeta}\,|\,d\zeta\,| = 0. \tag{8.11}$$

Now we know from (8.10) that

$$\left|\, C(E)^{-n}\,\Phi(\zeta)^{-n}\,\frac{T_n(\zeta)}{F_n(\zeta)}\,\right| \leqslant 1 + o(n^{1-2p}) \tag{8.12}$$

and so the expression in brackets in (8.11) has real part at most $o(n^{1-2p})$. It follows therefore from (8.11) that for any subset $e \subset E$ we have

$$\int_e \mathscr{R}\left\{C(E)^{-n}\Phi(\zeta)^{-n}\frac{T_{n,\rho}(\zeta)}{F_n(\zeta)}-1\right\}\prod \Psi(\zeta, z_j^*, z_j)\mid d\zeta\mid \geqslant -o(n^{1-2p}). \tag{8.13}$$

(Recall that our product of exponential Neumann functions is non-negative on E.)

Suppose that for some large n there is a point $\zeta_0 \in E$ for which

$$\left| C(E)^{-n}\Phi(\zeta_0)^{-n}\frac{T_{n,\rho}(\zeta_0)}{F_n(\zeta_0)}-1\right| \geqslant \eta. \tag{8.14}$$

It follows from this, using (8.12) and elementary geometry, that

$$\mathscr{R}\left\{C(E)^{-n}\Phi(\zeta_0)^{-n}\frac{T_{n,\rho}(\zeta_0)}{F_n(\zeta_0)}-1\right\} \leqslant -\frac{\eta^2}{4} \tag{8.15}$$

for n large enough.

Now by Lemma 4.1(1) we see that the functions

$$\Phi(z)^{-n}F_n(z)^{-1}$$

have uniformly bounded derivative on and near E. Since

$$C(E)^{-n}\, T_{n,\rho}(z)$$

are uniformly bounded on E, Lemma 2.1 tells us that their derivatives are at most $O(n)$ on E. Consequently

$$\left|\frac{\partial}{\partial t_\zeta}\mathscr{R}\left\{C(E)^{-n}\Phi(\zeta)^{-n}\frac{T_{n,\rho}(\zeta)}{F_n(\zeta)}-1\right\}\right| \leqslant An$$

for some $A > 0$. It follows from this and (8.15) that if e is the arc centered at ζ_0 and of length $\eta^2/4An$, then at every point of e we have

$$\mathscr{R}\left\{C(E)^{n}\Phi(\zeta)^{-n}\frac{T_{n,\rho}(\zeta)}{F_n(\zeta)}-1\right\} \leqslant -\frac{\eta^2}{8}. \tag{8.16}$$

If one uses Lemma 4.1(2) it is easy to show that

$$\int_e \prod_{j=1}^{p-1} \Psi(\Gamma, z_j^*, z_j)\mid d\zeta\mid$$

is at least a positive constant times the $2p - 1$ st power of the length of e. (Note that if $z_1, \ldots, z_p$ were all bounded away from E the integral would be at least a constant times the first power of the length; our assumption is only that $z_1, \ldots, z_q$ are bounded away from E.) This fact, together with (8.16), shows that

$$\int_e \mathscr{R} \left\{ C(E)^{-n} \Phi(\zeta)^{-n} \frac{T_{n,\rho}(\zeta)}{F_n(\zeta)} - 1 \right\} \prod \Psi(\zeta, z_j^*, z_j) \mid d\zeta \mid \leqslant -\eta_1 n^{1-2p}$$

for some positive η_1. This contradicts (8.13) and so (8.14) cannot occur.

We have shown

$$\left| C(E)^{-n} \Phi(z)^{-n} \frac{T_{n,\rho}(z)}{F_n(z)} - 1 \right| \to 0 \tag{8.17}$$

uniformly on E. Now let A be a closed subset of Ω such that all the zeros of F_n lie in A and are uniformly bounded away from the boundary of A, so that $F_n(z)^{-1}$ is uniformly bounded on A. We already know that (8.6) holds on A and so (8.17) hold on the boundary of A. But the boundary of $\bar{\Omega} - A$ consists of E and the boundary of A, on both of which we have (8.17). Consequently by the maximum modulus theorem (8.17), and so also (8.6), holds throughout $\bar{\Omega} - A$. This completes the proof of the theorem.

Although the determination of specific $\mu(\rho, \Gamma_n)$ is complicated, some general conclusions may be drawn concerning the sequence as a whole, or more exactly its set of limit points. This clearly depends on the set of limit points of Γ_n in $\mathscr{T}_p{}^0$, and since

$$\Gamma_n = (-n\omega_k(\infty)),$$

this is related to the arithmetic properties of the harmonic measures $\omega_k(\infty)$.

What comes immediately to mind is the theorem of Kronecker [see, for example, (7), Chapter XXIII], which says that, if the real numbers

$$r_1, \ldots, r_{p-1}, 1$$

are linearly independent over the rationals, then the $p - 1$-tuples of fractional parts

$$(\{nr_1\}, \ldots, \{nr_{p-1}\})$$

are dense in the $p - 1$-fold product of the unit interval. Thus if the numbers

$$\omega_1(\infty), \ldots, \omega_p(\infty)$$

are linearly independent over the rationals then Γ_n is dense in $\mathscr{T}_p{}^0$.

Clearly if all the $\omega_k(\infty)$ are multiples of q^{-1} (q an integer), then there are at most q^{p-1} different Γ_n. This is the extreme case in the other direction.

One can also get some idea of the form of the set of limit points of Γ_n in the general case. For this set (let us call it $\mathscr{L}$) is clearly a closed subsemigroup of the compact group $\mathscr{T}_p{}^0$, and so by an elementary argument is necessarily a subgroup. If we take the obvious mapping

$$\varphi: \mathscr{R}_p{}^0 \to \mathscr{T}_p^0$$

from the group of p-tuples of reals (r_k) satisfying $\Sigma r_k = 0$ onto $\mathscr{T}_p{}^0$, then $\varphi^{-1}(\mathscr{L})$ is a closed subgroup of $\mathscr{R}_p{}^0$ and so is the direct sum of a closed subspace and a discrete group. It follows that $\mathscr{L}$ is a finite union of compact connected sets (translates of the image of the subspace under φ).

Referring to Theorem 5.6, we can summarize our conclusions as follows. The assumptions here are those of the first part of Theorem 8.3.

Theorem 8.4. *The set $\mathscr{L}$ of limit points of the sequence*

$$M_{n,\rho}/C(E)^n$$

is a finite union of closed subintervals of the interval

$$\left[R(\infty), R(\infty) \exp\left\{\sum_{j=1}^{p-1} g(z_j^*)\right\}\right].$$

If the real numbers

$$\omega_1(\infty), \ldots, \omega_p(\infty)$$

are linearly independent over the rationals then $\mathscr{L}$ is the full interval. If these numbers are all integral multiples of q^{-1} for some integer q then $\mathscr{L}$ consists of at most q^{p-1} points.

9. Orthogonal Polynomials for a System of Jordan Curves

In this section $F_n(z)$ will denote the extremal function $F_{\rho,\Gamma_n}(z)$ for problem $(3, \rho, \Gamma_n)$. We hope this will not cause confusion with what went on in §8. As before, $\Gamma_n = \Gamma(\Phi^{-n})$. The quantities $\nu(\rho, \Gamma_n)$ are, as in §6, the extremal constants for problem $(3, \rho, \Gamma_n)$.

The weighted orthogonal polynomial $P_{n,\rho}(z)$ is that polynomial for which the minimum

$$m_{n,\rho} = \min_{a_1,\ldots,a_n} \int_E | \zeta^n + a_1\zeta^{n-1} + \cdots + a_n |^2 \rho(\zeta) \, | d\zeta |$$

is attained. The corresponding orthonormal polynomials are

$$m_{n,\rho}^{-1/2} P_{n,\rho}(z).$$

Theorem 9.1. *Suppose that $E \in C^{2+}$ and that ρ satisfies the conditions*

$$\int_E \rho(\zeta) \, | d\zeta | < \infty, \qquad \int_E \log \rho(\zeta) \, | d\zeta | > -\infty.$$

Then as $n \to \infty$ we have

$$m_{n,\rho} \sim C(E)^{2n} \nu(\rho, \Gamma_n), \tag{9.1}$$

$$\int_E | C(E)^{-n} P_{n,\rho}(\zeta) - \Phi(\zeta)^n F_n(\zeta)|^2 \rho(\zeta) \, | d\zeta | \to 0,$$

and

$$P_{n,\rho}(z) = C(E)^n \Phi(z)^n \{F_n(z) + o(1)\} \tag{9.2}$$

uniformly on each closed subset of Ω.

If $E \in C^{3+}$ and ρ is nonzero and of class C^{1+} then (9.2) *holds uniformly in $\bar{\Omega}$.*

Proof. Assume first that $E \in C^{\alpha+}$ ($\alpha \geqslant 2$) and ρ is nonzero and belongs to $C^{\alpha-2+}$.

If we refer to the form of F_n given in Theorem 6.2 and appeal to Lemma 4.1 (parts (3) and (4)), we see that $F_n(\zeta)$ is uniformly of class $C^{\alpha-2+}$. Consequently if we again introduce the polynomial

$$Q(z) = \frac{1}{2\pi i} \int_C F_n(\zeta) \, \Phi(\zeta)^{-n} \frac{d\zeta}{\zeta - z}$$

then Lemma 8.2 gives

$$Q(z) = F_n(z) \, \Phi(z)^n + o(n^{2-\alpha}), \qquad z \in \delta\Omega. \tag{9.3}$$

Since $\rho \in L_1$, this shows that the monic polynomial

$$C(E)^n Q(z)$$

satisfies

$$\int_E | C(E)^n Q(\zeta)|^2 \rho(\zeta) \, | d\zeta | = C(E)^{2n} \int_E | F_n(\zeta) + o(n^{2-\alpha})|^2 \rho(\zeta) \, | d\zeta |$$
$$= C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{2-\alpha})\}.$$

Thus

$$m_{n,\rho} \leqslant C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{2-\alpha})\}.$$

But for the polynomial $P_{n,\rho}(z)$, we have

$$| C(E)^{-n} P_{n,\rho} \Phi^{-n} | (\infty) = 1, \qquad \Gamma(C(E)^{-n} P_{n,\rho} \Phi^{-n}) = \Gamma_n ,$$

and so, by the definition of $\nu(\rho, \Gamma_n)$,

$$\nu(\rho, \Gamma_n) \leqslant \int_E | C(E)^{-n} P_{n,\rho}(\zeta) \, \Phi(\zeta)^{-n} |^2 \rho(\zeta) \, | d\zeta | = C(E)^{-2n} m_{n,\rho} .$$

Therefore

$$C(E)^{2n} \nu(\rho, \Gamma_n) \leqslant m_{n,\rho} \leqslant C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{2-\alpha})\}. \tag{9.4}$$

In particular, for any $\alpha \geqslant 2$ we obtain (9.1). To remove the restriction on ρ, let σ be any weight function which is nonzero and of class C^{0+}; let Q and F_n be as above, but associated with σ rather than ρ. We have

$$C(E)^{-2n} m_{n,\rho} \leqslant \int_E | Q(\zeta)|^2 \rho(\zeta) \, | d\zeta |$$
$$= \int_E \{| Q(\zeta)|^2 - | F_n(\zeta)|^2\} \rho(\zeta) \, | d\zeta |$$
$$+ \int_E | F_n(\zeta)|^2 \{\rho(\zeta) - \sigma(\zeta)\} \, | d\zeta | + \int_E | F_n(\zeta)|^2 \sigma(\zeta) \, | d\zeta |.$$

The first integral on the right tends to zero as $n \to \infty$ because of (9.3) and the fact that $\rho \in L_1$. The last integral is exactly $\nu(\sigma, \Gamma_n)$. As for the middle integral it may be seen, upon referring to the form of F_n given in Theorem 6.4, that F_n is at most a constant times the maximum of σ^{-1}. Consequently,

$$C(E)^{-2n} m_{n,\rho} \leqslant O(\max \sigma^{-1}) \int_E | \rho(\zeta) - \sigma(\zeta)| \, | d\zeta | + \nu(\sigma, \Gamma_n) + o(1). \tag{9.5}$$

If ρ is bounded away from zero we can find a sequence σ_k satisfying

$$\sigma_k^{-1} \text{ uniformly bounded,} \quad \int_E | \rho(\zeta) - \sigma_k(\zeta)| \, | d\zeta | \to 0,$$

and so necessarily also

$$\int_E | \log \rho(\zeta) - \log \sigma_k(\zeta)| \, | d\zeta | \to 0.$$

Hence from (9.5) and Lemma 6.4

$$C(E)^{-2n} m_{n,\rho} \leqslant \nu(\rho, \Gamma_n) + o(1).$$

Finally, even if ρ is not bounded away from zero,

$$C(E)^{-2n} m_{n,\rho} \leqslant C(E)^{-2n} m_{n,\rho+\delta} \leqslant \nu(\rho + \delta, \Gamma_n) + o(1).$$

If δ is chosen sufficiently small we shall have, again by Lemma 6.4,

$$\nu(\rho + \delta, \Gamma_n) \leqslant \nu(\rho, \Gamma_n) + \epsilon,$$

and so (9.1) is established in full generality.

The second statement of the theorem follows from the first by a standard Hilbert space argument which uses the parallelogram law. We have

$$\begin{aligned} 2 \int_E & | C(E)^{-n} P_{n,\rho}(\zeta)|^2 \rho(\zeta) \, | d\zeta | + 2 \int_E | \Phi(\zeta)^n F_n(\zeta)|^2 \rho(\zeta) \, | d\zeta | \\ &= \int_E | C(E)^{-n} P_{n,\rho}(\zeta) - \Phi(\zeta)^n F_n(\zeta)|^2 \rho(\zeta) \, | d\zeta | \\ &\quad + \int_E | C(E)^{-n} P_{n,\rho}(\zeta) + \Phi(\zeta)^n F_n(\zeta)|^2 \rho(\zeta) \, | d\zeta |. \end{aligned}$$

The left side is asymptotically $4\nu(\rho, \Gamma_n)$ as $n \to \infty$. The second integral on the right is

$$4 \int_E | \tfrac{1}{2}\{C(E)^{-n} \Phi(\zeta)^{-n} P_{n,\rho}(\zeta) + F_n(\zeta)\}|^2 \rho(\zeta) \, | d\zeta |,$$

and so, by the definition of $\nu(\rho, \Gamma_n)$, is at least $4\nu(\rho, \Gamma_n)$ for all n. This shows that the first integral on the right is at most $o(1)$, which is what we had to prove.

The third statement of the theorem follows from the second, by Corollary 7.4.

Finally, under the hypotheses of the last statement of the theorem, (9.4) gives

$$C(E)^{-2n}\, m_{n,\rho} = \nu(\rho, \Gamma_n) + o(n^{-1}).$$

Using this estimate and the parallelogram law once again we obtain

$$\int_E |\, C(E)^{-n}\, P_{n,\rho}(\zeta) - \Phi(\zeta)^n F_n(\zeta)|^2\, \rho(\zeta) \mid d\zeta \mid = o(n^{-1}). \tag{9.6}$$

Let us denote by A_n the maximum of the expression between the absolute value signs. Our object is to prove $A_n \to 0$. Since the $\Phi^n F_n$ are uniformly bounded,

$$\max_E C(E)^{-n} \mid P_{n,\rho}(\zeta)| = A_n + O(1)$$

and so by Lemma 2.1

$$\max_E C(E)^{-n} \mid P'_{n,\rho}(\zeta)| = O(n(A_n + 1)).$$

Since also (by Lemma 4.1 now)

$$\max_E \frac{d}{d\zeta} \mid \Phi(\zeta)^n F_n(\zeta)| = O(n),$$

we see that the expression between the absolute value signs in (9.6) has derivative at most $an(A_n + 1)$ for some constant a. Hence if ζ_0 is a point where the absolute value A_n is attained, and e is the arc centered at ζ_0 and of length $A_n/an(A_n + 1)$, then the expression between the absolute value signs is at least $A_n/2$ at each point of e. Consequently

$$\int_e |\, C(E)^{-n}\, P_{n,\rho}(\zeta) - \Phi(\zeta)^n F_n(\zeta)|^2\, \rho(\zeta) \mid d\zeta \mid \geqslant \frac{A_n{}^2}{2an(A_n + 1)} \min_E \rho.$$

The only way this can be reconciled with (9.6) is by having $A_n \to 0$.

We have shown that (9.2) holds uniformly on E. That this implies it holds uniformly on $\bar{\Omega}$ is shown in the same way as the corresponding statement for the Tchebycheff polynomials. (See the end of the proof of Theorem 8.3.)

In the case of a single contour, the first half of the theorem with a weaker assumption on E is due to Geroniums [(*6*), Theorem 7.1]; the last part, also with weaker assumptions on E and ρ, is contained in Theorem 2.3 of (*13*). This article of Suetin's, incidentally, gives a detailed account of various asymptotic properties of polynomials orthogonal on a single contour, and has an extensive bibliography. For the case of an analytic contour and continuous weight function see (*17*), Chapter XVI.

The analogue of Theorem 8.4, proved in exactly the same way, is the following. The assumptions are those of the first part of Theorem 9.1.

Theorem 9.2. *The set $\mathscr{L}$ of limit points of the sequence*

$$m_{n,\rho}/C(E)^{2n}$$

is a finite union of closed subintervals of the closed interval with end-points

$$2\pi R(\infty)\, C(E) \exp\left\{\pm \sum_{j=1}^{p-1} g(z_j^*)\right\}.$$

If the real numbers

$$\omega_1(\infty),\dots, \omega_p(\infty)$$

are linearly independent over the rationals then $\mathscr{L}$ is the full interval. If these numbers are all integral multiples of q^{-1} for some integer q then $\mathscr{L}$ consists of at most q^{p-1} points.

10. Moment Matrices for a System of Jordan Curves

These are

$$\left(\int_E \zeta^i \bar{\zeta}^j \rho(\zeta) \mid d\zeta \mid\right), \qquad i, j = 0,\dots, n.$$

The smallest eigenvalue $\lambda_{n,\rho}$ of the matrix is the minimum over all $a_0 ,\dots, a_n$, of

$$\frac{\sum_{i,j=0}^n a_i \bar{a}_j \int_E \zeta^i \bar{\zeta}^j \rho(\zeta) \mid d\zeta \mid}{\sum_{i=0}^n \mid a_i \mid^2} = \frac{\int_E \mid \sum_{i=0}^n a_i \zeta^i \mid^2 \rho(\zeta) \mid d\zeta \mid}{\sum_{i=0}^n \mid a_i \mid^2}.$$

Consequently

$$\lambda_{n,\rho}^{-1} = \max_P \frac{1}{2\pi} \int \mid P(e^{i\theta}) \mid^2 d\theta,$$

where the maximum is taken over all nth degree polynomials P satisfying

$$\int_E \mid P(\zeta) \mid^2 \rho(\zeta) \mid d\zeta \mid = 1. \tag{10.1}$$

The reason for the relevance of the Szegö kernel function is this. The function

$$F(z) = \Phi(z)^{-n} P(z) \tag{10.2}$$

belongs to class Γ_n and satisfies

$$\int_E |F(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid = 1. \tag{10.3}$$

Consequently,

$$|F(z)| \leqslant K_n(z, z)^{1/2} \tag{10.4}$$

(where we use the abbrevation K_n for K_{ρ,Γ_n}), and so

$$| P(z)| \leqslant K_n(z, z)^{1/2} e^{ng(z)}. \tag{10.5}$$

These estimates hold throughout Ω for all P satisfying (10.1). The last estimate suggests that $\lambda_{n,\rho}^{-1}$ is of the order of $e^{2\tau n}$ where

$$\tau = \max_{|s|=1} g(s).$$

We shall see that this is so.

We shall assume throughout this section that Ω contains points of the unit circle. Either the maximum τ of $g(s)$, for s on the intersection of Ω with the unit circle, is attained finitely often and with finite multiplicity each time or else the unit circle is contained in Ω and $g(s)$ is constantly τ on it. The reason is that g is real analytic on the unit circle and has limit 0 at each point of E.

The simplest case occurs when τ is attained at a single point $s_0 = e^{i\theta_0}$ of the unit circle. The multiplicity m of the zero of $g(e^{i\theta}) - \tau$ at $\theta = \theta_0$ is an even integer and we set

$$u_0 = -\frac{1}{m!} \frac{d^m}{d\theta^m} g(e^{i\theta})\bigg|_{\theta=\theta_0}$$

so that $u_0 > 0$. Then we have the following asymptotic results.

Theorem 10.1. *Suppose that $E \in C^{2+}$ and that ρ satisfies*

$$\int_E \rho(\zeta) \mid d\zeta \mid < \infty, \qquad \int_E \log \rho(\zeta) \mid d\zeta \mid > -\infty.$$

Then as $n \to \infty$ we have

$$\lambda_{n,\rho} \sim \pi\Gamma\left(\frac{1}{m} + 1\right)^{-1} K_n(s_0 , s_0)^{-1} (2u_0 n)^{1/m} e^{-2\tau n}. \tag{10.6}$$

If $p_n(z)$ denotes the corresponding extremal polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

normalized so that

$$\int_E | p_n(\zeta)|^2 \rho(\zeta) \, | d\zeta | = 1, \qquad p_n(s_0) \, \Phi(s_0)^{-n} > 0$$

then we have

$$\int_E | p_n(\zeta) - K_n(s_0 , s_0)^{-1/2} K_n(\zeta, s_0) \, \Phi(\zeta)^n |^2 \rho(\zeta) \, | d\zeta | \to 0 \tag{10.7}$$

and

$$p_n(z) = \Phi(z)^n \{K_n(s_0 , s_0)^{-1/2} K_n(z, s_0) + o(1)\}$$

uniformly on each closed subset of Ω.

Proof. Define

$$\Omega_\epsilon = \{z \in \Omega : g(z) > \epsilon\}.$$

The boundary of Ω_ϵ is a closed subset of Ω on which $g(z) = \epsilon$ and by (10.2), (10.3), and Corollary 7.3 we have the estimate

$$P(z) = O(e^{\epsilon n})$$

there. But then by the maximum modulus theorem the same estimate holds in the complement of Ω_ϵ in the plane, a bounded set with the same boundary as Ω_ϵ. Hence

$$\frac{1}{2\pi} \int | P(e^{i\theta})|^2 \, d\theta = O(e^{2\epsilon n}) + \frac{1}{2\pi} \int_{\Omega_\epsilon \cap C_1} | F(s)|^2 \, e^{2ng(s)} \, | ds | \tag{10.8}$$

where C_1 denotes the unit circle and F is given by (10.2).

The second integral is easily estimated by Laplace's method. We obtain without difficulty

$$\frac{1}{2\pi} \int_{\Omega_\epsilon \cap C_1} e^{2ng(s)} \, | ds | \sim \pi^{-1} \Gamma \left(\frac{1}{m} + 1\right) (2u_0 n)^{-1/m} \, e^{2\tau n}. \tag{10.9}$$

Now by (10.3) and Corollary 7.3 the functions F are uniformly bounded in each closed subset of Ω, and so the same is true of its derivatives. It follows that the functions

$$| F(s)|^2 - | F(s_0)|^2$$

are uniformly small in a neighborhood of s_0, which easily leads to the estimate

$$\int_{\Omega_\epsilon \cap C_1} \{|F(s)|^2 - |F(s_0)|^2\} e^{2ng(s)} \, |ds| = o(n^{-1/m} e^{2\tau n}).$$

If we combine this with (10.9), and recall (10.8), we find that

$$\lambda_{n,\rho}^{-1} \leqslant \pi^{-1} \Gamma\left(\frac{1}{m} + 1\right) K_n(s_0, s_0)(2u_0 n)^{-1/m} e^{2\tau n}(1 + o(1)). \tag{10.10}$$

To obtain the opposite inequality, assume first that ρ is nonzero and of class C^{0+} and consider the polynomial

$$Q(z) = \frac{1}{2\pi i} \int_C K_n(\zeta, s_0) \, \Phi(\zeta)^n \frac{d\zeta}{\zeta - z}.$$

By Theorem 7.3(3) and Lemma 8.2 we have

$$Q(z) = \Phi(z)^n \{K_n(z, s_0) + o(1)\}$$

uniformly on $\bar{\Omega}$. In particular

$$\begin{aligned} \int |Q(\zeta)|^2 \rho(\zeta) \, |d\zeta| &= \int_E |K_n(\zeta, s_0)|^2 \rho(\zeta) \, |d\zeta| + o(1) \\ &= K_n(s_0, s_0) + o(1) \end{aligned} \tag{10.11}$$

by (7.5).

Since the $K_n(z, s_0)$ are uniformly bounded in a neighborhood of s_0 the same is true of $K_n'(z, s_0)$. It follows that

$$\lim_{z \to s_0} K_n(z, s_0) = K_n(s_0, s_0)$$

uniformly in n. Thus if e is any arc of the unit circle, containing s_0 in its interior and entirely contained in Ω, we can apply Laplace's method to obtain

$$\begin{aligned} \frac{1}{2\pi} \int_E |Q(e^{i\theta})|^2 \, d\theta &\geqslant \frac{1}{2\pi} \int_e e^{2ng(s)} \, |K_n(s, s_0) + o(1)|^2 \, |ds| \\ &= \pi^{-1} \Gamma\left(\frac{1}{m} + 1\right) K_n(s_0, s_0)^2 \, (2u_0 n)^{-1/m} e^{2\tau n}(1 + o(1)). \end{aligned}$$

It follows from this and (10.11) that

$$\lambda_{n,\rho}^{-1} \geqslant \frac{1}{2\pi} \int |Q(e^{i\theta})|^2 \, d\theta \Big/ \int_E |Q(\zeta)|^2 \rho(\zeta) \, | d\zeta |$$

$$\geqslant \pi^{-1} \Gamma \left(\frac{1}{m} + 1 \right) K_n(s_0, s_0)(2u_0 n)^{-1/m} e^{2\tau n}(1 + o(1)).$$

This establishes the first statement of the theorem for ρ nonzero and of class C^{0+}. The passage to the general case proceeds almost exactly as in Theorem 9.1 and so the details will be omitted.

To see that the second statement of the theorem follows from the first, observe that

$$\lambda_{n,\rho}^{-1} = \frac{1}{2\pi} \int | p_n(e^{i\theta})|^2 \, d\theta.$$

Now if we did not have

$$| p_n(s_0)| \sim K_n(s_0, s_0)^n \, | \Phi(s_0)^n |, \tag{10.12}$$

in other words if in the inequality (10.5) the right side could be multiplied by $1 - \delta$ ($\delta > 0$) for a sequence $n \to \infty$, then the same would be true for the inequality (10.10) giving the upper bound for $\lambda_{n,\rho}^{-1}$. Since this is not the case, (10.12) must hold, and so our second normalization for p_n implies

$$p_n(s_0) = \Phi(s_0)^n \{K_n(s_0, s_0) + o(1)\}$$

The relation (10.7) is now obtained by expanding the integrand in the obvious way, using the characteristic reproducing property (7.3) of the kernel function and the first normalization for p_n. Finally the last statement of the theorem follows from the second by Corollary 7.4 applied to

$$\Phi(z)^{-n} \{p_n(z) - \Phi(z)^n K_n(s_0, s_0)^{-1/2} K_n(z, s_0)\}.$$

One can also prove, in analogy with the last part of Theorem 9.1, that the asymptotic formula for $p_n(z)$ holds on all of $\bar{\Omega}$ if E and ρ are appropriate. We omit the details.

From (10.6) and Theorem 7.3(2) we can easily obtain an analogue of Theorem 9.2.

Theorem 10.2. *The set $\mathscr{L}$ of limit points of the sequence*

$$n^{-1/m} e^{2\tau n} \lambda_{n,\rho}$$

is a finite union of closed subintervals of the closed interval with endpoints

$$\frac{1}{2}\Gamma\left(\frac{1}{m}+1\right)^{-1}(2u_0)^{1/m}R(\infty)^{-1}C(E)^{-1}\exp\left\{\pm\sum_{j=1}^{p-1}g(z_j^*)\right\}.$$

If the real numbers

$$\omega_1(\infty),\ldots,\omega_p(\infty)$$

are linearly independent over the rationals then $\mathscr{L}$ *is the full interval. If these numbers are all integral multiples of* q^{-1} *for some integer* q *than* $\mathscr{L}$ *consists of at most* q^{p-1} *points.*

We shall turn now to the more general case where $g(s)$ attains its maximum τ at finitely many points of the unit circle. Suppose that m is the largest of the multiplicities of the zeros of $g(s)-\tau$ on the unit circle and that $s_1,\ldots,s_k$ are the zeros with this multiplicity. Set

$$u_i=-\frac{1}{m!}\frac{d^m}{d\theta^m}g(e^{i\theta})\Big|_{\theta=\theta_i},\qquad i=1,\ldots,k.$$

If we go back go (10.8) and evaluate the integral on the right side in this case, we obtain the inequality

$$\frac{1}{2\pi}\int|P(e^{i\theta})|^2\,d\theta$$
$$\leqslant\pi^{-1}\Gamma\left(\frac{1}{m}+1\right)(2n)^{-1/m}e^{2\tau n}(1+o(1))\sum_{i=1}^{k}u_i^{-1/m}|F(s_i)|^2. \tag{10.13}$$

To see how large the right side can be, we consider the following problem:

For all $F\in H_2(\Omega,\rho,\Gamma_n)$ satisfying

$$\int_E|F(\zeta)|^2\rho(\zeta)\,|d\zeta|=1,$$

determine

$$\sup_F\sum_{i=1}^{k}v_i\,|F(s_i)|^2 \tag{10.14}$$

where $v_1,\ldots,v_k$ are given positive constants.

We have the following lemma.

Lemma 10.3. *In any Hilbert space, the supremum of*

$$\sum_{i=1}^{k} |(x, x_i)|^2 \tag{10.15}$$

taken over all x satisfying $\| x \| = 1$ is the largest eigenvalue of the Gramian

$$((x_i, x_j)), \qquad i, j = 1,..., k.$$

The maximum is attained for an appropriate linear combination of the x_i.

Proof. Because projection onto the subspace spanned by the x_i is norm-decreasing we may confine our attention to that subspace. Thus if

$$x = \sum_{i=1}^{k} a_i x_i ,$$

we have

$$\sum_{i=1}^{k} |(x, x_i)|^2 = \sum_i \Big| \sum_j (x_i , x_j)\, \bar{a}_j \Big|^2$$

and

$$\| x \|^2 = \sum_{i,j} a_i (x_i , x_j)\, \bar{a}_j .$$

It follows that if $\mathscr{G}$ denotes the Gramian matrix and

$$A = (\bar{a}_1 ,..., \bar{a}_k)$$

then we are asked to determine

$$\sup(\mathscr{G}A, \mathscr{G}A) \qquad \text{subject to} \quad (\mathscr{G}A, A) = 1.$$

If we set $B = \mathscr{G}^{1/2}A$ this becomes

$$\sup(\mathscr{G}B, B) \qquad \text{subject to} \quad \| B \| = 1,$$

and the result follows.

We can now find the asymptotic formula for $\lambda_{n,\rho}$ in this case.

Theorem 10.4. *Suppose E and ρ satisfy the assumption of Theorem* 10.1 *and the s_i and m are as above. Then we have*

$$\lambda_{n,\rho} \sim \pi\Gamma\left(\frac{1}{m} + 1\right)^{-1} \mu_n^{-1} (2n)^{1/m}\, e^{-2\tau n}$$

where μ_n *is the largest eigenvalue of the matrix*

$$((u_i u_j)^{-1/2m}\, K_n(s_j\, , s_i)) \qquad i, j = 1,..., k. \tag{10.16}$$

Proof. If we introduce cuts C so that $\Omega - C$ is simply connected then the representatives of $H_2(\Omega, \rho, \Gamma_n)$, as functions on $\Omega - C$, form a Hilbert space with inner product

$$\int_E F_1(\zeta)\, \overline{F_2(\zeta)}\, \rho(\zeta) \mid d\zeta \mid.$$

Because of the reproducing property (7.3) of the kernel function, the sum in (10.14) is exactly of the form (10.15) with

$$v_i^{1/2} K(z, s_i)$$

playing the role of x_i. It therefore follows from the lemma that the maximum of (10.14) is the largest eigenvalue of the matrix

$$\left((v_i v_j)^{1/2} \int_E K(\zeta, s_i)\, \overline{K(\zeta, s_j)}\, \rho(\zeta) \mid d\zeta \mid\right)$$

which, by (7.3) again, is equal to

$$((v_i v_j)^{1/2}\, K(s_j\, , s_i)).$$

Thus from (10.13) we obtain

$$\lambda_{n,\rho}^{-1} \leqslant \pi^{-1} \Gamma \left(\frac{1}{m} + 1\right) \mu_n (2n)^{-1/m}\, e^{2\tau n}(1 + o(1)).$$

The opposite inequality is obtained in the usual way. Let

$$F(z) = \sum a_i K_n(z, s_i) \tag{10.16}$$

be the linear combination of the $K_n(z, s_i)$ for which the maximum μ_n of

$$\sum_{i=1}^{k} u_i^{-1/m} \mid F(s_i) \mid^2$$

is attained, and set

$$Q(z) = \frac{1}{2\pi i} \int_\Omega F(\zeta)\, \Phi(\zeta)^n \frac{d\zeta}{\zeta - z}\,.$$

Then if ρ is nonzero and of class C^{0+}, Lemma 8.2 gives

$$Q(z) = \Phi(z)^n \left\{ \sum a_i K_n(z, s_i) + o(1) \right\}$$

uniformly on $\bar{\Omega}$, and we continue just as in Theorem 10.1. The details are omitted.

There is also an analogue in this situation of the statements in Theorem 10.1 concerning the extremal polynomials $p_n(z)$, but only if an extra rather ugly condition is satisfied. One expects of course that $p_n(z)$, suitably normalized, is asymptotically

$$\Phi(z)^n F(z)$$

where $F(z)$ is given by (10.16). In order to prove this we have to know that for the extremum problem of Lemma 10.3 any vector x for which (10.15) is close to its maximum is necessarily close to the extremal vector, and for this to occur the second largest eigenvalue of the Gramian must be substantially smaller than the largest; more exactly the ratio of the two eigenvalues must be bounded away from one. With this added assumption on the matrix, asymptotic formulas for $p_n(z)$ entirely analogous to those given in Theorem 10.1 can be established.

More interesting, and more difficult, is an analogue of Theorem 10.2 in this case. This would involve finding the range, as Γ varies over all of $\mathcal{T}_p{}^0$, of the largest eigenvalue of

$$((u_i u_j)^{-1/2m} K_{\rho,\Gamma}(s_j, s_i)), \qquad i, j = 1,\ldots, k.$$

Finally we state without proof the asymptotic formula for $\lambda_{n,\rho}$ in the last remaining case.

Theorem 10.5. *Suppose $g(s)$ is constantly equal to τ on the unit circle. Then we have*

$$\lambda_{n,\rho} \sim 2\pi\mu_n^{-1} e^{-2\tau n}$$

where μ_n is the largest eigenvalue of the kernel $K_n(s_1, s_2)$ on the unit circle.

11. The Tchebycheff Problem for a System of Jordan Curves and Arcs

In this and the following two sections we consider what happens to our previous results if one or more of the components E_k of E is a Jordan arc rather than a closed curve.

In the case of a single Jordan curve, with weight function $\rho(\zeta) \equiv 1$, we have Faber's result

$$M_n \sim C(E)^n.$$

But now consider the case of the unit interval $[-1, 1]$. Since

$$\Phi(z) = z + (z^2 - 1)^{1/2},$$

the capacity is $1/2$. However in this case the Tchebycheff polynomials are very well known [(*2*), p. 58]. They are

$$\begin{aligned} T_n(z) &= 2^{-n}(\{z + (z^2 - 1)^{1/2}\}^n + \{z - (z^2 - 1)^{1/2}\}^n) \\ &= 2^{-n+1} \cos(n \cos^{-1} z), \end{aligned}$$

and so

$$M_n = 2^{-n+1} = 2C(E)^n.$$

Thus M_n is asymptotically twice as large for an interval as for a closed curve of the same capacity. We conjecture that this is true generally; that is, if at least one of the E_k is an arc then the asymptotic formula for $M_{n,\rho}$ given in Theorem 8.3 must be multiplied by 2. Moreover we conjecture that the asymptotic formula (8.6) continues to hold (on closed subsets of Ω) in the case.

Unfortunately we cannot prove these statements and so they are nothing but conjectures. What we can prove is that the statement concerning $M_{n,\rho}$ holds if each E_k is an interval of the real axis; and that in any case $M_{n,\rho}$ is at most twice what is given in Theorem 8.3. Still lacking are a suitable lower bound for $M_{n,\rho}$ and a technique for deriving asymptotic formulas for $T_{n,\rho}$ from those for $M_{n,\rho}$. Even for the case of a general weight function on a single interval, the asymptotic form of $T_{n,\rho}$ seems not to have been established.

As a consequence of our work we obtain in Corollary 11.3 an asymptotic formula for the weighted Faber polynomials in the case of a Jordan arc. This extends a result of Suetin [(*14*), Theorem 12] for the case of an interval.

The reduction of the case of arcs to the case of closed curves is effected by a simple transformation which "opens up" the arc. Suppose E is a Jordan arc which for simplicity we assume has endpoints ± 1. Then if

$$(z^2 - 1)^{1/2}$$

denotes that branch of the square root in the complement of E which is asymptotically z near $z = \infty$ we set

$$s = z + (z^2 - 1)^{1/2}, \qquad z = \tfrac{1}{2}(s + s^{-1}). \tag{11.1}$$

Then the exterior of E corresponds to the exterior of a certain closed curve E' in the s-plane. This curve contains $s = 0$ in its interior. Moreover z, as a function of s, extends continuously to E' and as s traverses E' once z traverses E twice, once in each direction.

Lemma 11.1. *If $E \in C^{\alpha+}$ $(\alpha \geqslant 1)$, then the same is true of E'. Moreover if $F(z) \in C^{\beta+}$ $(\beta \leqslant \alpha)$ on E, then $F(\frac{1}{2}\{s + s^{-1}\}) \in C^{\beta+}$ on E'.*

Proof. The problem is a local one and the only difficulty occurs near $s = \pm 1$. Consider for example the neighborhood of $s = 1$. Denote arc length on E, measured from $z = 1$, by σ. Then $\eta = \sigma^{1/2}$ parametrizes E and we may also consider it as parametrizing E'. We shall show that $ds/d\eta$ is nonzero and of class $C^{\alpha-1+}$ (as a function of η) near $\eta = 0$, and this will imply the result. We have

$$s^{-1}\frac{ds}{d\eta} = 2\left\{\frac{z(\sigma)^2 - 1}{\sigma}\right\}^{-1/2}\frac{dz}{d\sigma}.$$

By Lemma 8.1 the expression in brackets, and so the entire right side, is of class $C^{\alpha-1+}$ as a function of σ and of course it is nonzero. It follows that the left side is nonzero and is of class C^{0+} as a function of η, since any Lipschitz function of σ is also a Lipschitz function of η. Differentiation of the above identity $\alpha - 1$ times with respect to η shows that in fact the left side is of class $C^{\alpha-1+}$. Hence $\log s$, and so also s, is of class $C^{\alpha+}$.

The second statement of the lemma is an easy consequence of the formula

$$\frac{d}{d\eta} = 2\eta\,\frac{d}{d\sigma},$$

and the fact that a Lipschitz function of σ is also a Lipschitz function of η.

Suppose we are given a function $F(z)$ analaytic and single-valued in a neighborhood of E, though not necessarily on E itself, and that we have an integral

$$\int_C F(z)\,dz$$

where C is a Jordan curve surrounding E. Then this integral can generally be written as an integral taken over E. Suppose for example that $F(z)$ is bounded. Then it has two sets of boundary values on E which we arbitrarily designate by $F_+(\zeta)$ and $F_-(\zeta)$ corresponding to approach to E from the two sides of E. The integral is then equal to a sum of two integrals over E; one the integral of $F_+(\zeta)\,d\zeta$ taken over E is one direction and the other of $F_-(\zeta)\,d\zeta$ taken over E in the other direction. For compactness of notation we shall write this as

$$\oint_E F(\zeta)\,d\zeta.$$

We shall be concerned with the situation where one or more of the E_k is an arc. We then perform a succession of transformations such as (11.1) which "opens up" each such arc into a closed curve. Denote the resulting transformation by $z = \varphi(s)$ and the resulting system of Jordan curves by $E' = E_1' \cup \cdots \cup E_p'$ with complement Ω' in the s-plane.

Associated with any function $F(z)$, analytic in a neighborhood of E in Ω, there is associated the function $F(\varphi(s))$ analytic in a neighborhood of E' in Ω'. The boundary values of $F(\varphi(s))$, when transferred back to E, give rise to the boundary values $F_\pm(\zeta)$ and we have

$$\oint_E F(\zeta)\,d\zeta = \int_{E'} F(z(s))\,\frac{d\varphi}{ds}\,ds.$$

The $\oint$ means that each closed curve E_k is described once and each arc twice, once in each direction.

The replacement for Lemma 8.2 in the present situation is the following. Observe that although the functions $\Phi(z, z_0)$, as well as $\Psi(z, z_1, z_2)$, were discussed in §4 under the assumption that E consisted entirely of closed curves, the extension to the present situation is no problem. For example one may use their conformal invariance and the function φ to reduce the situation to that of §4.

Lemma 11.2. *Suppose that $E \in C^{\alpha+}$ $(\alpha \geqslant 2)$, that*

$$F \in H_\infty(\Omega, \Gamma_n),$$

and that $F(\varphi(s))$ extends continuously to a function of class $C^{\beta+}$ $(\beta \leqslant \alpha - 2)$ on E'. Define the polynomial

$$Q(z) = \frac{1}{2\pi i}\int_C F(\zeta)\,\Phi(\zeta)^n\,\frac{d\zeta}{\zeta - z}$$

where C is a Jordan curve, described once counterclockwise, containing E and z in its interior.

Define

$$B(\zeta) = \begin{cases} F(\zeta)\,\Phi(\zeta)^n & \text{on each closed curve of } E, \\ F_+(\zeta)\,\Phi_+(\zeta)^n + F_-(\zeta)\,\Phi_-(\zeta)^n & \text{on each arc of } E. \end{cases}$$

Then we have

$$Q(\zeta) = B(\zeta) + o(n^{2-\alpha} + n^{-\beta}) \tag{11.2}$$

uniformly on E; and this holds uniformly for a family of F's for which $F(\varphi(s))$ are uniformly in $C^{\beta+}$.

Proof. We shall only give the details in the case where exactly one of the E_k is an arc. What follows could be used as the basis for an inductive proof of the result in the general case.

We shall prove this lemma by appealing to Lemma 8.2. More exactly what we shall need, and what was shown in the course of the proof of the lemma, is that the principal value integral

$$\frac{1}{2\pi i}\int_{E_k} F(\zeta)\,\Phi(\zeta)^n \frac{d\zeta}{\zeta - \zeta_0}$$

equals

$$\tfrac{1}{2}F(\zeta_0)\,\Phi(\zeta_0)^n + o(n^{2-\alpha} + n^{-\beta}) \tag{11.3}$$

uniformly for $\zeta_0 \in E_k$. The proof also gave

$$\frac{1}{2\pi i}\int_{E_k} F(\zeta)\,\Phi(\zeta)^n \frac{d\zeta}{\zeta - z} = o(n^{2-\alpha} + n^{-\beta}) \tag{11.4}$$

uniformly for z bounded away from E_k.

To use these things in our situation we apply the variable change $z = \varphi(s)$, and note that by the conformal invariance of Green's function, $\Phi(\varphi(s))$ is exactly the exponential Green's function for the region Ω' with pole at infinity. Let us assume that the one E_k which is an arc is E_1 and that its end-points are ± 1. Then φ is given by (11.1) and so we may write [here set $z = \varphi(s)$, $\zeta = \varphi(\sigma)$, $C' = \varphi^{-1}(C)$]

$$\begin{aligned} Q(z) &= \frac{1}{2\pi i}\int_{C'} F(\varphi(\sigma))\,\Phi(\varphi(\sigma))^n \frac{1 - \sigma^{-2}}{(\sigma + \sigma^{-1}) - (s + s^{-1})}\,d\sigma \\ &= \frac{1}{2\pi i}\int_{C'} F(\varphi(\sigma))\,\Phi(\varphi(\sigma))^n \frac{d\sigma}{\sigma - s} \\ &\quad + \frac{1}{2\pi i}\int_{C'} F(\varphi(\sigma))\,\Phi(\varphi(\sigma))^n \frac{d\sigma}{\sigma s(\sigma - s^{-1})}\,. \end{aligned} \tag{11.5}$$

Now suppose $z = \zeta_0 \in E_k$. This corresponds to $s = \sigma_0 \in E'_k$. We shall use the Plemelj formulas to write the integrals on the right side of (11.5) in terms of integrals over E'. The first integral is equal to

$$\frac{1}{2} F(\varphi(\sigma_0))\, \Phi(\varphi(\sigma_0))^n + \frac{1}{2\pi i} \int_{E'} F(\varphi(\sigma))\, \Phi(\varphi(\sigma)) \frac{d\sigma}{\sigma - \sigma_0},$$

where the part of the second integral taken over E'_k is a principal value, and by (11.3) and (11.4) this is

$$F(\varphi(\sigma_0))\, \Phi(\varphi(\sigma_0))^n + o(n^{2-\alpha} + n^{-\beta}).$$

As for the last integral in (11.5), it depends whether or not $k = 1$. If $k \neq 1$, then σ_0^{-1} belongs to $E_2'^{-1} \cup \cdots \cup E_p'^{-1}$ which is a closed set interior to E'_1. Thus the last integral, which is equal to the integral extended over E', is $o(n^{2-\alpha} + n^{-\beta})$ by (11.4), and (11.2) is established in this case. If $k = 1$ then σ_0^{-1} belongs to E'_1 and the last integral in (11.5) equals

$$\frac{1}{2} F(\varphi(\sigma_0^{-1}))\, \Phi(\varphi(\sigma_0^{-1}))^n + \frac{1}{2\pi i} \int_{E'_1} F(\varphi(\sigma))\, \Phi(\varphi(\sigma))^n \frac{d\sigma}{\sigma\sigma_0^{-1}(\sigma - \sigma_0^{-1})}$$
$$+ \sum_{k>1} \frac{1}{2\pi i} \int_{E'_k} F(\varphi(\sigma))\, \Phi(\varphi(\sigma))^n \frac{d\sigma}{\sigma\sigma_0^{-1}(\sigma - \sigma_0^{-1})},$$

where the integral over E'_1 is a principal value. By (11.3) the first two terms add up to

$$F(\varphi(\sigma_0^{-1}))\, \Phi(\varphi(\sigma_0^{-1}))^n + o(n^{2-\alpha} + n^{-\beta})$$

and by (11.4) the last term is $o(n^{2-\alpha} + n^{-\beta})$. If we observe that

$$F_{\pm}(\zeta_0)\, \Phi_{\pm}(\zeta_0)^n = F(\varphi(\sigma_0^{\pm 1}))\, \Phi(\varphi(\sigma_0^{\pm 1}))^n$$

we see that (11.2) has been established.

Corollary 11.3. *Suppose that E consists of a single C^{2+} Jordan arc and that ρ is of class C^{0+} on E. Then the weighted Faber polynomials*

$$\frac{1}{2\pi i} \int_C R(\zeta)\, \Phi(\zeta)^n \frac{d\zeta}{\zeta - z}$$

are given by the asymptotic formula

$$R_+(\zeta)\, \Phi_+(\zeta)^n + R_-(\zeta)\, \Phi_-(\zeta)^n + o(1)$$

uniformly for $\zeta \in E$.

Proof. By Lemma 11.1 the function $\rho(\varphi(s))$ belongs to C^{0+} on E', and so the same holds for $R(\varphi(s))$ and we may apply Lemma 11.2. $R(z)$ is, just as before, the function in Ω without zeros or poles, whose boundary values are of absolute value $\rho(\zeta)$.

We shall now use Lemma 11.2 to obtain an upper bound for the Tchebycheff constants $M_{n,\rho}$. The technique will be essentially the same as in Theorem 8.3 and so the proof will only be outlined.

Theorem 11.4. *Suppose that $E \in C^{2+}$ and that ρ is bounded above, upper semicontinuous, and satisfies*

$$\int \log \rho(\zeta) \mid d\zeta \mid > -\infty.$$

Then as $n \to \infty$ we have

$$M_{n,\rho} \leqslant \{2 + o(1)\}\, C(E)^n \,\mu(\rho, \Gamma_n).$$

Proof. As ususal it suffices to consider the case when ρ is nonzero and of class C^{0+}. Define $Q(z)$ by (8.8) exactly as was done in the proof of Theorem 8.3. In this case we obtain from Lemma 11.2, as a substitute for (8.9), the asymptotic formula

$$Q(\zeta) = B(\zeta) + o(1), \qquad \zeta \in E$$

where $B(\zeta)$ is the limiting value of

$$R(z)^{-1} \prod\nolimits_1 \Phi(z, z_j)^{-1}\, V_{\Gamma'}(z)\, \Phi(z)^n$$

on the closed curves of E and the sum of the two limiting values on the arcs of E. This gives the inequality

$$\max_E |Q(\zeta)| \leqslant 2e^{2\epsilon}.$$

The absolute value of the leading coefficient of Q is at least

$$e^{-\epsilon} C(E)^{-n}\, \mu_n(\rho, \Gamma)^{-1}$$

just as before, and so we obtain

$$M_{n,\rho} \leqslant 2e^{3\epsilon} C(E)^n\, \mu_n(\rho, \Gamma).$$

After the proof of Theorem 12.3 in the next section it will be indicated how a lower bound for $M_{n,\rho}$ can be obtained. The lower bound has a factor smaller than 2 and so does not give rise to an asymptotic formula. However if E is a union of intervals then oscillation properties may be used to obtain the correct lower bound.

Theorem 11.5. *Suppose each E_k is an interval on the real axis and that ρ is properly Riemann integrable and bounded away from zero. Then as $n \to \infty$ we have*

$$M_{n,\rho} \sim 2C(E)^n \mu(\rho, \Gamma_n).$$

Proof. The extremal function F_n for problem $(1, \rho, \Gamma_n)$ has all its zeros in the convex hull of E. This is true no matter what E is. For if z_0 were a zero of F_n, and z_0' the reflection of z_0 through a line separating z_0 from E, then

$$F_n(z) \frac{z - z_0'}{z - z_0}$$

would be of the same class Γ_n as F, have absolute value one at infinity, but have smaller absolute value on E than F_n. Thus F_n would not have been extremal.

In our case, if E_k is the interval $[\alpha_k, \beta_k]$ $(\alpha_1 < \alpha_2 < \cdots < \alpha_p)$, then all the zeros of F_n in Ω must lie in the complementary intervals (β_k, α_{k+1}). Moreover F_n can have no more than one zero in any of these intervals by the following simple argument, shown to us in the case of the Tchebycheff polynomials themselves by J. L. Ullman. If there were two zeros in the same interval, say at points $x_0 \pm \eta$, and if the distance from x_0 to E is $\eta_1 > \eta$, then

$$F_n(z) \frac{(z - x_0)^2 - \eta_1^2}{(z - x_0)^2 - \eta^2}$$

has smaller absolute value on E than F_n. Thus F_n could not have been extremal.

Assume as usual that ρ is nonzero and belongs to C^{0+}, and consider the function

$$R(z)^{-1} \prod_1 \Phi(z, z_j)^{-1} V_{\Gamma'}(z)\, \Phi(z)^n$$

appearing on the right side of (8.8). By this formula and Lemma 11.2

we know that on E, $Q(\zeta)$ is asymptotically equal to the sum of the two limiting values of this function. Let us write

$$H(z) = R(z)^{-1} \prod_1 \Phi(z, z_j)^{-1} V_{\Gamma'}(z)$$

so that the function under consideration is $H(z)\,\Phi(z)^n$. Since all the z_j are real we see by symmetry that this function takes conjugate values at conjugate points. On E we have for the absolute values,

$$| H_{\pm}(\zeta)\,\Phi(\zeta)^n |\, \rho(\zeta) = 1 + O(\epsilon)$$

where we have used the fact that for small δ we have

$$e^{-\epsilon} \leqslant | V_{\Gamma'}(z)| \leqslant e^{\epsilon}.$$

Let us look very carefully at the arguments.

To determine things uniquely, cut the plane along the real axis from α_1 to $+\infty$. Any function $h(z)$ harmonic in the cut plane and symmetric about the real axis satisfies

$$\partial h/\partial n_{\zeta} = 0, \qquad \zeta \text{ real}$$

and so also

$$\partial \tilde{h}(\zeta)/\partial \zeta = 0, \qquad \zeta \text{ real.}$$

Thus such as $h(z)$ has conjugate function which is constant on $(-\infty, \alpha_1)$. We may therefore determine all our conjugate functions by requiring them to be zero there; both $H(z)$ and $\Phi(z)$ have argument 0 on $(-\infty, \alpha_1)$ and their arguments on the entire cut plane are now uniquely determined.

We shall use the subscripts $+$ and $-$ to denote limiting values from above and below the cut. As ζ runs over E_1, the functions

$$\arg H_{\pm}(\zeta)\,\Phi_{\pm}(\zeta)^n$$

vary from 0 at α_1 to values

$$\arg H_{\pm}(\beta_1)\,\Phi_{\pm}(\beta_1)^n$$

at β_1. Since $\arg H\Phi^n$ takes negative values at conjugate points we have

$$\arg H_+(\beta_1)\,\Phi_+(\beta_1)^n = -\arg H_-(\beta_1)\,\Phi_-(\beta_1)^n.$$

But we also have

$$\arg H_-(\beta_1)\,\Phi_-(\beta_1)^n - \arg H_+(\beta_1)\,\Phi_+(\beta_1)^n = \underset{E_1}{\Delta} \arg H(z)\,\Phi(z)^n$$

(by this we mean the change in argument as z runs over a curve surrounding E_1) and since $H\Phi^n$ is single-valued in Ω this is an integral multiple of 2π, say $2m_1\pi$. Thus

$$\arg H_-(\beta_1)\,\Phi_-(\beta_1)^n = m_1\pi,$$
$$\arg H_+(\beta_1)\,\Phi_+(\beta_1)^n = -m_1\pi.$$

Let us continue now, just above and just below the cut, from β_1 to α_2. If no z_j (i.e., no zero of H) lies in (β_1, α_2) then by an argument already given, each of the functions

$$\arg H_-(\zeta)\,\Phi_-(\zeta)^n, \qquad \arg H_+(\zeta)\,\Phi_+(\zeta)^n$$

is constant; if there is a z_j in this interval then the first function increases by π and the second decreases by π. Therefore

$$\arg H_-(\alpha_2)\,\Phi_-(\alpha_2)^n = (m_1 + l_1)\,\pi,$$
$$\arg H_+(\alpha_2)\,\Phi_+(\alpha_2)^n = -(m_1 + l_1)\,\pi,$$

where

$$l_1 = \begin{cases} 0 & \text{if no } \; z_j \in (\beta_1, \alpha_2), \\ 1 & \text{otherwise.} \end{cases}$$

Continuing this process until we have passed all of E we obtain the formulas

$$\begin{aligned} \arg H_\pm(\beta_k)\,\Phi_\pm(\beta_k)^n &= \mp\left(\sum_{r=1}^{k} m_r + \sum_{r=1}^{k-1} l_r\right)\pi, \\ \arg H_\pm(\alpha_k)\,\Phi_\pm(\alpha_k)^n &= \mp\left(\sum_{r=1}^{k-1} m_r + \sum_{r=1}^{k-1} l_r\right)\pi, \end{aligned} \tag{11.6}$$

where

$$l_k = \begin{cases} 0 & \text{if no } \; z_j \in (\beta_k, \alpha_{k+1}), \\ 1 & \text{otherwise,} \end{cases}$$

and m_r are appropriate integers. Since $\tilde{g}_-(\zeta)$ increases on each E_k the m_r tend to $+\infty$ with n. We have

$$\begin{aligned} 2n\pi = \underset{E}{\Delta} \arg H(z)\,\Phi(z)^n &= \arg H_-(\beta_p)\,\Phi_-(\beta_p)^n - \arg H_+(\beta_p)\,\Phi_+(\beta_p)^n \\ &= 2\left(\sum_{k=1}^{p} m_k + \sum_{k=1}^{p-1} l_k\right)\pi \end{aligned}$$

and so

$$\sum_{k=1}^{p} m_k + \sum_{k=1}^{p-1} l_k = n.$$

Let us write

$$\psi(\zeta) = \arg H_-(\zeta)\,\Phi_-(\zeta)^n.$$

Then since

$$Q(\zeta) = H_+(\zeta)\,\Phi_+(\zeta)^n + H_-(\zeta)\,\Phi_-(\zeta)^n + o(1)$$

and

$$|\,H_\pm(\zeta)|\,\rho(\zeta) = 1 + O(\epsilon)$$

we have

$$Q(\zeta)\,\rho(\zeta) = 2\cos\psi(\zeta) + e_0$$

where the error term e_0 is $O(\epsilon)$. As ζ runs over E_k then we see from (11.6) that $\psi(\zeta)$ increases by $m_k\pi$ and so there are $m_k + 1$ points of E_k where $\cos\psi(\zeta)$ assumes the values ± 1, with alternating sign at consecutive points. Call these points

$$x_{k0}\,,\dots,\ x_{km_k} \qquad (x_{k0} = \alpha_k\,,\ x_{km_k} = \beta_k).$$

If $l_r = 0$ then $\psi(\zeta)$ has the same sign at β_r and α_{r+1}; otherwise the signs are opposite.

We shall now show that if P is any nth-degree polynomial with the same leading coefficient as Q then

$$\max_E |\,P(\zeta)|\,\rho(\zeta) \geqslant 2 - e_0\,.$$

For suppose not. Then there is a polynomial P, which can be assumed to be real on the real axis, such that

$$\max_E |\,P(\zeta)|\,\rho(\zeta) < 2 - e_0\,.$$

The polynomial $Q - P$, which has degree at most $n - 1$, has the same sign at each x_{ki} as Q itself. Thus in the interval E_k the polynomial has at least m_k zeros; if $l_k = 1$ then it also has a zero between β_k and α_{k+1}. Thus it has a total of at least

$$\sum_{k=1}^{p} m_k + \sum_{k=1}^{p-1} l_k = n$$

zeros and is therefore identically zero.

The absolute value of the leading coefficient of Q being at most

$$e^{\epsilon}C(E)^{-n}\,\mu(\rho, \Gamma_n)^{-1},$$

we have shown that

$$M_{n,\rho} \geqslant 2e^{-O(\epsilon)}C(E)^n\,\mu(\rho, \Gamma_n). \tag{11.7}$$

Since ϵ is arbitrarily small, this establishes the theorem for ρ nonzero and of class C^{0+}.

Now take any ρ which is properly Riemann integrable and bounded away from zero. Since ρ is continuous almost everywhere, it is equal almost everywhere to its upper semicontinuous modification. Thus if we apply Theorem 11.4 to this upper semicontinuous modification (which is everywhere $\geqslant \rho$) we deduce that

$$M_{n,\rho} \leqslant \{2 + o(1)\}\,C(E)^n\,\mu(\rho, \Gamma_n).$$

By the Reimann integrability of ρ, we can find a sequence ρ_m of nonzero C^{0+} functions satisfying

$$\rho_m(\zeta) \leqslant \rho(\zeta), \qquad \lim_{n\to\infty} \int_E \{\log \rho(\zeta) - \log \rho_m(\zeta)\}\,d\zeta = 0.$$

Then by (11.7) and Lemma 5.5 we have for a sufficiently large, but fixed, m

$$M_{n,o} \geqslant M_{n,\rho_m} \geqslant \{2 - o(1)\}\,C(E)^n\mu(\rho_m\,, \Gamma_n)\} \geqslant \{2 - o(1)\}\,C(E)^n\,\{\mu(\rho, \Gamma_n) - \epsilon\}$$

and the result follows.

In this case of a union of intervals on the real axis, the various Green's functions and harmonic measures that arise are easily expressible in terms of hyperelliptic integrals. The details will be worked out in §14.

12. Orthogonal Polynomials for a System of Jordan Curves and Arcs

The situation here is much better than in the Tchebycheff case and we get a complete analogue of Theorem 9.1. We shall retain the notation of §9, so that $\nu(\rho, \Gamma_n)$ and $F_n(z)$ are the extremal value and function for problem $(3, \rho, \Gamma_n)$.

What is crucial is finding the asymptotic form of $m_{n,\rho}$. Just as in the Tchebycheff case, finding the right upper bound is straightforward;

the difficulty arises in trying to establish the right lower bound. The obvious approach is the following. As in §10 let us denote by K_n the Szegö kernel function associated with ρ and Γ_n ; thus

$$F_n(z) = K_n(\infty, \infty)^{-1} K_n(z, \infty) \tag{12.1}$$

and

$$\nu(\rho, \Gamma_n) = K_n(\infty, \infty)^{-1}. \tag{12.2}$$

Then for any monic nth-degree polynomial P,

$$C(E)^n = \oint_E P(\zeta)\, \Phi(\zeta)^{-n} K_n(\zeta, \infty)\, \rho(\zeta) \mid d\zeta \mid \tag{12.3}$$

and so we obtain the inequality

$$C(E)^{2n} \leqslant \oint_E \mid P(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid \cdot \oint_E \mid K_n(\zeta, \infty)|^2 \rho(\zeta) \mid d\zeta \mid,$$

or

$$\oint_E \mid P(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid \geqslant C(E)^{2n} \nu(\rho, \Gamma_n).$$

This of course is nothing different from what went on in §9. What we are interested in however is not a lower bound on

$$\oint_E \mid P(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid,$$

in which the arcs of E are each traversed twice, but rather on

$$\int_E \mid P(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid,$$

where the arcs are traversed once. Naturally we get immediately the estimate

$$\int_E \mid P(\zeta)|^2 \rho(\zeta) \mid d\zeta \mid \geqslant \frac{1}{2} C(E)^{2n} \nu(\rho, \Gamma_n)$$

but we cannot expect this to be the best possible. In fact we shall see that the factor $\frac{1}{2}$ can be removed.

The point is we have not yet made use of the fact that P is single-valued, i.e., its limiting values from the two sides of any arc of E are equal. This fact allows us to make a preliminary modification of the integral in (12.3) after which the procedure just outlined gives the correct lower bound.

Let us write $E = E^{(1)} \cup E^{(2)}$ where $E^{(1)}$ is the union of those E_k which are closed curves and $E^{(2)}$ the union of the arcs. As in the preceding section we use subscripts $+$ and $-$ to denote limiting values from the two sides of the arcs of $E^{(2)}$. Then using the fact that $P(\zeta)$ is single-valued, (12.3) may be written

$$C(E)^n = \int_{E^{(1)}} P(\zeta)\, \Phi(\zeta)^{-n}\, \overline{K_n(\zeta, \infty)}\, \rho(\zeta) \mid d\zeta \mid$$
$$+ \int_{E^{(2)}} P(\zeta)\{\Phi_+(\zeta)^{-n}\, \overline{K_{n+}(\zeta, \infty)} + \Phi_-(\zeta)^{-n}\, \overline{K_{n-}(\zeta, \infty)}\}\, \rho(\zeta) \mid d\zeta \mid.$$

Let us introduce the notation

$$L_n(\zeta, z_0) = \begin{cases} \Phi(\zeta)^n\, K_n(\zeta, z_0) & \text{on } E^{(1)}, \\ \Phi_+(\zeta)^n\, K_{n+}(\zeta, z_0) + \Phi_-(\zeta)^n\, K_{n-}(\zeta, z_0) & \text{on } E^{(2)} \end{cases} \tag{12.4}$$

for z_0 an arbitrary point of Ω. We are here interested in $z_0 = \infty$ and the last identity may be written

$$C(E)^n = \int_E P(\zeta)\, \overline{L_n(\zeta, \infty)}\, \rho(\zeta) \mid d\zeta \mid.$$

Thus we obtain the lower bound

$$m_{n,n} \geqslant C(E)^{2n} \Big/ \int_E \mid L_n(\zeta, \infty)\mid^2 \rho(\zeta) \mid d\zeta \mid. \tag{12.5}$$

We shall now evaluate the last integral asymptotically, not only for the parameter value $z_0 = \infty$ but for general z_0. We have clearly

$$\int_E \mid L_n(\zeta, z_0)\mid^2 \rho(\zeta) \mid d\zeta \mid$$
$$= \int_{E^{(1)}} \mid K_n(\zeta, z_0)\mid^2 \rho(\zeta) \mid d\zeta \mid + \int_{E^{(2)}} \{\mid K_{n+}(\zeta, z_0)\mid^2 + \mid K_{n-}(\zeta, z_0)\mid^2\}\, \rho(\zeta) \mid d\zeta \mid$$
$$+ 2\mathscr{R} \int_{E^{(2)}} \Phi_+(\zeta)^n\, \Phi_-(\zeta)^{-n}\, K_{n+}(\zeta, z_0)\, \overline{K_{n-}(\zeta, z_0)}\, \rho(\zeta) \mid d\zeta \mid.$$

The first two integrals on the right add up to

$$\oint_E \mid K_n(\zeta, z_0)\mid^2 \rho(\zeta) \mid d\zeta \mid = K_n(z_0\,, z_0)$$

and so we have

$$\int_E |L_n(\zeta, z_0)|^2 \rho(\zeta) \, |d\zeta|$$
$$= K_n(z_0, z_0) + 2\mathscr{R} \int_{E^{(2)}} \Phi_+(\zeta)^n \Phi_-(\zeta)^{-n} K_{n+}(\zeta, z_0) \overline{K_{n-}(\zeta, z_0)} \rho(\zeta) \, |d\zeta|. \tag{12.6}$$

The following lemma shows that the last term tends to zero as $n \to \infty$. We state a slightly more general result which will be useful in the next section.

Lemma 12.1. *Suppose $E \in C^{\alpha+}$ $(\alpha \geqslant 1)$ and*

$$\int_E \log \rho(\zeta) \, |d\zeta| > -\infty.$$

Then as $n \to \infty$ we have

$$\int_{E^{(2)}} \Phi_+(\zeta)^n \Phi_-(\zeta)^{-n} K_{n+}(\zeta, z_1) \overline{K_{n-}(\zeta, z_2)} \rho(\zeta) \, |d\zeta| = o(n^{1-\alpha}).$$

uniformly for z_1 and z_2 in any closed subset of Ω.

Proof. We use the transformation $z = \varphi(s)$ introduced in the last section to transform E to a system E' consisting entirely of closed curves. If we write

$$\zeta = \varphi(\sigma), \qquad z_0 = \varphi(s_0), \qquad \text{etc.}$$

then it is easy to see that

$$K_n(\varphi(s), \varphi(s_0))$$

is the kernel function associated with the weight function

$$\rho_1(\sigma) = \rho(\varphi(\sigma)) \, |\varphi'(\sigma)|$$

on E'. Let us write it as $K_n^{(1)}(s, s_0)$. Then our integral is just

$$\int \Phi_+(\varphi(\sigma))^n \Phi_-(\varphi(\sigma))^{-n} K_n^{(1)}(\sigma, s_1) \overline{K_n^{(1)}(\sigma, s_2)} \rho^{(1)}(\sigma) \, |d\sigma| \tag{12.7}$$

integrated over a certain fixed set of subarcs of E'.

The last three factors in the integral may also be written

$$K_n^{(1)}(\sigma, s_1) R^{(1)}(\sigma)^{1/2} \overline{K_n^{(1)}(\sigma, s_2) R^{(1)}(\sigma)^{1/2}}$$

where $R^{(1)}(s)$ is the analytic function in Ω' corresponding to $\rho^{(1)}$. It follows from Theorem 7.3(3) that this is of class $C^{\alpha-1+}$ on E' uniformly in n, s_1, and s_2. Also, the function $\Phi(\varphi(s))$ is just the exponential Green's function for Ω' whose continuity properties are given in Lemma 4.1(1). In particular, if on any of the arcs of integration in (12.7) we introduce a new variable

$$\theta = \arg\{\Phi_+(\varphi(\sigma))/\Phi_-(\varphi(\sigma))\},$$

then the integral takes the form

$$\int_a^b e^{in\theta} u(\theta)\, d\theta$$

where $u \in C^{\alpha-1+}[a, b]$ uniformly in n, s_1 and s_2.

Since the functions

$$\Phi(z)^n K_n(z, z_i)$$

are single valued, the integrand in (12.7), together with each of its derivatives up to order $\alpha - 1$, takes the same value at the two ends of each arc of integration. Thus the function $e^{in\theta}u(\theta)$ in the last integral, together with its derivatives up to order $\alpha - 1$, takes the same values at a and b. The integral is therefore seen to be $o(n^{1-\alpha})$ upon integration by parts and using standard Fourier coefficient estimation [(*22*), p. 45].

The following lower bound now follows immediately from (12.5), (12.6), and Lemma 12.1, if we recall (12.2). The quantities $m_{n,\rho}$ and $\nu(\rho, \Gamma_n)$ are as before.

Lemma 12.2. *With the assumptions of Lemma* 12.1 *we have*

$$m_{n,\rho} \geqslant C(E)^{2n}\{\nu(\rho, \Gamma_n) - o(n^{1-\alpha})\}.$$

We can now prove the analogue of Theorem 9.1. The functions $P_{n,\rho}(z)$ and $F_n(z)$, are as before, and we set

$$H_n(\zeta) = \begin{cases} \Phi(\zeta)^n F_n(\zeta) & \text{on } E^{(1)}, \\ \Phi_+(\zeta)^n F_{n+}(\zeta) + \Phi_-(\zeta)^n F_{n-}(\zeta) & \text{on } E^{(2)}, \end{cases}$$

so that, in our previous notation,

$$H_n(\zeta) = K_n(\infty, \infty)^{-1} L_n(\zeta, \infty) = \nu(\rho, \Gamma_n) L_n(\zeta, \infty). \tag{12.8}$$

Theorem 12.3. *Suppose that $E \in C^{2+}$ and that ρ satisfies the conditions*

$$\int_E \rho(\zeta)\,|\,d\zeta\,| < \infty, \qquad \int_E \log \rho(\zeta)\,|\,d\zeta\,| > -\infty.$$

Then as $n \to \infty$ we have

$$m_{n,\rho} \sim C(E)^{2n} \nu(\rho, \Gamma_n),$$

$$\int_E | C(E)^{-n} P_{n,\rho}(\zeta) - H_n(\zeta)|^2 \rho(\zeta) \, | d\zeta | \to 0, \tag{12.9}$$

and

$$P_{n,\rho}(z) = C(E)^n \Phi(z)^n \{F_n(z) + o(1)\}$$

uniformly on each closed subset of Ω.

Suppose further that $E \in C^{3+}$ and that

$$\rho(\zeta) \prod |(\zeta - \alpha_k)(\zeta - \beta_k)|^{1/2} \tag{12.10}$$

is nonzero and of class C^{1+}, where α_k and β_k are the ends of the arcs of E. Then

$$P_{n,\rho}(\zeta) = C(E)^n \{H_n(\zeta) + o(1)\}$$

uniformly on E.

Proof. As usual we shall assume first that $E \in C^{\alpha+}$ ($\alpha \geqslant 2$) and that the function (12.10) is nonzero and of class $C^{\alpha-2+}$. Because of the way the function $\varphi(s)$ was defined it is easy to see that on E

$$| \varphi'(\varphi^{-1}(\zeta))| = \prod |(\zeta - \alpha_k)(\zeta - \beta_k)|^{1/2}$$
$$\times \text{ absolute value of nonzero analytic function.}$$

Thus our condition on ρ is equivalent to

$$\rho(\zeta) \, | \varphi'(\varphi^{-1}(\zeta))| \in C^{\alpha-2+}$$

and by Lemma 11.1 this implies that

$$\rho_1(\sigma) = \rho(\varphi(\sigma)) \, | \varphi'(\sigma)|$$

belongs to $C^{\alpha-2+}$ on E', and is nonzero.

Now we saw in the proof of Lemma 12.1 that $K_n(\varphi(s), \varphi(s_0))$ is the kernel function associated with the weight function $\rho_1(\sigma)$ on E' and so $F_n(\varphi(s))$ is the extremal function for problem $(3, \rho_1, \Gamma_n)$ on Ω'. Therefore the functions $F_n(\varphi(\sigma))$ are uniformly of class $C^{\alpha-2+}$ on E'. If we define as usual

$$Q(z) = \frac{1}{2\pi i} \int_C F_n(\zeta) \Phi(\zeta)^n \frac{d\zeta}{\zeta - z},$$

then by Lemma 11.2 we have

$$Q(\zeta) = H_n + o(n^{2-\alpha})$$

on E.

Now by (12.8),

$$\int |H_n(\zeta)|^2 \rho(\zeta) |d\zeta| = \nu(\rho, \Gamma_n)^2 \int |L_n(\zeta, \infty)|^2 \rho(\zeta) |d\zeta|,$$

and by (12.6) and Lemma 12.1 this is

$$\nu(\rho, \Gamma_n) + o(n^{1-\alpha}).$$

Consequently

$$\int |Q(\zeta)|^2 \rho(\zeta) |d\zeta| = \nu(\rho, \Gamma_n) + o(n^{2-\alpha}).$$

Since Q is a polynomial of degree n with leading coefficient $C(E)^{-n}$, this gives

$$m_{n,\rho} \leqslant C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{2-\alpha})\}.$$

If we combine this with the lower bound given in Lemma 12.2 we obtain

$$m_{n,\rho} = C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{2-\alpha})\}. \tag{12.11}$$

This proves the asymptotic formula for $m_{n,\rho}$ under the assumptions that $E \in C^{2+}$ and the function (12.10) is nonzero and of class C^{0+}. The restriction on ρ is removed exactly as in the proof of Theorem 9.1.

The second assertion of the lemma is proved as follows. We have

$$\begin{aligned}
&\int_E |C(E)^{-n} P_{n,\rho}(\zeta) - H_n(\zeta)|^2 \rho(\zeta) |d\zeta| \\
&\quad = C(E)^{-2n} m_{n,\rho} + \int_E |H_n(\zeta)|^2 \rho(\zeta) |d\zeta| \\
&\qquad - 2\mathscr{R} \int_E C(E)^{-n} P_{n,\rho}(\zeta) \overline{H_n(\zeta)} \rho(\zeta) |d\zeta| \\
&\quad = C(E)^{-2n} m_{n,\rho} + \nu(\rho\ \Gamma_n)^2 \int_E |L_n(\zeta, \infty)|^2 \rho(\zeta) |d\zeta| \\
&\qquad - 2\nu(\rho\ \Gamma_n) \mathscr{R} \oint_E C(E)^{-n} \Phi(\zeta)^{-n} P_{n,\rho}(\zeta) \overline{K_n(\zeta, \infty)} \rho(\zeta) |d\zeta| \\
&\quad = C(E)^{-2n} m_{n,\rho} + \nu(\rho, \Gamma_n)^2 \int_E |L_n(\zeta, \infty)|^2 \rho(\zeta) |d\zeta| - 2\nu(\rho, \Gamma_n)
\end{aligned}$$

where for the last identity we used the reproducing property of $K_n(\zeta, \infty)$. We know that

$$C(E)^{-2n} m_{n,\rho} = \nu(\rho, \Gamma_n) + o(1)$$

and by (12.6) and Lemma 12.1 we have also

$$\int_E |L_n(\zeta, \infty)|^2 \rho(\zeta) \, |d\zeta| = \nu(\rho, \Gamma_n)^{-1} + o(1).$$

The second assertion of the theorem follows.

To prove the last statement it suffices, by Corollary 7.4, to show that

$$\oint_E | C(E)^{-n} \Phi(\zeta)^{-n} P_{n,\rho}(\zeta) - F_n(\zeta)|^2 \rho(\zeta) \, |d\zeta| \to 0.$$

But since

$$F_n(\zeta) = \nu(\rho, \Gamma_n) K_n(\zeta, \infty)$$

the integral is equal to

$$C(E)^{-2n} m_{n,\rho} + \nu(\rho, \Gamma_n) - 2\nu(\rho, \Gamma_n),$$

where we have used the reproducing property of $K_n(\zeta,\infty)$. Thus the asymptotic formula for $m_{n,\rho}$ gives the result.

Finally, with the more severe restrictions of the last part of the theorem, we have by (12.11)

$$m_{n,\rho} = C(E)^{2n} \{\nu(\rho, \Gamma_n) + o(n^{-1})\}.$$

Following through the argument which gave us (12.9), we see that in this case

$$\int_E | C(E)^{-n} P_{n,\rho}(\zeta) - H_n(\zeta)|^2 \rho(\zeta) \, |d\zeta| = o(n^{-1}).$$

If we apply the variable change $\zeta = \varphi(\sigma)$, the integral becomes

$$\int | C(E)^{-n} P_{n,\rho}(\varphi(\sigma)) - H_n(\varphi(\sigma))|^2 \rho(\varphi(\sigma)) \, |\varphi'(\sigma)| \, |d\sigma|,$$

extended over certain subarcs of E'. Since this integral is at least half the integral extended over all of E', we deduce

$$\int_{E'} | C(E)^{-n} P_{n,\rho}(\varphi(\sigma)) - H_n(\varphi(\sigma))|^2 \rho(\varphi(\sigma)) \, |\varphi'(\sigma)| \, |d\sigma| = o(n^{-1}).$$

It is now easy to use the argument of Theorem 9.1 to show that this implies the integrand converges uniformly to zero. As was already seen, $F_n(\varphi(\sigma))$ is uniformly C^{1+} on E' and so very easily

$$\frac{d}{d\sigma} H_n(\varphi(\sigma)) = O(n).$$

Moreover it follows from the form of φ that $P_{n,\rho}(\varphi(s))$ is equal to a rational function of s (independent of $P_{n,\rho}$) without poles on E' times a polynomial in s of degree $2^r n$, where r is the number of arcs among the E_k. Hence by Lemma 2.1

$$\max_{E'} \left| \frac{d}{d\sigma} P_{n,\rho}(\varphi(\sigma)) \right| \leqslant An \max_{E'} | P_{n,\rho}(\varphi(\sigma))|.$$

These two estimates on the derivatives are all that one needs to carry through the argument of Theorem 9.1 and so establish uniform convergence.

This completes the proof of the theorem. Observe that the asymptotic form of $m_{n,\rho}$, and of $P_{n,\rho}(z)$ for z in Ω, are exactly the same for the case of arcs as they were for systems of closed curves exclusively. In particular Theorem 9.2 holds without change. The only difference occurs in the boundary behavior of $P_{n,\rho}(z)$.

We should like to point out here why the sort of argument we have just presented fails to go through in the case of Tchebycheff polynomials. Take for example the case of the unit interval. Then the analogue of the argument leading to (12.5) would be this:

For any monic nth-degree polynomial $P(z)$,

$$\begin{aligned} 2^{1-n} &= \frac{1}{\pi i} \oint P(z)\{z + (z^2 - 1)^{1/2}\}^{-n} \frac{dz}{(z^2-1)^{1/2}} \\ &= \frac{1}{\pi} \int_{-1}^{1} P(\zeta)(\{\zeta + i(1-\zeta^2)^{1/2}\}^n + \{\zeta - i(1-\zeta^2)^{1/2}\}^n) \frac{d\zeta}{(1-\zeta^2)^{1/2}} \\ &\leqslant \frac{1}{\pi} \max | P(\zeta)| \int_{-1}^{1} |\{\zeta + i(1-\zeta^2)^{1/2}\}^n + \{\zeta - i(1-\zeta^2)^{1/2}\}^n | \frac{d\zeta}{(1-\zeta^2)^{1/2}}. \end{aligned}$$

The change of variable $\zeta = \cos\theta$ reduces the last integral to

$$\int_0^\pi | 2 \cos n\theta | \, d\theta.$$

Unfortunately this is equal to 4, and so all we obtain is

$$\max |P(\zeta)| \geqslant \frac{\pi}{4} 2^{1-n}.$$

The best possible lower bound is of course 2^{1-n}, so a more refined argument is necessary. Naturally such an argument exists in the case of an interval (in fact there are several), but the problem is to find one that works for arcs.

13. Moment Matrices for a System of Jordan Curves and Arcs

The content of this section is the following assertion.

Theorem 13.1. *All Theorems stated in* §10 *hold without change in the more general case.*

The proof combines the techniques of §§10 and 12. There is no reason for going through the details. The strong form of Lemma 12.1, with different z_1 and z_2, is needed in the proof of the analogue of Theorem 10.3.

14. Some Special Cases

The theorems in the preceding sections give the asymptotic forms of various constants and polynomials in terms of corresponding constants and functions associated with certain extremum problems. These in turn were expressible in terms of Green's and Neumann's functions (see Theorems 5.4, 6.2, and 7.1). Moreover certain parameters (the z_j) entered which were solutions of systems of equations involving harmonic measures. It follows that in order to obtain really explicit asymptotic formulas, it is necessary to get explicit representations for the various functions and the parameters z_j.

Consider a system E of intervals

$$E_k = [\alpha_k, \beta_k], \qquad k = 1,\dots, p$$

on the real axis, ordered so that $\alpha_k < \alpha_{k+1}$. In this case it is easy to get explicit representations for all the functions and constants involved.

We shall write

$$q(z) = \prod_{k=1}^{p} (z - \alpha_k)(z - \beta_k)$$

and $q(z)^{1/2}$ will denote that branch of the square root of $q(z)$ which is asymptotically z^p near infinity. We see that $q(x)^{1/2}$ is real on the complement of E on the real axis, and the limiting values $q_{\pm}(x)^{1/2}$ on E are purely imaginary.

Suppose

$$h(z) = \sum_{n=0}^{p-1} h_n z^n$$

is a polynomial of degree $p - 1$ with real coefficients, and consider the function

$$\int_{\alpha_1}^{z} h(\zeta)\, q(\zeta)^{-1/2}\, d\zeta,$$

where the integration is performed along a path in the plane cut along E. The period of this integral around each E_k is purely imaginary, so the real part of the integral is a single-valued harmonic function, which near infinity has the form

$$h_{p-1} \log |z| + O(1).$$

Now the limiting values, from below and above E_k, of the real part of the integral are

$$\pm \sum_{j=1}^{k-1} \int_{\beta_j}^{\alpha_{j+1}} h(\zeta)\, |q(\zeta)|^{-1/2}\, d\zeta.$$

Thus if these are all zero, and $h_{p-1} = 1$, then the real part of the integral is Green's function for Ω with pole at ∞. Consequently we have

$$G(z) = \int_{\alpha_1}^{z} h(\zeta)\, q(\zeta)^{-1/2}\, d\zeta \tag{14.1}$$

where the coefficients of $h(z)$ are determined by the system of equations

$$\sum_{n=0}^{p-1} h_n \int_{\beta_j}^{\alpha_{j+1}} \zeta^n\, |q(\zeta)|^{-1/2}\, d\zeta = 0, \qquad j = 1,\ldots, p-1,$$

$$h_{p-1} = 1.$$

To show that this system does have a solution it suffices to prove that the corresponding homogeneous system has only the trivial solution.

But for the polynomial $h(z)$ corresponding to a solution of the homogeneous system the real part of (14.1) is harmonic in Ω including $z = \infty$, and zero on E. Thus the real part of the integral is identically zero and so the same is true of h itself.

It is easy to determine $C(E)$ from (14.1). Since

$$G(z) = \log\{z - (\alpha_1 + 1)\} + \int_{\alpha_1}^{z} \left\{ \frac{h(\zeta)}{q(\zeta)^{1/2}} - \frac{1}{\zeta - (\alpha_1 + 1)} \right\} d\zeta$$

we have

$$C(E) = \exp \int_{-\infty}^{\alpha_1} \left\{ \frac{h(x)}{q(x)^{1/2}} - \frac{1}{x - (\alpha_1 + 1)} \right\} dx.$$

The constant $R(\infty)$, appearing many places in our investigations, is determined by (5.11) and (14.1). In fact,

$$\log R(\infty) = \frac{1}{\pi} \int_E | h(x)\, q(x)|^{-1/2} \log \rho(x)\, dx.$$

The reason we have the factor $1/\pi$ rather than $\frac{1}{2}\pi$ is that the integral in (5.11) had to be taken over E twice, once in each direction.

Similarly for the harmonic measures we have [see (4.2)]

$$\omega_k(\infty) = \frac{1}{\pi} \int_{E_k} | h(x)\, q(x)^{-1/2} |\, dx.$$

As for the points z_j^*, the zeros of $G'(z)$, we see they are simply the zeros of the polynomial $h(z)$. It is clear from the behavior of $g(x)$ on the real axis that there is exactly one such z_j^* in the interval (β_j, α_{j+1}).

For the determination of the quantity

$$\sum_{j=1}^{p-1} g(z_j^*)$$

which also appeared several places, it is not necessary to locate the individual z_j^*. In fact we have

$$g(z_j^*) = g(z_j^*) - g(\beta_j) = \int_{\beta_j}^{z_j^*} h(x)\, q(x)^{-1/2}\, dx.$$

The integrand is positive in the interval of integration (since it is of one sign and the integral itself is positive). In the interval (z_j^*, α_{j+1})

the function $hq^{-1/2}$ is negative and the sum of the integrals over the two intervals is

$$\int_{\beta_j}^{\alpha_{j+1}} h(x)\, q(x)^{-1/2}\, dx = 0$$

by our construction of h. Hence we have

$$g(z_j^*) = \frac{1}{2} \int_{\beta_j}^{\alpha_{j+1}} | h(x)\, q(x)^{-1/2} |\, dx.$$

The points z_j ($j = 1, \ldots, q$) appearing in the statement of Theorem 5.4 and the z_j ($j = 1, \ldots, p$) of Theorem 6.2 are real since they all lie in the convex hull of E. (See the beginning of the proof of Theorem 11.5.) It is therefore of interest to determine Green's and Neumann's functions for real parameter values. The latter is easy; except for a constant factor we have

$$\Psi(z, x_1, x_2) = (z - x_2)/(z - x_1).$$

As for Green's function, we need only a slight modification of the construction in the case of parameter value infinity. The result is

$$G(z, x_0) = \int_{\alpha_1}^{z} \frac{k(\zeta)}{q(\zeta)^{1/2}} \frac{d\zeta}{\zeta - x_0},$$

where

$$k(z) = \sum_{n=0}^{p-1} k_n z^n$$

is determined by the system of equations

$$\sum_{n=0}^{p-1} k_n \int_{\beta_j}^{\alpha_{j+1}} \frac{\zeta^n}{q(\zeta)^{1/2}} \frac{d\zeta}{\zeta - x_0} = 0, \qquad j = 1, \ldots, p - 1,$$

$$\sum_{n=0}^{p-1} k_n x_0^{\,n} = -q(x_0)^{1/2}.$$

We obtain from this the representations

$$C_{x_0}(E) = | \alpha_1 - x_0 |^{-1} \exp \int_{-\infty}^{\alpha_1} \left\{ \frac{k(x)}{q(x)^{1/2}} + 1 \right\} \frac{dx}{x - x_0}$$

(see Theorem 7.1), and

$$\log |R(x_0)| = \frac{1}{\pi} \int_E \left| \frac{k(x)}{q(x)^{1/2}(x - x_0)} \right| \log \rho(x)\, dx,$$

$$\omega_k(x_0) = \frac{1}{\pi} \int_{E_k} \left| \frac{k(x)}{q(x)^{1/2}(x - x_0)} \right| dx,$$

$$g(z_j^*(x_0)) = \frac{1}{2} \int_{\beta_j}^{\alpha_{j+1}} \left| \frac{k(x)}{q(x)^{1/2}(x - x_0)} \right| dx$$

[see Theorem 7.3(2)].

In view of these last representations and Theorem 10.1 it is of interest to determine when

$$\max_{|s|=1} g(s, \infty)$$

occurs at a single point of the unit circle (which by symmetry would of necessity be $+1$ or -1); for then we could obtain a complete description of the asymptotic behavior of the eigenvalues $\lambda_{n,\rho}$ and corresponding polynomials $p_n(z)$. Unfortunately there seems to be no simple necessary and sufficient condition on E for the maximum to occur at a single point. However we shall show here that

$$\alpha_1 + \beta_1 > 0, \qquad \alpha_1 + \beta_1 + 2\alpha_1\beta_1 \geqslant 0$$

together give a sufficient condition; that in this case the maximum occurs at $s = -1$ and nowhere else. (In the case $p = 1$ the conditions are both necessary and sufficient.)

First note that

$$g(-x + iy) > g(x + iy), \qquad x > 0.$$

For the function

$$g(-x + iy) - g(x + iy)$$

is bounded and harmonic in the right half-plane cut along E (here we use $\alpha_1 + \beta_1 > 0$), is zero on the imaginary axis, and nonnegative at each point of E. It follows that it is nonnegative everywhere in the cut right half-plane. In fact it is everywhere positive except if it is identically zero. This will hold only if ($p = 1$ and) $\alpha_1 + \beta_1 = 0$ in which case our conditions are violated.

Thus the maximum can occur only for $\mathscr{R}s \leqslant 0$. We shall show that our conditions imply

$$\frac{d}{d\theta} g(e^{i\theta}) \neq 0, \qquad \frac{\pi}{2} \leqslant |\theta| < \pi$$

and this will suffice to show the maximum can occur only at $s = -1$. We have

$$\frac{d}{d\theta} G(e^{i\theta}) = isG'(s) \qquad (s = e^{i\theta})$$

and so

$$\frac{d}{d\theta} g(e^{i\theta}) = \mathscr{R} \frac{d}{d\theta} G(e^{i\theta}) = \mathscr{R} ish(s)\, q(s)^{-1/2}.$$

This will be zero if and only if

$$s^2 h(s)^2\, q(s)^{-1} \geqslant 0. \tag{14.2}$$

By symmetry we may confine attention to the upper half-plane. We shall determine all arguments by

$$0 < \arg(s - \gamma) < \pi, \qquad \gamma \text{ real}. \tag{14.3}$$

The zeros z_j^* of $h(z)$ satisfy

$$\beta_j < z_j^* < \alpha_{j+1}\,.$$

Therefore by elementary geometry we have for $\pi/2 \leqslant \theta < \pi$

$$\arg(s - z_j^*) < \begin{cases} \arg(s - \alpha_{j+1}) \\ \arg(s - \beta_{j+1}) \end{cases} \qquad j = 1, \dots, p - 1.$$

Hence

$$\arg s^2 h(s)^2\, q(s)^{-1} = \arg \frac{s^2}{(s - \alpha_1)(s - \beta_1)} \prod_{j=1}^{p-1} \frac{(s - z_j^*)^2}{(s - \alpha_{j+1})\,(s - \beta_{j+1})}$$

$$\leqslant \arg \frac{s^2}{(s - \alpha_1)(s - \beta_1)}\,.$$

Now this last argument is negative at $s = i$ (this follows from $\alpha_1 + \beta_1 > 0$). We shall show it is always negative for $\pi/2 < \theta < \pi$. For otherwise there will be such a θ for which

$$\frac{e^{2i\theta}}{(e^{i\theta} - \alpha_1)(e^{i\theta} - \beta_1)}$$

is purely real. This implies

$$0 = \mathscr{I}(1 - \alpha_1 e^{-i\theta})(1 - \beta_1 e^{-i\theta}) = \sin\theta\,\{(\alpha_1 + \beta_1) - 2\alpha_1\beta_1 \cos\theta\}.$$

Since $0 > \cos\theta > -1$, our conditions imply that this cannot be zero.

We have shown that with arguments computed by the convention (14.3) in the upper half-plane,

$$\arg s^2 h(s)^2\, q(s)^{-1} < 0, \qquad \pi/2 < \theta < \pi.$$

However since

$$\arg(s - z_j^*) > \begin{cases} \arg(s - \alpha_j) \\ \arg(s - \beta_j) \end{cases} \qquad j = 1, \ldots, p - 1,$$

we also have

$$\arg s^2 h(s)^2\, q(s)^{-1} \geqslant \arg \frac{s^2}{(s - \alpha_p)(s - \beta_p)} > -2\pi.$$

Consequently

$$-2\pi < \arg s^2 h(s)^2\, q(s)^{-1} < 0,$$

so (14.2) cannot occur and our assertion is established.

The representations obtained above for the potential-theoretic functions and quantities can be used to determine the corresponding things for any system E for which we can find a conformal correspondence between Ω and the exterior of a union of real intervals with ∞ corresponding to ∞. In the cases $p = 1, 2$ there are always such correspondences, although of course finding them is another matter.

Thus far in this section we have said nothing about how to determine the parameters z_j and q that occur in Theorem 5.4 or the z_j of Theorem 6.2. It turns out that the corresponding problems for an annulus are very easy, and so in the case $p = 2$ everything can be determined if we know the conformal correspondence, which always exists, between Ω and an annulus.

To see why the situation is so easy for an annulus

$$A : r_1 < |z| < r_2$$

observe that the harmonic measures of the inner and outer boundaries are

$$\omega_1(z) = \log r_2 |z| / \log r_1 r_2, \qquad \omega_2(z) = \log r_1 |z| / \log r_1 r_2$$

and that the level curves of $\tilde{\omega}_i$ are the rays

$$\arg z = \text{constant}.$$

Suppose our correspondence between Ω and A is such that ∞ corresponds to a point on the negative real axis. Then the single z_j^* is on the positive real axis [because $g(z)$ is symmetric about the real axis and vanishes at r_1, r_2].

Now consider Theorem 5.4. There is a single point z_1 to find and either $q = 1$ or $q = 0$. (In the former case z_1 is a zero of the extremal function F; in the latter case, not.) The first equation determining z_1 is equivalent to the assertion that z_1 and z_1^* lie on the same level curve, so it says z_1 is real and positive. (Note that although there are $p = 2$ equations, only $p - 1 = 1$ of them are independent; that is, the pth equation follows from the first $p - 1$. The same holds for the second system.)

Consider the second equation with $k = 1$. This is satisfied for $q = 0$ if and only if the right side is zero (mod 1). This may happen, but very rarely. When the right side is not congruent to zero, let η_1 be the real number in the interval (0, 1) congruent to it. Then we must have

$$\omega_1(z_1) = \eta_1$$

which is equivalent to

$$| z_1 | = r_2^{-1} \exp(\eta_1 \log r_1 r_2).$$

Since z_1 is real and positive,

$$z_1 = r_2^{-1} \exp(\eta_1 \log r_1 r_2).$$

In the case of Theorem 6.2 we again have $z_1 > 0$. We must determine, in addition to z_1, the constant $\epsilon_1 = \pm 1$ which tells us whether or not the extremal function F is zero at z_1. Let η_1 be congruent (mod 2) to the right side of the last equation and satisfy

$$-1 \leqslant \eta_1 \leqslant 1.$$

Then the last equation is equivalent to

$$\epsilon_1 \omega_1(z_1) = \eta_1 .$$

If η_1 is 0 or ± 1 then z_1 is r_2 or r_1 respectively, and it makes no difference what ϵ_1 is. Otherwise

$$z_1 = r_2^{-1} \exp(| \eta_1 | \log r_1 r_2), \qquad \epsilon_1 = \operatorname{sgn} \eta_1 .$$

Notice that in this last case, corresponding to the orthogonal polynomials, there is as likely to be a zero in Ω as not. In fact, if we recall the equivalent Jacobi inversion problem for the case of general p (see the end of §6), there is no reason to expect more z_j' on one half of the Riemann surface S than on the other (except in rather special cases). Thus the expected number of zeros of the orthogonal polynomials $P_{n,\rho}(z)$ in Ω (under the assumptions of the last part of Theorem 9.1) is asymptotically $(p-1)/2$; of course there are in any case at most $p-1$ zeros.

The case of Tchebycheff polynomials is different. We just saw that in the case $p=2$ we almost always have one zero (the maximum possible under the assumptions of the last part of Theorem 8.3); a dimensionality argument leads one to the conclusion that, for general p, there are almost always $p-1$ zeros in Ω.

References

1. N. Akhiezer, *Izv. Akad. Nauk SSSR*, 1932–33.
2. N. Akhiezer, "Theory of Approximation." Ungar, New York, 1956.
3. G. Faber, *J. Reine Angew. Math.* v. **150**, 79 (1919).
4. M. Fekete, *Math. Zeit.*, **17**, 228 (1923). (1923), pp. 228–249.
5. P. R. Garabedian, *Trans. Am. Math. Soc.* **67**, 1 (1949).
6. J. L. Geroniums, *Mat. Sborn.* **31**, 3 (1952).
7. G. H. Hardy and E. M. Wright. "An introduction to the Theory of Numbers." Oxford University Press, 1954.
8. M. Kac, W. L. Murdock, and G. Szegö, *J. Ratl. Mech. Anal.* **2**, 767 (1952).
9. N. I. Muskhelishvili, "Singular Integral Equations." Noordhoff, Groningen, The Netherlands, 1953.
10. W. F. Osgood, "Lehrbuch der Functionentheorie." Vol. II, pp. 671–675. B. G. Teubner, Leipzig, (1932).
11. W. Rudin, *Trans. Am. Math. Soc.* **78**, 46 (1955).
12. G. Springer, "Introduction to Riemann Surfaces." Addison-Wesley, Reading, Massachusetts, 1957.
13. P. K. Suetin, *Math. Surveys* **19**, 121 (1964).
14. P. K. Suetin, *Russian Math. Surveys* **21**, 35 (1966).
15. G. Szegö, *Math. Zeit.* **21**, 203 (1924).
16. G. Szegö, *Math. Zeit.* **23**, 45 (1925).
17. G. Szegö, *Am. Math. Soc. Colloq. Publ.*, Vol. 23.
18. G. Tamarkin and S. J. Havinson, *Mat. Sborn.* **46**, 195 (1958).
19. S. Warschawski, *Math. Zeit.* **35**, 321 (1932).
20. H. Widom and N. Wilf, *Proc. Am. Math. Soc.* **17**, 338 (1966).
21. H. Widom, *J. Math. Mech.* **16**, 997 (1967).
22. A. Zygmund. "Trigonometric Series." Vol. I (2nd ed.). Cambridge University Press, 1959.

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