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Non-Local Equivariant Star Product on the Minimal Nilpotent Orbit

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We construct a unique G-equivariant graded star product on the algebra $S(\mathfrak{g})/I$ of polynomial functions on the minimal nilpotent coadjoint orbit $\mathcal{O}_{\min}$ of G where G is a complex simple Lie group and $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C})$. This strengthens the result of Arnal, Benamor and Cahen.

Our main result is to compute, for G classical, the star product of a momentum function μ_x with any function f. We find $\mu_x \star f = \mu_x f + \frac{1}{2}\{\mu_x, f\}t + \Lambda^x(f)t^2$. For $\mathfrak{g}$ different from $\mathfrak{sp}(2n, \mathbb{C})$, Λ^x is not a differential operator. Instead Λ^x is the left quotient of an explicit order 4 algebraic differential operator D^x by an order 2 invertible diagonalizable operator. Precisely, $\Lambda^x = -\frac{1}{4}\frac{1}{E'(E'+1)}D^x$ where E' is a positive (half-form) shift of the Euler vector field. Thus $\mu_x \star f$ is not local in f.

Using $\star$ we construct a positive definite hermitian inner product on $S(\mathfrak{g})/I$. The Hilbert space completion of $S(\mathfrak{g})/I$ is then a unitary representation of G. This quantizes $\mathcal{O}_{\min}$ in the sense of geometric quantization and the orbit method. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

The fundamental problem in equivariant quantization is the G-equivariant quantization of the coadjoint orbits of G, where G is a simply

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connected Lie group. In deformation quantization, there is a nice set of axioms for the star product $\star$ and then G-equivariance of $\star$ is a relation involving the momentum functions μ_x, $x \in \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G)$. In fact, this amounts to G-equivariance of the corresponding quantization map (see Section 2).

It was already recognized by Fronsdal [8] that the locality axiom for star products must be modified in order to accommodate equivariance. The locality axiom means, in either the smooth or algebraic setting, that the operators which define the star product are bidifferential.

One could simply exclude any constructions that are not local. But this would cast aside equivariant constructions (such as [6], [7], [8, Section 9, p. 124] and, as we show, [1]) which are unique and very natural. In fact these constructions retain a strong flavor of locality. Figuring out what this "flavor" is and how to axiomatize it is a very interesting problem. It seems to involve "pseudo-differential" operators.

In this paper, we investigate the unique G-equivariant graded star product on the algebra $\mathscr{R}$ associated to the minimal (nonzero) nilpotent coadjoint orbit $\mathcal{O}_{\mathrm{min}}$ in $\mathfrak{g}^*$, where $\mathfrak{g}$ is a simple complex Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$. Here $\mathscr{R} = S(\mathfrak{g})/I$ is the algebra of polynomial functions on $\mathcal{O}_{\mathrm{min}}$. The star product was constructed for $\mathfrak{g}$ different from $\mathfrak{sl}(n, \mathbb{C})$ by Arnal *et al.* [1]. We strengthen their result in our Proposition 3.1, after some preliminary work in Section 2. We find an analog of the Joseph ideal for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $n \geqslant 3$. (There is a 1-dimensional family of candidates, but only one of them produces a star product with parity.) We prove uniqueness whenever $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C})$.

We show (Proposition 4.1) that the star product of a momentum function μ_x with any function $f \in \mathscr{R}$ is the three-term sum

$$\mu_x \star f = \mu_x f + \tfrac{1}{2}\{\mu_x, f\} t + \Lambda^x(f) t^2, \tag{1.1}$$

where Λ^x are graded operators on $\mathscr{R}$ of degree -1. We compute Λ^x for $\mathfrak{g}$ classical. For $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, we find (Section 5) some familiar order 2 differential operators (which appear in the Fock space model of the oscillator representation).

Our main result (Theorem 6.3) is a formula for Λ^x when $\mathfrak{g}$ is classical but different from $\mathfrak{sp}(2n, \mathbb{C})$, i.e., when $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ $(n \geqslant 2)$ or $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ $(n \geqslant 6)$. We find that Λ^x $(x \neq 0)$ is not a differential operator but instead is the left quotient of an order 4 algebraic differential operator D^x by an order 2 invertible diagonalizable operator. Precisely,

$$\Lambda^x = -\frac{1}{4} \frac{1}{E'(E'+1)} D^x,$$

where E' is a positive (half-form) shift of the Euler vector field. So $\mu_x \bigstar f$ is not local as an operator on f. Thus $\bigstar$ is not local.

The differential operators D^x were constructed by us earlier (for this purpose) in [2]. It would be very interesting now to find formulas for the operators $C_p(f,g)$ that define $f \bigstar g$. For $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, some progress toward this is made in [5] using results of Lecomte and Ovsienko [11]. Also we think that the method of Levasseur and Stafford [12], which gave a new elegant construction of our D^x for $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, might be extended to give the Λ^x and the $C_p(\cdot, \cdot)$. These approaches are based on the fact that $\mathscr{R}$ identifies with the algebra of regular functions on $T^*(\mathbb{CP}^n)$.

The star product defines a representation π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathscr{R}$. We write this out in Corollary 4.3 using the Λ^x. In Section 9, we show that $\bigstar$ gives rise to a positive definite hermitian inner product on $\mathscr{R}$ compatible with π and the grading on $\mathscr{R}$. In this way, $\mathscr{R}$ becomes the Harish-Chandra module of a unitary representation of G on the Hilbert space completion $\mathscr{H} = \hat{\mathscr{R}} = \hat{\oplus}_{d=0}^{\infty} \mathscr{R}^d$. This quantizes $\mathcal{O}_{\min}$, regarded as a real symplectic manifold, in the sense of geometric quantization. We compute the reproducing kernel of $\mathscr{H}$ and deduce that $\mathscr{H}$ is a Hilbert space of holomorphic functions on $\mathcal{O}_{\min}$.

2. EQUIVARIANT GRADED STAR PRODUCTS ON $S(\mathfrak{g})/I$

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. The symmetric algebra $\mathscr{S} = S(\mathfrak{g})$ is the algebra of polynomial function on $\mathfrak{g}^*$. Then $\mathscr{S} = \oplus_{d=0}^{\infty} \mathscr{S}^d$ is a graded Poisson algebra in the natural way, where $\{\mathscr{S}^d, \mathscr{S}^p\} \subseteq \mathscr{S}^{d+p-1}$. Let $I = \oplus_{d=0}^{\infty} I^d$ be a graded Poisson ideal in $\mathscr{S}$. We are most interested in the case when I is the ideal $\mathscr{I}(\mathcal{O})$ of functions vanishing on a nilpotent coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$. The term "nilpotent" means that the corresponding adjoint orbit consists of nilpotent elements; this happens if and only if $\mathcal{O}$ is stable under dilations.

Let $\mathscr{R} = \mathscr{S}/I$ and $\mathscr{R}^d = \mathscr{S}^d/I^d$. Then $\mathscr{R} = \oplus_{d=0}^{\infty} \mathscr{R}^d$ is again a graded Poisson algebra. If $I = \mathscr{I}(\mathcal{O})$, then $\mathscr{R}$ is the algebra of polynomial functions on the closure $\bar{\mathcal{O}}$. In the sense of algebraic geometry, $\bar{\mathcal{O}}$ is a closed complex algebraic subvariety of $\mathfrak{g}^*$ and $\mathscr{R}$ is its algebra $\mathbb{C}[\bar{\mathcal{O}}]$ of regular functions. The elements $x \in \mathfrak{g}$ define momentum functions μ_x in $\mathscr{R}^1$ and $\{\mu_x, \mu_y\} = \mu_{[x,y]}$. The natural graded linear G-action on $\mathscr{R}$ corresponds to the $\mathfrak{g}$-representation given by the operators $\{\mu_x, \cdot\}$.

A *graded star product* on $\mathscr{R}$ is an associative $\mathbb{C}[t]$-linear product $\bigstar$ on $\mathscr{R}[t]$ with the following properties. For $f, g \in \mathscr{R}$ we can write $f \bigstar g = \sum_{p=0}^{\infty} C_p(f,g) t^p$ and then

(i) $C_0(f,g) = fg$

(ii) $C_1(f,g) - C_1(g,f) = \{f,g\}$

(iii) $C_p(f, g) = (-1)^p C_p(g, f)$
(iv) $C_p(f, g) \in \mathscr{R}^{k+l-p}$ if $f \in \mathscr{R}^k$ and $g \in \mathscr{R}^l$

Notice that (ii) and (iii) imply $C_1(f, g) = \frac{1}{2}\{f, g\}$. Axiom (iii) is called the parity axiom.

Given $\bigstar$, we define a new noncommutative product on $\mathscr{R}$ by $f \circ g = f \bigstar g|_{t=1}$. Because of (iv), we can completely recover $\bigstar$ from $\circ$. It is easy to see that (iii) amounts to the relation $(f \circ g)^\alpha = g^\alpha \circ f^\alpha$ where $f \mapsto f^\alpha$ is the Poisson algebra anti-involution of $\mathscr{R}$ defined by $f^\alpha = (-1)^d f$ if $f \in \mathscr{R}^d$.

The star bracket is given by $[f, g]_\bigstar = f \bigstar g - g \bigstar f$. We say $\bigstar$ is $\mathfrak{g}$-*covariant* if $[\mu_x, \mu_y]_\bigstar = t\mu_{[x,y]}$. We say $\bigstar$ is *G-equivariant* (or *strongly* $\mathfrak{g}$-*invariant*) if we have the much stronger relation $[\mu_x, f]_\bigstar = t\{\mu_x, f\}$. We say that a G-equivariant graded star product on $\mathscr{R}$ is a *G-equivariant deformation quantization of* $\mathscr{R}$.

Suppose $\bigstar$ is a graded $\mathfrak{g}$-covariant star product on $\mathscr{R}$. Let $\mathscr{U} = \mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ equipped with its canonical filtration $\{\mathscr{U}_d\}_{d=0}^\infty$. Then $\operatorname{gr}\mathscr{U}$ identifies naturally with $\mathscr{S}$. We have a noncommutative algebra homomorphism $\Psi : \mathscr{U} \to \mathscr{R}$ defined by

$$\Psi(x_1 \cdots x_d) = \mu_{x_1} \circ \cdots \circ \mu_{x_d}. \tag{2.1}$$

Then Ψ is surjective in a filtered way, i.e., $\Psi(\mathscr{U}_p) = \oplus_{d=0}^p \mathscr{R}^d$. The kernel of Ψ is a 2-sided ideal J such that $\operatorname{gr} J = I$, and so $\operatorname{gr}(\mathscr{U}/J)$ identifies naturally with $\mathscr{S}/I$.

Thus we get a vector space isomorphism $\mathbf{q} : \mathscr{R} \to \mathscr{U}/J$ defined by

$$\mathbf{q}(\mu_{x_1} \circ \cdots \circ \mu_{x_d}) = x_1 \cdots x_d + J. \tag{2.2}$$

Then $\mathbf{q}$ is a *quantization map*, i.e., $\mathbf{q}$ induces the identity maps $\mathscr{R}^d \to \mathscr{S}^d/I^d$. We can recover $\circ$ from $\mathbf{q}$ by the formula $f \circ g = \mathbf{q}^{-1}((\mathbf{q}f)(\mathbf{q}g))$. Then $\bigstar$ is given by $f \bigstar g = \mathbf{q}_t^{-1}((\mathbf{q}_t f)(\mathbf{q}_t g))$ where $\mathbf{q}_t(f) = \mathbf{q}(f)t^d$ if $f \in \mathscr{R}^d$.

Let τ be the algebra anti-involution of $\mathscr{U}$ defined by $x^\tau = -x$ for $x \in \mathfrak{g}$; this is the so-called principal anti-automorphism. The parity axiom (iii) implies that J is stable under τ. So τ descends to $\mathscr{U}/J$, and also $\mathbf{q}(f^\alpha) = \mathbf{q}(f)^\tau$.

Clearly $\bigstar$ is G-equivariant if and only if $\mathbf{q}$ is $\mathfrak{g}$-equivariant, i.e., $\mathbf{q}(\{\mu_x, f\}) = x\mathbf{q}(f) - \mathbf{q}(f)x$. This amounts to $\mathbf{q}$ being G-equivariant. In summary, this discussion gives

PROPOSITION 2.1. *Suppose $\bigstar$ is a graded G-equivariant star product on $\mathscr{R} = \mathscr{S}/I$. Then we obtain a 2-sided ideal J in $\mathscr{U}$ and a G-equivariant quantization map $\mathbf{q} : \mathscr{R} \to \mathscr{U}/J$ given by* (2.2).

3. CONSTRUCTION OF $\bigstar$ WHEN $\mathcal{O} = \mathcal{O}_{\min}$

From now on we assume that $\mathfrak{g}$ is simple. Let $\mathcal{O}_{\min}$ be the minimal nonzero nilpotent coadjoint orbit in $\mathfrak{g}^*$. So $\mathcal{O}_{\min}$ corresponds to the adjoint orbit of highest weight vectors. We put $\mathcal{R} = \mathcal{S}/I$ where I is the ideal of $\mathcal{O}_{\min}$.

PROPOSITION 3.1. *Assume $\mathfrak{g}$ is different from $\mathfrak{sl}(2, \mathbb{C})$. Then $\mathcal{R}$ admits a unique G-equivariant graded star product $\bigstar$.*

This strengthens the result in [1] where the authors showed that, if $\mathfrak{g}$ is different from $\mathfrak{sl}(n+1, \mathbb{C})$ for $n \geqslant 1$, then $\mathcal{R}$ admits a G-equivariant graded star product which is unique up to equivalence of star products. We need to exclude $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ because $\mathfrak{sl}(2, \mathbb{C})$ admits infinitely many such star products (parameterized by $\mathbb{C}$).

Proof. The discussion in Section 2 reverses easily to give a converse to Proposition 2.1. Precisely, suppose J is a 2-sided τ-stable ideal of $\mathcal{U}$ such that $\operatorname{gr} J = I$ and $\mathbf{q} : \mathcal{R} \to \mathcal{U}/J$ is a G-equivariant quantization map such that $\mathbf{q}(f^\alpha) = \mathbf{q}(f)^\tau$. Then the formula $f \bigstar g = \mathbf{q}_t^{-1}((\mathbf{q}_t f)(\mathbf{q}_t g))$ defines a G-equivariant graded star product on $\mathcal{R}$. Thus it suffices to prove the following two statements.

(i) There exists a unique 2-sided ideal J of $\mathcal{U}$ such that $\operatorname{gr} J = I$ and $J^\tau = J$.

(ii) For such J, there exists a unique G-equivariant quantization map $\mathbf{q} : \mathcal{R} \to \mathcal{U}/J$.

Notice that in (ii), $\mathbf{q}(f^\alpha) = \mathbf{q}(f)^\tau$ follows automatically by uniqueness.

The proof of (i) breaks into two cases. If $\mathfrak{g}$ is different from $\mathfrak{sl}(n+1, \mathbb{C})$, then as in [1] we take J to be the Joseph ideal constructed in [10, Section 5]. We may characterize J as the unique 2-sided ideal in $\mathcal{U}$ whose associated graded is I. This is not the most familiar characterization, but it follows immediately by combining the fact [10, Proposition 10.2] that J is the only completely prime 2-sided ideal such that $\sqrt{\operatorname{gr} J} = I$ with the equality [9] $\operatorname{gr} J = I$. Then uniqueness of J implies that $J = J^\tau$.

Now suppose that $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geqslant 1$. Let $\mathfrak{D}^\lambda(\mathbb{CP}^n)$ be the algebra of global sections of the sheaf of twisted differential operators acting on local sections of the λth power of the canonical bundle on complex projective space; this makes sense for any complex number λ. We have a natural algebra homomorphism $\Phi^\lambda : \mathcal{U} \to \mathfrak{D}^\lambda(\mathbb{CP}^n)$. It is easy to write nice formulas for the twisted vector fields Φ_x^λ, $x \in \mathfrak{g}$, in local coordinates on the big cell $\mathbb{C}^n$; see e.g., [13].

Let J^λ be the kernel of Φ^λ. Then Φ^λ is surjective and $\operatorname{gr} J^\lambda = I$. (This follows by [3, Lemma 1.4]—their result goes through to the twisted case with the same proof.) The principal anti-automorphism τ carries J^λ to $J^{1-\lambda}$. So

$J = J^{1/2}$ satisfies the two conditions in (i). Assume $n \geqslant 2$. Then we claim that J^λ is τ-stable if and only if $\lambda = \frac{1}{2}$. To show this, we consider copies of the adjoint representation $\mathfrak{g}$.

Since $\mathfrak{g}$ appears (exactly) once in $\mathcal{S}^2$, we see that $\mathfrak{g}$ occurs twice in $\mathcal{U}_2$ and once in $J_2^\lambda = J^\lambda \cap \mathcal{U}_2$. The copy of $\mathfrak{g}$ in $\mathcal{S}^2$ corresponds, uniquely up to scaling, to a G-equivariant map $r : \mathfrak{g} \to \mathcal{S}^2$, $x \mapsto r^x$. Put $a^x = \mathbf{s}(r^x)$ where $\mathbf{s} : \mathcal{S} \to \mathcal{U}$ is the usual symmetrization map. Then the copy of $\mathfrak{g}$ in J_2^λ consists of elements $b_\lambda^x = a^x + c_\lambda x$, where c_λ is some function of λ. A simple computation using the formulas for Φ_x^λ from [13] gives (for an appropriate scaling of r) $c_\lambda = \lambda - \frac{1}{2}$. We have $\tau(a^x) = a^x$ while $\tau(x) = -x$. So $\tau(b_\lambda^x) = b_\lambda^x + (1 - 2\lambda)x$. Thus, if $\lambda \neq \frac{1}{2}$ then the unique copy of $\mathfrak{g}$ in J_2^λ is not τ-stable and consequently J^λ is not τ-stable. This proves the claim. We can show with some extra work that $J^{1/2}$ is the only 2-sided ideal J in $\mathcal{U}(\mathfrak{g})$ such that J contains the $b_{1/2}^x$ and also $\operatorname{gr} J = I$. This finishes the proof of (i). (See also Remark 7.4.)

To prove (ii) we need only the fact that the natural G-representation on $\mathcal{R}$ is multiplicity free. This fact is immediate since $\mathcal{O}_{\min}$ is the orbit of highest weight vectors in $\mathfrak{g}^*$. Indeed, $\mathcal{R}^d$ is irreducible and carries the dth Cartan power $\mathfrak{g}^{\boxtimes d}$ of the adjoint representation.

We can find a G-stable graded complement $F = \oplus_{p=0}^\infty F^p$ to I in $\mathcal{S}$. This follows because the G-action on $\mathcal{S}$ is completely reducible. (In fact, F is unique since the representation $\mathfrak{g}^{\boxtimes d}$ occurs just once in $\mathcal{S}^d$.) Then the natural projection $\mathcal{S} \to \mathcal{R}$ identifies F with $\mathcal{R}$. Since $\operatorname{gr} J = I$, we also have a vector space isomorphism $F \xrightarrow{\mathbf{s}} \mathcal{U} \to \mathcal{U}/J$. Let $\mathbf{q}$ be the corresponding map from $\mathcal{R}$ to $\mathcal{U}/J$. Then clearly $\mathbf{q}$ is a G-equivariant quantization map. Suppose $\mathbf{h}$ is another such map, and put $L = \mathbf{h}^{-1}\mathbf{q}$. Then $f \in \mathcal{R}^d$ implies $L(f) = f + g$ where $g \in \mathcal{R}^{\leqslant (d-1)}$. But L is G-equivariant and so the multiplicity-free G-decomposition of $\mathcal{R}$ forces $L(\mathcal{R}^d) = \mathcal{R}^d$. Thus $L(f) = f$. So $\mathbf{h} = \mathbf{q}$. ∎

COROLLARY 3.2. *Suppose* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. *Then* $J^\lambda = \operatorname{Ker} \Phi^\lambda$ $(\lambda \in \mathbb{C})$ *corresponds to a G-equivariant graded star product* $\star_\lambda$ *on* $\mathcal{R}$. *All such star products arise in this way, and* $\star_\lambda = \star_\mu$ *if and only if* μ *is equal to* λ *or* $1 - \lambda$.

Proof. J^λ is generated by $Q - s_\lambda$ where Q is the casimir in $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$ and $s_\lambda = 4\lambda(\lambda - 1)$. Clearly then J^λ is τ-stable. Also $J^\lambda = J^\mu$ if and only if $s_\lambda = s_\mu$. ∎

PROPOSITION 3.3. *In Proposition* 3.1, *the noncommutative algebra* $\mathcal{U}/J$ *obtained by specializing* $\star$ *at* $t = 1$ *is a simple ring.*

Proof. The Joseph ideal is maximal by [10, Theorem 7.4], and this means $\mathcal{U}/J$ is simple. If $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geqslant 2$, then $\mathcal{U}/J$ is isomorphic to $\mathfrak{D}^{1/2}(\mathbb{CP}^n)$, which is simple by [14]. ∎

4. THE OPERATORS Λ^x

PROPOSITION 4.1. *The star product of a momentum function* μ_x, $x \in \mathfrak{g}$, *with an arbitrary function* $f \in \mathscr{R}$ *is the three-term sum*

$$\mu_x \bigstar f = \mu_x f + \tfrac{1}{2}\{\mu_x, f\}t + \Lambda^x(f)t^2 \tag{4.1}$$

where the Λ^x *are operators on* $\mathscr{R}$. *The* Λ^x *commute, are graded of degree* -1, *and transform in the adjoint representation of* G.

Proof. We have $\mu_x \bigstar f = \mu_x f + \frac{1}{2}\{\mu_x, f\}t + \sum_{1=2}^{\infty} M_p^x(f)t^p$ where M_p^x is graded of degree $-p$. Then $x \otimes f \mapsto M_p^x(f)$ defines a G-equivariant map $M_p : \mathfrak{g} \otimes \mathscr{R}^d \to \mathscr{R}^{d-p}$. We know $\mathscr{R}^d \simeq \mathfrak{g}^{\boxtimes d}$—see the proof of Proposition 3.1. An easy fact about representations (from highest weight theory) is that if $\mathfrak{g}^{\boxtimes k}$ appears $\mathfrak{g} \otimes \mathfrak{g}^{\boxtimes d}$ then k lies in $\{d+1, d, d-1\}$. So $M_p = 0$ if $p \geqslant 2$. Thus we get (4.1) where $\Lambda^x = M_1^x$.

We have $(\mu_x \bigstar f) \bigstar \mu_y = \mu_x \bigstar (f \bigstar \mu_y)$. Computing the coefficients of t^4, we find $\Lambda^x \Lambda^y(f) = \Lambda^y \Lambda^x(f)$. Computing the coefficients of t^3, we get the relation $[\eta^x, \Lambda^y] = \Lambda^{[x,y]}$ where $\eta^x = \{\mu_x, \cdot\}$; this means the Λ^x transform in the adjoint representation of $\mathfrak{g}$. ∎

COROLLARY 4.2. (i) *The operators* Λ^x, $x \in \mathfrak{g}$, *completely determine* $\bigstar$.

(ii) *The* Λ^x *generate a graded commutative subalgebra* $\mathscr{A}$ *of* $\operatorname{End} \mathscr{R}$ *isomorphic to* $\mathscr{R}$.

Proof. (i) Once we know (4.1), it is easy to compute $\mu_{x_1} \cdots \mu_{x_k} \bigstar f$ by induction on k.

(ii) This is easy, in fact $\Lambda^{x_1} \cdots \Lambda^{x_k}(f)$ is the coefficient of t^{2k} in $\mu_{x_1} \cdots \mu_{x_k} \bigstar f$. Notice that $\mathscr{A} = \oplus_{d=0}^{\infty} \mathscr{A}^{-d}$ is graded in negative degrees, so that $\mathscr{A}^{-d}$ corresponds to $\mathscr{R}^d$. ∎

We have a representation π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathscr{R}$ defined by $\pi^{x,y}(f) = \mu_x \circ f - f \circ \mu_y$.

COROLLARY 4.3. *The representation* π *is irreducible and we have*

$$\pi^{x,y}(f) = \mu_{x-y} f + \tfrac{1}{2}\{\mu_{x+y}, f\} + \Lambda^{x-y}(f). \tag{4.2}$$

Proof. π is equivalent to the natural representation Π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathscr{U}/J$; indeed $\mathbf{q}$ is an intertwining map. Proposition 3.3 says that J is maximal, and so Π is simple. ∎

Remark 4.4. Once we know the Λ^x, we can construct J directly as the kernel of the algebra homomorphism $\mathscr{U} \to \operatorname{End} \mathscr{R}$ defined by $x \mapsto \pi^{x,0} =$

$\mu_x + \frac{1}{2}\{\mu_x, \cdot\} + \Lambda^x$. This is a noncommutative deformation of the fact that I is the kernel of the algebra homomorphism $\mathscr{S} \to \operatorname{End}\mathscr{R}$ defined by $x \mapsto \mu_x$.

The rest of this paper is devoted to computing the operators Λ^x when $\mathfrak{g}$ is classical.

5. THE CASE $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$

Suppose $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geqslant 1$. Let $\mathscr{P}$ be the Poisson algebra $\mathbb{C}[z_1, w_1, \ldots, z_n, w_n]$ where $\{z_i, z_j\} = \{w_i, w_j\} = 0$ and $\{z_i, w_j\} = \delta_{ij}$. We have a Poisson algebra grading $\mathscr{P} = \oplus_{k=0}^{\infty} \mathscr{P}^k$ where $\mathscr{P}^k$ is the space of homogeneous polynomials of total degree k. Then $\mathscr{P}^2$ is a Lie subalgebra, and this is a model for $\mathfrak{g}$ (i.e., $\mathscr{P}^2$ is isomorphic to $\mathfrak{g}$). Moreover, $\mathscr{P}^{even} = \oplus_{k=0}^{\infty} \mathscr{P}^{2k}$ is a model for $\mathscr{R}$. The Moyal star product on $\mathscr{P}$ restricts to $\mathscr{P}^{even}$; in this way we get a Moyal star product on $\mathscr{R}$.

We find a strengthened version of [1, Proposition 6].

PROPOSITION 5.1. *Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ $(n \geqslant 1)$. The Moyal star product on $\mathscr{R}$ is the unique G-equivariant graded star product on $\mathscr{R}$. If $n \geqslant 2$, then it corresponds to the Joseph ideal; if $n = 1$, then it corresponds to the ideal $J^{1/4}$. The Λ^x are order 2 algebraic differential operators and*

$$\Lambda^{z_i z_j} = \frac{1}{4} \frac{\partial^2}{\partial w_i \partial w_j}, \qquad \Lambda^{w_i w_j} = \frac{1}{4} \frac{\partial^2}{\partial z_i \partial z_j}, \qquad \Lambda^{z_i w_j} = -\frac{1}{4} \frac{\partial^2}{\partial w_i \partial z_j}. \tag{5.1}$$

6. COMPUTATION OF Λ^x

We assume from now on that $\mathfrak{g}$ is a classical complex simple Lie algebra different from $\mathfrak{sp}(2n, \mathbb{C})$, $n \geqslant 1$. This falls into two cases: (I) $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ where $n \geqslant 2$, or (II) $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ where $n \geqslant 6$. It turns out that we can deal with both cases simultaneously by introducing a parameter ε and setting $\varepsilon = 0$ in (I) or $\varepsilon = 1$ in (II).

We put $G = \mathrm{SL}(n+1, \mathbb{C})$ in (I) or $G = \mathrm{Spin}(n, \mathbb{C})$ in (II). Notice that there is one coincidence between (I) and (II), namely $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C}) = \mathfrak{so}(6, \mathbb{C})$.

We define m by $\dim \mathcal{O}_{\min} = 2m + 2$; so $m = n - 1$ in (I) or $m = n - 4$ in (II). We can choose a triple X, h, Y in $\mathfrak{g}$ such that X and Y correspond to elements of $\mathcal{O}_{\min}$, h is semisimple and we have the bracket relations $[X, Y] = h$, $[h, X] = 2X$, $[h, Y] = -2Y$.

We put $p_\varepsilon = p + \varepsilon$ and $p_{-\varepsilon} = p - \varepsilon$. In this same setting we proved the following. Let $\mathscr{D}(\mathcal{O}_{\min})$ be the algebra of algebraic differential operators on $\mathcal{O}_{\min}$.

THEOREM 6.1 (Astashkevich and Brylinski [2]). *Let $\mathcal{D}_{4;-1}(\mathcal{O}_{\min})$ denote the subspace of $D \in \mathcal{D}(\mathcal{O}_{\min})$ such that D has order at most 4 and D is graded of degree -1, i.e., $D(\mathcal{R}^p) \subseteq \mathcal{R}^{p-1}$.*

Then $\mathcal{D}_{4;-1}(\mathcal{O}_{\min})$ contains a unique copy of the adjoint representation of G. In other words, there is a nonzero G-equivariant complex linear map $\mathfrak{g} \to \mathcal{D}_{4;-1}(\mathcal{O}_{\min})$, $x \mapsto D^x$, and this map is unique up to scaling. For $x \neq 0$, D^x has order exactly 4.

We can normalize the map $x \mapsto D^x$ so that, for $p \geqslant 0$,

$$D^Y(\mu_X^p) = \gamma_p \mu_X^{p-1}, \tag{6.1}$$

where $\gamma_p = p(p + \frac{m-1}{2})p_\varepsilon(p_{-\varepsilon} + \frac{m}{2})$. For $p \geqslant 1$, $D^Y(\mu_X^p) \neq 0$.

Finally, the operators D^x generate a graded commutative subalgebra of $\mathcal{D}(\mathcal{O}_{\min})$ which is isomorphic to $\mathcal{R}$. Thus we get a G-equivariant algebra embedding $\mathcal{R} \to \mathcal{D}(\mathcal{O}_{\min})$, $f \mapsto D^f$, where $D^{\mu_x} = D^x$.

Proof. This is a summary of the following results in [2]: Theorems 3.2.1 and 3.2.3, Corollary 3.2.4, Propositions 4.2.3 and 4.3.3, and Corollary 3.2.5. ∎

Remark 6.2. (i) If $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C}) = \mathfrak{so}(6, \mathbb{C})$, then we can equally well choose $\varepsilon = 0$ or $\varepsilon = 1$ in computing γ_p. In either case we end up with the same answer. (ii) D^x $(x \in \mathfrak{g})$ extends to an algebraic differential operator on $\bar{\mathcal{O}}_{\min}$.

Let E be the Euler vector field on $\mathcal{O}_{\min}$ so that E operates on $\mathcal{R}$ and $\mathcal{R}^d$ is its d-eigenspace. We put $E' = E + \frac{m+1}{2}$. Notice that E' is diagonalizable on $\mathcal{R}$ with positive spectrum. Thus $E' + k$ is invertible for any $k \geqslant 0$.

THEOREM 6.3. *For $x \in \mathfrak{g}$ we have $\Lambda^x = -\frac{1}{4E'(E'+1)} D^x$.*

Proof. This occupies Section 7. ∎

We found this formula for Λ^x because we expected this shape $\Lambda^x = P^{-1}D^x$ where P is a quantization of $4\lambda^2$ and λ is the symbol of E; see [2, Sect. 1].

Remark 6.4. We can fit the case $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ discussed in Section 5 into this framework formally by putting $D^x = -4E'(E'+1)\Lambda^x$ where the Λ^x were given in (5.1) and again $E' = E + \frac{m+1}{2}$ for $m = \frac{1}{2}\dim \mathcal{O}_{\min} - 1 = n - 1$. Then the formula $D^Y(\mu_X^p) = \gamma_p \mu_X^{p-1}$ in Theorem 6.1 still holds if we compute γ_p when $\varepsilon = -\frac{1}{2}$. Here we may choose $X = -\frac{1}{2}w_1^2$, $Y = \frac{1}{2}z_1^2$, $h = z_1 w_1$.

7. PROOF OF THEOREM 6.3

LEMMA 7.1. *We have $\Lambda^x = \phi D^x$ where ϕ is a diagonalizable linear operator on $\mathcal{R}$. The operator ϕ is unique and given by scalars ϕ_d $(d \geqslant 0)$ so that $\phi(f) = \phi_d f$ if $f \in \mathcal{R}^d$.*

Proof. We know Λ^x and D^x kill $\mathscr{R}^0$. Let $p \geqslant 1$. We have two G-equivariant maps $\mathfrak{g} \otimes \mathscr{R}^p \to \mathscr{R}^{p-1}$ defined by $\alpha_p(x \otimes f) = \Lambda^x(f)$ and $\beta_p(x \otimes f) = D^x(f)$. These must be proportional because $\mathrm{Hom}_G(\mathfrak{g} \otimes \mathfrak{g}^{\boxtimes p}, \mathfrak{g}^{\boxtimes (p-1)})$ is 1-dimensional. We know that β_p is nonzero by Theorem 6.1. So there is a unique scalar ϕ_{p-1} such that $\alpha_p = \phi_{p-1}\beta_p$. ∎

At this point, there is no guarantee that ϕ_p will be a nice function of p, in the sense that ϕ is a reasonable function of E. But Theorem 6.3 asserts $\phi = -\frac{1}{4}\frac{1}{E'(E'+1)}$.

To prove this, we will write down a series of recursion relations for the ϕ_p. To derive the recursions, we start with the bracket relation $[\pi^{x,-x}, \pi^{y,-y}] = \pi^{z,z}$ where $z = [x, y]$. By (4.2) we have $\pi^{x,-x} = 2\mu_x + 2\Lambda^x$ and $\pi^{z,z} = \eta^z$ where $\eta^z = \{\mu_z, \cdot\}$. Since the operators Λ^x (like the operators $f \mapsto \mu_x f$) commute among themselves, we get

$$[\mu_x, \Lambda^y] + [\Lambda^x, \mu_y] = \tfrac{1}{4}\eta^{[x,y]}. \tag{7.1}$$

We choose $x = X$ and $y = Y$ so that $[x, y] = h$. Writing $\Lambda^x = \phi D^x$ and applying the operator identity (7.1) to a test function $f \in \mathscr{R}^p$, $p \geqslant 1$, we find

$$\phi_{p-1}\mu_X D^Y(f) - \phi_p D^Y(\mu_X f) + \phi_p D^X(\mu_Y f) - \phi_{p-1}\mu_Y D^X(f) = \tfrac{1}{4}\eta^h(f). \tag{7.2}$$

The recursions will arise by evaluating this for $f = \mu_X^s \mu_Y^t$, with $s + t = p$.

Before we can write down the recursions, we need some auxiliary computations, provided by the next result. (Unfortunately, (7.2) is not sufficient to determine all ϕ_p.)

LEMMA 7.2. *For $s, t \geqslant 0$ we have*

$$D^Y(\mu_X^s \mu_Y^t) = \alpha_{s,t}\mu_X^{s-1}\mu_Y^t + \beta_{s,t}\mu_X^{s-2}\mu_Y^{t-1}\mu_h^2, \tag{7.3}$$

$$D^X(\mu_X^t \mu_Y^s) = \alpha_{s,t}\mu_X^t \mu_Y^{s-1} + \beta_{s,t}\mu_X^{t-1}\mu_Y^{s-2}\mu_h^2, \tag{7.4}$$

where $\alpha_{s,t} = \gamma_s + \frac{1}{2}st(2s + t + m)$ *and* $\beta_{s,t} = -\frac{1}{4}(s-1)st(2s + t + m)$.

Proof. We have to go back into our explicit construction of D^Y in [2, Sect. 4]. We worked over the Zariski open dense set $\mathcal{O}^*_{\min} = (\mu_Y \neq 0)$ in $\mathcal{O}_{\min}$. We constructed D^Y as the quotient $D^Y = \frac{1}{\mu_Y} S$ where S is a certain differential operator on $\mathcal{O}^*_{\min}$. More precisely, $S = \frac{1}{4}(T - q(\eta^Y)^2)$ where $q = (E + \frac{m}{2} + \varepsilon)(E + \frac{m}{2} - \varepsilon)$ and T is an explicit noncommutative polynomial in some vector fields on $\mathcal{O}^*_{\min}$ which annihilate μ_Y. Also η^Y annihilates μ_Y. It follows that for any $g \in \mathbb{C}[\mathcal{O}^*_{\min}]$ we have $T(g\mu_Y^t) = T(g)\mu_Y^t$ and $\eta^Y(g\mu_Y^t) = \eta^Y(g)\mu_Y^t$.

Now we can compute $D^Y(\mu_X^s\mu_Y^t)$. We have $D^Y = A - B$ where $A = \frac{1}{4\mu_Y}T$ and $B = \frac{1}{4\mu_Y}q(\eta^Y)^2$. Then we find $D^Y(g\mu_Y^t) = D^Y(g)\mu_Y^t + B(g)\mu_Y^t - B(g\mu_Y^t)$.

Let $g = \mu_X^s$. Then $D^Y(\mu_X^s) = \gamma_s \mu_X^{s-1}$ by (6.1). Also, since $\eta^Y(\mu_X) = \mu_{[Y,X]} = -\mu_h$ and η^Y is a vector field we find (as in [2, (67)])

$$(\eta^Y)^2(\mu_X^s) = (-2s\mu_X\mu_Y + s(s-1)\mu_h^2)\mu_X^{s-2}.$$

Using this we find

$$B(\mu_X^s\mu_Y^t) = \tfrac{1}{4} s q_{s+t}(-2\mu_X + (s-1)\mu_Y^{-1}\mu_h^2)\mu_X^{s-2}\mu_Y^t,$$

where $q_p = (p + \frac{m}{2} + \varepsilon)(p + \frac{m}{2} - \varepsilon)$. Now we obtain

$$\begin{aligned} D^Y(\mu_X^s\mu_Y^t) &= \gamma_s\mu_X^{s-1}\mu_Y^t - \tfrac{1}{4}s(q_{s+t} - q_s)(-2\mu_X + (s-1)\mu_Y^{-1}\mu_h^2)\mu_X^{s-2}\mu_Y^t \\ &= \alpha_{s,t}\mu_X^{s-1}\mu_Y^t + \beta_{s,t}\mu_X^{s-2}\mu_Y^{t-1}\mu_h^2, \end{aligned}$$

where $\alpha_{s,t} = \gamma_s + \frac{1}{2}s(q_{s+t} - q_s)$ and $\beta_{s,t} = -\frac{1}{4}(s-1)s(q_{s+t} - q_s)$. This proves (7.3).

We can prove (7.4) by applying a certain automorphism. Let $\chi\colon \mathrm{SL}(2,\mathbb{C}) \to G$ be the Lie group homomorphism corresponding to the Lie algebra inclusion $\mathfrak{s} \to \mathfrak{g}$ where $\mathfrak{s}$ is the span of X, h, and Y. The adjoint action of $\chi\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ defines a Lie algebra automorphism ϑ of $\mathfrak{g}$. Then $\vartheta(X) = -Y$, $\vartheta(Y) = -X$ and $\vartheta(h) = -h$. Clearly ϑ preserves $\mathcal{O}_{\min}$ and hence induces algebra automorphisms of $\mathcal{R}$ and of $\mathcal{D}(\bar{\mathcal{O}}_{\min})$ which we again call ϑ. Then $\vartheta(\mu_x) = \mu_{\vartheta(x)}$ and $\vartheta(D^x) = D^{\vartheta(x)}$. Now applying ϑ to (7.3) we get (7.4). ∎

Remark 7.3. For $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, these calculations become much easier if we use the formulas for D^x found in [12]. But there are no such formulas known when $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$.

Now we can obtain the recursions by plugging in $f = \mu_X^s\mu_Y^t$, where $p = s+t$, into (7.2). We evaluate using (7.3), (7.4) and the fact $\eta^h(f) = 2(s-t)f$. The result only involves two functions, namely f and $g = \mu_X^{s-1}\mu_Y^{t-1}\mu_h^2$. We find, for $s, t \geqslant 0$,

$$\begin{aligned} &\phi_{p-1}[(\alpha_{s,t} - \alpha_{t,s})f + (\beta_{s,t} - \beta_{t,s})g] - \phi_p[(\alpha_{s+1,t} - \alpha_{t+1,s})f + (\beta_{s+1,t} - \beta_{t+1,s})g] \\ &\quad = \tfrac{1}{2}(s-t)f. \end{aligned}$$

Equating coefficients of f and g we obtain the two recursions

$$\phi_{p-1}(\alpha_{s,t} - \alpha_{t,s}) - \phi_p(\alpha_{s+1,t} - \alpha_{t+1,s}) = \tfrac{1}{2}(s-t), \tag{7.5}$$

$$\phi_{p-1}(\beta_{s,t} - \beta_{t,s}) - \phi_p(\beta_{s+1,t} - \beta_{t+1,s}) = 0. \tag{7.6}$$

Both recursions are valid for $s, t \geqslant 1$, since f and g are linearly independent functions on $\mathcal{O}_{\min}$. Moreover (7.5) is valid for all $s, t \geqslant 0$, since $\beta_{i,j} = 0$ if $i = 0$, $i = 1$ or $j = 0$.

First we consider (7.6). Our formula for $\beta_{s,t}$ in Lemma 7.2 yields

$$\begin{aligned} \beta_{s,t} - \beta_{t,s} &= -\tfrac{1}{4} st(s-t)(2s+2t+m-1), \\ \beta_{s+1,t} - \beta_{t+1,s} &= -\tfrac{1}{4} st(s-t)(2s+2t+m+3). \end{aligned} \tag{7.7}$$

For $p \geqslant 3$ we can write $p = s + t$ with $s, t \geqslant 1$ and $s \neq t$. Then (7.6) and (7.7) give

$$\phi_p = \frac{2p+m-1}{2p+m+3}\phi_{p-1}, \qquad p \geqslant 3, \tag{7.8}$$

This is a very simple recursion with solution

$$\phi_p = \phi_2 \frac{(m+5)(m+7)}{(2p+m+1)(2p+m+3)}, \qquad p \geqslant 2. \tag{7.9}$$

Our aim is to prove $\phi = -\frac{1}{4}\frac{1}{E'(E'+1)}$, which amounts to $\phi_p = -\frac{1}{4d_p(d_p+1)}$, $p \geqslant 0$, where $d_p = p + \frac{m+1}{2}$. So we are pleased that (7.9) gives

$$\phi_p = \frac{\omega}{4d_p(d_p+1)}, \qquad p \geqslant 2, \tag{7.10}$$

where ω is the constant $(m+5)(m+7)\phi_2$.

To determine ϕ_p at $p = 0, 1, 2$, we implement (7.5) for $t = 0$ and $p = s$. Since $\alpha_{p,0} = \gamma_p$ and $\alpha_{0,p} = 0$ we get

$$\phi_{p-1}\gamma_p - \phi_p(\gamma_{p+1} - \alpha_{1,p}) = \tfrac{1}{2}p, \qquad p \geqslant 1. \tag{7.11}$$

To use this, we observe $\gamma_p = p d_p v_p$, $p \geqslant 0$, where $v_p = p_\varepsilon(p_{-\varepsilon} + \frac{m}{2})$. After a little work, we find $\gamma_{p+1} - \alpha_{1,p} = p(d_p+1)(v_p + 2d_p)$. Now (7.11) gives

$$\phi_p = \frac{v_p(d_p-1)\phi_{p-1} - \frac{1}{2}}{(v_p+2d_p)(d_p+1)}, \qquad p \geqslant 1. \tag{7.12}$$

If we put $\lambda_p = 4d_p(d_p+1)\phi_p$ $(p \geqslant 0)$, then this simplifies nicely to give

$$v_p(\lambda_p - \lambda_{p-1}) = -2d_p(\lambda_p + 1), \qquad p \geqslant 1. \tag{7.13}$$

Plugging in $\lambda_p = \omega$ for $p \geqslant 2$, we get $\omega = -1$. Then (7.13) gives $\lambda_1 = \lambda_0 = -1$. Thus, for all $p \geqslant 0$, $\lambda_p = -1$ and so $\phi_p = -\frac{1}{4d_p(d_p+1)}$.

Remark 7.4. In this proof, we only used the fact that there exists some J such that $\operatorname{gr} J = I$ and $J = J^\tau$. But now (see Remark 4.4) we can recover J

as the kernel of the algebra homomorphism $\mathscr{U} \to \operatorname{End} \mathscr{R}$ defined by $x \mapsto \mu_x + \frac{1}{2}\{\mu_x, \cdot\} - \frac{1}{4E'(E'+1)} D^x$. This gives a different proof that J is unique.

8. CONSEQUENCES OF THEOREM 6.3

Theorem 6.3 gives the following. We may rescale the complex Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ of $\mathfrak{g}$ so that $\langle X, Y \rangle_{\mathfrak{g}} = \frac{1}{2}$.

COROLLARY 8.1. (i) *We have* $\Lambda^Y(\mu_X^p) = \zeta_p \mu_X^{p-1}$ *where*

$$\zeta_p = -\frac{\gamma_p}{(2p+m-1)(2p+m+1)}$$

(ii) *The map* $\Lambda^x : \mathscr{R}^p \to \mathscr{R}^{p-1}$ *is nonzero if* $p \geqslant 1$ *and* $x \neq 0$.

(iii) $\Lambda^x(y) = c\langle x, y \rangle_{\mathfrak{g}}$ *where* c *is a nonzero scalar*; *in fact* $c = 2\zeta_1$.

Proof. (i) is immediate using (6.1). This gives (ii) if $x = Y$ (since $\gamma_p \neq 0$ if $p \geqslant 1$). Since the Λ^x transform in the adjoint representation, we get (ii) for all x. Finally (iii) follows because the map $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, $x \otimes y \mapsto \Lambda^x(\mu_y)$, is G-invariant and so must be a multiple $c\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ of our normalized Killing form (see Section 7). Then c is nonzero by (ii). By choosing $x = Y$ and $y = X$, we find $c = 2\zeta_1$. ∎

COROLLARY 8.2. *For* $x \neq 0$, Λ^x *fails to be a differential operator on* $\mathcal{O}_{\min}$. *In fact, neither factor* E' *nor* $E' + 1$ *left divides* D^x.

Proof. Suppose one of E' or $E' + 1$ left divides D^x so that the quotient is a differential operator A^x on $\mathcal{O}_{\min}$. Since D^x has order 4 (Theorem 6.1), A^x has order 3. But then the A^x ($x \in \mathfrak{g}$) span a copy of the adjoint representation in $\mathscr{D}_{4;-1}(\mathcal{O}_{\min})$ which is different from the copy spanned by the D^x. This contradicts the uniqueness part of Theorem 6.1. ∎

Notice that the corollary implies that Λ^x fails to be a differential operator on $\mathscr{R}$ (since otherwise Λ^x would be a differential operator on $\bar{\mathcal{O}}_{\min}$).

Remark 8.3. Theorem 6.3 suggests that $C_2(\mu_x, \cdot) = \Lambda^x$ is "pseudo-differential" in some sense. This is different in character from the often cited example of "pseudo-differential" star product found [8, Sect. 9, p. 124] for coadjoint orbits of the Euclidean group $E(2)$. There Fronsdal obtains a star product where the operator $f \mapsto \mu_x \bigstar f$ is an infinite series of differential operators $C_k(\mu_x, \cdot)t^k$ with increasing order.

There is a unique (up to scaling) casimir in $\mathscr{U}_2$, namely $Q=\sum_{i=1}^{\dim\mathfrak{g}} x_i^2$, where $\{x_i\}_{i=1}^{\dim\mathfrak{g}}$ is a basis of $\mathfrak{g}$ such that $\langle x_i, x_j\rangle_{\mathfrak{g}}=\delta_{ij}$. We next compute how Q acts on $\mathscr{U}/J$ with respect to the left multiplication action of $\mathscr{U}$.

COROLLARY 8.4. *Q acts on $\mathscr{U}/J$ by the scalar $s=-\frac{(1+\varepsilon)(m+2-2\varepsilon)}{2(m+3)}\dim\mathfrak{g}$.*

Proof. Let $N=\dim\mathfrak{g}$. By (2.2), the image of Q in $\mathscr{U}/J$ is $\mathbf{q}(\sum_{i=1}^{N}\mu_{x_i}\circ\mu_{x_i})$. But the function $\sum_{i=1}^{N}\mu_{x_i}^2$ is G-invariant and so it vanishes on $\mathcal{O}_{\min}$. So (4.1) and Corollary 8.1 give $\sum_{i=1}^{N}\mu_{x_i}\circ\mu_{x_i}=\sum_{i=1}^{N}(\mu_{x_i}^2+c\langle x_i,x_i\rangle_{\mathfrak{g}})=2\zeta_1 N$. Thus the image of Q in $\mathscr{U}/J$ is equal to $2\zeta_1 N$. ∎

Remark 8.5. We conjecture that for the five exceptional simple Lie algebras, Λ^x again has the form $-\frac{1}{4E'(E'+1)}D^x$ where D^x are some (as yet unknown) order 4 algebraic differential operators on $\mathcal{O}_{\min}$.

9. HERMITIAN INNER PRODUCT ON $\mathscr{R}$

We assume that $\mathfrak{g}$ is a complex simple Lie algebra different from $\mathfrak{sl}(2,\mathbb{C})$. Let U be a maximal compact subgroup of G. Let σ be the corresponding Cartan involution of $\mathfrak{g}$; so σ is $\mathbb{C}$-antilinear. Then $\mathfrak{g}^\sharp=\{(x,\sigma(x))\mid x\in\mathfrak{g}\}$ is a real form of $\mathfrak{g}\oplus\mathfrak{g}$ isomorphic to $\mathfrak{g}$. We have a U-invariant $\mathbb{C}$-antilinear graded algebra involution $f\mapsto\bar{f}$ on $\mathscr{R}$ defined by $\bar{f}(z)=\overline{f(\sigma(z))}$; see [2, Sect. 2.3].

THEOREM 9.1. *The formula*

$$(f|g) = \text{constant term in } f\circ\bar{g} \tag{9.1}$$

defines a U-invariant positive definite hermitian inner product on $\mathscr{R}$ with $(1|1)=1$. In addition $(\cdot|\cdot)$ is $\mathfrak{g}^\sharp$-invariant, i.e., the operators $\pi^{x,\sigma(x)}$ $(x\in\mathfrak{g})$ are skew-adjoint.

Proof. The pairing $(\cdot|\cdot)$ is clearly sesquilinear and U-invariant with $(1|1)=1$. The spaces $\mathscr{R}^j$ $(j\geqslant 0)$ carry inequivalent irreducible representations of G, and so it follows by U-invariance that $\mathscr{R}^j$ is orthogonal to $\mathscr{R}^k$ if $j\neq k$. We can compute $(\cdot|\cdot)$ on $\mathscr{R}^j$ in the following way. If $g\in\mathscr{R}^j$ then using Corollary 4.3 we find

$$(\mu_{x_1}\cdots\mu_{x_j}|g)=(\mu_{x_1}\circ\cdots\circ\mu_{x_j}|g)=\Lambda^{x_1}\cdots\Lambda^{x_j}(\bar{g}). \tag{9.2}$$

By Corollary 4.2(ii), we have a G-equivariant algebra embedding $\mathscr{R}\to\operatorname{End}\mathscr{R}$, $f\mapsto\Lambda^f$, where $\Lambda^{\mu_x}=\Lambda^x$. It follows now that for any $f,g\in\mathscr{R}$ we have

$$(f|g)=\text{constant term in } \Lambda^f(\bar{g}). \tag{9.3}$$

This formula easily implies that the adjoint of (ordinary) multiplication by μ_x is $\Lambda^{\sigma(x)}$. Hence the operators $\mu_x - \Lambda^{\sigma(x)}$ are skew-adjoint. But also the operators $\{\mu_{x+\sigma(x)}, \cdot\}$ are skew-adjoint since they correspond to the action of U. So using formula (4.2) for $\pi^{x,y}$, we see the operators $\pi^{x,\sigma(x)}$ are skew-adjoint.

To show $(\cdot|\cdot)$ is hermitian positive definite, it suffices to show that each number $||\mu_Y^p||^2 = (\mu_Y^p|\mu_Y^p)$, $p \geqslant 1$, is positive. We will use Remark 6.4 so that we can treat all cases $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C})$ simultaneously. We may assume now, by rechoosing (X, h, Y) if needed, that $\sigma(Y) = -X$; see [2, Sect. 2.3]. Using 8.1(i), we compute

$$||\mu_Y^p||^2 = (-1)^p \Lambda^{Y^p}(\mu_X^p) = (-1)^p \zeta_1 \cdots \zeta_p = \prod_{i=1}^{p} \frac{\gamma_i}{(2i+m-1)(2i+m+1)}. \tag{9.4}$$

This number is positive, since γ_p is positive by Theorem 6.1. ∎

Notice that, since π is irreducible by Corollary 4.3, $(\cdot|\cdot)$ is the unique $\mathfrak{g}^\sharp$-invariant hermitian pairing on $\mathscr{R}$ such that $(1|1) = 1$.

COROLLARY 9.2. *The operators* $\pi^{x,\sigma(x)}$ *on* $\mathscr{R}$ *exponentiate to give a unitary representation of G on the Hilbert space direct sum* $\mathscr{H} = \hat{\bigoplus}_{d=0}^{\infty} \mathscr{R}^d$. *Then* $\mathscr{R}$ *is the Harish-Chandra module of this unitary representation.*

Proof. This follows by a theorem of Harish-Chandra since $\mathscr{R}$ is an admissible $(\mathfrak{g} \oplus \mathfrak{g}, U)$-module where $\mathfrak{g} \oplus \mathfrak{g}$ acts by π. ∎

This quantizes $\mathcal{O}_{\min}$ in the sense of geometric quantization and the orbit method.

Remark 9.3. The shift from E to E' is a half-form shift just as in [4, Proposition 5].

COROLLARY 9.4. *The unitary representation of* G *on* $\mathscr{H}$ *admits a reproducing kernel* $\mathscr{K}$. *Explicitly,* $\mathscr{K}$ *is the function* $\mathscr{K}(x, y)$ *on* $\mathcal{O}_{\min} \times \mathcal{O}_{\min}$ *given by the hypergeometric function*

$$\mathscr{K} = {}_1F_2\left(\frac{m+3}{2}; 1+\varepsilon, 1-\varepsilon+\frac{m}{2}; 2T\right), \tag{9.5}$$

where $T(x, y) = -\langle x, \sigma(y) \rangle_{\mathfrak{g}}$. *So* $\mathscr{K}(x, y)$ *is holomorphic in* x *and anti-holomorphic in* y. *Consequently,* $\mathscr{H}$ *is a Hilbert space of holomorphic functions on* $\mathcal{O}_{\min}$.

Proof. Going back to (9.4), we find

$$||\mu_Y^p||^2 = \frac{p!(1+\varepsilon)_p(1-\varepsilon+\frac{m}{2})_p}{4^p(\frac{m+3}{2})_p}, \tag{9.6}$$

where we are using the classical notation $(a)_p = a(a+1)\cdots(a+p-1)$. By definition, $\mathscr{K} = \sum_{i=0}^{\infty} f_i \otimes \overline{f}_i$ where $f_0, f_1, \ldots$ is an orthonormal basis of $\mathscr{R}$ with respect to $(\cdot|\cdot)$. On the other hand, $T = \sum_{i=0}^{N} s_i \otimes \bar{s}_i$ where $s_0, \ldots, s_N$ is an orthonormal basis of $\mathscr{R}^1$ with respect to the hermitian inner product $\langle \mu_x | \mu_y \rangle = -\langle x, \sigma(y) \rangle_{\mathfrak{g}}$. This is positive definite since $\langle \mu_Y | \mu_Y \rangle = \langle Y, X \rangle_{\mathfrak{g}} = \frac{1}{2}$. It follows, as in [4, Sect. 8], that

$$\mathscr{K} = \sum_{p=0}^{\infty} \frac{1}{||\mu_Y^p||^2} \left(\frac{T}{2}\right)^p.$$

So (9.6) gives (9.5). ∎

Remark 9.5. In the case $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, our Hilbert space $\mathscr{H}$ is just the classical Fock space of even holomorphic functions $f(z_1, w_1, \ldots, z_n, w_n)$ with reproducing kernel $\cosh(2\psi)$ where $\psi = \sum_{i=1}^{n} (|z_i|^2 + |w_i|^2)$. Indeed, $T = \frac{1}{2}\psi^2$ and $\varepsilon = -\frac{1}{2}$ (by Remark 6.4). The hypergeometric series collapses so that

$$\mathscr{K} = {}_1F_2\left(\frac{m+3}{2}; \frac{1}{2}, \frac{m+3}{2}; 2T\right) = \cosh(\sqrt{8T}) = \cosh(2\psi). \tag{9.7}$$

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