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An Algebra of Pseudodifferential Operators and the Asymptotics of Quantum Mechanics

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We discuss in detail the regularity properties of a class of pseudodifferential operators on $\mathbb{R}^l$ introduced by Grossmann, Loupias, and Stein, which are more regular and more symmetrical than usual pseudodifferential operators. Those operators that are self-adjoint form a suitable class of smooth observables for a nonrelativistic quantum theory. If their symbols are allowed to depend smoothly upon Planck's constant $\hbar$, those operators provide the framework for regular asymptotics expansions as $\hbar \to 0$ of quantum mechanics around classical mechanics.

INTRODUCTION

This article studies the relevance of a ring of pseudodifferential operators on $\mathbb{R}^l$, introduced by Grossmann *et al.* [1], to the usual formulation of nonrelativistic quantum mechanics using self-adjoint operators on a Hilbert space. The symbols in [1] lead to operators more regular than the usual pseudodifferential operators [2, 3]; they satisfy global L^2 estimates, and some operators with real semi-bounded symbols admit Friedrichs extensions. When the symbols are allowed to depend explicitly and smoothly upon $\hbar$, the relationship between symbol calculus and the classical limit of quantum mechanics is recovered [1, 4, 5]. Moreover, we thus obtain a framework for *regular* asymptotic expansions of quantum mechanics as $\hbar \to 0$; the discussion of relevant physical examples is started in [5].

We use the following notation: $Q = \mathbb{R}^l$ is configuration space; our pseudo-differential operators will act on various function or distribution spaces $\mathscr{S}(Q)$, $L^2(Q)$, $\mathscr{S}'(Q)$,... (always complex-valued unless stated otherwise, and noted as in [6]); their symbols will be functions on the phase space $X = Q \oplus P$ $(P = Q^*)$ which carries the canonical symplectic 2-form ω; $\omega(x_1, x_2) = q_1 p_2 - q_2 p_1$ (vector indices will usually be suppressed and qp will denote the scalar product).

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In any dimension n, we shall use multi-indices α, $\beta \in \mathbb{N}^n$ with the usual notation:

$$|\alpha| = \sum_1^n \alpha_j; \qquad \alpha \pm \beta = (\alpha_1 \pm \beta_1, ..., \alpha_n \pm \beta_n);$$

$$\alpha \geqslant \beta \qquad \text{if} \quad \alpha - \beta \in \mathbb{N}^n; \qquad \alpha! = \prod_1^n \alpha_j!;$$

if $r \in \mathbb{R}^n$, $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$; $\partial_r^{\alpha} = (\partial/\partial r)^\alpha = \partial^{\alpha_1}/\partial r_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial r_n^{\alpha_n}$.

The Hilbert space $L^2(Q, dq)$ is represented by $\mathscr{H}$ and its scalar product is represented by $\langle , \rangle$.

1. The Weyl Correspondence

The starting idea in any pseudodifferential operator theory is to establish a 1–1 correspondence between polynomials on X and differential operators on Q: $a(q, p) \leftrightarrow \hat{a}(q, -i\partial/\partial q)$ and then to extend this correspondence to larger classes of functions; the function a is called the (full) *symbol* of the operator $\hat{a}$. The purpose of this correspondence is to study the operators $\hat{a}$ via their symbols.

The symbol map is usually such that if $\hat{a}$ has constant coefficients, $\hat{a} = P(-i\partial/\partial q_j)$, then its symbol is the same function as that of the scalar variables p_j ($a = P(p_j)$) (thus, since the symbol of $-i\partial/\partial q_j$ is p_j, we can write $\hat{p}$ for $-i(\partial/\partial q)$).

The extension of the symbol map to differential operators with variable coefficients involves some arbitrariness. For operators on an affine space Q, the *Weyl* prescription [7] has important symmetry advantages. Given a C^∞ function $a(q, p)$ polynomial in p, it defines $\hat{a}$ by its action on the functions $u \in C_0^\infty(Q)$ as

$$(\hat{a}u)(q) = (2\pi)^{-l} \int_{P \times Q} a\left(\frac{q + q'}{2}, p\right) u(q')\, e^{ip(q-q')}\, dp\, dq'. \tag{1.1}$$

This formula defines a differential operator: Let

$$a(q, p) = \sum_{|\alpha| \leqslant m} a_\alpha(q)\, p^\alpha \qquad \text{and} \qquad q' = q + r.$$

Then

$$(\hat{a}u)(q) = (2\pi)^{-l} \sum_\alpha \int a_\alpha(q + r/2)\, p^\alpha u(q + r)\, e^{-ipr}\, dp\, dr,$$

$$\text{(by parts)} = (2\pi)^{-l} \sum_\alpha \int \left(-i \frac{\partial}{\partial r}\right)^\alpha \left[a_\alpha\left(q + \frac{r}{2}\right) u(q + r)\right] e^{-ipr}\, dp\, dr$$

$$= \sum_\alpha \left(-i \frac{\partial}{\partial r}\right)^\alpha \left[a_\alpha\left(q + \frac{r}{2}\right) u(q + r)\right]\Big|_{r=0}$$

(by the Fourier inversion theorem), and the result has the form $\sum_{|\alpha|\leqslant m} c_\alpha(q)\, \partial_q{}^\alpha u(q)$ by the Leibniz formula.

Equation (1.1) defines $\hat{a}$ as an integral operator of kernel

$$\langle q \mid \hat{a} \mid q' \rangle = (2\pi)^{-l} \int_P a\left(\frac{q+q'}{2}, p\right) e^{ip(q-q')}\, dp. \tag{1.2}$$

Let $q'' = (q+q')/2$, $r = q' - q$, corresponding to the decomposition $Q \oplus Q = \Delta \oplus R$ (Δ = diagonal subspace of $Q \oplus Q$; R = antidiagonal subspace of $Q \oplus Q$). Equation (1.2) reads

$$\langle q'' - r/2 \mid \hat{a} \mid q'' + r/2 \rangle = (2\pi)^{-l} \int_P a(q'', p)\, e^{-ipr}\, dp; \tag{1.2'}$$

hence

$$\langle q \mid \hat{a} \mid q' \rangle = (\mathscr{F} a)(q, q'),$$

where $\mathscr{F}$ is a partial Fourier transformation between the conjugate spaces P and R, not acting on the space Δ.

In the form (1.2), the Weyl correspondence can be considerably extended [5, 8]: $\mathscr{F}$ is a continuous isomorphism of

$$\mathscr{S}'(X)\ (= \mathscr{S}'(Q) \times \mathscr{S}'(P)) \qquad \text{onto} \qquad \mathscr{S}'(Q \times Q)\ (= \mathscr{S}'(\Delta) \times \mathscr{S}'(R));$$

hence to any $a \in \mathscr{S}'(X)$ it associates an "operator" $\hat{a}$ of integral kernel in $\mathscr{S}'(Q \times Q)$: $\hat{a}$ makes sense as continuous sesquilinear form on $\mathscr{S}(Q) \times \mathscr{S}(Q)$: $(u, v) \to \langle v \mid \hat{a} \mid u \rangle$, or equivalently as a continuous operator $\mathscr{S}(Q) \to \mathscr{S}'(Q)$.

The map $a \to \hat{a}$ was introduced by Weyl [7] to "quantize" classical observables; in this context we have here Planck's constant $\hbar = 1$, until stated otherwise. The map $\hat{a} \to a$ is a Wigner transformation [9], and we call $a(x)$ the *Wigner symbol* of the operator $\hat{a}$.

Mathematicians commonly associate with a given symbol $a(x)$ a *different* operator a^M, which is such that [1, 3]

$$(a^M u)(q) = (2\pi)^{-l} \int_{P \times Q} a(q, p)\, u(q')\, e^{ip(q-q')}\, dp\, dq' \qquad (u \in C_0{}^\infty(Q));$$

hence

$$\left(\sum_{|\alpha| \leqslant m} a_\alpha(q)\, p^\alpha \right)^M = \sum_{|\alpha| \leqslant m} a_\alpha(q) \left(-i \frac{\partial}{\partial q} \right)^\alpha.$$

Another most interesting operator prescription is the Wick (or "normal") ordering $a \to {:}a{:}$ [10, 11]. Some properties of the correspondences $a \to a^M$ and $a \to {:}a{:}$ are listed in the Appendix.

1.1. *Some Properties of the Weyl Correspondence* [12, 13]

It is immediately checked that in the sense of operators on $\mathscr{S}(Q)$,

$$\hat{1} = \mathbb{1} \qquad \text{(the identity operator)},$$

$$\widehat{q_j} = q_j \qquad \text{(multiplication by the coordinate function } q_j\text{)},$$

$$\widehat{p_j} = -i \frac{\partial}{\partial q_j}.$$

If $T(p) + V(q) \in O_M(X)$ we can define $\widehat{T(p) + V(q)}$ by Eq. (1.2) and $T(\hat{p}) + V(\hat{q})$ by the operator functional calculus [14]: These two definitions coincide (the latter is the Schrödinger quantization for Hamiltonians of the standard type $p^2/2m + V(q)$).

The most interesting are the various *symmetry* properties (not satisfied by other symbol maps).

—For all $m \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$,

$$\widehat{(\alpha q + \beta p)^m} = (\alpha \hat{q} + \beta \hat{p})^m, \tag{1.3}$$

and this property characterizes the restriction of the Weyl correspondence to polynomials (proof in [13])

—For all $\alpha, \beta \in \mathbb{R}$,

$$\widehat{\exp i(\alpha q + \beta p)} = \exp i(\alpha \hat{q} + \beta \hat{p}). \tag{1.4}$$

Proof. The operator $(\alpha \hat{q} + \beta \hat{p})$ admits the Hermite functions as entire vectors [14]; for any finite linear combinations u and v of such vectors, $\langle u \mid \exp i(\alpha \hat{q} + \beta \hat{p}) \mid v \rangle$ is defined (as $\sum_0^\infty (i^m/m!) \langle u \mid (\alpha \hat{q} + \beta \hat{p})^m \mid v \rangle$), and

$$\begin{aligned}
\langle u \mid \exp i(\alpha \hat{q} + \beta \hat{p}) \mid v \rangle &= \langle u \mid e^{i\alpha\beta/2} \exp i\alpha\hat{q} \exp i\beta\hat{p} \mid v \rangle \qquad \text{(cf. [7])} \\
&= e^{i\alpha\beta/2} \int_Q u^*(q)\, e^{i\alpha q} v(q + \beta)\, dq \\
&= (2\pi)^{-l} \int_X \mathscr{F}^{-1}(v \otimes u^*)\, e^{i(\alpha q + \beta p)}\, dx \\
&= \int_{Q \times Q} (v \otimes u^*)\, \mathscr{F} e^{i(\alpha q + \beta p)} = \langle u \mid \widehat{e^{i(\alpha q + \beta p)}} \mid v \rangle
\end{aligned}$$

and this extends by continuity to $u, v \in \mathscr{S}$ (the closed linear span of the Hermite functions).

Conversely, extending (1.4) by linearity to superpositions of exponentials, one can recover the Weyl correspondence [1].

— $\widehat{a^*} = \hat{a}^\dagger$ (in the sense of sesquilinear forms), since the kernel given by (1.2) trivially satisfies $\langle q \mid \widehat{a^*} \mid q' \rangle = \langle q' \mid \hat{a} \mid q \rangle^*$. By contrast, the mathematicians' symbol for the adjoint of an operator is rather complicated [2, 3] (see Appendix).

In particular if $a(x)$ is real-valued, the form $\hat{a}$ is symmetric (Hermitian). There is no known characterization of those real symbols for which a self-adjoint operator (densely defined) on the Hilbert space $L^2(Q)$ can be associated to the form $\hat{a}$ (a symmetric operator as simple as $\hat{q}\hat{p}\hat{q}$ has no self-adjoint extension [15]). We shall give a partial solution to this problem for a class of symbols.

1.2. *Metaplectic Covariance*

Let $\mathrm{Mp}(\mathscr{H}) \to^{\pi} \mathrm{Sp}(X)$ denote the metaplectic representation of the symplectic group of X [16]: It is a double covering of $\mathrm{Sp}(X)$ by a subgroup of the unitary group of $\mathscr{H}$; every $\mathscr{U} \in \mathrm{Mp}(\mathscr{H})$ is not only a unitary map on $L^2(Q)$ but it extends to a continuous isomorphism of $\mathscr{S}'(Q)$ [17]. The Weyl quantization has the interesting covariance property

$$\forall \mathscr{U} \in \mathrm{Mp}(\mathscr{H}), \qquad \forall a \in \mathscr{S}'(X), \qquad \mathscr{U}\hat{a}\mathscr{U}^\dagger = \widehat{a \circ \pi(\mathscr{U})^{-1}}, \tag{1.5}$$

which implies that Weyl quantizations with respect to different symplectic frames on X are isomorphically equivalent.

With minor changes in notation, we have given a proof of Eq. (1.5) in [5]. Here we mention it only as another symmetry of the Weyl quantization, which is not shared by the other ordering procedures listed in the Appendix.

1.3. *Hilbert–Schmidt Operators*

The Fourier transformation $\mathscr{F}$ introduced above is a unitary map from $L^2(X, dx/(2\pi)^l)$ onto $L^2(Q \times Q, dq \times dq)$ by Parseval's identity. But the operators with L^2 kernels are the Hilbert–Schmidt operators [14] and they form a Hilbert space $\mathscr{L}^2(\mathscr{H})$ for the norm

$$\langle B, A \rangle_{HS} = \mathrm{Tr}\, B^\dagger A = \int_{Q \times Q} \langle q \mid B \mid q' \rangle^* \langle q \mid A \mid q' \rangle \, dq \, dq'.$$

Hence the Weyl correspondence restricted to $L^2(X, dx/(2\pi)^l)$ is a unitary map onto $\mathscr{L}^2(\mathscr{H})$.

2. An Algebra of Symbols

Grossmann, Loupias, and Stein [1] have defined a space of symbols, denoted here GLS symbols, which give rise to pseudodifferential-like operators under Weyl quantization. We review and strengthen their results here. Symbols presenting some analogies have been defined by other authors [18].

2.1. GLS *Symbols of Order* $\leqslant m$

A GLS symbol of order $\leqslant m$ is any function $a \in C^\infty(X)$ satisfying (for a real number m)

$$\forall \alpha \in \mathbb{N}^{2l}\ \exists C_\alpha > 0, \qquad |\partial^\alpha a(x)| \leqslant C_\alpha (1 + \|x\|^2)^{(m-|\alpha|)/2}, \tag{2.1}$$

where we let $\|x\|^2 = q^2 + p^2$ (but any other quadratic norm on X would do).

The space S_m of these symbols can be topologized by the seminorms $\|a\|_\alpha = \sup_{x\in X} (|\partial^\alpha a(x)|/(1 + \|x\|^2)^{(m-|\alpha|)/2})$; it is then a Fréchet space.

S_m is stable under complex conjugation; any operator $\partial_x{}^\alpha$ maps S_m into $S_{m-|\alpha|}$ continuously; multiplication of functions sends $S_m \times S_{m'}$ into $S_{m+m'}$. We have $S_{m'} \subset S_m$ (a continuous inclusion) if $m' \leqslant m$. The space $S_{-\infty} = \bigcap_{m\in\mathbb{R}} S_m$ (with the coarsest topology making all inclusions $S_{-\infty} \subset S_m$ continuous) is the Schwartz space $\mathscr{S}(X)$. We denote $S = \bigcup_{m\in\mathbb{R}} S_m$ the space of all GLS symbols.

The main differences from the usual definitions of symbols [2, 3] are: (1) The latter involve a growth condition in the p coordinates only (while we take stronger growth conditions, isotropic in phase space and Sp(X)-invariant); (2) That growth condition is expressed [3] by an asymptotic series

$$a(q, p) \sim \sum_{j=0}^{\infty} a_{m-j}(q, p)\ (\|p\| \to +\infty),$$

where each a_{m-j} is positively homogeneous in p of degree $= m - j$, while our condition (2.1) is only an inequality. Here we can also define subspaces of symbols $\mathbf{S}_m \subset S_m$ by a condition stronger than (2.1):

$$\left.\begin{aligned} & a(x) \sim \sum_{j=0}^{\infty} a_{m-j}(x) \text{ in the sense that} \\ \forall \alpha \in \mathbb{N}^{2l}, \forall n \in \mathbb{N}, \quad & \left|\partial^\alpha\left(a - \sum_{j=0}^{n-1} a_{m-j}\right)\right| \leqslant \mathcal{O}(\|x\|^{m-n-|\alpha|}) \end{aligned}\right\} \quad (\|x\| \to +\infty), \tag{2.2}$$

where each $a_{m-j}(x)$ is positively *homogeneous* in x of degree $(m - j)$; but this requirement is too strong in view of realistic physical applications (see Section 4). (Note: By convention, any sum of the form $\sum_{j=0}^{-1} (\ldots)$ has value 0.)

2.2. *Symbols Defined by Asymptotic Series*

We say that the symbol $a \in S_m$ admits an asymptotic expansion $\sum_{j=0}^{\infty} a_{m-j}$ with $a_{m-j} \in S_{m-j}$, if for all $n \in \mathbb{N}$, $(a - \sum_{j=0}^{n-1} a_{m-j}) \in S_{m-n}$; this is denoted $a \sim \sum_j a_{m-j}$ (more general descending asymptotic expansions could be allowed). A given symbol admits an infinity of expansions of this form, but the converse is more interesting: to find a symbol having a given expansion.

THEOREM 2.2.1. (i) *Given* any *sequence* $\{a_{m-j} \in S_{m-j}\}_{j\in\mathbb{N}}$ *there exists* $\mathbf{a} \in S_m$ *such that* $\mathbf{a} \sim \sum_j a_{m-j}$.

(ii) *The set of symbols a such that* $a \sim \sum_j a_{m-j}$ *is the class of* $\mathbf{a}$ *modulo* $S_{-\infty}$.

Proof. Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers increasing to $+\infty$, and let $\chi \in C^\infty(\mathbb{R})$ be a monotonic function such that $\chi(r) = 0$ for $r \leqslant 1$ and $\chi(r) = 1$ for $r \geqslant 2$. We define

$$\mathbf{a}(x) = \sum_{j=0}^{\infty} \chi\left(\frac{\| x \|}{\lambda_j}\right) a_{m-j}(x) = \sum_{j=0}^{\infty} \mathbf{a}_{m-j}(x).$$

In any compact subset of X this sum is actually finite, so it defines a C^∞ function $\mathbf{a}(x)$. Now we must control the behavior of $\mathbf{a}(x)$ when $\| x \| \to +\infty$ for a suitable choice of the sequence $\{\lambda_j\}$. For any $\alpha \in \mathbb{N}^{2l}$ and $j \in \mathbb{N}$,

$$\partial^\alpha \left[\chi\left(\frac{\| x \|}{\lambda}\right) a_{m-j}(x)\right] = \sum_{\beta\leqslant\alpha} \frac{c_{j\alpha\beta}}{\lambda^{|\beta|}} (\partial^\beta\chi)\left(\frac{\| x \|}{\lambda}\right) \partial^{\alpha-\beta} a_{m-j}(x),$$

by the Leibniz formula; estimate (2.1) then yields

$$\left|\partial^\alpha \left[\chi\left(\frac{\| x \|}{\lambda}\right) a_{m-j}(x)\right]\right| \leqslant \sum_{\beta\leqslant\alpha} \frac{c'_{j\alpha\beta}}{\lambda^{|\beta|}} \left[(\partial^\beta\chi)\left(\frac{\| x \|}{\lambda}\right)\right] \| x \|^{m-j-|\alpha|+|\beta|}.$$

In the RHS we can replace each $\| x \|^{|\beta|}/\lambda^{|\beta|}$ by $2^{|\beta|}$ (obvious if $|\beta| = 0$; if $|\beta| > 0$, it is due to $(\partial^\beta\chi)(\| x \|/\lambda) \equiv 0$ if $\| x \| > 2\lambda$). We now choose

$$\lambda_j \geqslant \sup_{|\alpha|\leqslant j} \left(\sum_{|\beta|\leqslant\alpha} 2^{|\beta|} c'_{j\alpha\beta} \cdot \sup_{r\in\mathbb{R}} |\partial^\beta\chi(r)|\right);$$

let $\chi(\| x \|/\lambda_j)\, a_{m-j}(x) = \mathbf{a}_{m-j}(x)$. Whenever $|\alpha| \leqslant j$,

$$|\partial^\alpha \mathbf{a}_{m-j}(x)| \leqslant \lambda_j \| x \|^{m-j-|\alpha|} \leqslant \| x \|^{m-j-|\alpha|+1},$$

since the left-hand side vanishes when $\| x \| \leqslant \lambda_j$. The series $\sum_j \partial^\alpha \mathbf{a}_{m-j}(x)$ is thus uniformly bounded by a geometric series if $\| x \| > 1 + \epsilon$, and we have

$$\begin{aligned} |\partial^\alpha \mathbf{a}(x)| &= \left|\partial^\alpha \mathbf{a}_m(x) + \sum_{j=1}^{\infty} \partial^\alpha \mathbf{a}_{m-j}(x)\right| \\ &\leqslant C_\alpha \| x \|^{m-|\alpha|} + \frac{\| x \|^{m-|\alpha|+1}}{\| x \| - 1} = \mathcal{O}(\| x \|^{m-|\alpha|}) \text{ as } \| x \| \to \infty, \end{aligned}$$

where we have used (2.1) for $\mathbf{a}_m(x)$ ($=a_m(x)$ for $\| x \| > 2\lambda_0$). So we have $\mathbf{a}(x) \in S_m$. The same argument shows that

$$\mathbf{a}(x) - \sum_{j=0}^{n-1} \mathbf{a}_{m-j}(x) \in S_{m-n}.$$

Now:

$$\mathbf{a}(x) - \sum_{j=0}^{n-1} a_{m-j}(x) = \left(\mathbf{a}(x) - \sum_{0}^{n-1} \mathbf{a}_{m-j}\right) + \sum_{0}^{n-1} (\mathbf{a}_{m-j} - a_{m-j})$$

and the second term vanishes identically for $\| x \| \geqslant 2\lambda_{n-1}$, owing to the form of the cutoff function χ. Hence $\mathbf{a} - \sum_{j=0}^{n-1} a_{m-j} \in S_{m-n}$, proving (i).

(ii) $a \sim \sum_j a_{m-j}$ means that for all n, $a - \sum_{j=0}^{n-1} a_{m-j} \in S_{m-n}$, which is equivalent to $(a - \sum_{j=0}^{n-1} a_{m-j}) - (\mathbf{a} - \sum_{j=0}^{n-1} a_{m-j}) \in S_{m-n}$, or $(a - \mathbf{a}) \in S_{m-n}$ for all n, i.e., $a - \mathbf{a} \in S_{-\infty}$.

This motivates the following:

DEFINITION 2.2.2. We say that $a_1 \sim a_2$ if $a_1 - a_2 \in S_{-\infty}$; the class of $a \in S_m$ in the quotient space $\tilde{S}_m = S_m/S_{-\infty}$, or *asymptotic class* of a, is denoted $\tilde{a}$; we let $\tilde{S} = S/S_{-\infty} = \bigcup_m \tilde{S}_m$. (In this language, Theorem 2.2.1 just states that any formal series $\sum_{j=0}^{\infty} a_{m-j}$ with $a_{m-j} \in S_{m-j}$ defines a (unique) element of $\tilde{S}_m$.)

2.3. *Quantization of Symbols*

Every GLS symbol $a(x)$ is an element of $O_M(X) \subset \mathscr{S}'(X)$; hence by Weyl quantization it defines a continuous map $\hat{a}$: $\mathscr{S}(X) \to \mathscr{S}'(X)$; we call $\hat{S}_m$ (resp. $\hat{S}$) the image of S_m (resp. S) under $\hat{}$. The properties of the operators $\hat{a} \in \hat{S}$ are discussed in Section 3, but we give a preliminary result here.

THEOREM 2.3.1. (i) *$\hat{a}$ maps $\mathscr{S}(Q)$ into $\mathscr{S}(Q)$ continuously; hence by duality it extends to a continuous operator $\mathscr{S}'(Q) \to \mathscr{S}'(Q)$ (also denoted $\hat{a}$).*

(ii) *If $a \in S_m$ with $m < -l$, $\hat{a}$ maps $\mathscr{H} = L^2(Q)$ into itself, and $\hat{a} \in \mathscr{L}^2(\mathscr{H})$.*

(iii) *If $a \in S_{-\infty}$, $\hat{a}$ maps $\mathscr{S}'(Q)$ into $\mathscr{S}(Q)$ continuously.*

Proof. (i) Let $a \in S_m$ and $u \in \mathscr{S}(Q)$. In the sense of distributions,

$$(\hat{a}u)(q) = (2\pi)^{-l} \int_{P\times Q} dp\, dr\, a(q + r/2, p)\, u(q + r)\, e^{-ipr}$$

$$= \int_{P\times Q} dp\, dr\, I(q, p, r)\, e^{-ipr} \tag{2.3}$$

and $\hat{a}u \in \mathscr{S}(Q)$ if every distribution $q^\alpha \partial^\beta(\hat{a}u)$ is actually a bounded function. Since $\partial^\beta(\hat{a}u) = (2\pi)^{-l} \int dp\, dr\, \partial_q{}^\beta(a(q + r/2, p) \cdot u(q + r))\, e^{-ipr}$ is a sum of integrals of the same type as (2.3), it suffices to prove that $q^\alpha(\hat{a}u)(q)$ is a bounded function of q.

To do this, we shall exploit the oscillatory character of the integrand (due to e^{-ipr}) as $\| p \|$ and $\| r \| \to \infty$, by the well-known method of successive integrations by parts.

First, for any integer k,

$$\hat{a}u(q) = \int_Q dr\, I(q, p, r)\, e^{-ipr} = \int_Q dr \frac{I(q, p, r)}{(1 + \| p \|^2)^k} (1 + \| p \|^2)^k e^{-ipr}$$

$$= \int_Q dr \frac{I(q, p, r)}{(1 + \| p \|^2)^k} [(1 - \Delta_q)^k e^{-ipr}] = \int_Q dr (1 - \Delta_q)^k \frac{I(q, p, r)}{(1 + \| p \|^2)^k} e^{-ipr}$$

(by parts, using the rapid decrease of all $\partial_r^\alpha I(q, p, r)$ as $\| r \| \to \infty$). We take k large enough to let $(1 - \Delta_r)^k (I(q, p, r)/(1 + \| p \|^2)^k) = J(q, p, r)$ be absolutely integrable in p; this is possible because $a(x)$ satisfies condition (2.1).

Then, for any integer k' we get, similarly,

$$\int_{P\times Q} dp\, dr\, J(q, p, r)\, e^{-ipr} = \int_{P\times Q} dp\, dr (1 - \Delta_p)^{k'} \frac{J(q, p, r)}{(1 + \| r \|^2)^{k'}} e^{-ipr}. \quad (2.4)$$

Thus we have reduced ourselves to (definite) integrals of the form

$$\int K(q, p, r)\, dp\, dr = \int dp\, dr \frac{a'(q + r/2, p)(\partial^\beta u)(q + r)\, e^{-ipr}}{(1 + \| p \|^2)^k (1 + \| r \|^2)^{k'}} \qquad (a' \in S_m),$$

since (2.4) is a sum of such terms.

We now bound the integrand $K(q, p, r)$ from above. If $a' \in S_m$, let $m_+ = \max\{m/2, 0\}$; by (2.1): $| a'(q, p) | \leqslant \mathcal{O}[(1 + \| q \|^2 + \| p \|^2)^{m_+}] \leqslant \mathcal{O}[(1 + \| q \|^2)^{m_+} (1 + \| p \|^2)^{m_+}]$. Also, for every $N \in \mathbb{N}$: $| \partial^\beta u(q) | \leqslant C'_{N\beta}(1 + \| q \|^2)^{-N}$. Lemma 2.3.2 (below) then yields, for $\| q \|$ or $\| r \| \to \infty$,

$$| a'(q + r/2, p) | \leqslant \mathcal{O}[(1 + \| q \|^2)^{m_+} (1 + \| r \|^2)^{m_+} (1 + \| p \|^2)^{m_+}],$$

$$| \partial^\beta u(q + r) | \leqslant \mathcal{O}[C'_{N\beta}(1 + \| q \|^2)^{-N} (1 + \| r \|^2)^N];$$

hence,

$$| K(q, p, r) | \leqslant \mathcal{O}[C'_{N\beta}(1 + \| q \|^2)^{m_+ - N} (1 + \| p \|^2)^{m_+ - k} (1 + \| r \|^2)^{m_+ + N - k'}$$

$$\Rightarrow \int K\, dp\, dr \leqslant \mathcal{O}[C'_{N\beta}(1 + \| q \|^2)^{-M}] \qquad \text{for all} \quad M \in \mathbb{N},$$

by a suitable choice of N ($\geqslant M + m_+$) and of k and k' (depending on N). More generally we would obtain, for all M, α,

$$|(1 + \| q \|^2)^M \partial^\alpha (\hat{a}u)(q)| \leqslant c C'_{N\beta} = c \cdot \sup_{q \in Q} \{|(1 + \| q \|^2)^N \partial^\beta u(q)|\},$$

for some N, β; hence $\mathcal{S} \to^{\hat{a}} \mathcal{S}$ is continuous.

The adjoint operator of $\mathcal{S} \to^{\widehat{a^*}} \mathcal{S}$ is then an extension of $\hat{a}$ to a continuous operator $\mathcal{S}' \to^{\hat{a}} \mathcal{S}'$.

Proof of (ii). Trivial, since $m < -l$ implies $S_m \subset L^2(X)$, and by Section 1.3 Weyl quantization sends $L^2(X)$ onto $\mathscr{L}^2(\mathscr{H})$.

Proof of (iii). If $a \in S_{-\infty} = \mathscr{S}(X)$, the kernel of $\hat{a}$ is in $\mathscr{S}(Q \times Q)$ by Fourier transformation, it then defines a continuous sesquilinear functional on $\mathscr{S}'(Q)$, i.e., a continuous map $\mathscr{S}'(Q) \to \mathscr{S}(Q)$. Operators of this type are thus "infinitely regularizing."

The proof of (i) required the following:

LEMMA 2.3.2. *For all* $q, r \in Q$,

$$(1 + \| q + r \|^2)^{\pm 1} < 2(1 + \| q \|^2)^{\pm 1}(1 + \| r \|^2). \tag{2.5}$$

Proof. For (+1) we have

$$(1 + \| q + r \|^2) < 2(1 + \| q \|^2 + \| r \|^2) \leqslant 2(1 + \| q \|^2)(1 + \| r \|^2).$$

For (−1) the result follows from

$$(1 + \| q + r \|^2) \geqslant 1 + (\| q \| - \| r \|)^2 > \frac{1}{2}\left(\frac{1 + \| q \|^2}{1 + \| r \|^2}\right),$$

since

$$\begin{aligned} &2[1 + (\| q \| - \| r \|^2)](1 + \| r \|^2) - (1 + \| q \|^2) \\ &\quad = 1 + (\| q \| - 2\| r \|)^2 + 2\| r \|^2(\| q \| - \| r \|)^2 > 0. \end{aligned}$$

2.4. *The Twisted Product of Symbols*

By Theorem 2.3.1(i), the operator product $\hat{a}\hat{b}$ of two GLS operators makes sense as a continuous mapping $\mathscr{S}(Q) \to \mathscr{S}(Q)$. $\hat{a}\hat{b}$ has a kernel in $\mathscr{S}'(Q \times Q)$, and by isomorphism (1.2) there is a unique $c \in \mathscr{S}'(X)$ such that $\hat{c} = \hat{a}\hat{b}$. c is called the twisted product of a and b; it can be computed by expressing $(\hat{a}\hat{b})u = \hat{c}u$ (for $u \in \mathscr{S}(Q)$) via the transformation $\mathscr{F}$ (Eq. (1.2)). By a straightforward computation,

$$\begin{aligned} c(x) &= \mathscr{F}^{-1}(\langle q \mid \hat{a}\hat{b} \mid q' \rangle) = \mathscr{F}^{-1}\left[\int_Q dq'' \langle q \mid \hat{a} \mid q'' \rangle \langle q'' \mid \hat{b} \mid q' \rangle\right] \\ &= \mathscr{F}^{-1}\left(\int_Q dq'' (\mathscr{F}a)(q, q'')(\mathscr{F}b)(q'', q')\right) \\ &= \pi^{-2l} \int_{X^l} dq_1\, dp_1\, dq_2\, dp_2\, a(q + q_1, p + p_1)\, b(q + q_2, p + p_2)\, e^{2i(q_1 p_2 - q_2 p_1)} \\ &= \pi^{-2l} \int_{X^l} dx_1\, dx_2\, a(x + x_1)\, b(x + x_2)\, e^{2i\omega(x_1, x_2)}, \end{aligned} \tag{2.6}$$

which illustrates the special role played by the symplectic form ω of X in the Weyl correspondence [1, 8, 9]. The integral (2.6) is defined in the sense of distributions.

THEOREM 2.4.1. (i) *S is an associative *-algebra for twisted multiplication, which is a continuous map from $S_{m_1} \times S_{m_2}$ into $S_{m_1+m_2}$ (for any $m_1, m_2 \geqslant -\infty$). The twisted product c of a and $b \in S$ is given by Eq.* (2.6).

(ii) *$c(x)$ admits the asymptotic expansion* (*Groenewold* [19])

$$c(x) \sim \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n [(\partial_{q_1}\partial_{p_2} - \partial_{q_2}\partial_{p_1})^n a(x + x_1)\, b(x + x_2)]_{x_1=x_2=0} \qquad (2.7)$$

denoted $c(x) \sim a(x)[\exp(i/2)(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)]\, b(x)$.

(iii) *If a or b is a polynomial in x, expansion* (2.7) *is a finite sum which is exactly equal to $c(x)$.*

COROLLARY 2.4.2. (i) *S is a Lie algebra for the twisted commutator $(a, b) \to \{\{a, b\}\} = c$ defined by*

$$\hat{c} = [\hat{a}, \hat{b}]_-/i, \qquad \text{and} \qquad \{\{S_{m_1}, S_{m_2}\}\} \subset S_{m_1+m_2-2}\,.$$

(ii) $\{\{a, b\}\}(x) \sim a(x)[2 \sin((\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)/2)]\, b(x)$. (2.8)

(iii) *If a or b is a polynomial in x of degree $\leqslant 2$,*

$$\{\{a, b\}\} \equiv \{a, b\} \qquad (\textit{the ordinary Poisson bracket}).$$

(The operation $\{\{,\}\}$ is the *Moyal bracket* [20].)

LEMMA 2.4.3. *For any polynomial $Q(x_1, x_2)$ there exists a differential operator $P(\partial_{x_1}, \partial_{x_2})$ (with constant coefficients) such that*

$$Q(x_1, x_2)\, e^{2i\omega(x_1,x_2)} \equiv P(\partial_{x_1}, \partial_{x_2})\, e^{2i\omega(x_1,x_2)}.$$

Proof. For convenience we denote $Y = X^2$; let $y = (x_1, x_2) \in Y$. The statement to be proved is invariant by linear transformations of Y; but there exists such a transformation under which $e^{2i\omega(x_1,x_2)}$ becomes a nondegenerate gaussian $\exp(i \sum_{j=1}^{4l} \epsilon_j y_j^2)$ ($\forall j$: $\epsilon_j = \pm 1$) (this follows solely from the nondegeneracy of the 2-form ω). Then it is known that the Hermite-like polynomials

$$\left\{\exp\left(-i \sum \epsilon_j y_j^2\right) \partial_y^\alpha \exp\left(i \sum \epsilon_j y_j^2\right)\right\} \qquad \alpha \in \mathbb{N}^{4l}$$

generate all polynomials in y; hence the result.

Proof of Theorem 2.4.1. (i) We need only prove the continuity of the mapping (2.6): $(a, b) \to c$ from $S_{m_1} \times S_{m_2}$ into $S_{m_1+m_2}$; the other algebraic properties are those of the corresponding operator multiplication. We thus want

$$\partial^\alpha c(x) = \pi^{-2l} \int_{X^2} dx_1\, dx_2\, \partial_x^\alpha [a(x + x_1)\, b(x + x_2)]\, e^{2i\omega(x_1,x_2)} \qquad (2.9)$$

and

$$\sup_{x\in X} \frac{\| \partial^\alpha c(x)\|}{(1 + \| x \|^2)^{(m_1+m_2-|\alpha|)/2}}$$

bounded by some seminorms of a and b.

We shall prove both statements by defining the RHS of (2.9) as an oscillatory integral (the derivation under $\int$ in (2.9) is then legitimate). The proof is essentially the same as for Theorem 2.3.1. By the Leibniz formula, the RHS of (2.9) is a sum of terms of the form

$$\int dx_1\, dx_2\, \partial_x^{\alpha_1} a(x + x_1)\, \partial_x^{\alpha_2} b(x + x_2)\, e^{2i\omega(x_1,x_2)} \qquad (\alpha_1 + \alpha_2 = \alpha)$$

$$= \int dx_1\, dx_2 \frac{\partial_x^{\alpha_1} a(x + x_1)}{(1 + \| x_1 \|^2)^{k_1}} \cdot \frac{\partial_x^{\alpha_2} b(x + x_2)}{(1 + \| x_2 \|^2)^{k_2}} [P_{k_1,k_2}(\partial_{x_1}, \partial_{x_2})\, e^{2i\omega(x_1,x_2)}] \tag{2.10}$$

for all $k_1, k_2 \in \mathbb{N}$, where P_{k_1,k_2} is the polynomial in ∂_{x_1} and ∂_{x_2} such that $P_{k_1,k_2}(\partial_{x_1}, \partial_{x_2})\, e^{2i\omega(x_1,x_2)} = (1 + \| x_1 \|^2)^{k_1}(1 + \| x_2 \|^2)^{k_2} e^{2i\omega(x_1,x_2)}$ (by Lemma 2.4.3).

In the sense of distributions, (2.10) means

$$\int_{X^1} dx_1\, dx_2\, P_{k_1,k_2}(-\partial_{x_1}, -\partial_{x_2}) \left[\frac{\partial_x^{\alpha_1} a(x + x_1)}{(1 + \| x_1 \|^2)^{k_1}} \cdot \frac{\partial_x^{\alpha_2} b(x + x_2)}{(1 + \| x_2 \|^2)^{k_2}}\right] e^{2i\omega(x_1,x_2)}$$

and this integral converges for k_1 and k_2 large enough, since a and b are GLS symbols. Since the derivations ∂_{x_1} and ∂_{x_2} can only improve the behavior of the integrand $L(x, x_1, x_2)$, the latter can be bounded by

$$\mathcal{O}\left[\frac{\partial_x^{\alpha_1} a(x + x_1)}{(1 + \| x_1 \|^2)^{k_1}} \cdot \frac{\partial_x^{\alpha_2} b(x + x_2)}{(1 + \| x_2 \|^2)^{k_2}}\right]$$

$$\leqslant \mathcal{O}\left[\frac{\| a \|_{\alpha_1} \| b \|_{\alpha_2} (1 + \| x + x_1 \|^2)^{(m_1-|\alpha_1|)/2}}{(1 + \| x_1 \|^2)^{k_1}} \times \frac{(1 + \| x + x_2 \|^{(m_2-|\alpha_2|)/2})}{(1 + \| x_2 \|^2)^{k_2}}\right]$$

and using Lemma 2.3.2,

$$\leqslant \mathcal{O}[\| a \|_{\alpha_1} \| b \|_{\alpha_2} (1 + \| x \|^2)^{(m_1+m_2-|\alpha|)/2}$$

$$\times (1 + \| x_1 \|^2)^{|(m_1-|\alpha_1|)/2|-k_1} (1 + \| x_2 \|^2)^{|(m_2-|\alpha_2|)/2|-k_2}].$$

This upper bound is also (x_1, x_2)-integrable for k_1 and k_2 large enough, so that we get (with m: $m_1 + m_2$)

$$\| c \|_\alpha = \sup_x \frac{| \partial^\alpha c(x)|}{(1 + \| x \|^2)^{(m-|\alpha|)/2}} \leqslant \text{const} \times \max_{\alpha_1+\alpha_2=\alpha} \{\| a \|_{\alpha_1} \cdot \| b \|_{\alpha_2}\}. \quad \text{Q.E.D.}$$

The proof of (ii) essentially amounts to a stationary phase expansion of integral (2.6) around the critical point $x_1 = x_2 = 0$ (we recall (cf. Lemma 2.4.3) that $e^{2i\omega(x_1, x_2)}$ is a Gaussian phase function around $y = (x_1, x_2) = 0$).

We first replace $a(x + x_1)\, b(x + x_2)$ by its Taylor polynomial to some order n:

$$[a(x + x_1)\, b(x + x_1)]_n = \sum_{|\alpha| \leqslant n} \frac{y^\alpha}{\alpha!} \partial_y{}^\alpha [a(x + x_1)\, b(x + x_2)]_{y=0}\,.$$

The remainder $R_n(x, y)$ has the form

$$\sum_{|\alpha|=n+1} \frac{y^\alpha}{\alpha!} \partial_y{}^\alpha [a(x + \theta x_1)\, b(x + \theta x_2)] \qquad \text{for some} \quad \theta \in [0, 1];$$

hence by Lemma 2.3.2, $|R_n(x, y)| \leqslant \mathcal{O}(\| x \|^{m-n-1} \| y \|^k)$ for some $k > 0$; similarly, $| \partial_x{}^\beta R_n(x, y)| \leqslant \mathcal{O}(\| x \|^{m-n-|\beta|-1} \| y \|^{k(\beta)})$ for some $k(\beta)$, because $\partial_x{}^\beta R_n$ is the remainder of the Taylor expansion in y of $\partial_x{}^\beta(a(x + x_1)\, b(x + x_2))$. By the same argument as that in the proof of (i), all this implies

$$\int_{X^2} R_n(x, y)\, e^{2i\omega(x_1\, x_2)}\, dx_1\, dx_2 \in S_{m-n-1}\,.$$

It remains to compute $\pi^{-2l} \int f(x, y)\, e^{2i\omega(x_1, x_2)}\, dy$ for f polynomial in y of degree $\leqslant n$. For each monomial y^β $(\beta \in \mathbb{N}^{4l})$, $\int (y^\beta/\beta!)\, e^{2i\omega(x_1, x_2)}\, dy$ is the coefficient of $(i\xi)^\beta$ in

$$\int e^{i\xi y}\, e^{2i\omega(x_1\, x_2)}\, dy = \int e^{i(\xi_1 x_1 + \xi_2 x_2 + 2\omega(x_1\, x_2))}\, dy$$

$$= e^{-(i/2)\omega(\#\xi_1\, \#\xi_2)} \int e^{2i\omega(x_1 + \#\xi_2/2 .\, x_2 - \#\xi_1/2)}\, dy$$

$$= \pi^{2l}\, e^{-(i/2)(\zeta_1 \eta_2 - \zeta_2 \eta_1)}$$

(where $\xi = (\xi_1, \xi_2) = (\zeta_1, \eta_1, \zeta_2, \eta_2) \in \mathbb{R}^{4l}$ is dual to $y = (q_1, p_1, q_2, p_2)$, and $\#\colon X^* \to X$ is the isomorphism such that $\xi \cdot x \equiv \omega(\#\xi, x)\ \forall x \in X\ \forall \xi \in X^*$). Hence,

$$\pi^{-2l} \int \frac{y^\beta}{\beta!}\, e^{2i\omega(x_1\, x_2)}\, dy = \frac{1}{\beta!} \left[-i \frac{\partial}{\partial \xi}\right]^\beta [e^{-(i/2)(\zeta_1 \eta_2 - \zeta_2 \eta_1)}]_{\xi=0}\,.$$

By linearity we get for any $f(x, y)$ polynomial in y of degree n:

$$\pi^{-2l} \int f(x, y)\, e^{2i\omega(x_1, x_2)} = \sum_{|\beta| \leqslant n} \frac{1}{\beta!} \left[-i \frac{\partial}{\partial \xi}\right]^\beta [e^{-(i/2)(\zeta_1 \eta_2 - \zeta_2 \eta_1)}] \cdot \partial_y{}^\beta f(x, y)_{y=0}$$

which is formally equal (by Taylor's formula) to

$$[e^{-(i/2)(\zeta_1\eta_2-\zeta_2\eta_1)}\,|_{\xi=-i\partial_y}]\cdot f_{y=0} = \exp(i/2)(\partial_{q_1}\partial_{p_2}-\partial_{q_2}\partial_{p_1})\,f(x,y)_{y=0}$$

$$= \sum_{\substack{k\in\mathbb{N}\\ k\leqslant n/2}} \frac{1}{k!}\left(\frac{i}{2}\right)^k [(\partial_{q_1}\partial_{p_2}-\partial_{q_2}\partial_{p_1})^k f(x,y)]_{y=0}\,.$$

Summing up: For each $n\in\mathbb{N}$ we have

$$\pi^{-2l}\int dx_1\,dx_2\,a(x+x_1)\,b(x+x_2)\,e^{2i\omega(x_1,x_2)}$$

$$=\pi^{-2l}\int dx_1\,dx_2[a(x+x_1)\,b(x+x_2)]_n\,e^{2i\omega(x_1,x_2)} \qquad (\text{mod } S_{m-n-1}]$$

$$=\sum_{k\leqslant n/2}\frac{1}{k!}\left(\frac{i}{2}\right)^k(\partial_{q_1}\partial_{p_2}-\partial_{q_2}\partial_{p_1})^k[a(x+x_1)\,b(x+x_2)]_{x_1=x_2=0} \qquad [\text{mod } S_{m-n-1}]$$

$$=\sum_{k\leqslant n/2} c_k(x) \qquad [\text{mod } S_{m-n-1}].$$

Hence the twisted product satisfies $c(x)\sim\sum_{k=0}^{\infty}c_k(x)$ with $c_k\in S_{m-2k}$, in the sense of Section 2.2.

This is precisely what is meant by Eq. (2.7). Q.E.D.

(iii) If $a(x)$ is a polynomial, $a(x+x_1)$ is a polynomial in x_1, and $c(x)$ is the sum of the finite series

$$\pi^{-2l}\sum_{\gamma,\gamma'}\int dx_1\,dx_2\,\partial_q^{\gamma}\partial_p^{\gamma'}a(x)\,\frac{q_1^{\gamma}p_1^{\gamma'}}{\gamma!\gamma'!}\,b(x+x_2)\,e^{2i(q_1p_2-q_2p_1)}$$

$$=(\text{by parts})\;\pi^{-2l}\sum_{\gamma\gamma'}\frac{\partial_q^{\gamma}\partial_p^{\gamma'}a(x)}{\gamma!\gamma!}\int dx_1\,dx_2$$

$$\times\left[\left(\frac{i}{2}\frac{\partial}{\partial p_2}\right)^{\gamma}\left(-\frac{i}{2}\frac{\partial}{\partial q_2}\right)^{\gamma'}b(x+x_2)\right]e^{2i(q_1p_2-q_2p_1)}$$

$$=\sum_{\gamma\gamma'}\frac{\partial_q^{\gamma}\partial_p^{\gamma'}a(x)}{\gamma!\gamma!}\int dx_2\left[\left(\frac{i}{2}\frac{\partial}{\partial p_2}\right)^{\gamma}\left(-\frac{i}{2}\frac{\partial}{\partial q_2}\right)^{\gamma'}b(x+x_2)\right]\delta(p_2)\,\delta(q_2)$$

$$=\left(\sum_{\gamma}\frac{1}{\gamma!}\left(\frac{i}{2}\,\partial_{q_1}\partial_{p_2}\right)^{\gamma}\right)\left(\sum_{\gamma'}\frac{1}{\gamma'!}\left(-\frac{i}{2}\,\partial_{p_1}\partial_{q_2}\right)^{\gamma'}\right)[a(x+x_1)\,b(x+x_2)]_{x_1=x_2=0}\,,$$

$$c(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n(\partial_{q_1}\partial_{p_2}-\partial_{q_2}\partial_{p_1})^n[a(x+x_1)\,b(x+x_2)]_{x_1=x_2=0}$$

(a finite sum) Q.E.D.

Remarks. A formal proof of (ii) (which also works for any stationary phase expansion) is to expand $e^{2i\omega(x_1,x_2)}/(2\pi)^l$ in a series of distributions

$$(2\pi)^{-l} e^{2i\omega(x_1,x_2)} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n (\partial_{q_1}\partial_{p_2} - \partial_{q_2}\partial_{p_1})^n \,\delta(x_1)\,\delta(x_2),$$

understood as the formal Fourier transform of the Taylor series

$$e^{-(i/2)(\zeta_1\eta_2-\zeta_2\eta_1)} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\right)^n (\zeta_1\eta_2 - \zeta_2\eta_1)^n.$$

that series is then substituted into (2.6) to yield expansion (2.7).

—The use of a different symbol map (see Appendix) leads to a different (less symmetrical) operation on symbols corresponding to operator multiplication.

The proof of Corollary 2.4.2 is straightforward.

PROPOSITION 2.4.4. *The following subspaces of S are subalgebras for twisted multiplication*:

—*the subspace* $\mathbf{S}$ *defined by condition* (2.2),

—*the subspace of polynomials in* x,

—*the subspace of symbols* $a(q, p)$ *polynomial in* p, *i.e., symbols of standard partial differential operators on* $\mathscr{S}(Q)$ (*resp. polynomial in* q).

PROPOSITION 2.4.5. $S_{-\infty}$ *is a two-sided ideal of* S, *hence* $\tilde{S} = S/S_{-\infty}$ *is an algebra for "twisted multiplication modulo* $S_{-\infty}$" (*defined by* (2.7) [1].

We omit the proofs (elementary).

2.5. *The Inversion of Symbols*

We seek conditions under which the inverse $\hat{a}^{-1}$ of a GLS operator $\hat{a} \in \hat{S}$ exists in $\hat{S}$ and can be computed. We shall actually solve this problem modulo $S_{-\infty}$. If $\hat{a}^{-1} = \hat{b}$, the asymptotic class $\tilde{b}$ of the symbol b is the inverse for twisted multiplication mod $\widehat{S_{-\infty}}$ of $\tilde{a}$ in the sense that

$$\tilde{a} \exp \frac{i}{2} (\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)\, \tilde{b} = 1. \tag{2.11}$$

We look for $\tilde{b}$ in the form of a descending expansion $\tilde{b}(x) = \sum_{j=0}^{\infty} b_{-2j}(x)$ with $b_{-2j}(x) \in S_{m'-2j}$ (for some m') computed recursively so as to satisfy (the *parametrix* method):

$$ab_0 = 1,$$

$$ab_{-2} + (i/2)\, a(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)\, b_0 = 0,$$

$$ab_{-2j} + \sum_{k=0}^{j-1} \mathscr{A}_{j-k}(x, \partial_x)\, b_{-2k} = 0,$$

where $\mathscr{A}_k(x, \partial_x)$ is the differential operator on X defined by

$$\mathscr{A}_k(x, \partial_x)u = a\left[\frac{1}{k!}\left(\frac{i}{2}\right)^k (\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)^k\right] u \qquad (\forall u \in \mathscr{S});$$

i.e.,

$$\mathscr{A}_k(x, \partial_x) = \sum_{|\gamma|+|\gamma'|=k} \frac{1}{\gamma!\gamma'!}\left(\frac{i}{2}\right)^k (-1)^{|\gamma'|} (\partial_q^{\gamma}\partial_p^{\gamma'} a)\, \partial_q^{\gamma'}\partial_p^{\gamma}. \tag{2.12}$$

We can thus choose

$$b_0(x) = \frac{1}{a(x)}, \ldots, b_{-2j}(x) = -\frac{1}{a(x)}\left[\sum_{k=0}^{j-1} \mathscr{A}_{j-k}(x, \partial_x)\, b_{-2k}\right]. \tag{2.13}$$

THEOREM 2.5.1. *A sufficient condition for the existence in* $\tilde{S}$ *of the asymptotic inverse* $\tilde{b}$ *is that* $b_{-2j} \in S_{m'-2j}$ *for some* $m' \in \mathbb{R}$ *(then* $\tilde{b} = \sum_{j=0}^{\infty} b_{m'-2j} \in \tilde{S}_{m'}$*). This is realized under the following condition:* $a(x) \neq 0$ *for all* x, *and*

$$\forall \alpha \in \mathbb{N}^{2l}, \qquad |\partial^\alpha a(x)| \leqslant c_\alpha\, |a(x)|\, \| x \|^{-|\alpha|} \qquad \textit{for} \quad \| x \| \to \infty. \tag{2.14}$$

Remark. This condition states not only that $a(x)$ should not vanish for finite x, but that it should not tend to 0 too rapidly as $\| x \| \to \infty$. This condition is not necessary: Given any expansion of $\tilde{a}$ such as $\tilde{a} = \sum_0^{\infty} a_{m-j} (a_{m-j} \in S_{m-j})$, the previous method can be adapted in an obvious way provided $a_m(x) \neq 0$ $(\forall x)$ and $|\partial^\alpha a_{m-j}(x)| \leqslant c_{j\alpha}\, |a_m(x)|\, \| x \|^{-|\alpha|-j}$ $(\forall j \in \mathbb{N}, \forall \alpha \in \mathbb{N}^{2l})$, the latter condition being satisfied, for instance, if a_m is homogeneous of degree m (then, moreover, $\tilde{b} \in S_{-m}$).

Proof of the theorem. Condition (2.14) with $|\alpha| = 1$ implies $\| \vec{\nabla}_x(\log |a(x)|)\| \leqslant m'/\| x \|$ for $m' = \max_{|\alpha|=1} \{c_\alpha\}$; hence $|1/a(x)| \leqslant \mathcal{O}(\| x \|^{m'})$ as $\| x \| \to \infty$. A repeated use of Eqs. (2.12)–(2.13) allows us to show recursively on j and $|\alpha|$ that

$$\forall j \in N, \qquad \forall \alpha \in \mathbb{N}^{2l}, \qquad |\partial^\alpha b_{-2j}(x)| \leqslant c_{j\alpha}\, |b_0(x)|\, \| x \|^{-|\alpha|-2j};$$

hence

$$|\partial^\alpha b_{-2j}(x)| \leqslant \mathcal{O}(\| x \|^{m'-|\alpha|-2j}) \qquad \text{and} \qquad b_{-2j} \in S_{m'-2j}. \qquad \text{Q.E.D.}$$

This almost settles the question of the existence of the inverse operator $\hat{a}^{-1}$. For if $b(x)$ is any GLS symbol admitting the previously found $\tilde{b}$ as an asymptotic class, we have

$$\begin{aligned} \hat{a}\hat{b} \quad \mathbb{1} + \hat{c} \qquad &\text{for some} \quad \hat{c} \in S_{-\infty} \\ \to \hat{a}^{-1} = b(\mathbb{1} + \hat{c})^{-1}. \end{aligned} \tag{2.15}$$

Operator $\hat{c}$ is compact on $\mathscr{H}$; except if by accident (-1) happens to be an eigenvalue of $\hat{c}$, $(\mathbb{1} + \hat{c})^{-1}$ exists (by the Fredholm alternative). Writing

$(\mathbb{1}+\hat{c})^{-1}=\mathbb{1}+\hat{d}$: $\mathscr{S}\to^{\hat{d}}\mathscr{S}$ is continuous; hence by the relation $\hat{d}=-\hat{c}-\hat{d}\hat{c}$, $\mathscr{S}'\to^{\hat{d}}\mathscr{S}$ is continuous, so that $\hat{d}\in S_{-\infty}\Rightarrow\hat{a}^{-1}=\hat{b}(\mathbb{1}+\hat{d})\in\hat{S}_{m'}$ and its symbol class is $\tilde{b}$.

Remark. Procedure (2.13) can be simplified by noting that $\hat{a}\hat{b}=\mathbb{1}\Rightarrow\hat{b}\hat{a}=\mathbb{1}$; hence Eq. (2.11) implies

$$\tilde{a}[\exp-(i/2)(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)]\tilde{b}=1. \tag{2.16}$$

System (2.11)–(2.16) splits as

$$\tilde{a}[\cos((\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)/2)]\tilde{b}=1, \tag{2.17$^+$}$$

$$\tilde{a}[\sin((\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)/2)]\tilde{b}=0. \tag{2.17$^-$}$$

Equation (2.17$^+$) alone determines the b_{-2j} by separate recursions on even and odd j:

$$b_0=\frac{1}{a},\ldots,b_{-4j}=-\frac{1}{a}\left[\sum_{k=0}^{j-1}\mathscr{A}_{2(j-k)}b_{-4k}\right],$$

$$b_{-2}=0,\ldots,b_{-2(2j+1)}=-\frac{1}{a}\left[\sum_{k=0}^{j-1}\mathscr{A}_{2(j-k)}b_{-2(2k+1)}\right]=0;$$

hence all b_{-2j} with j odd are actually *zero*, and Eq. (2.17$^-$) then yields an infinite set of identities satisfied by the b_{-4j}.

2.6. *The Twisted Square Root of a Symbol*

In Section 2.7, we shall seek conditions under which a GLS operator $\hat{a}\in\hat{S}$ is positive. For this it is natural to check whether $\hat{a}$ admits a symmetric operator square root $\hat{a}=\hat{b}\hat{b}$, with $b\in S$ real-valued. We can solve this problem mod $S_{-\infty}$ along the same lines as those in Section 2.5:

THEOREM 2.6.1. *If* $a\in S_m$ *satisfies* (2.14) *and* $a(x)>0$ *for all* x, *there exists a real-valued* $\tilde{b}\in\tilde{S}_{m/2}$ *such that*

$$\tilde{a}=\tilde{b}[\exp(i/2)(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)]\tilde{b}.$$

Proof. Since all $\tilde{b}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)^{2k+1}\tilde{b}=0$, by symmetry,

$$\tilde{a}=\tilde{b}[\cos(\overleftarrow{\partial}_q\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_q)/2)]\tilde{b}.$$

We try $\tilde{b}=\sum_{j=0}^{\infty}b_{-4j}$ with $b_{-4j}\in S_{m-4j}$. We can choose $b_0(x)=(a(x))^{1/2}$, and b_{-4j} recursively as the solution of the equation

$$\sum_{j'+j''+k=j}b_{-4j'}\left[\frac{(-1)^k}{(2k)!}\left(\frac{\overleftarrow{\partial}_p\overrightarrow{\partial}_q-\overleftarrow{\partial}_p\overrightarrow{\partial}_q}{2}\right)^{2k}\right]b_{-4j''}=0;$$

i.e.,

$$b_{-4j} = \frac{-1}{2a^{1/2}}\left(\sum_{\substack{j'+j''+k=j\\ j',j''\leqslant j}} b_{-4j'}\left[\frac{(-1)^k}{(2k)!}\left(\frac{\tilde\partial_q\tilde\partial_p - \tilde\partial_p\tilde\partial_q}{2}\right)^{2k}\right] b_{-4j''}\right).$$

b_{-4j} is real-valued, and condition (2.14) implies recursively on j and $|\alpha|$:

$$|\partial^\alpha b_{-4j}(x)| \leqslant c_{j\alpha}\,|b_0(x)|\cdot\|x\|^{-|\alpha|-4j};$$

hence

$$|\partial^\alpha b_{-4j}(x)| \leqslant \mathcal{O}(\|x\|^{m/2-|\alpha|-4j}) \qquad \text{and} \qquad \tilde b \in \tilde S_{m/2}.$$

If $b \in S_{m/2}$ is any real symbol of asymptotic class $\tilde b$, then

$$\hat a = \hat b\hat b + \hat c \qquad \text{for some real-valued } c \in S_{-\infty}. \tag{2.18}$$

Remark. As in Section 2.5, we can relax the hypothesis of Theorem 2.6.1: If $\tilde a = \sum_0^\infty a_{m-j}$ with $a_{m-j} \in S_{m-j}$, the previous method of finding b can be adapted if we only require: $a_m(x) > 0$, a and all a_{m-j} real-valued, and $|\partial^\alpha a_{m-j}(x)| \leqslant c_{j\alpha}\,|a_m(x)|\,\|x\|^{-|\alpha|-j}$ $(\forall j, \alpha)$.

2.7. *Some L^2 Properties of GLS Operators*

We begin by introducing a class of comparison operators: the powers of the quantized harmonic oscillator. Let $H = \hat h$ with $h(x) = \|x\|^2/2 = \frac{1}{2}(q^2 + p^2)$. Then for all $n \in \mathbb{Z}$: $(1 + H)^n \in S_{2n}$ and its symbol behaves like $(1 + \|x\|^2/2)^n \times (1 + o(1))$ as $\|x\| \to \infty$. The proof is tedious but simple, using Theorems 2.4.1 (for $n > 0$) and 2.5.1 (for $n < 0$).

We now consider $A \in \hat S_m$ as a (possibly unbounded) operator on $\mathscr{H} = L^2(Q)$.

THEOREM 2.7.1. (i) *If $m < 0$, A is compact.*

(ii) *If $m < -l$, A is of Hilbert–Schmidt class.*

(iii) *If $m < -2l$, A is of trace class.*

Proof. (ii) see Theorem 2.3.1(ii). (i) If $m < 0$, some power $A^k \in \hat S_{km}$ (with $k \in \mathbb{N}$, $k > l/|m|$) is Hilbert–Schmidt by (ii); hence A is compact. (iii) If $m < -2l$, A is the product $B(B^{-1}A)$ with $B = \widehat{(1 + h)^{m/4}} \in S_{m/2}$ and $B^{-1} \in \hat S_{-m/2}$, hence $B^{-1}A \in \hat S_{m/2}$; both B and $B^{-1}A$ are Hilbert–Schmidt, so that A is of trace class.

THEOREM 2.7.2. (i) *If $a \in S_m$ is a semibounded function: $a(x) \geqslant M > -\infty$ and satisfies* (2.14), *then $\hat a$ is a semibounded symmetric operator for which the Friedrichs (self-adjoint) extension exists.*

(ii) *The operators in $\hat S_0$ are bounded.*

(iii) *For all $m \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geqslant m/2$: If $A \in \hat S_m$, $A(1 + H)^{-n}$ is a bounded operator; hence A is defined on the dense domain of H^n in $\mathscr{H}$.*

Proof. (i) We let $M > 0$ (the general case follows trivially). Then, hypothesis (2.14) is automatically satisfied if $a \in S_0$ (this is needed for proving (ii)). Using Theorem 2.6.1, we can write Eq. (2.18): $\hat{a} = \hat{b}^2 + \hat{c}$ in the sense of continuous operators $\mathscr{S} \to \mathscr{S}$, with b and c real-valued, $c \in S_{-\infty}$. Then $\hat{a}$ is a densely defined symmetric operator on $\mathscr{H}$, whose associated quadratic form is semibounded ($\hat{b}^2$ defines a positive form and c is a symmetric bounded operator by Theorem 2.7.1(i)). The Friedrichs extension theorem [21] then holds and defines a semibounded self-adjoint extension A of $\hat{a}$: $A \geqslant -\| \hat{c} \| \mathbb{1}$.

Note that $a(x) > 0$ does not imply that A is a positive operator; as a one-dimensional counterexample, let $a(x) = p^2q^2 + \epsilon$ with $0 < \epsilon < \frac{1}{4}$; then $\hat{a} = (\widehat{pq})^2 + (\epsilon - \frac{1}{4})\mathbb{1}$ admits $(\epsilon - \frac{1}{4}, +\infty)$ as spectrum and is not a positive operator. Our proof provides no estimate for the lower bound of $\hat{a}$ in the general case. The result is mainly useful as a criterion of self-adjointness. (Note: Berezin and Shubin [31] prove that if $a(x)$ is real-valued and if $a(x) \pm i$ satisfies (2.14), then $\hat{a}$ has deficiency indices (0, 0); hence its closure is self-adjoint.)

(ii) If $a \in S_0$, then Re $a(x)$ is bounded above and below; (i) then implies that $\widehat{\text{Re}\, a}$ is a bounded operator, and similarly for $\widehat{\text{Im}\, a}$; hence $\hat{a}$ is bounded.

(iii) If $a \in S_m$ and $n \geqslant m/2$, $a(\mathbb{1} + H)^{-n} \in \hat{S}_0$ and is a bounded operator by (ii).

We can somewhat refine the result of (iii). For each $n \in \mathbb{Z}$, let $\mathscr{H}_n$ be the Hilbert space obtained by completion of the domain of $(1 + H)^n$ in $\mathscr{H}$ for the inner product $(u, v) \to \langle u, v \rangle_n = \langle (1 + H)^n u, (1 + H)^n v \rangle$. We obtain a sequence of spaces

$$\cdots \subset \mathscr{H}_{+k} \subset \cdots \subset \mathscr{H}_{+1} \subset \mathscr{H}_0 = \mathscr{H} \subset \mathscr{H}_{-1} \subset \cdots \subset \mathscr{H}_{-k} \subset \cdots$$

($k > 0$), where $\mathscr{H}_k$ and $\mathscr{H}_{-k}$ are dual of each other for the inner product of $\mathscr{H}$. Moreover, $\mathscr{S}(Q) = \bigcap_{k \in \mathbb{Z}} \mathscr{H}_k$ and its topology is given by the directed family of seminorms $\| \; \|_n$; hence $\mathscr{S}'(Q) = \bigcup_{k \in \mathbb{Z}} \mathscr{H}_k$ [14, 22].

PROPOSITION 2.7.3. *Let $A \in S_m$. For every $n \in \mathbb{Z}$ with $n \geqslant m/2$, and for all $k \in \mathbb{Z}$, A is a continuous map $\mathscr{H}_k \to \mathscr{H}_{k-n}$.*

Proof. It suffices to show that A is bounded on the domain of $(1 + H)^k$ in $\mathscr{H}$, which is dense in $\mathscr{H}_k$ for the topology of $\mathscr{H}_k$. For any vector u in the domain, $Au = (1 + H)^{n-k}[(1 + H)^{k-n} A(1 + H)^{-k}](1 + H)^k u$. But $(1 + H)^k$ is an isometry of $\mathscr{H}_k$ into $\mathscr{H}$, $[(1 + H)^{k-n} A(1 + H)^{-k}] \in \hat{S}_0$; hence it is a bounded operator on $\mathscr{H}$, and $(1 + H)^{n-k}$ is an isometry of $\mathscr{H}$ into $\mathscr{H}_{k-n}$.

Letting $k \to +\infty$ (resp. $-\infty$) we would recover the continuity of A from $\mathscr{S}$ to $\mathscr{S}$ (resp. $\mathscr{S}'$ to $\mathscr{S}'$).

3. Asymptotic Quantum Mechanics and Admissible Operators

We refer to "asymptotic quantum mechanics" when dealing with power series expansions of quantum mechanical quantities in Planck's constant $\hbar$. The mathematical structure of quantum mechanics [23] depends explicitly on $\hbar \in \mathbb{R}^+\backslash\{0\}$, which can be considered as a variable parameter of the theory. Although $\hbar$ is a constant in nature, this viewpoint can be useful because certain quantum mechanical quantities tend as $\hbar \to 0^+$ to their "analogs" of classical Hamiltonian mechanics [24], and in many physical instances the actual value of $\hbar$ is relatively so small as to make a perturbative computation "around" classical mechanics a sensible operation. The works of Maslov [25], Leray [26], and Duistermaat [27] have brought out the analogy between these small-$\hbar$ expansions and the ordinary asymptotic expansions in pseudodifferential operator theory. A specific pseudodifferential algebra can be introduced [4, 5] to deal directly with $\hbar$-expansions. It is analogous to the GLS algebra, but with additional requirements on the explicit behavior in $\hbar$ of the symbols. We thus need to rewrite Sections 1 and 2 showing the explicit dependence in $\hbar$ everywhere ($\hbar > 0$).

3.1. *The Weyl Quantization*

It associates to a classical function $a(q, p)$ the integral operator on $\mathscr{H} = L^2(Q)$ of kernel (cf. Eq. (1.2)):

$$\langle q \mid \hat{a}_\hbar \mid q' \rangle = (\mathscr{F}_\hbar a)(q, q') = (2\pi\hbar)^{-l} \int_P a((q+q')/2, p)\, e^{ip(q-q')/\hbar}\, dp. \quad (3.1)$$

The conditions of validity and the properties of this Weyl quantization are the same as those in Section 1 (up to some normalization constants); in particular, $\mathscr{F}_\hbar$ is a continuous isomorphism $\mathscr{S}'(X) \to \mathscr{S}'(Q \times Q)$ and also an isometry $L^2(X, dx/(2\pi\hbar)^l) \to L^2(Q \times Q) \approx \mathscr{L}^2(\mathscr{H})$. The coordinate functions become quantized as

$$\hat{q}_j = q_j \qquad \text{(multiplication by } q_j\text{)},$$
$$\hat{\partial}_j = -i\hbar \frac{\partial}{\partial q_j};$$

hence they generate the well-known Schrodinger representation of the canonical commutation rules: $[\hat{q}_j, \hat{p}_k]_- = i\hbar\, \delta_{jk}\mathbb{1}$ $(= i\hbar\, \delta_{jk}\hat{1})$. The other properties listed in Section 1 stay unchanged.

3.2. *Admissible Symbols*

An admissible symbol of order $\leqslant m$ ($m \in \mathbb{R}$) is any C^∞ function $a \in C^\infty([0, \hbar_0) \times X)$ (for some unspecified $0 < \hbar_0 \leqslant \infty$) such that for all $j \in \mathbb{N}$ the mapping $\hbar \to (\partial^j/\partial\hbar^j)\, a(\hbar, \cdot)$ is C^∞ from $[0, \hbar_0)$ into S_{m-2j} (note that regularity in $\hbar$ is required up to $\hbar = 0$).

The space of such symbols is denoted $\mathcal{A}_m$; we put on it the topology of uniform convergence on compacts of $[0, \hbar_0)$ of every derivative $\partial^j a/\partial \hbar^j$ in S_{m-2j}. We have $\mathcal{A}_{m'} \subset \mathcal{A}_m$ if $m' < m$; we let $\mathcal{A}_{-\infty} = \bigcap_m \mathcal{A}_m$ and $\mathcal{A} = \bigcup_m \mathcal{A}_m$.

The Taylor series at $\hbar = 0$ of a symbol $a \in \mathcal{A}_m$,

$$a(\hbar, \cdot) \sim \sum_0^\infty a_j \hbar^j, \qquad a_j \in S_{m-2j}, \tag{3.2}$$

defines an asymptotic expansion both in the sense of Section 2.2 and in the sense of functions of $\hbar$ as $\hbar \to 0^+$; for all $n \in \mathbb{N}$,

$$\left(a(\hbar, \cdot) - \sum_{k=0}^{n-1} a_k \hbar^k\right) \Big/ \hbar^n$$

stays in a bounded set of S_{m-2n} as $\hbar \to 0^+$.

The leading term $a_0(x)$ is the *principal symbol.*

The asymptotic study for $\hbar \to 0$ will identify as zero any symbol admitting the expansion $\sum_0^\infty 0 \cdot \hbar^n$ in the sense of (3.2). We are thus led to define:

—The subspace $\mathcal{N}$ of *negligible* symbols as the space of symbols $a \in \mathcal{A}$ such that $a \sim 0$ in the sense of (3.2); i.e., $a(\hbar, \cdot) \in S_{-\infty}$ and for all $n \in \mathbb{N}$: $a(\hbar, \cdot)/\hbar^n$ stays in a bounded set of $S_{-\infty}$ as $\hbar \to 0$.

—The quotient spaces $\tilde{\mathcal{A}}_m = \mathcal{A}_m/\mathcal{N}$, $\tilde{\mathcal{A}}_{-\infty} = \mathcal{A}_{-\infty}/\mathcal{N}$, $\tilde{\mathcal{A}} = \mathcal{A}/\mathcal{N}$ of asymptotic classes of symbols modulo $\mathcal{N}$; the class of a is noted $\tilde{a}$. Note that here we let the space $\mathcal{N}$ (and *not* $\mathcal{A}_{-\infty}$) play the same role as the space $S_{-\infty}$ in the GLS case.

THEOREM 3.2.1. *Every formal series* $\sum_0^\infty a_j \hbar^j$ *with* $a_j \in S_{m-2j}$ *defines an element of* $\tilde{\mathcal{A}}_m$ *uniquely.*

The proof can be done similarly to the proof of the analogous theorem, Theorem 2.2.1.

3.3. *Quantization of Admissible Symbols*

For every $0 < \hbar < \hbar_0$, $\widehat{a(\hbar, \cdot)}_\hbar$ is an operator in $\hat{S}_m$ with all the properties listed in Section 2.3; we call it an *admissible operator*.

Many important operators of quantum mechanics are indeed admissible operators: for instance ($l = 3$), the components of the angular momentum $\hat{\mathbf{q}} \times \hat{\mathbf{p}}$, the squared angular momentum $(\hat{\mathbf{q}} \times \hat{\mathbf{p}})^2$ (it has the symbol $(\mathbf{q} \times \mathbf{p})^2 - 3\hbar^2/4$), and all Hamiltonians $\hat{p}^2/2 + \widehat{V(q)}$ with polymonial interaction potentials. Nonpolynomial potentials ($V \in C^\infty(Q)$ are *never* admissible ($V \notin S$) [33];[1] but this restriction disappears if we replace condition (2.1) by: $\exists m, n \in \mathbb{R}\ \forall \alpha, \beta \in \mathbb{N}^l$, $| \partial_q{}^\alpha \partial_p{}^\beta a(x)| \leqslant C_{\alpha\beta}(1 + \| q \|)^{m-|\alpha|} (1 + \| p \|)^{n-|\beta|}$

[1] We thank J. Chazarain for calling our attention to this fact.

3.4. *The Twisted Multiplication of Admissible Symbols*

As in Section 2.4, it is defined so as to correspond to the operator multiplication under the now $\hbar$-dependent Weyl quantization.

THEOREM 3.4.1. (i) *$\mathcal{O}\!\!\ell$ is an associative *-algebra for twisted multiplication, which is a continuous map from $\mathcal{O}\!\!\ell_{m_1} \times \mathcal{O}\!\!\ell_{m_2}$ into $\mathcal{O}\!\!\ell_{m_1+m_2}$ (for all $m_1, m_2 \geqslant -\infty$): $(a, b) \to c$, given by*

$$c(\hbar, x) = (\pi\hbar)^{-2l} \int_{X^2} dx_1\, dx_2\, a(\hbar, x + x_1)\, b(\hbar, x + x_2)\, e^{2i\omega(x_1, x_2)/\hbar}. \tag{3.3}$$

(ii) *c admits the asymptotic expansion in $\mathcal{O}\!\!\ell$:*

$$c(\hbar, x) \sim \tilde{a}(\hbar, x)[\exp(i\hbar/2)(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{p_2} - \overleftarrow{\partial}_{p_1}\overrightarrow{\partial}_{q_2})]\, \tilde{b}(\hbar, x). \tag{3.4}$$

(Its principal symbol is $c_0(x) = a_0(x)\, b_0(x)$.)

(iii) *If a or b is polynomial in x, expansion* (2.7) *is a finite sum which equals $c(x)$ exactly.*

The proof is identical to that of Theorem 2.4.1. All other statements in Section 2.4 have easy to find analogs; in particular, $\tilde{\mathcal{O}\!\!\ell}$ is a *-algebra for the "twisted multiplication modulo $\mathcal{N}$" defined in Eq. (3.4).

3.5. *The "Twisted Inverse" of an Admissible Symbol*

Here we can make a stronger statement than the analog of Theorem 2.5.1:

THEOREM 3.5.1. *Let $a \in \mathcal{O}\!\!\ell$, $a \sim \sum_0^\infty a_j\hbar^j$ in the sense of Section 3.2, with $a_0(x) \neq 0$ for all x and $a_0(x)$ satisfying, as $\| x \| \to \infty$,*

$$\forall\alpha \in \mathbb{N}^{2l}, \qquad | \partial^\alpha a_j(x)| \leqslant c_{j\alpha}\, | a_0(x)|\, \| x \|^{-|\alpha|-2j}. \tag{3.5}$$

Then for all $\hbar$ small enough, $\hat{a}_\hbar$ admits an operator inverse of the form $\hat{b}_\hbar$, with $b \in \mathcal{O}\!\!\ell$ of principal symbol $1/a_0(x)$.

Proof. We begin, as in Theorem 2.5.1, by computing the asymptotic class $\tilde{b} = \sum_0^\infty b_j\hbar^j$ using the parametrix method; in particular, $b_0(x) = 1/a_0(x)$. Condition (3.5) is needed to keep $a_0(x)$ from tending to zero too fast as $\| x \| \to \infty$ and it guarantees that for some $m' \in \mathbb{R}$ and all $j \in \mathbb{N}$: $b_j \in S_{m'-2j}$. For any $b \in \mathcal{O}\!\!\ell$ having this asymptotic class $\tilde{b}$, we then have

$$\hat{a}_\hbar \hat{b}_\hbar = \mathbb{1} + \hat{c}_\hbar \qquad \text{for some} \quad c \in \mathcal{N},$$

hence:

$$\hat{a}_\hbar^{-1} = \hat{b}_\hbar(\mathbb{1} + \hat{c}_\hbar)^{-1}. \tag{3.6}$$

The new fact here is that not only is $\hat{c}_\hbar$ ($\in \hat{S}_{-\infty}$) a Hilbert–Schmidt operator for all $\hbar$, but its operator norm is a rapidly decreasing function of $\hbar$ as $\hbar \to 0$, since for all $n \in \mathbb{N}$,

$$\| \hat{c}_\hbar \| \leqslant \operatorname{Tr} \hat{c}_\hbar^{\dagger} \hat{c}_\hbar = (2\pi\hbar)^{-l} \| c(\hbar, \cdot) \|_{L^2(X)}$$

$$\leqslant (2\pi\hbar)^{-l} C_n \left\| \frac{\hbar^n}{(1 + \| x \|^2)^n} \right\|_{L^2} = \mathcal{O}(\hbar^{n-l}).$$

This guarantees that for $\hbar$ smaller than some (undetermined) upper bound, the operator inverse $(\mathbb{1} + \hat{c}_\hbar)^{-1}$ exists (and can be computed by its Neumann series or by the Fredholm method); also, $(\mathbb{1} + \hat{c}_\hbar)^{-1} = \mathbb{1} + \hat{d}_\hbar$ with $d \in \mathcal{N}$; hence $\hat{a}^{-1}$ given by (3.6) is an admissible operator with $\tilde{b}$ as asymptotic class.
Q.E.D.

Remark. The reader can check that condition (3.5) is preserved under twisted multiplication and twisted inversion. (The role of this condition is suggested by its implication that, in the sense of operator theory, $(\widehat{a(\hbar, \cdot)_\hbar} - \widehat{a_{0\hbar}})$ is a regular perturbation of $\widehat{a_{0\hbar}}$, namely, that $(\widehat{a(\hbar, \cdot)_\hbar} - \widehat{a_{0\hbar}})\,(\widehat{a_{c\hbar}})^{-1} \in \hat{S}_0$ and its operator norm is $\mathcal{O}(\hbar)$).

3.6. *Some L^2 Properties of Admissible Operators*

We consider for every $\hbar \in (0, \hbar_0)$ fixed, and for $a \in \mathcal{Cl}_m$, the operator $\hat{a}_\hbar$ on $\mathcal{H} = L^2(Q)$. Since $a(\hbar, \cdot) \in S_m$, Theorems 2.7.1 and 2.7.2 hold without modification, but we can also strengthen point (i) of Theorem 2.7.2.

THEOREM 3.6.1. *Let $a \in \mathcal{Cl}$, $a \sim \sum_0^\infty a_j \hbar^j$ in the sense of Section* 3.2, *with $a_0(x) > 0$, $a(\hbar, x)$ and $a_j(x) \in \mathbb{R}$ for all $x \in X$, $j \in \mathbb{N}$. If a satisfies* (3.5), *then for all $\hbar$ small enough, $\hat{a}_\hbar$ is a positive operator.*

Proof. We begin by assuming that $a_0(x) \geqslant k > 0$. Then $(a - k/2) \in \mathcal{Cl}$ also satisfies (3.5) (use $| a_0(x) | \leqslant 2 \mid a_0(x) - k/2 \mid$) and has a positive principal symbol. As in Theorem 2.6.1, we can compute a real-valued symbol $b \sim \sum_0^\infty b_j \hbar^j$ with $b_j \in S_{m/2-2j}$ (if $a \in \mathcal{Cl}_m$) such that $\widehat{(a - k/2)_\hbar} = \hat{b}_\hbar \hat{b}_\hbar + \hat{c}_\hbar$ with $c \in \mathcal{N}$ real-valued. Then $\hat{a}_\hbar = \hat{b}_\hbar^2 + (k/2)\mathbb{1} + \hat{c}_\hbar$, with $\hat{b}_\hbar$ symmetric and $\hat{c}_\hbar$ self-adjoint with operator norm $= o(\hbar^\infty)$: for $\hbar$ small enough this norm is $< k/2$, and then $\hat{a}_\hbar$ is a positive operator.

If $\inf_{x \in X} \{a_0(x)\} = 0$, we know by condition (3.5) that $a_0(x) \geqslant K \| x \|^{-m'}$ still holds for some $m' \in R$. Fix $n \in \mathbb{N}$, $n \geqslant m'/4$; let $h(x) = \| x \|^2$ and define the operator $A' = (1 + \hat{h}_\hbar)^n \hat{a}_\hbar (1 + \hat{h}_\hbar)^n$. By Theorem 3.4.1, $A' = \widehat{a_\hbar'}$ with $a' \in \mathcal{Cl}$, $a' \sim \sum_0^\infty a'_j \hbar^j$, and $a'_0 = (1 + \| x \|^2)^{2n} a_0(x)$. We can check that a' satisfies the hypothesis of Theorem 3.6.1 (using the fact that (3.5) is satisfied by a and by $(1 + h)$, and is preserved under twisted multiplication), and also $a'_0(x) \geqslant k > 0$. By the previous argument $\widehat{a_\hbar'}$ is a positive operator for small $\hbar$, but then so is $\hat{a}_\hbar = (1 + \hat{h}_n)^{-n} \hat{a}'_\hbar(1 + \hat{h}_\hbar)^{-n}$.

4. Admissible Functionals and Physical Applications

Assume that we are interested in the spectrum and in the eigenfunctions of a given self-adjoint admissible operator $A = \hat{a}_\hbar$. It is convenient, in order to have the discrete and continuous spectra on the same footing, to work in the "Dirac formalism," i.e., with Gelfand triplets [28]. Since the algebra of admissible operators admits $\mathscr{S}(Q)$ as a common dense invariant domain such that $\mathscr{S} \to^A \mathscr{S}$ is continuous (Theorem 2.3.1), we can use the Gelfand triplet $\mathscr{S}(Q) \subset L^2(Q) \subset \mathscr{S}'(Q)$: The "eigenfunctions" of the continuous spectrum are tempered distributions (those of the discrete spectrum lie in $L^2(Q)$).

Is it then reasonable to devise a regular perturbation scheme to compute the eigenfunctions of A in power series of $\hbar$? Even if we ignore the difficulties linked with the discrete spectrum (which becomes continuous in the classical limit $\hbar = 0$), the answer is certainly no; even in the simplest case (cf. the WKB method), the eigenfunctions exhibit very bad caustic singularities. An alternative approach eliminates this difficulty: We associate to every $u \in \mathscr{S}'(Q)$ the linear functional on $\mathcal{A}_{-\infty}$ (for fixed $\hbar > 0$),

$$c \in \mathcal{A}_{-\infty} \to \langle u(q) \otimes u^*(q'), (\mathscr{F}_\hbar c)(q, q')\rangle,$$

in the sense of the inner product between $\mathscr{S}(Q \times Q)$ and $\mathscr{S}'(Q \times Q)$. Let $\rho_u(\hbar) = (2\pi\hbar)^{-l} \mathscr{F}_\hbar^{-1}(u \otimes u^*) \in \mathscr{S}'(X)$. By the Parseval identity for distributions,

$$\langle u \otimes u^*, (\mathscr{F}_\hbar c)\rangle_{\mathscr{S}'(Q\times Q)\times\mathscr{S}(Q\times Q)} = \langle \rho_u(\hbar), c(\hbar, \cdot)\rangle_{\mathscr{S}'(X)\times\mathscr{S}(X)} \tag{4.1}$$

(ρ_u is the "Wigner function" of the distribution u [9]).

In many solvable cases, $\langle \rho_u(\hbar), c(\hbar, \cdot)\rangle$ (of Eq. (4.1)) is a continuous linear functional of $c(\hbar, \cdot) \in S_{-\infty}$ (for $\hbar \neq 0$) whose value is a C^∞ function of $\hbar$ (up to $\hbar = 0$). Since $\mathcal{A}_{-\infty}$ consists of the C^∞ maps $\hbar \to c(\hbar, \cdot) \in \mathscr{S}(X)$, an equivalent statement is that ρ_u must be a C^∞ map: $\hbar \in [0, \hbar_0) \to \rho_u(\hbar) \in \mathscr{S}'(X)$; we call all such C^∞ maps $\hbar \in [0, \hbar_0) \to \rho(\hbar) \in \mathscr{S}'(X)$ *admissible functionals* (acting on $\mathcal{A}_{-\infty}$); they form the space $\widehat{\mathscr{S}'}(X)$.

Twisted multiplication extends to a mapping $\mathcal{A} \times \widehat{\mathscr{S}'} \to \widehat{\mathscr{S}'}$ by duality [8]: If $a \in \mathcal{A}$, $b \in \widehat{\mathscr{S}'}$, the twisted product of a and b is the functional $d \in \mathcal{A}_{-\infty} \to \langle b, e\rangle_{\mathscr{S}'(X)\times\mathscr{S}(X)}$, where $e \in \mathcal{A}_{-\infty}$ is the twisted product of a^* by d: $\hat{e}_\hbar = \hat{a}_\hbar^\dagger \hat{d}_\hbar$. Equations (3.3) and (3.4) remain true when $a \in \mathcal{A}$, $b \in \widehat{\mathscr{S}'}$ [5].

A reasonable perturbative scheme is then to compute the "coefficients" $\rho_n \in \mathscr{S}'(X)$ such that $\rho_u(\hbar) \sim \sum_0^\infty \rho_n \hbar^n$ (in the weak sense) as $\hbar \to 0^+$.

Unfortunately we have no theorem stating that for a given admissible operator, suitable eigenfunctions $u \in \mathscr{S}'(Q)$ define functionals $\rho_u(\hbar)$ that are admissible (we know this to be true only for some exactly solvable cases; see examples below). What we know is that in many other cases the eigenvalue equations for ρ_u, $A\rho_u = \rho_u A = E\rho_u$, can be *formally* solved to any order $\hbar^n$, yielding very reasonable coefficients $\rho_n \in \mathscr{S}'(X)$; no singularities are ever encountered. Moreover, in completely integrable situations the equation $\rho_u(\hbar) \sim \sum_0^\infty \rho_n \hbar^n$ can then be solved algebraically (again formally, to any order) to yield an eigenfunction u of the Maslov (WKB) type $u(q) = \alpha(\hbar, q)\, e^{iS(q)/\hbar}$ (away from caustics); in the case of discrete eigenvalues corresponding to this situation, they, too, can be computed.

This whole scheme uses properties of admissible functionals which follow by duality from those of admissible operators; we refer the reader to [5] for details.

Here are two explicit examples of admissible functionals in one dimension:

—if u is the ground state of the harmonic oscillator

$$\widehat{\frac{\| x \|_\hbar^2}{2}} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + \frac{q^2}{2} \qquad \left(\text{of eigenvalue } \frac{\hbar}{2}\right),$$

then

$$\rho_u(\hbar) = \frac{1}{\pi\hbar}\, e^{-(q^2+p^2)/\hbar} \sim \sum_0^\infty \left[\frac{1}{n!}\left(\frac{\partial^2/\partial q^2 + \partial^2/\partial p^2}{4}\right)^n \delta(q)\, \delta(p)\right] \hbar^n.$$

—If u is the eigenfunction of the "Airy operator"

$$\widehat{\left(\frac{p^2}{2} - q\right)_\hbar} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} - q$$

with "eigenvalue" 0 (in the continuous spectrum),

$$\rho_u(\hbar) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\tau \cdot e^{-i\tau^3/24\hbar} e^{i\tau(q-p^2/2)\hbar} \sim \left[\exp\left(\frac{\hbar^2}{24}\frac{\partial^3}{\partial q^3}\right)\right] \delta\left(q - \frac{p^2}{2}\right),$$

whereas the function u itself ($u(q, \hbar) = 2^{+1/3}(2\pi)^{1/2}\, \hbar^{-1/6} Ai(-2^{+1/3}\hbar^{-2/3} q)$) has a caustic singularity at $q = 0$ and cannot be regularly expanded in powers of $\hbar$.

In conclusion, although the mathematical theory is not yet complete, the theory of admissible operators and functionals seems to lead to methods for solving pseudodifferential equations asymptotically that are more regular, and probably more general, than the usual methods.

APPENDIX: OTHER SYMBOL MAPS

Different correspondences between symbols and operators can be viewed as different prescriptions for ordering the noncommuting operators q_j and $-i(\partial/\partial q_j)$ in a formal expression given as $a(q, -i\partial/\partial q)$.

The usual ordering in mathematics associates to a function $a(x)$ an operator a^M such that [2, 3]

$$(a^M u)(q) = (2\pi)^{-l} \int_{P\times Q} a(q, p)\, u(q')\, e^{ip(q-q')}\, dp\, dq'. \tag{A1}$$

If $a(x) = b(q)\, c(p)$, then $a^M = b(q)\, c(-i\partial/\partial q)$; this ordering puts all differentiations to the right. In particular, as compared with Eq. (1.4), $[\exp i(\alpha q+\beta p)]^M = \exp(i\alpha q^M) \cdot \exp(i\beta p^M)$. Note that for any $a(x) = f(q) + g(p)$, $a^M \equiv \hat{a}$, in particular $q_j{}^M = q_j$ and $p_j{}^M = -i\partial/\partial q_j$ (on $C_0{}^\infty(Q)$).

The adjoint $(a^M)^\dagger$ (in the sense of forms) arises from the symbol

$$b(q, p) = (2\pi)^{-l} \int_{P\times Q} a^*(q + r, p')\, e^{i(p-p')r}\, dp'\, dr. \tag{A2}$$

In the sense of oscillatory integrals, the mapping $a^*(x) \to b(x)$ defined by (A2) is a continuous isomorphism of S_m (same style of proof as that for Theorem 2.4.1) Equation (A2) admits the formal asymptotic expansion

$$b(q, p) \sim a^*\left(q - i\frac{\partial}{\partial p}, p\right) = \sum_{\alpha\in\mathbb{N}^l} \frac{1}{\alpha!} (-i)^{|\alpha|} \partial_p{}^\alpha \partial_q{}^\alpha a^*(q, p). \tag{A2'}$$

The mapping $a \to a^M$ is 1–1 from S_m onto $\hat{S}_m$; for every $a \in S_m$, $a^M = \widehat{a'}$ with the symbol $a' \in S_m$ defined by

$$a'(q, p) = (2\pi)^{-l} \int_{P\times Q} a\left(q - \frac{r}{2}, p'\right) e^{i(p-p')r}\, dp'\, dr, \tag{A3}$$

and (A3) defines a continuous isomorphism $S_m \to S_m$; the inverse formula of (A3) is

$$a(q, p) = (2\pi)^{-l} \int_{P\times Q} a'\left(q + \frac{r}{2}, p'\right) e^{i(p-p')r}\, dp'\, dr. \tag{A4}$$

The corresponding asymptotic formulas are

$$\begin{aligned} a'(q, p) &\sim a\left(q + \frac{i}{2}\frac{\partial}{\partial p}, p\right) = \sum_{\alpha\in\mathbb{N}^l} \frac{1}{\alpha!}\left(\frac{i}{2}\right)^{|\alpha|} \partial_q{}^\alpha\, \partial_p{}^\alpha\, a(q, p), \\ a(q, p) &\sim a'\left(q - \frac{i}{2}\frac{\partial}{\partial p}, p\right) = \sum_{\alpha\in\mathbb{N}^l} \frac{1}{\alpha!}\left(\frac{-i}{2}\right)^{|\alpha|} \partial_q{}^\alpha\, \partial_p{}^\alpha\, a'(q, p). \end{aligned} \tag{A5}$$

Thus the Weyl ordering and this ordering lead to the same operator spaces.

But the operator relation $c^M = a^M b^M$ defines a symbol multiplication $(a, b) \to c$ which is different from twisted multiplication (Eqs. (2.6 and 2.7)):

$$c(q, p) = (2\pi)^{-l} \int_X a(q, p + p')\, b(q + q', p)\, e^{-ip'q'}\, dp'\, dq' \tag{A6}$$

$$\sim a(q, p - i\partial_{q'})\, b(q + q', p)_{q'=0} = \sum_{\alpha \in \mathbb{N}^l} \frac{1}{\alpha!} (-i)^{|\alpha|}\, \partial_p{}^\alpha\, a(q, p)\, \partial_q{}^\alpha b(q, p)$$

$$= a[\exp(-i\overleftarrow{\partial}_p \overrightarrow{\partial}_q)]b. \tag{A7}$$

An inverse ordering (putting all differentiations to the left) can similarly be defined. The two orderings clearly do not share the Weyl ordering's invariance property with respect to linear symplectic automorphisms of X.

A Wick (or "normal") ordering can be associated to every Euclidean structure on X. By a symplectic automorphism, any positive definite quadratic form on X can be put in the "normal" form $h(x) = \frac{1}{2}\sum_1^l \omega_j(q_j^2 + p_j^2)$ (all $\omega_j > 0$). We introduce the isotropic coordinates $\bar{z}_j = (q_j + ip_j)/2^{1/2}$, $z_j = (q_j - ip_j)/2^{1/2}$; the quantized operators $\hat{z}_j$ (resp. $\hat{\bar{z}}_j$) are the creation (resp. annihilation) operators of the harmonic oscillator $\widehat{h(x)}$. The Wick quantization of a classical function expressed as $a(z, \bar{z})$ is an operator $:a:$ such that if $a(z, \bar{z}) = \sum b_j(z)\, c_j(\bar{z})$ then $:a: = \sum b_j(\hat{z})\, c_j(\hat{\bar{z}})$. This prescription determines $:a:$ for all polynomials $a(z, \bar{z})$, for instance. To quantize general functions, in a class of symbols for instance, the most convenient way is to define $:a:$ as a pseudodifferential operator on analytic functions $u(z)$ in the Bargmann–Fock–Segal (or holomorphic) representation [29], where $:z_j:$ is multiplication by z_j and $:\bar{z}_j: = \partial/\partial z_j$; then Wick ordering in this representation is analogous to the mathematical ordering (all differentiations $\partial/\partial z_j$ pushed to the right). But since the Bargmann function spaces have a quite different structure, this analogy is rigorous in a nontrivial way; here we simply give the formal analogs of Eqs. (A2′) and (A7) wtihout their rigorous derivation:

—For the adjoint $:a(z, \bar{z}):^\dagger = :a^*(\bar{z}, z):$,

—for the product of "normal" symbols (in the sense that $:c: = :a::b:$),

$$c(z, \bar{z}) \sim a(z, \bar{z})[\exp(\overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)]\, b(z, \bar{z});$$

i.e.,

$$:a::b: \sim \sum_{\alpha \in \mathbb{N}^l} \frac{1}{\alpha!}\, {}_{\xi}\partial_{\bar{z}}{}^\alpha a \cdot \partial_z{}^\alpha b: .$$

This last formula is known as Wick's theorem for normal-ordered products [10]. It is an exact relation for polynomials.

Further properties of Wick orderings are discussed in [11].

Some other quantization procedures, with their associated symbols calculus, are discussed in [30].

Geometric quantization [32], when performed in the canonical coordinates (q, p) of X, also associates an operator a^Q on $C_0^\infty(Q)$ to suitable functions $a(x)$. According to Gawedzki [32, p. 215]: $a^Q = (a - (i/2)\,\partial_q \partial_p a)^M$ (with our notation), hence formally: $a^Q = \widehat{a'}$ with $a'(q,p) = \sum_{|\alpha| \geqslant 2} (1/\alpha!)(i/2)^{|\alpha|} \partial_q^\alpha \partial_p^\alpha a(q,p)$, by (A5), and $a^Q \equiv \hat{a}$ only for $a(x)$ polynomial of degree $\leqslant 2$; otherwise the algebraic structure is not very attractive.

Note added in proof. When this manuscript was completed we received an article (previously unknown to us) by Berezin and Shubin [31] which deals with the same spaces of symbols and gives several of the results proved here in Section 2. We have nevertheless kept our text unchanged with the idea that more detailed proofs than those in Ref. [31] might be useful.

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