Clifford Algebras for Algebras with Involution of Type D

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In this paper we shall give a natural generalization of the even (or second) Clifford algebra of a quadratic form so as to apply to any finite dimensional central simple algebra $\mathfrak{A}$ with involution $J$ of type $D$. The corresponding even Clifford algebra $\mathcal{C}^+(\mathfrak{A}, J)$ has a canonical involution $\iota$. One can distinguish two classes of algebras with involution of type $D$ called type $D_1$ and $D_2$ (definition in Section I). Besides this, if the dimensionality of $\mathfrak{A}$ is $(2l)^2$, then one distinguishes the case $l$ even and $l$ odd. Accordingly, one has four possibilities for the structure of $(\mathcal{C}^+(\mathfrak{A}, J), \iota)$. The most interesting of these is that of type $D_1$, $l$ even. In this case, we obtain a decomposition of $\mathcal{C}^+(\mathfrak{A}, J)$ as a direct sum of two ideals $\mathfrak{C}_1$ and $\mathfrak{C}_2$ each of which is invariant under $\iota$. The $\mathfrak{C}_i$ are central simple and we show that $\mathfrak{A} \otimes \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim 1$ in the Brauer group. The proof of this involves an extension of the classical principle of triality.

The first two sections of the paper are almost wholly expository. In Section III we define the even Clifford algebra $\mathcal{C}^+(\mathfrak{A}, J)$ and we show that this coincides with the usual even Clifford algebra when the latter is defined. In Section IV we study the case of algebras of type $D_1$ with $l$ even and in Section V we define an extension of the notion of the Clifford group to the present case of an algebra with involution of type $D$.

I. CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

By an algebra with involution we mean a pair $(\mathfrak{A}, J)$ where $\mathfrak{A}$ is a finite dimensional associative algebra with $1 \neq 0$ over a field $\Phi$ of characteristic $\neq 2$, together with an involution $J$ (antiautomorphism of period two) in $\mathfrak{A}$. Homomorphism, isomorphism, etc. for such pairs are defined as usual for groups with operators. For example, a homomorphism $\eta$ of $(\mathfrak{A}, J)$ into the algebra with involution $(\mathfrak{B}, K)$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $J\eta = \eta K$. The pair $(\mathfrak{A}, J)$ is simple if the only ideals of $\mathfrak{A}$ fixed by $J$ are $\mathfrak{A}$ and $0$. Then either $\mathfrak{A}$ is simple or it is a direct sum of two simple ideals which are exchanged by $J$. If $(\mathfrak{A}, J)$ is simple then the subset of $J$-symmetric
elements of the center of \( \mathfrak{A} \) is a subfield and \( (\mathfrak{A}, J) \) is called central if this subfield coincides with the base field \( \Phi \). If \( P \) is an extension of the base field then we obtain the algebra with involution \( (\mathfrak{A}_P, J) \) where

\[
\mathfrak{A}_P = P \otimes_\Phi \mathfrak{A}
\]

and \( J \) in \( \mathfrak{A}_P \) is the \( P \)-linear extension of \( J \) to \( \mathfrak{A}_P \). \( (\mathfrak{A}, J) \) is central simple if and only if \( (\mathfrak{A}_P, J) \) is simple for every extension field \( P \). The simple (necessarily central) pairs over an algebraically closed field \( \Omega \) are readily classified. They are

1. \( A_l \). \( \mathfrak{A} = \text{Hom}_\Phi (\check{M}, \check{M}) \oplus \text{Hom}_\Phi (\check{M}, \check{M}) \) where \( \check{M} \) is an \( l \) dimensional vector space over \( \Omega \) and \( J \) is an involution exchanging the two factors.

2. \( B_{l-1} \). \( \mathfrak{A} = \text{Hom}_\Phi (\check{M}, \check{M}) \), \( \dim \check{M} = 2l + 1 \), \( J \) the adjoint mapping relative to a nondegenerate symmetric bilinear form \( (x, y) \) in \( \check{M} \).

3. \( C_{l-1} \). \( \mathfrak{A} = \text{Hom}_\Phi (\check{M}, \check{M}) \), \( \dim \check{M} = 2l \), \( J \) the adjoint mapping relative to a nondegenerate skew bilinear form \( [x, y] \) in \( \check{M} \).

4. \( D_{l-1} \). \( \mathfrak{A} = \text{Hom}_\Phi (\check{M}, \check{M}) \), \( \dim \check{M} = 2l \), \( J \) the adjoint mapping relative to a nondegenerate symmetric bilinear form \( (x, y) \) in \( \check{M} \).

If \( \Phi \) is arbitrary and \( (\mathfrak{A}, J) \) is central simple then \( (\mathfrak{A}_\Omega, J) \) is an algebra \( X_t \), \( X = A, B, C \) or \( D \), if \( \Omega \) is the algebraic closure of \( \Phi \). Then we say that \( (\mathfrak{A}, J) \) is of type \( X_t \). If \( (\mathfrak{A}, J) \) is of type \( A_t \) then either \( \mathfrak{A} \) is a direct sum of two ideals and \( J \) exchanges these or the center of \( \mathfrak{A} \) is a quadratic extension of \( \Phi \) and \( J \) is of second kind. Accordingly, one says that the type is \( A_{tt} \) or \( A_{ttt} \). In either case \( \dim \mathfrak{A} = 2 \cdot l^2 \) and the dimensionality of the space \( \mathfrak{S}(\mathfrak{A}, J) \) of \( J \)-symmetric elements is \( l^2 \). If \( (\mathfrak{A}, J) \) is of type \( B_t \) then necessarily \( \mathfrak{A} \sim 1 \) (in the Brauer group) so \( \mathfrak{A} = \text{Hom}_\Phi (M, M) \) where \( M \) is \( 2l + 1 \) dimensional and \( J \) is the adjoint relative to some nondegenerate symmetric bilinear form. Here \( \dim \mathfrak{A} = (2l + 1)^2 \) and \( \dim \mathfrak{S}(\mathfrak{A}, J) = (l - 1) (2l + 1) \). If \( (\mathfrak{A}, J) \) is of type \( C_t \) then \( \dim \mathfrak{A} = (2l)^2 \) and \( \dim \mathfrak{S}(\mathfrak{A}, J) = l(2l - 1) \). In addition if \( \mathfrak{A} \sim 1 \) then the involution in \( \text{Hom}_\Phi (M, M) \) is given by a nondegenerate skew bilinear form. If the type is \( C_t \) and \( \mathfrak{A} \sim 1 \) or if the type is \( D_t \) and \( \mathfrak{A} \) is unrestricted then it is known that \( \mathfrak{A} \) is isomorphic to the algebra of linear transformations in a finite dimensional vector space over a division algebra \( D \) which has an involution of first kind. Moreover, there exists a nondegenerate hermitian form \( h(u, v) \) relative to the involution in \( D \) such that \( J \) is the adjoint mapping relative to \( h \). We can have \( D = \Phi \) and \( h(u, v) \) is a symmetric bilinear form only if the type is \( D_t \). We have \( \dim \mathfrak{A} = (2l)^2 \) and

\[
\dim \mathfrak{S}(\mathfrak{A}, J) = l(2l - 1)
\]

if the type is \( D_t \).

If \( (\mathfrak{A}, J) \) is of type \( X_t \), \( X = A, B, C \) or \( D \) then there exists a finite dimensional Galois extension \( P/\Phi \) such that \( (\mathfrak{A}_P, J) \) is an algebra such as described in the list for the algebraically closed case and moreover, if the type is \( B_t \) or
D_1$, then the symmetric bilinear form $(x, y)$ is of maximal Witt index. An extension field having these properties will be called a splitting field for $(\mathfrak{A}, J)$. If $P/\Phi$ is a finite dimensional Galois splitting field, then for each $s$ in the Galois group $G$ of $P/\Phi$ we let $\tau_s$ be the $s$-semilinear mapping in $\mathfrak{A} = \mathfrak{A}_P$, which is the identity on $\mathfrak{A}$ (considered a subset of $\mathfrak{A}$). Then $\tau_s$ is an $s$-semiautomorphism of $(\mathfrak{A}, J)$ (that is, automorphism of $(\mathfrak{A}/\Phi, J)$ which is $s$-semilinear). It is clear that $\tau_s \tau_t = \tau_{st}$ and $\mathfrak{A}$ is the subset of $\mathfrak{A}$ of elements which are fixed under every $\tau_s$.

Conversely, let $(\mathfrak{A}, J)$ be an algebra with involution over $P$ and assume that for each $s$ in the Galois group $G$ we have an $s$-semilinear automorphism $\tau_s$ of $(\mathfrak{A}, J)$ such that $\tau_s \tau_t = \tau_{st}$. Let $\mathfrak{A}$ be the set of fixed points under all the $\tau_s$. Then $\mathfrak{A}$ is an algebra over $\Phi$ which is linearly disjoint with $P/\Phi$ and satisfies $P\mathfrak{A} = \mathfrak{A}$. Also $J$ maps $\mathfrak{A}$ into itself. Hence $(\mathfrak{A}, J)$ is an algebra with involution such that $(\mathfrak{A}_P, J) = (\mathfrak{A}, J)$.

From now on we restrict our attention to the algebras $(\mathfrak{A}, J)$ of type $D_1$ with base field $\Phi$. If $P$ is a finite dimensional Galois splitting field then we have a $2l$ dimensional vector space $M$ over $\Phi$ with a nondegenerate symmetric bilinear form $(x, y)$ of maximal Witt index such that $\mathfrak{A}_P = \text{Hom}_P(\mathfrak{M}, \mathfrak{M})$, $\mathfrak{M} = M_\rho$, and $J$ is the adjoint mapping relative to $(x, y)$ (extended to $\mathfrak{M}$). We remark that if $\mathfrak{M}/P$ is equipped with a nondegenerate bilinear form $f$ and $J$ is the adjoint mapping relative to $f(f(xA, y) = f(x, yA^t)$ then $f$ is determined up to a scalar factor by $J$. Also if $\mathfrak{M}'/P$ is a second vector space with nondegenerate bilinear form $f'$ and $J'$ is the adjoint mapping relative to $f'$ then an isomorphism $\sigma$ of $(\text{Hom}_P(\mathfrak{M}, \mathfrak{M}), J)$ onto $(\text{Hom}_P(\mathfrak{M}', \mathfrak{M}'), J')$ necessarily has the form $X \mapsto S^{-1}XS$ where $S$ is an isometry of $(\mathfrak{M}, \rho, f)$ onto $(\mathfrak{M}', f')$ for a suitable $\rho$ in $P$. In this sense the pair $(\mathfrak{M}, (,.))$ is determined up to equivalence by $\mathfrak{A}_P$. If $\tau_s$ is determined as indicated then there exists an $s$-semisimilarity $T_s$ in $\mathfrak{M}$ such that

$$X^{\tau_s} = T_s^{-1}XT_s \quad (1)$$

for $X \in \mathfrak{M} = \mathfrak{M}_P$. Thus we have

$$(xT_s, yT_s) = \mu_s(x, y)s, \quad (2)$$

$x, y \in \mathfrak{M}, \mu_s \in P$. The transformation $T_s$ is determined up to a multiplier in $P$ and we have

$$T_sT_t = T_{s\rho_t}, \quad (3)$$

where $\rho = \{\rho_{s, t}\}$ is a factor set. The set of endomorphisms of the form

\footnote{See [2, Lemma 2, p. 295].}
\[ \sum_{\xi \in G} T_s \xi_s \cdot \xi_s \in P, \text{ can be identified with the crossed product } (P, G, \rho) \text{ which is central simple over } \Phi. \text{ Writing } Q(x) \text{ for } (x, x), \text{ we have} \]
\[ Q(x_T) = \mu_s Q(x)^s \]
and
\[ \rho_s^2 Q(x_T s) = Q(x_T s \rho_s) = Q(x_T T_s) \]
which gives \[ \rho_s^2 = \mu_{s1} \mu_{s1}' \]. Hence \[ \rho^2 \sim 1 \] and \((P, G, \rho)\) has an antiautomorphism. It is clear from (1) that \( \mathfrak{A} \) is the set of linear transformations in \( \tilde{M}/P \) which commute with the \( T_s \). Hence \( \mathfrak{A} \) is the set of endomorphisms in \( \tilde{M} \) which commute with the endomorphisms of the form \( \sum T_s \xi_s \). Since \((P, G, \rho)\) has an antiautomorphism this implies that
\[ \mathfrak{A} \sim (P, G, \rho), \] (4)
that is, \( \mathfrak{A} \) and \((P, G, \rho)\) give the same element of the Brauer group over the field \( \Phi \).

Let \( T \) be an \( s \)-semisimilarity in \( \tilde{M} \) and let \((e_1, e_2, \ldots, e_{2l})\) be a basis for \( M/\Phi \), hence for \( \tilde{M}/\tilde{P} \). Then
\[ \beta_{ij} = (e_i, e_j) \in \Phi \quad \text{and} \quad (e_i T, e_j T) = \mu(e_i, e_j)^s = \mu(e_i, e_j) \]
gives \((t)(\beta)(t)^s = \mu(\beta)\) where \((t)\) is the matrix of \( T \) relative to \((e_1, e_2, \ldots, e_{2l})\) and the ' denotes the transposed matrix. Taking determinants we obtain
\[ \det(t) = \mu'. \]
We shall call \( T \) proper or improper according as \( \det(t) = \mu \) or \( \det(t) = -\mu \). It is clear that this does not depend on the choice of the basis for \( M \) and we shall see that it is independent also of the choice of the \( \Phi \)-subspace \( M \) such that \( M_\rho = \tilde{M} \). This notion is well known for similarities \((s = 1)\). The semisimilarities form a group and one checks that the proper ones form a subgroup of index two in the group of semisimilarities.

Again let \( \mathfrak{A} \), the \( \tau_s \) and \( T_s \) be as before. If every \( T_s \cdot s \in G \), is proper we shall say that \((\mathfrak{A}, J)\) is of type \( D_{II} \); otherwise of type \( D_{III} \). In the latter case, the subset of \( s \) such that \( T_s \) is proper is a subgroup \( H \) of index two in \( G \). It is easy to see that the notion of type \( D_{II} \) or \( D_{III} \) is independent of the choice of the Galois splitting field \( P \). Since this will be a consequence of another result which we shall establish later we omit the proof at this point.

### II. Clifford Algebras of Forms of Maximal Witt Index

Let \( M \) be a \( 2l \)-dimensional vector space over \( \Phi \), \((x, y)\) a nondegenerate symmetric bilinear form on \( M \) of maximal Witt index, \( Q \) the associated quadratic form \( Q(x) = (x, x) \). Let \( C(M, Q) \) be the Clifford algebra based on \( M \) and \( Q \), \( C^+(M, Q) \) the even Clifford algebra, that is, the subalgebra of even elements of
$C = C(M, Q)$. It is well known that $C(M, Q)$ can be identified with $\text{Hom}_\mathbb{C}(S, S)$ where $S$ (the space of spinors) is a 2$^t$ dimensional vector space over $\Phi$ [I, p. 70]. The vector space $C$ has a natural graded structure whose subspace $M_{[k]}$ of homogeneous elements of degree $k$ is the space spanned by all elements of the form $[x_1, x_2, \ldots, x_k]$ where this is defined inductively by

$$[x_1] = x_1$$
$$[x_1, \ldots, x_{2j-1}, x_{2j}] - \begin{bmatrix} x_1, \ldots, x_{2j-1}, x_{2j} \end{bmatrix}$$
$$[x_1, \ldots, x_{2j}, x_{2j-1}] - \begin{bmatrix} x_1, \ldots, x_{2j}, x_{2j-1} \end{bmatrix}$$

and $[ab] = ab - ba, \{ab\} = ab + ba$ [4, p. 47]. The subspace $M_{[k]}$ can be characterized as the subspace spanned by all products of $k$ mutually orthogonal vectors in $M$. We have $M_{[1]} = M$ and

$$C = \Phi \oplus M_{[1]} \oplus M_{[2]} \oplus \cdots \oplus M_{[2t]}$$
$$C^+ = \Phi \oplus M_{[2]} \oplus M_{[1]} \oplus \cdots \oplus M_{[2t]}$$

and $\dim M_{[k]} = \binom{2^t}{k}$. In particular, $\dim M_{[2t]} = 1$ and this space is spanned by any vector $c = u_1u_2 \cdots u_{2t}$ where $(u_1, u_2, \ldots, u_{2t})$ is an orthogonal basis for $M/\Phi$. We have $c^2 = (-1)^t \beta_1\beta_2 \cdots \beta_{2t}$ if $Q(u_i) = \beta_i$. Since $\beta_i$ has maximal Witt index we may choose the $u_i$ so that $\beta_i = (-1)^i$. Then $c^2 = 1$ and $\Phi[c] = \Phi c_1 \oplus \Phi c_2$ where the $c_i$ are the orthogonal idempotents $\neq 0, 1$ in $\Phi[c]$.

The subalgebra $\Phi[c]$ is the center of $C^+$ and $c$ is determined up to sign by the conditions that $c$ is in this center and $c^2 = 1$. We have $C^+ = C_i(M, Q) \oplus C_2(M, Q)$ where $C_i = C_i(M, Q) = C^+c_i$ is an ideal in $C^+$. It is known that $C_i$ is isomorphic to the algebra of linear transformations in a $2^{t-1}$ dimensional vector space [I, p. 70]. Since $C^+ = C_1 \oplus C_2$ we have $S = S_1 \oplus S_2$ where $S_i = SC_i$. We have $S_iC_j = 0$ if $i \neq j$ and the restriction of $C_i$ to $S_i$ is $\text{Hom}_\Phi(S_i, S_i)$. If $x \in M$ we have $xc = -cx$. Hence if $Q(x) \neq 0$ then the inner automorphism $X \rightarrow x^{-1}ax$ in $C$ maps $C^+$ into itself and exchanges the two components $C_i$ of $C^+$. Hence

$$S_1x = SC_1x = S^{-1}C_1x = S_2.$$

Similarly, $S_2x = S_1$. It follows that

$$S_1x \subseteq S_2, \quad S_2x \subseteq S_1$$

for any $x \in M$. The space $S_i$ is $2^{t-1}$-dimensional.

We recall that there is an involution $\iota$ in $C$ called the main involution which is characterized by $x^\iota = x$ for $x \in M$. If $u_1, u_2, \ldots, u_k$ are orthogonal elements of $M$ then

$$(u_1u_2 \cdots u_k)^\iota = u_ku_{k-1} \cdots u_1 = (-1)^{k(k-1)/2} u_1 \cdots u_k.$$

Hence $\iota$ is the identity in the spaces $M_{[4k]}, M_{[4k+1]}$ and is $-1$ in the spaces
\[ M_{[4k+2]} \), \( M_{[4k+3]} \), \( k = 0, 1, \ldots \). It follows that the dimension of the space of \( \iota \)-symmetric elements of \( C \) is

\[
\begin{align*}
2^{l-1}(2^l - 1) & \quad \text{if} \quad l \equiv 2 \text{ or } 3 \pmod{4} \\
2^{l-1}(2^l + 1) & \quad \text{if} \quad l \equiv 0 \text{ or } 1 \pmod{4}.
\end{align*}
\]

Accordingly, \( \iota \) is an involution of type \( C \) or \( D \) according as \( l \equiv 2, 3 \) or \( 0, 1 \pmod{4} \). In the first case, \( \iota \) is the adjoint mapping relative to a non-degenerate skew bilinear form \( f \) on \( S \) and in the second it is the adjoint relative to a nondegenerate symmetric bilinear form \( f \) on \( S \).\(^2\) In either case \( f(ux, v) = f(u, xv) \) if \( u, v \in S, x \in M \) since \( x^4 = x \).

We have \( c^l = c \) if \( l \) is odd and \( c^l = -c \) if \( l \) is even. Hence in the first case \( \iota \) exchanges the two components \( C_i \) of \( C^+ \) and \( (C^+, \iota) \) is of type \( A \). In the second case, \( C_i^+ = C_i \). This implies that the spaces \( S_i \) are orthogonal relative to \( f \) and the restriction \( (\iota)_i \) of \( f \) to \( S_i \) is nondegenerate. The involution \( \iota \) in \( C_i \) is the adjoint mapping relative to \( (\cdot) \) so \( (C_i, \iota) \) is of type \( D \) or \( C \) according as \( l \equiv 0 \) or \( 2 \pmod{4} \). In either case we introduce a trilinear mapping of the space \( M, S_1 \) and \( S_2 \) into \( \Phi \) by

\[
(x, x_1, x_2) = (x_1, x, x_2) = (x_1, x_2, x_1^\iota) \quad (10)
\]

for \( x \in M, x_i \in S_i \).

III. EVEN CLIFFORD ALGEBRA FOR AN ALGEBRA WITH INVOLUTION OF TYPE D

We now tensor all the spaces and algebras we considered in the last section with a finite dimensional Galois extension field \( P/\Phi \) with Galois group \( G \) and we observe that the expected commutativities hold. For example, we can identify \( \mathcal{C} = C(M, Q), M = M_P \) with \( C_P, C_P^+ \), \( C_i \) with \( C_i^P \).

Let \( T \) be an \( s \)-semisimilarity in \( M \). Then it is known that \( T \) determines a unique \( s \)-semiautomorphism \( \eta(T) \) of \( (\mathcal{C}^+, \iota) \) which is characterized by

\[
(x_1x_2 \cdots x_{2^l})^\iota = \mu^{-k}(x_1T)(x_2T) \cdots (x_{2^l}T) \quad (11)
\]

where \( \mu \) is the multiplier of \( T \) [4, p. 48]. Replacement of \( T \) by a multiple \( Ty, y \) in \( P \), gives the same semiautomorphism \( \eta \) in \( \mathcal{C}^+ \). If \( c \) is chosen in \( M_{[2]} \) as before then \( c^\iota = c \) or \( -c \) according as \( T \) is proper or improper. Since \( \pm c \) are the only elements of the center of \( \mathcal{C}^+ \) satisfying \( x^2 = 1 \) this gives a characterization of the condition: \( T \) a proper semisimilarity, independent of the choice of the \( \Phi \)-subspace \( M \). If \( T \) is proper we have \( C_i^\eta \subseteq C_i \).

If in addition \( l \) is even then the restriction \( \eta^{(i)} \) of \( \eta \) to \( C_i \) is an \( s \)-semiautomorphism of \( (C_i, \iota) \). It follows that \( \eta^{(i)} \) has the form \( X_i \rightarrow T_i^{-1}X_iT_i \) where \( T_i \) is an \( s \)-semisimilarity in \( S_i \) relative to \( (\cdot) \).

\(^2\) It can seen that \( f \) has maximum Witt index.
Now suppose $(\mathfrak{A}, J)$ is an algebra with involution of type $D$ and $P$ is a Galois splitting field, so that $\mathfrak{A} = \mathfrak{A}_P = \text{Hom}_P(\tilde{M}, \tilde{M})$ and $J$ in $\mathfrak{A}$ is the adjoint mapping relative to $(,)$. For each $s \in G$ let $\tau_s$ and $T_s$ be as in Section I. Then (3) and (4) hold. Let $\eta_s$ be the $s$-semiautomorphism in $(\tilde{C}, i)$ determined by $T_s$ as above. Then $\eta_s$ depends only on $\tau_s$, hence only on $(\mathfrak{A}, J)$ and the splitting field $P$. Also (3) implies that $\eta_s\eta_t = \eta_{st}$. Now let $\mathfrak{C}(\mathfrak{A}, J)$ be the subset of $\tilde{C}^+$ of elements which are fixed under all the $\eta_s$. Then $\mathfrak{C}(\mathfrak{A}, J)$ is a $\Phi$-subalgebra of $\tilde{C}^+$ which is linearly disjoint to $P/\Phi$ and satisfies $P\mathfrak{C} = \tilde{C}^+$, so $\mathfrak{C}_P = \tilde{C}^+$. Since $\eta_s$ is a semiautomorphism of $(\tilde{C}, i)$, $i$ maps $\mathfrak{C}(\mathfrak{A}, J)$ into itself so its restriction to $\mathfrak{C}(\mathfrak{A}, J)$ is an involution which we shall call the main involution in $\mathfrak{C}$. It is clear also that each $\eta_s$ maps the space $\tilde{M}_{[2k]}$ into itself. Hence

$$\mathfrak{M}_{[2k]} = \tilde{M}_{[2k]} \cap \mathfrak{C}$$

is a $\Phi$ subspace of $\tilde{M}_{[2k]}$ such that $\mathfrak{M}_{[2k]}P = \tilde{M}_{[2k]}$. We have

$$\mathfrak{C}^+ = \Phi \oplus \mathfrak{M}_{[2]} \oplus \cdots \oplus \mathfrak{M}_{[2k]}.$$

We shall now show that all of this is independent of the splitting field $P$ and of the choice of $\tilde{M}$ and $(,)$. We note first that if $\tilde{M}'$ is another vector space over $P$ with nondegenerate symmetric bilinear form $(,)'$ and $S$ is a 1-1 linear mapping of $\tilde{M}$ onto $\tilde{M}'$ such that $(xS, yS)' = \rho(x, y)$ then the argument establishing the existence of $\eta$ satisfying (11) shows that there exists an isomorphism $\tilde{C}^+$ onto the even Clifford algebra defined by $(,)'$ which maps $x_1 \cdots x_{2k}$ into $\rho^{-1}(x_1S)(x_2S) \cdots (x_{2k}S)$. It follows from the remark on the determination of $\tilde{M}$, $(,)$ by $\mathfrak{A}_P$ made in Sec. I that $\mathfrak{C}(\mathfrak{A}, J)$, $i$ and the space $\tilde{M}_{[2k]}$ are determined up to equivalence by $(\mathfrak{A}, J)$ and $P$. We consider next the invariance under change of the Galois splitting field $P$. Since any two Galois splitting fields are contained in a single one it is enough to show invariance under extension of $P$ to a finite dimensional Galois field $\Sigma / \Phi$. We have $\tilde{M}_\Sigma = (M_P)_\Sigma = M_\Sigma$, $\tilde{M}_\Sigma = \mathfrak{M}_\Sigma$, $\tilde{C}_\Sigma = C_\Sigma$, etc. We observe also that an $s$-semilinear transformation in $\tilde{M}$ can be extended in exactly $(\Sigma : P)$ ways to a semilinear transformation in $M_\Sigma$. These correspond to the extensions of $s$ to automorphisms of $\Sigma / \Phi$. The same remark applies to the other spaces. The extensions obtained in this way from the $\tau_s$, $s \in G$, are the semilinear automorphisms of $(\mathfrak{M}_\Sigma, J)$ associated with the elements of the Galois groups $K$ of $\Sigma / \Phi$ and $\mathfrak{A}$ is the set of fixed points under all of these. If $u \in K$ is an extension of $s \in G$ and $\tau_u$ is the associated $u$-semilinear mapping of $\mathfrak{M}_\Sigma$ then $\tau_u$ has the form $X \mapsto T_u^{-1}XT_u$ where $T_u$ is a semilinear extension of $T_s$. It follows that $\eta_u$ is a $u$-semilinear extension of $\eta_s$. Then the set of $\eta_u$, $u \in K$, is the set of semilinear extensions to $C_\Sigma$ of the $\eta_s$ in $C_P$. It follows that the algebra of fixed points relative to these is $\mathfrak{C}(\mathfrak{A}, J)$. This shows that $\mathfrak{C}(\mathfrak{A}, J)$ is independent of $P$. In a similar fashion one sees that $i$ and the subspaces $\mathfrak{M}_{[2k]}$ are independent of $P$. 

Next suppose \( \mathfrak{A} \approx 1 \). This is equivalent to assuming that \( \mathfrak{A} = \text{Hom}_\Phi(N, N) \) where \( N \) is \( 2l \)-dimensional over \( \Phi \) and \( J \) is the adjoint relative to a non-degenerate symmetric bilinear form in \( N \). If \( R \) is the corresponding quadratic form then we can construct the Clifford algebras \( C(N, R) \) and \( C^+(N, R) \). We proceed to show that \( C^+(N, R) \) is isomorphic to the algebra \( C^+(\mathfrak{A}, J) \) which we constructed. Since \( \mathfrak{A} \approx 1 \) we have \( \rho \approx 1 \) which means that we may suppose \( T_s T_t = T_{st} \). Then if \( Q(x T_s) = \mu_s Q(x)^s \) we have \( \mu_s \mu_t = \mu_{st} \) so \( \mu_s = \nu^{-1}s^s \), \( \nu \in \Phi \). We can replace \( (, ) \) in \( \mathcal{M} \) by \( v,(, ) \). Then, changing notation, we have \( Q(x T_s) = Q(x)^s \). Let \( N \) be the \( \Phi \)-subspace of \( \mathcal{M} \) of \( T_s \)-fixed elements for all \( s \in G \). Then \( \mathcal{M} = P N = N_p \) and if \( x \in N, Q(x) = Q(x)^s \). Hence the restrictions of \( Q \) and \( (, ) \) to \( N \) have values in \( \Phi \). It follows that \( (, ) \) is \( \text{Hom}_\Phi(N, N) \) and \( J \) is the adjoint relative to the form \( (, ) \) on \( N \). Hence we may assume these are as given at the outset and we have to show that \( C^+(\mathfrak{A}, J) \) is the same as \( C(N, R) \). Now \( T_s \) extends in a unique fashion to an \( s \)-semilinear automorphism \( \eta_s \) in \( \mathcal{C} \) and in \( \mathcal{C}^+ \), and it is clear that \( C(N, R) \) and \( C^-(N, R) \) can be identified with the \( \Phi \)-subalgebras of \( \mathcal{C} \) and \( \mathcal{C}^+ \) of fixed points under all the \( \eta_s \), \( s \in G \). On the other hand, the definition of \( \eta_s \) shows that \( \eta_s^2 = \eta_s \) on \( \mathcal{C}^+ \). Hence \( C^+(N, R) = C^+(\mathfrak{A}, J) \) by the definition of the latter algebra. This proves

**Theorem 1.** If \( \mathfrak{A} = \text{Hom}_\Phi(N, N) \), \( \dim N = 2l \), and \( J \) is the adjoint mapping relative to a non-degenerate symmetric bilinear form with associated quadratic form \( R \) then the even Clifford algebra \( C^+(\mathfrak{A}, J) \) is isomorphic to the even Clifford algebra \( C^+(N, R) \).

We now sort out the various possibilities for the structure of \( C^+(\mathfrak{A}, J) \) and its main involution \( \iota \). It is clear from the structure of \( \mathcal{C}^+ \) and \( \mathcal{C}^+ = \mathcal{C}^+ \) that \( \mathcal{C}^+ \) is either simple with center a quadratic extension of \( \Phi \) or it is a direct sum of two ideals. The second is the case if and only if every \( \eta_s \) is the identity on the center \( P[e] \) of \( \mathcal{C}^+ \). We have seen that this is so if and only if every \( T_s \) is proper. Hence we have

**Theorem 2.** The even Clifford algebra \( C^+(\mathfrak{A}, J) \) of the algebra with involution \( (\mathfrak{A}, J) \) of type \( D_l \) is a direct sum of two simple ideals or is simple according as the type is \( D_{II} \) or \( D_{III} \).

This, of course, implies that the type is independent of the Galois splitting field \( P/\Phi \).

It is well known that \( C^+(N, R) \) is a direct sum of two simple ideals or is simple according as \((-1)^l \delta, \delta \) the discriminant of \( R \), is a square or not a square in \( \Phi \). Hence Theorems 1 and 2 give the

**Corollary.** If \( \mathfrak{A} = \text{Hom}_\Phi(N, N) \), \( \dim N = 2l \), \( J \) the adjoint relative to a symmetric nondegenerate bilinear form with quadratic form \( R \) then \( (\mathfrak{A}, J) \) is of
type $D_{1/2}$ or type $D_{1/2}$ according as $(-1)^l \delta$, $\delta$ the discriminant of $R$, is a square or not a square in $\Phi$.

If we take into account the structure of $C^+(Q, f)$ and the action of $i$ in $\tilde{C}$, we can see that we have the following four cases: (1) type $D_l$, $l$ odd. Then $(C^+, i)$ is central simple of type $A_l$. (2) type $D_l$, $l$ even. Here $C^+ = \oplus_{i} C_i$, where the $C_i$ are central simple and $i$ maps $C_i$ into itself. Also $(C_i, i)$ is of type $D$ or $C$ according as $k = l/2$ is even or odd. (3) type $D_{1/2}$, $l$ odd. Here $(C^+, i)$ is of type $A_{1/2}$. (4) type $D_{1/2}$, $l$ even. Here $C^+$ is simple with center $\Phi$ and $i$ is involution of type $D$ or $C$ according as $k = l/2$ is even or odd in $\Phi/P$.

IV. THE CASE $l$ EVEN AND TYPE $D_l$

Again let $M$ be $2l$-dimensional over $\Phi$, $(,)$ a nondegenerate symmetric bilinear form of maximal Witt index, $Q$ the associated quadratic form, $C^+(M, Q)$ the even Clifford algebra of $M$ and $Q$. We have $C^+_i = C_1 \oplus C_2$, $C_i$ simple ideals. We assume that $l = 2k$ is even. Then $C_i \subseteq C_i$, $i = 1, 2$. We have seen that if $S$ is the space of spinors and $S_i = SC_i$ then $C_i = \text{Hom}_Q (S_i, S_i)$ and $i$ is the adjoint mapping relative to a nondegenerate symmetric or skew bilinear form $(,)$ in $S_i$. Finally, we have the trilinear mapping of $M, S_1$ and $S_2$ satisfying (10). If $P/\Phi$ is a finite dimensional Galois extension field with Galois group $G$ then we obtain $\tilde{M} = M_P, \tilde{C}^+ = \oplus_i C^+_i = C^+(\tilde{M}, Q)$, $\tilde{C}_i$, etc. Moreover, if $T$ is an s-semisimilarity of $\tilde{M}$ relative to $(,)$ (extended to $\tilde{M}$) then this determines an s-semiautomorphism $\eta$ of $\tilde{C}^+$ and if $T$ is proper then $\tilde{C}^+_i \subseteq \tilde{C}_i$, Then the restriction $\eta^{(i)}$ of $\eta$ to $\tilde{C}_i$ has the form

$$X_i \rightarrow T_i^{-1}X_iT_i$$

where $T_i$ is an s-semisimilarity in $\tilde{S}_i$ relative to $(,)_i$. We shall call the s-semisimilarities $(T, T_1, T_2)$ in $\tilde{M}, \tilde{S}_1$ and $\tilde{S}_2$ related if

$$(xT, x_1T_1, x_2T_2) = (x, x_1, x_2)^{\lambda}$$

for all $x \in \tilde{M}, x_i \in \tilde{S}_i$ where $\lambda$ is a fixed nonzero element in $P$. We shall now prove

**Theorem 3.** The s-semisimilarity $T$ and the $T_i$ determined up to multipliers in $P$ by the semiautomorphisms $\eta^{(i)}$ in $\tilde{C}_i$ are related. Moreover, if $(T, T_1, T_2)$ are related then $T'_i$ is a scalar multiple of $T_i$, $i = 1, 2$.

**Proof.** We note first that if $(T, T_1, T_2)$ and $(U, U_1, U_2)$ are related then so are $(TU, T_1U_1, T_2U_2)$ and $(T^{-1}, T_1^{-1}, T_2^{-1})$. Next let $U$ be the s-semisimilarity in $\tilde{M}$ which is the identity on $M$. Then it is clear that the semi-
automorphism determined by $U$ in $\mathcal{C}_i$ has the form $X_i \rightarrow U_i^{-1}X_iU_i$ where $U_i$ is the $s$-semisimilarity in $\mathcal{S}_i$ which is the identity on $S_i$. We have

$$(xU, x_1U_1, x_2U_2) = (x, x_1, x_2)^s$$

so $(U, U_1, U_2)$ are related. It is now clear that it is enough to prove the first statement of the theorem for $T$ a proper similarity. We consider next the uniqueness question. Thus let $(T, T'_1, T'_2)$ be related. Then from (12) and (10) we obtain

$$(x_1T'_1, (x_2T'_2)(xT)) = \lambda(x_1, x_2x'_1)^s.$$  

If

$$(x_1T'_1, y_1T'_1) = \mu_1(x_1, y_1)^s,$$

then this leads to

$$(\lambda s^{-1}(x_2x)) T'_1 = \mu_1 s^{-1}(x_2T'_2)(xT)$$

(13)

Similarly, we have

$$(\lambda^{-1}(x_1x)) T'_2 = \mu_2^{-1}(x_1T'_1)(xT)$$

(14)

if $\mu_2$ is the multiplier of $T'_2$. If we combine (13) and (14) we obtain

$$(x_1T'_1)((xT)(yT)) \mu^{-1} = \gamma(x_1(xy)) T'_1$$

(15)

where $\gamma$ is the multiplier of $T$ and $\gamma \in P$.

This implies that

$$X_1^{(11)} = \gamma (T'_1)^{-1} X_1 T'_1$$

(16)

if $\eta^{(1)}$ is the $s$-semiautomorphism in $\mathcal{C}_1$ defined by $T$. Since $X_1^{(11)} = T_1 T_1^{-1} X_1 T_1$ it follows from this that $\gamma = 1$ and $T'_1$ is a scalar multiple of $T_1$. Similarly, $T'_2$ is a scalar multiple of $T_2$. Now let $O$ be a proper orthogonal transformation in $\tilde{\mathcal{M}}$ (relative to $Q$). Then $O$ determines a unique automorphism $\sigma$ of $(\mathcal{C}_i, i)$ such that $x^\sigma = xO$, $x \in \tilde{\mathcal{M}}$. It follows that there exists a similarity $O'$ in $\tilde{\mathcal{S}}$ such that $X^{\sigma} = (O')^{-1}XO'$ for $X \in \tilde{\mathcal{S}}$. Also since $O$ is proper $c^{\sigma} = c$ and $\mathcal{C}^{\sigma} = \mathcal{C}_i$. It follows that $O'$ maps $\tilde{\mathcal{S}}_1$ into itself so its restriction $O_i$ to $\tilde{\mathcal{S}}_i$ is a similarity relative to $(,)_i$. If $x_1 \in \tilde{\mathcal{S}}_1$ and $x \in \tilde{\mathcal{M}}$ then $((x_1(O')^{-1}x)O' = x_1 x^{\sigma}$ and the fact that $\tilde{S}_1 x \subseteq \tilde{S}_2$ and $\tilde{S}_2 x \subseteq \tilde{S}_1$ give the relation

$$((x_1O_1^{-1}) x)O_2 = x_1(xO).$$

(17)

Hence if $x_2 \in \tilde{S}_2$ then

$$(((x_1O_1^{-1}) x)O_2, x_2)_2 = (x_1(xO), x_2)_2$$

so

$$(((x_1O_1^{-1}) xO^{-1}), x_2O_2^{-1})_2 = \mu(x_1x, x_2)_2$$
and
\[(xO^{-1}, x_1O_1^{-1}, x_2O_2^{-1}) = \mu(x, x_1, x_2).\]
It is clear from this that if \(O\) is any proper orthogonal transformation then there exist similarities \(O_i\) so that \((O, O_1, O_2)\) are related. Now let \(T\) be a proper similarity in \(\tilde{M}\) and let \(T_i, i = 1, 2\), be the similarity in \(\tilde{S}_i\) determined by the automorphism \(\gamma^{(i)}\) in \(\tilde{C}_i\). We extend the base field \(P\) to the algebraic closure \(\Omega\) and make the corresponding extension of the spaces and forms. Now we have a \(\gamma \in \Omega\) such that \(O = \gamma T\) is proper orthogonal. Hence there exist similarities \(O_i\) in \(S_{i\Omega}\) such that \((xO, x_1O_1, x_2O_2) = \rho(x, x_1, x_2), x \in \tilde{M}_{i\Omega}, x_i \in \tilde{S}_{i\Omega}\). Also the uniqueness result we established shows that \(O, O_1, O_2\) are multiples in \(\Omega\) of \(T, T_1, T_2\) respectively. Hence we have
\[(xT, x_1T_1, x_2T_2) = \lambda(x, x_1, x_2)\]
for \(x \in \tilde{M}, x_i \in \tilde{S}_i\) where \(\lambda\) is a fixed element in \(\Omega\). Since \((xT, x_1T_1, x_2T_2)\) and \((x, x_1, x_2) \in P\), and \((x, x_1, x_2) \neq O\), we have \(\lambda \in P\) and this completes the proof.

**Remark.** It is easy to see—for example, by using the fact that the rotation group in the algebraically closed case has no subgroup of index two—that the \(T_i\) are necessarily proper.

We can now prove

**Theorem 4.** Let \((\mathfrak{A}, J)\) be an algebra with involution of type \(D_1\) where \(\dim \mathfrak{A} = (2l)^2\) and \(l\) is even and let \(\mathfrak{C}_1\) and \(\mathfrak{C}_2\) be the simple components of the even Clifford algebra \(\mathfrak{C}^+(\mathfrak{A}, J)\). Then \(\mathfrak{A} \otimes \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim \mathfrak{I}\).

**Proof.** Let \(P\) be a finite dimensional Galois splitting field and let the \(\tau_s, T_s\) and \(\eta_s, s \in G\) the Galois group, be as in Section III. The restriction \(\gamma_{s(i)}\) of \(\eta_s\) to \(\tilde{C}_i\) has the form \(X \rightarrow (T_{s(i)}^{-1})^{-1} X T_i^{(i)}\) where \(T_i^{(i)}\) is an \(s\)-semi-similarity and
\[(xT_s, x_1T_s^{(1)}, x_2T_s^{(2)}) = \lambda_s(x, x_1, x_2)^s \tag{18}\]
for \(x \in \tilde{M}, x_i \in \tilde{S}_i\). Suppose
\[T_s T_t = \rho_{s,t}T_{st}, \quad T_{s(i)}^{(i)} T_{t(i)}^{(i)} = \rho_{s,t(i)}^{(i)} T_{st(i)}^{(i)}. \tag{19}\]
Then \(\mathfrak{A} \sim (P, G, \rho)\) and \(\mathfrak{C}_1 \sim (P, G, \rho^{(1)})\) and so
\[\mathfrak{A} \otimes \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim (P, G, \rho) \otimes (P, G, \rho^{(1)}) \otimes (P, G, \rho^{(2)}).\]
Now it follows from (18) that
\[\rho_{s, i} \rho_{s,t}^{(1)} \rho_{s,t}^{(2)} = \lambda_t \lambda_i \lambda_{st}^{-1}. \tag{20}\]
Hence \(\rho \times \rho^{(1)} \times \rho^{(2)} \sim \mathfrak{I}\) and consequently \(\mathfrak{A} \otimes \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim \mathfrak{I}\).
If \((\mathfrak{A}, J)\) is an algebra with involution of type \(D\) then the element \(u \in \mathfrak{A}\) is called a \(J\)-similarity (\(J\)-unitary) if \(uu^J = \beta I \neq 0\) (\(uu^J = 1\)). If \(P/\Phi\) is a Galois splitting field and \(\hat{M}, \tau_s, T_s\) are as before then \(u\) is a \(J\)-similarity (\(J\)-unitary) if and only if \(u\) is a similarity (orthogonal) transformation in \(\hat{M}/P\) relative to \((\cdot, \cdot)\) and \(T_s^{-1}uT_s = u\), \(s \in G\). We shall call \(u\) proper if \(u\) is a proper similarity in \(\hat{M}/P\). It is clear that the \(J\)-similarities (\(J\)-unitary elements) form a group under multiplication and the proper ones form a subgroup of index \(\leq 2\) in the group of \(J\)-similarities (\(J\)-unitary elements).

Now let \(u\) be a proper \(J\)-similarity and let \(\zeta\) be the automorphism in \((\mathcal{C}^+, \iota)\) determined by \(u\) as a proper similarity of \(\hat{M}\). Thus if \(x, y\) are orthogonal elements of \(\hat{M}\) then \((xy)^\zeta = v^{-1}(xu)(yu)\) where \(v\) is the multiplier of \(u\). Then the condition \(T_s^{-1}uT_s = u\) implies that \(\zeta \eta_s = \eta_s \zeta\) if \(\eta_s\) is the \(s\)-semisimilarity in \((\mathcal{C}^+, \iota)\) determined by \(T_s\). It follows that \(\zeta\) maps the even Clifford algebra \(\mathcal{C}^+(\mathfrak{A}, J)\) of \((\mathfrak{A}, J)\) into itself. Hence the restriction of \(\zeta\) to \(\mathcal{C}^+(\mathfrak{A}, J)\) is an automorphism of \((\mathcal{C}^+, \iota)\). Since \(u\) is proper it follows also that \(\zeta\) is the identity on the center of \(\mathcal{C}^+\). Hence this is an inner automorphism of \(\mathcal{C}^+\) and consequently there exists a \(v \in \mathcal{C}^+\) such that \(\zeta^v = v^{-1}zv\), \(z \in \mathcal{C}^+\). Since \(\zeta = \iota^z\) we have \(vv^v = \gamma I, \gamma \neq 0\) in the center of \(\mathcal{C}^+\).

Conversely, let \(v\) be a regular element of \(\mathcal{C}^+\) such that \(v^{-1}\mathfrak{M}_{[2]}v \subseteq \mathfrak{M}_{[2]}\). Since \(\hat{M}_{[2]} = P\mathfrak{M}_{[2]}\) it follows that \(v^{-1}\hat{M}_{[2]}v = \hat{M}_{[2]}\). Then it has been shown by Wonenburger ([4], p. 53) that there exists a similarity \(u\) in \(\hat{M}\) whose associated automorphism \(\zeta\) in \(\hat{C}^+\) is the mapping \(z \rightarrow v^{-1}zv\). Moreover, if \(l > 1\) then \(u\) is determined up to a multiplier in \(P\). Since \(v \in \mathcal{C}^+\) it follows that \(T_s^{-1}uT_s = \delta_s u\) where \(\delta_s \in P\). Then \(\delta_s^\zeta = \delta_s\) and this implies that we can replace \(u\) by a multiple and after a change of notation we may suppose that \(T_s^{-1}uT_s = u\). Hence \(u \in \mathfrak{A}\) and this element is a \(J\)-similarity. Finally, since the automorphism \(\zeta\) leaves fixed the elements of the center of \(\hat{C}^+\) it is clear that \(u\) is proper.

These considerations suggest the following

**Definition.** Let \((\mathfrak{A}, J)\) be an algebra with involution of type \(D\), \(\mathcal{C}^+(\mathfrak{A}, J)\) the associated even Clifford algebra. Then multiplicative group of regular elements \(v\) of \(\mathcal{C}^+\) such that \(v^{-1}\mathfrak{M}_{[2]}v \subseteq \mathfrak{M}_{[2]}\) is called the **even Clifford group** \(\theta^+\) of \((\mathfrak{A}, J)\).

We have seen that if \(v \in \theta^+\) then \(vv^v\) is in the center of \(\mathcal{C}^+\). It follows that the mapping \(v \rightarrow vv^v\) is a homomorphism into a commutative group. The kernel of this mapping will be called the **reduced even Clifford group** of \((\mathfrak{A}, J)\). We hope to study the Clifford groups in a later paper.
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