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Parallelogram Polyominoes and Corners

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We give an equation satisfied by the generating function for parallelogram polyominoes according to the area, the width and the number of left path corners. Next, we give an explicit formula for the generating function of these polyominoes according to the area, the width and the number of right and left path corners.

1. Introduction

In the Cartesian plane $\mathbb{N} \times \mathbb{N}$, a polyomino is a finite connected union of elementary cells (unit squares) without a cut point and defined up to translation. Studied for a long time in combinatorics, they also appear in statistical physics. Usually, physicists consider equivalent objects which are named animals, obtained by taking the center of the cells of a polyomino (see Dhar, 1988; Hakim and Nadal, 1983).

Several parameters are defined for a polyomino (or an animal). The area is the number of elementary cells, the width (respectively height) is the number of columns (respectively rows) of the polyomino, the (bond) perimeter is the length of the perimeter and the site perimeter is the number of cells of the outside along the boundary. No exact formula is known for the general case but many results exist concerning certain classes of polyominoes. Surveys can be found in Delest (1991), Guttmann (1992) and Viennot (1992). A polyomino is called column-convex (respectively row-convex) if all its columns (respectively rows) are connected. A convex polyomino is both row- and column-convex. The parallelogram polyominoes are a particular case of this family. They are defined by a pair of paths only made with north and east steps and such that the paths are disjoint, except at their common ending points (see Figure 1). The path beginning by a north (respectively east) step is called left (respectively right) path.

These polyominoes have been enumerated for the first time according to the area by Polya (1969) and by Gessel (1980). In order to get the results of Section 3, we use a bijection between Dyck words of length $2n$ and parallelogram polyominoes of perimeter $2n + 2$, found by Delest and Viennot (1984). There are also many other bijections. Let us quote Viennot's one between parallelogram polyominoes of perimeter $2n + 2$ and bi-coloured Motzkin words of length $n + 1$ (see Delest and Viennot, 1984). There is also a bijection between parallelogram polyominoes of perimeter $2n$ and Motzkin words according to a simple criterion (see Dubernard, 1993). Delest, Gouyou–Beauchamps and

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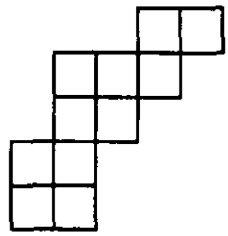

Figure 1. A parallelogram polyomino.

Vauquelin (1987) have counted the parallelogram polyominoes according to the bond and site perimeters. In 1989, Delest and Fédou have shown that the generating function of parallelogram polyominoes according to the area and the width could be written using the quotient of a q-analog of the Bessel functions J_0 and J_1 (see Delest and Fédou, 1989). They have also proved that the study of this generating function could be reduced to the study of the following recurrence

$$\begin{aligned} \beta_1 &= 1 \\ \beta_2 &= 1 \\ \beta_{n+1} &= (1+q^n)\frac{\beta_n}{[n+1]}\frac{\lambda_{n+1}}{\lambda_n} + \sum k = 2^{n-1}\frac{\lambda_{n+1}}{[n+1]\lambda_k\lambda_{n-k+1}}\beta_k\beta_{n-k+1}q^k \end{aligned}$$

where $\lambda_n = \prod_{i=1}^{n}[i]^{\lfloor \frac{n}{i} \rfloor}$ and $[i] = \sum_{k=0}^{i-1} q^k$.

On the other hand, Delest, Gouyou–Beauchamps and Vauquelin (1987) have shown that the site perimeter can be deduced from the bond perimeter using another parameter, the number of corners. A corner is a double step east–north (respectively north–east) on the left (respectively right) path. Thus, the site perimeter is equal to the difference between the bond perimeter and the number of corners. The number of corners is thus also an important statistic.

Using objects grammars (Dutour and Fédou, 1994) and a result of Bousquet-Mélou (1993), we give an explicit formula for a generalization of the generating function of parallelogram polyominoes. Note that, at the same time and independently, Fédou and Rouillon (1994) have found another expression for this generating function, using a method based on a bijection between certain paths of the Cartesian plane. In the second section of this article, we refine the result by Delest and Fédou, introducing the parameter corner. In fact, the properties of the last paragraph were studied before obtaining Theorem 2.3 which seems more general.

2. A New Generating Function for Parallelogram polyominoes

In this paragraph, we use "objects grammars", developed by Dutour and Fédou (1994) with the aim to give a recursive description of the objects from which one can deduce a functional equation satisfied by the generating function. For example, a column-convex polyomino can be obtained by successively "gluing" columns, in a certain way which depends on the studied polyominoes class. A particular case of this method has been frequently used in statistical physics and is called "Temperley methodology".

Bousquet-Mélou (1993) uses this description of column-convex polyominoes from which she obtains functional equations. Whatever the column-convex polyominoes class is, these

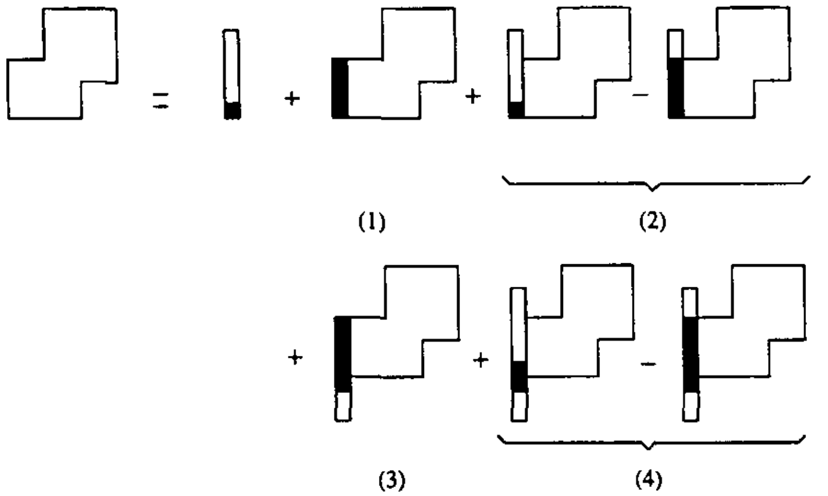

Figure 2. Decomposition of parallelogram polyominoes.

equations are of the same type. She proves a lemma that solves systematically such q-equations. We give below an outline of the result.

Let $\mathcal{R} = \mathbb{R}[\![s,t,x,y,y';q]\!]$ be the algebra of formal power series in the variables s,t,x,y,y',q with real coefficients and $\mathcal{A}$ a sub-algebra of $\mathcal{R}$ such that the series are convergent for $s=1$. If $X(s,t,x,y,y';q)$ is such a series, we will often denote it $X(s)$.

In the application to column-convex polyominoes (Bousquet-Mélou, 1993), the different ways of gluing a new column to a directed polyomino imply that we have to deal with affine equations expressing $X(s)$ in terms of $X(sq)$ and $X(1)$. The function $X(sq)$ appears when the first column of the polyomino is duplicated; the number of cells of the first column is added to the area of the polyomino (see for example, case (1) of Figure 2). The function $X(1)$ appears when we duplicate only the lowest cell of the first column; the height of the first column becomes equal to 1 (see for example, case (2) of Figure 2).

This type of equations can be solved using the following lemma proved in Bousquet-Mélou (1993).

LEMMA 2.1. *Let $X(s,t,x,y,y';q)$ be a formal power series in $\mathcal{A}$. Suppose that:*

$$X(s) = te(s) + tf(s)X(1) + tg(s)X(sq),$$

where $e(s)$, $f(s)$ and $g(s)$ are some given power series in $\mathcal{A}$. Then

$$X(s) = \frac{E(s) + E(1)F(s) - E(s)F(1)}{1 - F(1)},$$

where

$$E(s) = \sum_{n\geq 0} t^{n+1} g(s)g(sq)\dots g(sq^{n-1})e(sq^n),$$

and

$$F(s) = \sum_{n\geq 0} t^{n+1} g(s)g(sq)\dots g(sq^{n-1})f(sq^n).$$

In particular:

$$X(1) = \frac{E(1)}{1 - F(1)}.$$

Let P be a parallelogram polyomino. Its left-height is the height of its leftmost column. We denote:

— its left-height by $l(P)$,
— its width (respectively height) by $w(P)$ (respectively $h(P)$),
— its area by $a(P)$,
— its number of left (respectively right) path corners by $n_1(P)$ (respectively $n_2(P)$).

Let $\mathcal{P}$ be the set of parallelogram polyominoes. Its generating function is the following formal power series:

$$\sum_{P \in \mathcal{P}} s^{l(P)} t^{w(P)} x^{h(P)} y^{n_1(P)} y'^{n_2(P)} q^{a(P)}.$$

Let $P(s, t, x, y, y'; q)$ be the generating function of parallelogram polyominoes. We must study two cases. The first is the case of parallelogram polyominoes of width 1 which are enumerated by $tsxq/(1 - sxq)$. The second is the case of parallelogram polyominoes of width > 1 which are obtained by gluing a new column to parallelogram polyominoes of width ≥ 1.

We denote t_1 (respectively b_1) the ordinate of the top (respectively bottom) of the first column and t_n (respectively b_n) the top (respectively bottom) of the new column.

For parallelogram polyominoes of width > 1, we consider four different cases in the process of gluing the new first column (see Figure 2):

1 $t_n = t_1$ and $b_n = b_1$, then no corner is created,
2 $t_n < t_1$ and $b_n = b_1$, then only one left path corner is created,
3 $t_n = t_1$ and $b_n < b_1$, then only one right path corner is created,
4 $t_n < t_1$ and $b_n < b_1$ then one left path corner and one right path corner are created.

Thus, we obtain an object grammar for parallelogram polyominoes (Figure 2) from which we can directly deduce an equation satisfied by their generating function.

LEMMA 2.2. *The generating function $P(s, t, x, y, y'; q)$ for parallelogram polyominoes satisfies the functional equation:*

$$\begin{aligned} P(s) = {} & \frac{tsxq}{1 - sxq} + tP(sq) + y\frac{t}{1 - sq}(sqP(1) - P(sq)) \\ & + y'\frac{tsxq}{1 - sxq}P(sq) + yy'\frac{tsxq}{(1 - sq)(1 - sxq)}(sqP(1) - P(sq)). \end{aligned}$$

Using Lemma 2.1, we get:

THEOREM 2.3. *The generating function $P(s, t, x, y, y'; q)$ for parallelogram polyominoes is given by*

$$P(s, t, x, y, y'; q) = x\frac{J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)}{J_0(1)}$$

where

$$J_0(s) = 1 - ys \sum_{n\geq 1} \frac{t^n q^n ((1-y')sxq)_n \prod_{i=1}^{n-1} (1-y-sq^i)}{(sq)_n (sxq)_n}$$

and

$$J_1(s) = s \sum_{n\geq 1} \frac{t^n q^n ((1-y')sxq)_{n-1} \prod_{i=1}^{n-1} (1-y-sq^i)}{(sq)_{n-1} (sxq)_n}.$$

If we substitute s by 1, we obtain from Theorem 2.3

$$P(1, t, x, y, y'; q) = x \frac{J_1(1)}{J_0(1)}.$$

REMARK 2.4. *In this formula, the symmetrical role of y and y' does not become apparent.*

An expression for $c(t, y; q)$, the generating function of parallelogram polyominoes according to the area, the width and the number of left path corners, can be deduced from $P(s, t, x, y, y'; q)$ by putting the variables s, x and y' to 1.

In the special case of $s = q = 1$, Kreweras (1986) gives an exact formula for this enumeration. This formula can also be derived from a result of Krattenthaler and Sulanke (1993).

3. Parallelogram Polyominoes and Left Path Corners

At first, we explain the enumeration of parallelogram polyominoes according to the area, the width and the number of left path corners. Next, we give the main results of the enumeration when we consider the two types of corners.

Delest and Viennot (1984) give a bijection between parallelogram polyominoes having perimeter $2n+2$ and Dyck words of length $2n$. A parallelogram polyomino can be defined as two sequences of integers $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_{n-1})$, where a_i denotes the number of cells belonging to the ith column and $(b_i + 1)$ denotes the number of edges shared by columns i and $i+1$. The corresponding Dyck word is the Dyck word with n peaks, whose height of peaks are $a_1, \ldots, a_n$ and height of valleys are $b_1, \ldots, b_{n-1}$. Then, the number of $x\bar{x}$ factors in a Dyck word is the width of the parallelogram polyomino associated to the word in the bijection and the sum of the height of the peaks is the area of the polyomino. Moreover, it has been proved in Delest, Gouyou-Beauchamps and Vauquelin (1987) that the number of the left path corners is equal to the number of $\bar{x}xx$ factors in the corresponding Dyck word.

Let μ be the morphism of $\{x, \bar{x}, y, t\}^*$ defined by

$$\mu(x) = \mu(y) = x, \mu(\bar{x}) = \bar{x} \quad \text{and} \quad \mu(t) = \epsilon.$$

Let C be the set of the words w with letters from $\{w, \bar{x}, y, t\}$ satisfying the following conditions:

$\mu(w)$ is a Dyck word;

$w = x\bar{x}$ or $w = w_1 x t \bar{x} w_2 x t \bar{x} \cdots x t \bar{x} w_k$ with

— $w_1 \in x^*$,

— $w_k \in \bar{x}^*$,
— for $2 \leq i \leq k-1$, $w_i \in \bar{x}^* \cup \bar{x}^* y x^*$.

The following algebraic grammar:

(1) $C \to xt\bar{x}$,
(2) $C \to xC\bar{x}$,
(3) $C \to Cxt\bar{x}$,
(4) $C \to CyC\bar{x}$.

generates these words.

Clearly, there is a bijection between the words of length $2n$ having m letters t and p letters y, and the Dyck words of length $2n$ having m factors $x\bar{x}$ and p factors $\bar{x}xx$.

Computing the area is done using the technique of attribute grammars (see Delest and Fédou, 1992). We use the attribute τ associated to the corresponding rules

(1) $\tau(C) = qxt\bar{x}$,
(2) $\tau(C) = q^{|\tau(C)|_t} x\tau(C)\bar{x}$,
(3) $\tau(C) = q\tau(C)xt\bar{x}$,
(4) $\tau(C) = q^{|\tau(C_2)|_t}\tau(C_1)y\tau(C_2)\bar{x}$,

where $|w|_t$ is the number of letters t in the word w.

This attribute computes, recursively on the derivation trees, the sum of the height of the peaks of the words generated from C. Figure 3 illustrates the fourth rule of the attributes system.

Using COM_QGRAM, a Maple package handling algebraic grammars and q-grammars (see Delest and Dubernard, 1994), we obtain the following q-equation

$$c(t,y;q) = qxt\bar{x} + x\bar{x}\,c(qt,y;q) + qxt\bar{x}\,c(t,y;q) + y\bar{x}\,c(qt,y;q)\,c(t,y;q).$$

From here, $q, x, \bar{x}, t$ and y are commutating variables and no longer (non-commutating) letters.

Then, erasing the variables x and $\bar{x}$, we get

PROPOSITION 3.1. *The generating function of parallelogram polyominoes according to the area, the width and the number of left path corners, $c(t,y;q)$, satisfies the functional equation*

$$c(t,y;q) = qt + c(qt,y;q) + qt\,c(t,y;q) + y\,c(qt,y;q)\,c(t,y;q),$$

where t encodes the number of columns, q the area and y the number of left path corners.

Figure 3. Illustration of the rule $\tau(c) = q^{|\tau(C_2)|_t}\tau(C_1)y\tau(C_2)\bar{x}$.

Let

$$c(t, y; q) = \sum_{n \geq 1} a_n t^n,$$

where a_n is the generating function of parallelogram polyominoes having width n according to the area and the number of left path corners. Let $a_{n,i,j}$ be the coefficient of $y^i q^j$ in a_n. From Proposition 3.1, we obtain

$$\begin{cases} a_1 = q + qa_1 \\ a_n = q^n a_n + qa_{n-1} + y \sum_{k=1}^{n-1} a_k a_{n-k} q^k & \text{for } n > 1. \end{cases}$$

Define now

$$\alpha_n = \frac{(1-q)^{2n-1}}{q^n} a_n,$$

then we find $\alpha_1 = 1$, $\alpha_2 = \frac{1-q(1-y)}{1+q}$, and for $n > 2$

$$[n]\alpha_n = (1 - q(1-y) + yq^{n-1})\alpha_{n-1} + y \sum_{k=2}^{n-2} \alpha_k \alpha_{n-k} q^k.$$

So, using the notation $\theta_n = \frac{\lambda_n \alpha_n}{(1-q+qy)}$, where λ_n is defined in the introduction, we finally get:

Proposition 3.2. $\theta_2 = 1$ *and for* $n \geq 2$

$$\theta_{n+1} = (1 - q(1-y) + yq^n)\theta_n \frac{\lambda_{n+1}}{[n+1]\lambda_n} + y(1-q(1-y)) \sum_{k=2}^{n-1} q^k \theta_k \theta_{n-k+1} \frac{\lambda_{n+1}}{[n+1]\lambda_k \lambda_{n-k+1}}.$$

The polynomial θ_n appears naturally in the Maple computation after a factorization of a_n.

We note that the denominators of the fractions

$$\frac{\lambda_{n+1}}{[n+1]\lambda_n} \quad \text{and} \quad \frac{\lambda_{n+1}}{[n+1]\lambda_k \lambda_{n-k+1}}$$

cancel out and polynomials in q remain. Thus, we easily show by induction the following.

Property 3.3. *θ_n is a polynomial in y and q.*

Note that any characterization of θ_n induces one on a_n since

$$\theta_n = \frac{\lambda_n}{(1-q+qy)} \frac{(1-q)^{2n-1}}{q^n} a_n.$$

The first values of θ_n are displayed, in Figure 4, in a matrix form in which the intersection of the $(i+1)$th row and the $(j+1)$th column is equal to the coefficient of $y^i q^j$ in θ_n. We will denote it by $\theta_{n,i,j}$. We also define

$$\theta_{n,i,\bullet} = \sum_j \theta_{n,i,j} q^j.$$

These matrices are obtained using the Maple package COM_QGRAM (see Delest and Dubernard, 1994). Studying θ_n, the properties which are displayed in Figure 5 are found.

```
θ2     θ3                θ4                              θ5

[1] [1 -1 0] [1 -1 -1  1  0  0 0] [1 -2 0  2 -1  0  0  0  0  0 0]
    [0  1 1] [0  2  2 -1 -1 -2 0] [0  3 0 -4  1 -2  0  1  1  0 0]
             [0  0  1  3  3  3 1] [0  0 3  6  2  2 -2 -4 -4 -3 0]
                                  [0  0 0  1  4  6  8  8  6  4 1]

                                  θ6

[1 -1 -2  1  2  1 -2  -1   1   0   0   0   0   0   0   0   0  0 0]
[0  4  4 -5 -7 -4  3   0   0   6   3  -1  -1   0  -2   0   0  0 0]
[0  0  6 18 18  8 -8 -16 -22 -30 -17  -3   8  11  16   8   3  0 0]
[0  0  0  4 20 43 63  69  58  36  -5 -41 -64 -66 -59 -37 -17 -4 0]
[0  0  0  0  1  7 22  47  79 110 135 144 135 110  79  47  22  7 1]
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Figure 4. The first values of θ_n.

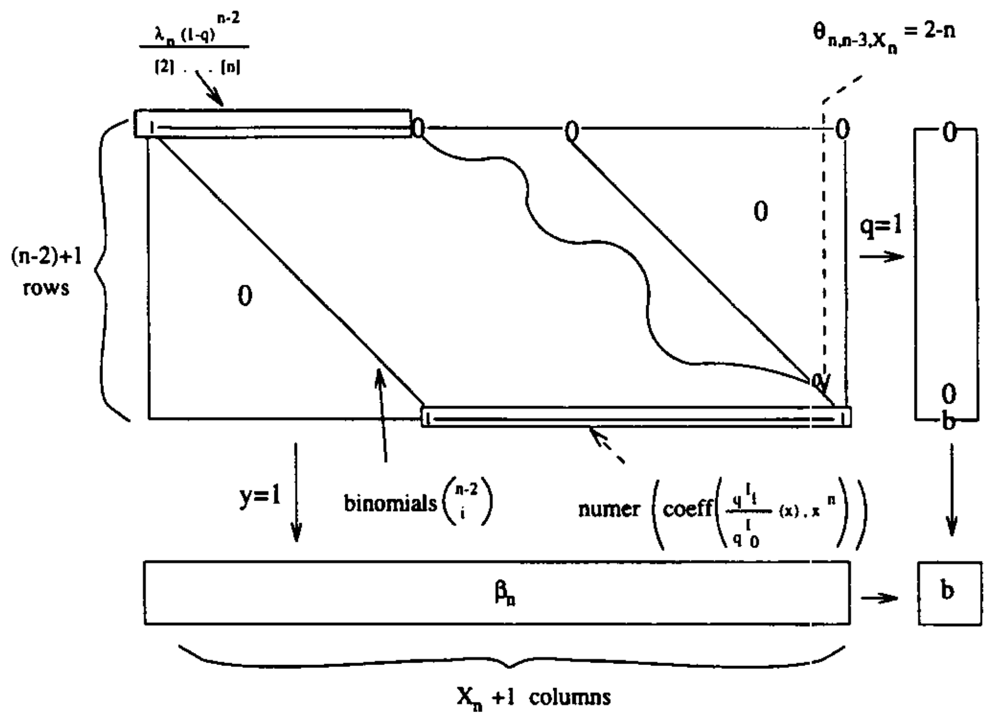

Figure 5. The matrix θ_n.

PROPERTY 3.4. *Let i be a positive integer. Then, the degree of $\theta_{n,i,\bullet}$ in q is at most $X_n - (n-2-i)$ where $X_n = \sum_{k=1}^{n}(k-1)\lfloor \frac{n}{k} \rfloor - 1$.*

Remark that X_n is also the degree of the polynomial β_n which appears in the enumeration of the parallelogram polyominoes according to the area and the width (see Delest and Fédou, 1993).

PROOF. From Proposition 3.2, we deduce that, for all $i \in [1, n-1]$,

$$\begin{aligned}\theta_{n+1,i,\bullet} &= (1-q)\frac{\lambda_{n+1}}{[n+1]\lambda_n}\theta_{n,i,\bullet} + (q+q^n)\frac{\lambda_{n+1}}{[n+1]\lambda_n}\theta_{n,i-1,\bullet} \\ &+ \frac{1-q}{[n+1]}\sum_{k=2}^{n-1} q^k \sum_{l=0}^{i-1} \theta_{k,l,\bullet}\theta_{n-k+1,i-l-1,\bullet}\frac{\lambda_{n+1}}{\lambda_k\lambda_{n-k+1}} \\ &+ \frac{q}{[n+1]}\sum_{k=2}^{n-1} q^k \sum_{l=0}^{i-2} \theta_{k,l,\bullet}\theta_{n-k+1,i-l-2,\bullet}\frac{\lambda_{n+1}}{\lambda_k\lambda_{n-k+1}}.\end{aligned}$$

If we suppose that $\deg_q(\theta_{n,i,\bullet}) \leq X_n - (n-2-i)$, we obtain by induction that $\deg_q(\theta_{n+1,i,\bullet}) \leq X_n - (n-1-i)$ □

Let

$$T_n = X_n - \frac{(n-1)(n-2)}{2}.$$

From the numerical values of θ_n, we notice that $\theta_{n,0,i}$ and $\theta_{n,0,T_n-i}$ have the same absolute value and the same sign (respectively opposite sign) if n is even (respectively odd). So we find

PROPOSITION 3.5. *For all $j \in [0, T_n]$, the polynomial $\theta_{n,0,\bullet}$ satisfies*

$$\theta_{n,0,j} = (-1)^n \theta_{n,0,T_n-j}.$$

PROOF. From Proposition 3.2, we get

$$\theta_{n+1,0,\bullet} = (1-q)\theta_{n,0,\bullet}\frac{\lambda_{n+1}}{[n+1]\lambda_n}.$$

Let

$$\frac{\lambda_{n+1}}{[n+1]\lambda_n} = \sum_{i=0}^{D_n} l_{n,i}q^i$$

where $D_n = X_{n+1} - X_n - n$ is the degree of polynomial. We easily show that

$$\frac{\lambda_{n+1}}{[n+1]\lambda_n}$$

is symmetrical because it is equal to products of q-factorials which are symmetrical polynomials. So

$$l_{n,i} = l_{n,D_n-i}.$$

Using this notation, $\theta_{n,0,\bullet}$ can be written

$$\theta_{n+1,0,\bullet} = (1-q)\left(\sum_{j=0}^{T_n}\theta_{n,0,j}q^j\right)\left(\sum_{k=0}^{D_n} l_{n,k}q^k\right)$$

As $T_{n+1} = T_n + D_n + 1$, we deduce that

$$\theta_{n+1,0,\bullet} = \sum_{i=0}^{T_{n+1}}\left(\left(\sum_{j=0}^{i}\theta_{n,0,j}q^j l_{n,i-j}q^{i-j}\right) - q\left(\sum_{j=0}^{i-1} q^j\theta_{n,0,j}l_{n,i-1-j}q^{i-j-1}\right)\right)$$

$$= \sum_{i=0}^{T_{n+1}} \left(\left(\sum_{j=0}^{i} \theta_{n,0,j} l_{n,i-j} - \sum_{j=0}^{i-1} \theta_{n,0,j} l_{n,i-1-j} \right) \right) q^i.$$

So, we find

$$\theta_{n+1,0,i} = \sum_{j=0}^{i} \theta_{n,0,j} l_{n,i-j} - \sum_{j=0}^{i-1} \theta_{n,0,j} l_{n,i-1-j}.$$

In the same way, we have

$$\begin{aligned} \theta_{n+1,0,T_{n+1}-i} &= \sum_{j=0}^{r} \theta_{n,0,T_n-j} l_{n,(T_{n+1}-i)-(T_n-j)} - \sum_{j=0}^{s} \theta_{n,0,T_n-j} l_{n,(T_{n+1}-i)-(T_n-j)-1} \\ &= \sum_{j=0}^{r} \theta_{n,0,T_n-j} l_{n,D_n+j-i+1} - \sum_{j=0}^{s} \theta_{n,0,T_n-j} l_{n,D_n-i+j}. \end{aligned}$$

We deduce from this that $r = i - 1$ and $s = i$.

As $l_{n,i} = l_{n,D_n-i}$ and, by induction hypothesis,

$$\theta_{n,0,i} = (-1)^n \theta_{n,0,T_n-i},$$

we obtain

$$\begin{aligned} \theta_{n+1,0,T_{n+1}-i} &= \sum_{j=0}^{i-1} (-1)^n \theta_{n,0,j} l_{n,i-1-j} - \sum_{j=0}^{i} (-1)^n \theta_{n,0,j} l_{n,i-j} \\ &= (-1)^{n+1} \left(\sum_{j=0}^{i} \theta_{n,0,j} l_{n,i-j} - \sum_{j=0}^{i-1} \theta_{n,0,j} l_{n,i-1-j} \right) \\ &= (-1)^{n+1} \theta_{n+1,0,i}. \quad \square \end{aligned}$$

If we extract the values $\theta_{n,i,i}$, we find the $(n-2)$th row of the Pascal triangle. So, we have the following.

PROPERTY 3.6. *For all integer k between 0 and $n-2$,*

$$\theta_{n,k,k} = \binom{n-2}{k}.$$

PROOF. As

$$\theta_n = \frac{\lambda_n}{(1-q+qy)} \frac{(1-q)^{2n-1}}{q^n} a_n,$$

we can obtain that

$$\begin{cases} a_{n,i,n+i} = \theta_{n,i,i} + \theta_{n,i-1,i-1} & \text{if } 0 < i < n-1, \\ a_{n,0,n} = \theta_{n,0,0}. \end{cases}$$

Thus, let us compute $a_{n,i,i+n}$. There is only one parallelogram polyomino of area n, with n columns, and no corner. It is the "rectangle" with n columns of height 1.

To build a parallelogram polyomino with n columns, i left path corners and area $n+i$, we must choose i columns among the last $n-1$ of the rectangle with n columns. On each of these columns, we insert by the bottom a cell, pushing all that is upper and at

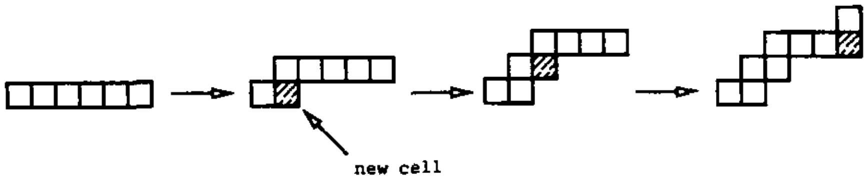

Figure 6. Building of a parallelogram polyomino with 6 columns, 3 left path corners and area 9.

the right of this cell. For instance, we build a parallelogram polyomino with 6 columns, 3 left path corners and area 9 in Figure 6.

So, we can build $\binom{n-1}{i}$ parallelogram polyominoes with n columns, i left path corners and of area $n+i$, which gives

$$a_{n,i,i+n} = \binom{n-1}{i}$$

Now, let us prove the property by induction. Suppose that

$$\theta_{n,i,i} = \binom{n-2}{i}.$$

As $a_{n,i+1,n+i+1} = \theta_{n,i,i} + \theta_{n,i+1,i+1}$, we obtain

$$\begin{aligned}\theta_{n,i+1,i+1} &= \binom{n-1}{i+1} - \binom{n-2}{i} \\ &= \binom{n-2}{i+1}. \quad \square\end{aligned}$$

We derive from θ_n the polynomial β_n defined by Delest and Fédou (1993) by taking $y = 1$,

$$\beta_{n,k} = \sum_{i=0}^{n-2} \theta_{n,i,k},$$

and we find the following.

PROPERTY 3.7. *For all k, $\sum_{i=0}^{n-2} \theta_{n,i,k} = \sum_{i=0}^{n-2} \theta_{n,i,X_n-k}$.*

In the same way, if we substitute the value 1 for q, we obtain:

PROPERTY 3.8. *For all i in $[0, n-3]$, we have $\sum_{k=0}^{X_n} \theta_{n,i,k} = 0$.*

Thus, by substituting $q = 1$ we see that $\beta_n(1) = \theta_{n,n-2,\bullet}(1)$, hence $\theta_{n,n-2,\bullet}$ and β_n are q-analogs of a same quantity. We are thus led to formulate:

CONJECTURE 3.9. *$\theta_{n,n-2,\bullet}$ is the numerator of the coefficient of x^n in the expansion of ${}_qI_1(x)/{}_qI_0(x)$ where ${}_qI_\nu(x)$ is the classical q-analog of the Bessel function defined by Ismail (1982),*

$${}_qI_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n x^{n+\nu}}{[n]![n+\nu]!}.$$

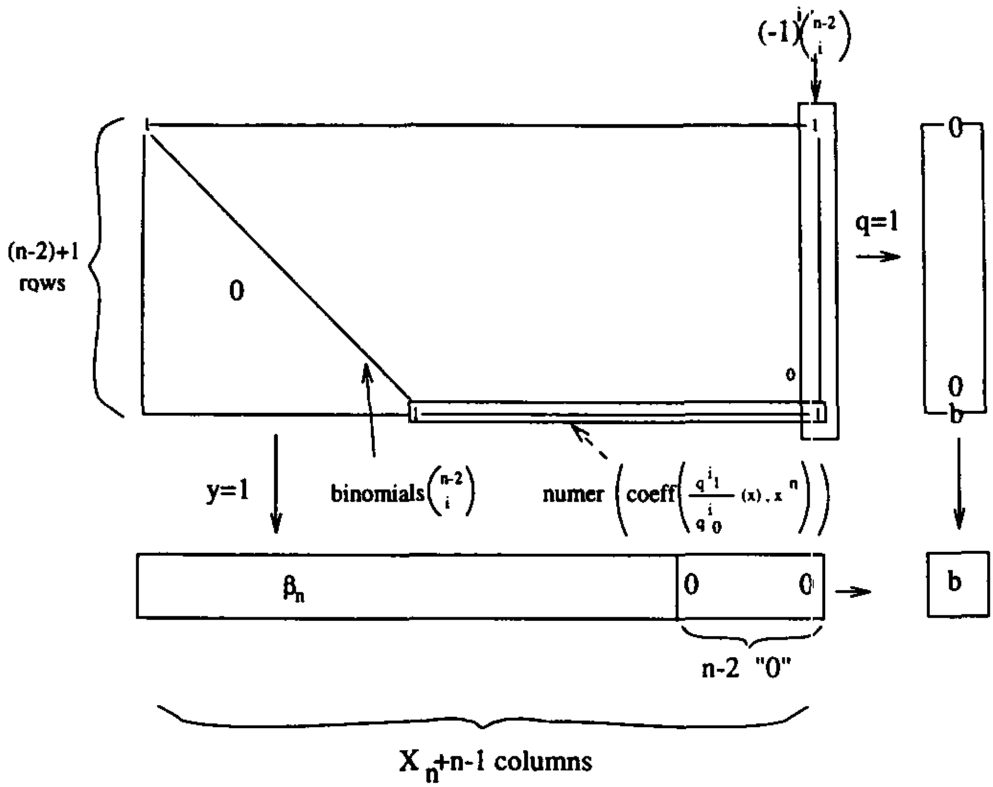

Figure 7. The matrix in the case where both types of corners are counted.

This conjecture has been found empirically. The first numerical values suggest the following Property, which can also be proved by induction.

PROPERTY 3.10. *$\theta_{n,n-2,\bullet}$ is a symmetrical polynomial in q.*

REMARK 3.11. *Before doing all this study, we also studied the two types corners case. Employing the same method, we proved that the study of the generating function according to the area, the width and the total number of corners could be reduced to the study of a polynomial recurrence. Each polynomial can be described in a matrix form as displayed in Figure 7. Note that in this case, we have no conjecture on ${}_qi_\nu(x)$ which is a q-analog of the Bessel function. We get similar results, but they are not as interesting as in the case studied above.*

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