Asymptotic stability of impulsive stochastic partial differential equations with infinite delays

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Abstract

In this paper, we study the existence and asymptotic stability in $p$th moment of mild solutions to nonlinear impulsive stochastic partial differential equations with infinite delay. By employing a fixed point approach, sufficient conditions are derived for achieving the required result. These conditions do not require the monotone decreasing behaviour of the delays.

1. Introduction

Many real world problems in science and engineering can be modelled by nonlinear stochastic partial differential equations. The existence, uniqueness and asymptotic behaviour of solutions of the stochastic partial differential equations have been considered by many authors (see [1,3,6,8,11,21] and references therein). Caraballo and Liu [2], Liu and Mao [10], and Taniguchi [20] discussed the exponential stability of the strong solutions and mild solutions, by the method of coercivity condition, by the Lyapunov method and by the estimate of solutions, respectively. In particular, the Lyapunov direct method has some difficulties with the theory and application to specific problems when discussing the asymptotic behavior of solutions in stochastic differential equations. More recently, Luo [13] has studied the asymptotic stability of mild solutions of stochastic partial differential equations with finite delays using fixed point approach which shows that some of these difficulties are rectified when applying the fixed point theory.

On the other hand, the impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as finance, economics, mechanics, electronics and telecommunications, etc. (see [18]). The theory of impulsive differential equations has been studied extensively (see [16,17] and references therein). However, in addition to impulsive effects, stochastic effects likewise exist in real systems. It is well known that a lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. The stability investigation of stochastic differential equations with delays have been discussed by several authors [5,9,12,14,15,19,22].

Moreover, systems with infinite delay deserve a study because they describe a kind of system present in the real world. For example, in a predator–prey system, the predation decreases the average growth rate of the prey species, linearly, with an infinite delay—for the predator cannot hunt prey when the predators are infants, and predators have to mature for a...
duration of time which for simplicity in the mathematical analysis has been assumed to be infinite, before they are capable of decreasing the average growth rate of the prey species. Therefore, it is interesting to study the stability problems for stochastic systems with infinite delays. However, to the authors best knowledge no work has been reported on existence and stability problems for impulsive stochastic systems with infinite delays. Motivated by the above consideration, in this paper we obtain sufficient conditions for ensuring the existence and asymptotic stability in $p$th moment of mild solutions to impulsive stochastic partial differential equations with infinite delays.

2. Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with probability measure $\mathbb{P}$ on $\Omega$ and a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, that is the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Let $X, Y$ be two real separable Hilbert spaces and we denote by $\langle \ldots \rangle_X, \langle \ldots \rangle_Y$ their inner products and by $\| \cdot \|_X, \| \cdot \|_Y$ their vector norms, respectively. $L(Y, X)$ be the space of bounded linear operators mapping $Y$ into $X$ equipped with the usual norm $\| \cdot \|_\mathcal{L}$. Let $\|w(t)\| = 0$ denote an $X$-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance operator $Q$, that is $E(w(t), x)_{Y}(w(s), y)_Y = (t \wedge s)(Q x, y)_Y$ for all $x, y \in Y$, where $Q$ is a positive, self-adjoint, trace class operator on $Y$. In particular, we denote $w(t)$ an $X$-valued $Q$-Wiener process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

In order to define stochastic integrals with respect to the $Q$-Wiener process $w(t)$, we introduce the subspace $Y_0 = Q^{1/2}(Y)$ of $Y$ which is endowed with the inner product $(u, v)_Y = (Q^{-1/2}u, Q^{-1/2}v)_Y$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i \geq 1}$ in $Y$, a bounded sequence of nonnegative real numbers $\lambda_i$ such that $Q e_i = \lambda_i e_i, i = 1, 2, \ldots, $ and a sequence $\{\beta_i\}_{i \geq 1}$ of independent Brownian motions such that

$$\langle w(t) \rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \langle e_i, e_i \rangle \beta_i(t), e \in Y,$$

and $\mathcal{F}_t = \mathcal{F}_t^\infty$, where $\mathcal{F}_t^\infty$ is the sigma algebra generated by $\{w(s): 0 \leq s \leq t\}$. Let $L_0^2 = L_2(Y_0, X)$ denote the space of all Hilbert–Schmidt operators from $Y_0$ into $X$. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|_{L_2}^2 = \text{tr}(\mu Q^{1/2} \mu Q^{1/2})$ for any $\mu \in L_0^2$. Clearly for any bounded operators $\mu \in L(Y, X)$ this norm reduces to $\|\mu\|_{L_2}^2 = \text{tr}(\mu Q \mu^*)$.

In this article, we consider a mathematical model given by the following impulsive stochastic differential equations with infinite delays

$$dx(t) = [Ax(t) + f(t, x(t - \tau(t)))]dt + g(t, x(t - \delta(t)))dw(t), t \geq 0, t \neq t_k, (1)$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, k = 1, 2, \ldots, m,$$

$$x(0) = \varphi \in D_{\mathcal{F}_0}^{\mathcal{F}_0}(\hat{m}(0), 0), X,$$

where $f: R_+ \times X \rightarrow X, g: R_+ \times X \rightarrow L(Y, X)$ are all Borel measurable, $A$ is the infinitesimal generator of a bounded linear operators $S(t), t \geq 0$, in $X, I_k: X \rightarrow X$. Furthermore the fixed moments of time $t_k$ satisfies $0 < t_1 < \cdots < t_m < \lim_{k \rightarrow \infty} t_k = \infty, x(t_k^+) \text{ and } x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Also $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state $x$ at time $t_k$ with $I_k$ determining the size of the jump. Moreover, let $\tau(t), \delta(t) \in C([0, \infty) \cap R_+, R_+]$ satisfy $t - \tau(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $\hat{m}(0) = \max(\inf(s - \tau(s), s \geq 0), \inf(s - \delta(s), s \geq 0))$. Here $D_{\mathcal{F}_0}^{\mathcal{F}_0}(\hat{m}(0), 0), X$ be the family of all almost surely bounded, $\mathcal{F}_0$-measurable, continuous random variables from $[\hat{m}(0), 0]$ to $X$. Denote the norm $\|\cdot\|$ by $\|\|_D = \sup_{s \in [\hat{m}(0), 0]} E(\|\varphi(t)\|_X).$

3. Stability of stochastic delay equations with impulses

In this section, we shall formulate and prove the conditions for the asymptotic stability in the $p$th moment of mild solutions to Eqs. (1)–(3) by using a fixed point approach. To prove the result, on the functions $f$ and $g$ we impose some Lipschitz and linear growth conditions. Also for the purpose of stability, we shall assume that $f(t, 0) = 0, g(t, 0) = 0$ and $I_k(0) = 0$ ($k = 1, 2, \ldots, m$). Then Eqs. (1)–(3) have a trivial solution when $\varphi = 0$.

Further, let $H$ be the Banach space of all $\mathcal{F}_0$-adapted process $\varphi(t, \hat{w}): [\hat{m}(0), \infty) \times \Omega \rightarrow R$ which is almost surely continuous in $t$ for fixed $\hat{w} \in \Omega$. Moreover $\varphi(s, \hat{w}) = \varphi(s) \text{ for } s \in [\hat{m}(0), 0]$ and $E(\|\varphi(t, \hat{w})\|_X) \rightarrow 0 \text{ as } t \rightarrow \infty$.

Further, we impose the following assumptions on data of the problem:

1. $A$ is the infinitesimal generator of a semigroup of bounded linear operators $S(t), t \geq 0$, in $X$ satisfying $\|S(t)\|_X \leq Me^{-\alpha t}, t \geq 0$, for some constants $M \geq 1$ and $0 < \alpha \in R_+.$
To prove this result, we use the contraction mapping principle. In order to apply the contraction mapping principle, first we present Lemma 3.1.

Lemma 3.1. For any $r \geq 1$ and for arbitrary $L^2$-valued predictable process $\Phi(\cdot)$,

$$\sup_{s \in [0,t]} E \left[ \int_0^s \Phi(u) dW(u) \right]^{2r} \leq (r(2r-1)) \left( \int_0^t (E \left| \Phi(s) \right|^{2r})^{1/r} ds \right)^r.$$

Let us recall the definition of mild solution for the stochastic differential equations (1)–(3).

Definition 3.2. A stochastic process $\{x(t), t \in [0, T]\}$ ($0 \leq T < \infty$) is called a mild solution of Eqs. (1)–(3) if

(i) $x(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$;

(ii) $x(t) \in X$ has càdlàg paths on $t \in [0, T]$ a.s and for each $t \in [0, T]$, $x(t)$ satisfies the integral equation

$$x(t) = S(t)\psi(0) + \int_0^t S(t-s)f(s,x(s-\tau(s)))ds + \int_0^t S(t-s)g(s,x(s-\delta(s)))dw(s) + \sum_{0<k\leq t} S(t-t_k)I_k(x(t_k^-)),$$

and

$$x_0(.) = \psi \in D^1_{\mathcal{F}_0}([\hat{m}(0), 0], X).$$

Definition 3.3. Let $p \geq 2$ be an integer. Eq. (4) is said to be stable in $p$th moment if for arbitrarily given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\psi\|_D < \delta$ guarantees that

$$E\left\{ \sup_{t \geq 0} \|x(t)\|^p_X \right\} = \epsilon.$$

Definition 3.4. Let $p \geq 2$ be an integer. Eq. (4) is said to be asymptotically stable in $p$th moment if it stable in $p$th moment and for any $\psi \in D^1_{\mathcal{F}_0}([\hat{m}(0), 0], X)$,

$$\lim_{r \to \infty} E\left\{ \sup_{t \geq 0} \|x(t)\|^p_X \right\} = 0.$$

Theorem 3.5. Assume the conditions (I)–(III) hold. Let $p \geq 2$ be an integer. If the inequality $3^{p-1}M^p(L_1^2a^{-p} + L_1^p(p(p-1)/2)^{p/2}) < 1$ is satisfied, then the impulsive stochastic differential equations (1)–(3) is asymptotically stable in $p$th moment; here $\hat{\lambda} = e^{-\alpha_T} E(\sum_{k=1}^m \|q_k\|^p_X)$.

Proof. Define a nonlinear operator $\Psi : H \to H$ by $\Psi(x)(t) = \psi(t)$ for $t \in [\hat{m}(0), 0]$ and for $t \geq 0$,

$$\Psi(x)(t) = S(t)\psi(0) + \int_0^t S(t-s)f(s,x(s-\tau(s)))ds + \int_0^t S(t-s)g(s,x(s-\delta(s)))dw(s) + \sum_{0<k\leq t} S(t-t_k)I_k(x(t_k^-))$$

$$= \sum_{i=1}^4 F_i(t).$$

(5)

As mentioned in Luo [13], to prove the asymptotic stability it is enough to show that the operator $\Psi$ has a fixed point in $H$. To prove this result, we use the contraction mapping principle. In order to apply the contraction mapping principle, first we verify the mean square continuity of $\Psi$ on $[0, \infty)$.

Let $x \in H$, $t_1 \geq 0$ and $\|r\|$ be sufficiently small then

$$E \left\| \Psi(x)(t_1 + r) - \Psi(x)(t_1) \right\|^p_X \leq 4^{p-1} \sum_{i=1}^4 E \left\| F_i(t_1 + r) - F_i(t_1) \right\|^p_X.$$
It can be easily obtain that \( E\|F_i(t_1 + r) - F_i(t_1)\|_X^p \to 0, i = 1, 2, 4 \), as \( r \to 0 \). Moreover by using Holders inequality and Lemma 3.1, we obtain

\[
E \left\| F_3(t_1 + r) - F_3(t_1) \right\|_X^p \leq 2^{p-1} c_p \left[ \int_0^{t_1} \left( E \left\| (S(t_1 + r - s) - S(t_1 - s)) g(s, x(s - \delta(s))) \right\|_X \right)^{2/p} ds \right]^{(p/2)}
\]

\[
+ 2^{p-1} c_p \left[ \int_{t_1}^{t_1 + r} \left( E \left\| S(t_1 + r - s) g(s, x(s - \delta(s))) \right\|_X \right)^{2/p} ds \right]^{(p/2)} \to 0 \text{ as } r \to 0.
\]

(6)

where \( c_p = (p(p-1)/2)^{p/2} \). Thus \( \Psi \) is continuous in \( p \)th moment on \([0, \infty)\).

Next we show that \( \Psi(H) \subset H \). From (5), we obtain

\[
E \left\| \Psi(x)(t) \right\|_X^p \leq 4^{p-1} E \left\| S(t) \phi(0) \right\|_X^p + 4^{p-1} E \left\| \int_0^t S(t-s) f(s, x(s - \tau(s))) ds \right\|_X^p
\]

\[
+ 4^{p-1} E \left\| \int_0^t S(t-s) g(s, x(s - \delta(s))) dw(s) \right\|_X^p + 4^{p-1} \sum_{0 < t_k < t} E \left\| S(t - t_k) I_k(x(t_k^-)) \right\|_X^p.
\]

(7)

Now we estimate the terms on the R.H.S. of (7). Using (I) and (III) we get

\[
4^{p-1} E \left\| S(t) \phi(0) \right\|_X^p \leq 4^{p-1} M^p e^{-p\alpha t} \|\phi\|_D^p \to 0 \text{ as } t \to \infty.
\]

(8)

\[
4^{p-1} \sum_{0 < t_k < t} E \left\| S(t - t_k) I_k(x(t_k^-)) \right\|_X^p \leq 4^{p-1} M^p e^{-p\alpha t} \|I_k(x(t_k^-))\|_X^p \to 0 \text{ as } t \to \infty.
\]

(9)

Now from (I), (II) and Holder’s inequality, we have

\[
4^{p-1} E \left\| \int_0^t S(t-s) f(s, x(s - \tau(s))) ds \right\|_X^p \leq 4^{p-1} M^p L_{\tau}^p \left[ \int_0^t e^{-a(t-s)} ds \right]^{p-1} \left[ \int_0^t e^{-a(t-s)} E \left\| x(s - \tau(s)) \right\|_X^p ds \right]
\]

\[
\leq 4^{p-1} M^p L_{\tau}^p a^{1-p} \left[ \int_0^t e^{-a(t-s)} E \left\| x(s - \tau(s)) \right\|_X^p ds \right].
\]

(10)

For any \( x(t) \in H \) and any \( \epsilon > 0 \) there exists a \( t_1 > 0 \) such that \( E\|x(s - \tau(s))\|_X^p < \epsilon \) for \( t > t_1 \). Thus from (10) we obtain

\[
4^{p-1} E \left\| \int_0^t S(t-s) f(s, x(s - \tau(s))) ds \right\|_X^p \leq 4^{p-1} M^p L_{\tau}^p a^{1-p} e^{-at} \left[ \int_0^{t_1} e^{at} E \left\| x(s - \tau(s)) \right\|_X^p ds + 4^{p-1} M^p L_{\tau}^p a^{-p} \epsilon \right].
\]

(11)

As \( e^{-at} \to 0 \) as \( t \to \infty \) and by assumption on Theorem 3.5, there exists \( t_2 \geq t_1 \) such that for any \( t \geq t_2 \) we obtain

\[
4^{p-1} M^p L_{\tau}^p a^{1-p} e^{-at} \int_0^{t_1} e^{at} E \left\| x(s - \tau(s)) \right\|_X^p ds \leq \epsilon - 4^{p-1} M^p L_{\tau}^p a^{-p} \epsilon.
\]

(12)

From (11) and (12), we obtain for any \( t \geq t_2 \),

\[
4^{p-1} E \left\| \int_0^t S(t-s) f(s, x(s - \tau(s))) ds \right\|_X^p < \epsilon.
\]

That is, to say,

\[
4^{p-1} E \left\| \int_0^t S(t-s) f(s, x(s - \tau(s))) ds \right\|_X^p \to 0 \text{ as } t \to \infty.
\]

(13)

Now for any \( x(t) \in H, t \in \hat{m}(0, \infty) \), we obtain

\[
4^{p-1} E \left\| \int_0^t S(t-s) g(s, x(s - \delta(s))) dw(s) \right\|_X^p \leq 4^{p-1} c_p M^p L_{\tau}^p \left[ \int_0^t e^{-2\alpha(t-s)} E \left\| x(s - \delta(s)) \right\|_X^p \right]^{2/p} ds \right]^{p/2}.
\]

(14)
Further, similar to the proof of (13), from (14) we get
\[
4^{-1}E \left| \int_0^t S(t-s)g(s,x(s-\delta(s)))\,dw(s) \right|^p_x \to 0 \quad \text{as } t \to \infty.
\]  
(15)

Using (8), (5), (13) and (15) in (7), we obtain \( E\|\Psi(x)(t)\|^p_X \to 0 \) as \( t \to \infty \). So we conclude that \( \Psi(H) \subset H \).

Finally, we prove that \( \Psi \) is a contraction mapping. To see this let \( x, y \in H \) so for \( s \in [0, T] \), we obtain
\[
\sup_{s \in [0, T]} E\|\Psi(x)(t)-(\Psi y)(t)\|_X^p \leq 3\sup_{s \in [0, T]} E\left| \int_0^t S(t-s)(f(s,x(s-\tau(s)))-f(s,x(s-\tau(s))))\,ds \right|^p_X \\
+ 3\sup_{s \in [0, T]} E\left| \int_0^t S(t-s)(g(s,x(s-\delta(s)))-g(s,y(s-\delta(s))))\,dw(s) \right|^p_X \\
+ 3\sup_{s \in [0, T]} E\left| \sum_{0<\tau_k<t} S(t-t_k)l_k(x(t_k^-)) - l_k(y(t_k^-)) \right|^p_X \\
\leq [3M^pL^p_c+3M^pL^p_c(\theta^p_a+\theta^p_b)+3M^pL^p_c(\theta^p_a+\theta^p_b+\theta_a^p+\theta_b^p)]E\|x(t)-y(t)\|_X^p,
\]  
where \( \tilde{L} = e^{-aqT}E(\sum_{k=1}^m \|l_k\|_X^2) \). Therefore, \( \Psi \) is a contraction mapping and hence there exists a unique fixed point \( x(\cdot) \) in \( H \) which is a solution of Eqs. (1)–(3) with \( x(s) = \phi(s) \) on \([0,0]) \) and \( E\|x(t)\|_X^p \to 0 \) as \( t \to \infty \).

To obtain the asymptotic stability, we have to prove that the mild solution of (1)–(3) is stable in pth moment. Let \( \epsilon > 0 \) be given and choose \( \delta > 0 \) and \( \delta > \epsilon \) satisfying the condition \( 4^{-1}M^p\delta + 4^{-1}M^p(L^p_c+a^p+L^p_c(2a)^{-p/2}+\tilde{L})\epsilon < \epsilon \).

If \( x(t) = x(t,0,\varphi) \) is a mild solution of (1)–(3) with \( \|\varphi\|_X^p < \delta \) then \( \Psi(x(t)) = x(t) \) defined in (4). We claim that \( E\|x(t)\|_X^p < \epsilon \) for all \( t \geq 0 \). Notice that \( E\|x(t)\|_X^p \in [\hat{m}(0),0] \). If there exists \( t^* > 0 \) such that \( E\|x(t^*)\|_X^p = \epsilon \) and \( E\|x(s)\|_X^p < \epsilon \) for \( s \in [0,t^*] \), then it follows from (7) that \( E\|x(t^*)\|_X^p \leq 4^{-1}M^p\epsilon + 4^{-1}M^p(L^p_c+a^p+L^p_c(2a)^{-p/2}+\tilde{L})\epsilon < \epsilon \) which contradicts the definition of \( t^* \). This shows that the mild solution of (1)–(3) is asymptotically stable in pth moment if assumption in Theorem 3.5 holds. This completes the proof. \( \square \)

Remark 3.6. If the hypotheses (1)–(III) hold, then the impulsive stochastic system (1)–(3) is mean square asymptotically stable if \( 3M^2(l_1^2 + l_3^2 + a\tilde{L}) < a \); here \( \tilde{L} = e^{-aqT}E(\sum_{k=1}^m \|l_k\|_X^2) \).

Note 3.7. It should be pointed out that for the proof of above Theorem 3.5, we do not require the monotone decreasing behaviour of the delays, i.e. \( \tau^*(t) \leq 0, \delta(t) < 0, \forall t \geq 0 \).

Remark 3.8. In many applications, due to the complex random nature of situation, the stochastic problem should be considered in a stochastic integro-differential framework. In this remark, we consider the following perturbed impulsive stochastic integro-differential equation with infinite delay
\[
dx(t) = \left[ A\left( x(t) + \int_0^t \tilde{q}(t-s)x(s)\,ds \right) + f(t,x(t-\tau(t))) + \tilde{F}(t,x(t-\tau(t))) \right] dt \\
+ g(t,x(t-\delta(s)))\,dw(t) + \tilde{G}(t,x(t-\delta(s)))\,dw(t), \quad t \in J, \ t \neq t_k, \\
\Delta x(t_k) = x(t_k^+) \neq x(t_k^-) = l_k(x(t_k^-)) \quad \text{for } t = t_k, \ k = 1,2,\ldots,m, \\
x_0(\cdot) = \varphi \in D^p_{\tilde{L}}(\hat{m}(0),0), \ X.
\]  
(17)
(18)
(19)
where \( A : D(A) \subset X \to X \) is a linear, closed and densely-defined operator; \( \tilde{q} : [0,b] \times \Omega \to R \) is a stochastic kernel; \( f, \tilde{F} : R_+ \times X \to X, g, \tilde{G} : R_+ \times X \to L(Y, X) \) are measurable, locally bounded mappings.

It is important to note that the above Eqs. (17)–(19) are more general than (1)–(3). On the other hand, from the point of view of practical applications, (17)–(19) allow some long-range dependence of the noise in the models under consideration. Mild solution of the above equations is
\[
x(t) = R(t)\varphi(0) + \int_0^t R(t-s)[f(s,x(s-\tau(s)))+\tilde{F}(s,x(s-\tau(s)))]\,ds \\
+ \int_0^t R(t-s)g(s,x(s-\tau(s)))\,dw(s) + \int_0^t R(t-s)\tilde{G}(s,x(s-\tau(s)))\,dw(s) + \sum_{0<\tau_k<t} R(t-t_k)l_k(x(t_k^-)).
\]
where $R(t)$ is a resolvent family for stochastic systems which is defined in [7]. For details related to resolvent of operator associated to stochastic integro-differential equations and additional background, we refer the reader to [7] and the references therein. One can easily prove that by adopting and employing the method used in Theorem 3.5, the impulsive stochastic equations (17)–(19) is asymptotically stable in $p$th moment.

References