Stability of functional equations in the spaces of distributions and hyperfunctions ✩

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Abstract

Making use of the fundamental solution of the heat equation we prove the stability theorems of quadratic functional equation and d’Alembert equation in the spaces of Schwartz distributions and Sato hyperfunctions.

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1. Introduction

We consider the stability of the following functional equations in the spaces of distributions and hyperfunctions:

\[ f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0, \quad (1.1) \]
\[ f(x + y) + f(x - y) - 2f(x)f(y) = 0. \quad (1.2) \]

We call Eq. (1.1) the quadratic functional equation and (1.2) the d’Alembert equation.

The concept of stability for a functional equation arises when the equation is replaced by an inequality which acts as a perturbation of the equation, i.e.,

\[ \| f(x + y) + f(x - y) - 2f(x) - 2f(y) \|_{L^\infty} \leq \epsilon, \quad (1.3) \]
\[ \| f(x + y) + f(x - y) - 2f(x)f(y) \|_{L^\infty} \leq \epsilon. \quad (1.4) \]

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The stability question is that how do the solutions of inequalities (1.3) and (1.4) differ from those of Eqs. (1.1) and (1.2), respectively.

In this paper we reformulate and prove the above stability theorems in the spaces generalized functions such as the space $S'$ of Schwartz tempered distributions which is the dual space of the Schwartz space $S$ of rapidly decreasing functions and the space $F'$ of Fourier hyperfunctions which is the dual space of the Sato space $F$ of analytic functions of exponential decay.

Note that the above inequalities (1.3) and (1.4) themselves make no sense in the spaces of generalized functions. Making use of the tensor product and pullback of generalized functions as in [1,4,5,7] we extend inequality (1.3) and (1.4) to the spaces of generalized functions as follows: Let $A$, $B$, $P_1$, and $P_2$ be the functions

$$A(x, y) = x + y, \quad B(x, y) = x - y,$$
$$P_1(x, y) = x, \quad P_2(x, y) = y, \quad x, y \in \mathbb{R}^n.$$

Then inequalities (1.3) and (1.4) can be naturally extended as

$$\|u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2\| \leq \epsilon, \quad (1.3')$$
$$\|u \circ A + u \circ B - 2u \otimes u\| \leq \epsilon. \quad (1.4')$$

Here $\otimes$ denotes the tensor product of generalized functions and $u \circ A$, $u \circ B$, $u \circ P_1$, and $u \circ P_2$ the pullbacks of $u$ by $A$, $B$, $P_1$, and $P_2$, respectively and $\|v\| \leq \epsilon$ means that $|\langle v, \varphi \rangle| \leq \epsilon \|\varphi\|_{L^1}$ for all test functions $\varphi$.

As results, we prove that every solution $u$ of inequality (1.3') can be written uniquely in the form

$$u = q(x) + \mu,$$

where $q(x)$ is a quadratic function and $\mu$ is a bounded measurable function such that

$$\|\mu\|_{L^\infty} \leq \frac{7}{6} \epsilon.$$

Also, every solution $u$ of inequality (1.4') is either a bounded measurable function such that

$$\|u\|_{L^\infty} \leq \frac{1}{2} (1 + \sqrt{1 + 2\epsilon})$$

or else the trigonometric function

$$u = \cos(a \cdot x).$$

2. Distributions and hyperfunctions

We first introduce briefly some spaces of generalized functions such as the space $S'$ of tempered distributions and the space $F'$ of Fourier hyperfunctions which is a natural generalization of $S'$. Here we use the multi-index notations for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. 
\[ |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \]
\[ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \]
where \( \mathbb{N}_0 \) is the set of nonnegative integers and \( \partial_j = \partial / \partial x_j \).

**Definition 2.1.** We denote by \( \mathcal{S} \) or \( \mathcal{S}(\mathbb{R}^n) \) the Schwartz space of all infinitely differentiable functions \( \varphi \) in \( \mathbb{R}^n \) such that
\[
\| \varphi \|_{\alpha, \beta} = \sup_{x} |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}
\]
for all \( \alpha, \beta \in \mathbb{N}_0^n \), equipped with the topology defined by the seminorms \( \| \cdot \|_{\alpha, \beta} \). The elements of \( \mathcal{S} \) are called rapidly decreasing functions and the elements of the dual space \( \mathcal{S}' \) are called tempered distributions.

As a matter of fact, it is known in \([2]\) that (2.1) is equivalent to
\[
\sup_{x \in \mathbb{R}^n} |x^\alpha \varphi(x)| < \infty, \quad \sup_{\xi \in \mathbb{R}^n} |\xi^\beta \hat{\varphi}(\xi)| < \infty \tag{2.1'}
\]
for all \( \alpha, \beta \in \mathbb{N}_0^n \).

Imposing growth conditions on \( \| \cdot \|_{\alpha, \beta} \) in (2.1) Sato and Kawai introduced the space \( \mathcal{F} \) of test functions for the Fourier hyperfunctions as follows.

**Definition 2.2.** We denote by \( \mathcal{F} \) or \( \mathcal{F}(\mathbb{R}^n) \) the Sato space of all infinitely differentiable functions \( \varphi \) in \( \mathbb{R}^n \) such that
\[
\| \varphi \|_{A, B} = \sup_{x, \alpha, \beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.2}
\]
for some positive constants \( A, B \).

We say that \( \varphi_j \to 0 \) as \( j \to \infty \) if \( \| \varphi_j \|_{A, B} \to 0 \) as \( j \to \infty \) for some \( A, B > 0 \), and denote by \( \mathcal{F}' \) the strong dual of \( \mathcal{F} \) and call its elements Fourier hyperfunctions.

It can be verified that (2.2) is equivalent to
\[
\| \varphi \|_{h, k} = \sup_{x \in \mathbb{R}^n} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|} \alpha!} < \infty \tag{2.2'}
\]
for some \( h, k > 0 \). Furthermore it is proved in \([3]\) that inequality (2.2') is equivalent to
\[
\sup_{x \in \mathbb{R}^n} |\varphi(x)| \exp k|x| < \infty, \quad \sup_{\xi \in \mathbb{R}^n} |\hat{\varphi}(\xi)| \exp h|\xi| < \infty \tag{2.2''}
\]
for some \( h, k > 0 \).

It is easy to see the following topological inclusions:
\[ \mathcal{F} \hookrightarrow \mathcal{S}, \quad \mathcal{S}' \hookrightarrow \mathcal{F}'. \]

From now on a test function means an element in the Schwartz space \( \mathcal{S} \) or the Sato space \( \mathcal{F} \) and a generalized function means a tempered distribution or a Fourier hyperfunction.

Now we briefly introduce the tensor product of generalized functions.
Definition 2.3. Let \( u_j \in S'(\mathbb{R}^n) \) (respectively, \( F'(\mathbb{R}^n) \)) for \( j = 1, 2 \). Then the tensor product \( u_1 \otimes u_2 \) of \( u_1 \) and \( u_2 \) is defined on \( S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) (respectively, \( F(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \)) by
\[
\langle u_1 \otimes u_2, \varphi(x_1, x_2) \rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2) \rangle \rangle
\]
for \( \varphi(x_1, x_2) \in S(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \) (respectively, \( F(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \)).

Now for pullbacks of generalized functions we refer to [7, Chapter VI]. As a matter of fact, the pullbacks \( u \circ A, u \circ B, u \circ P_1, \) and \( u \circ P_2 \) involved in (1.3') and (1.4') can be written in a more transparent way as
\[
\langle u \circ A, \varphi(x, y) \rangle = \left\langle u, \int \varphi(x - y, y)\, dy \right\rangle,
\]
\[
\langle u \circ B, \varphi(x, y) \rangle = \left\langle u, \int \varphi(x + y, y)\, dy \right\rangle,
\]
\[
\langle u \circ P_1, \varphi(x, y) \rangle = \left\langle u, \int \varphi(x, y)\, dy \right\rangle,
\]
\[
\langle u \circ P_2, \varphi(x, y) \rangle = \left\langle u, \int \varphi(x, y)\, dx \right\rangle
\]
for all test functions \( \varphi \) defined on \( \mathbb{R}^{2n} \).

3. Main theorems

We employ the \( n \)-dimensional heat kernel, that is, the fundamental solution \( E_t(x) \) of the heat operator \( \partial_t - \Delta_x \) in \( \mathbb{R}^n \times \mathbb{R}_t^+ \) given by
\[
E_t(x) = \begin{cases} 
(4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\
0, & t \leq 0.
\end{cases}
\]
The semigroup property
\[
(E_t * E_s)(x) = E_{t+s}(x)
\]
(3.1)
of the heat kernel will be very useful later. Now let a generalized function \( u \) be given. Then its Gauss transform
\[
Gu(x, t) = (u * E)(x, t) = u_y \{ E(x - y, t) \}, \quad x \in \mathbb{R}^n, \; t > 0,
\]
is a \( C^\infty \)-function in \( \mathbb{R}^n \times (0, \infty) \). As a matter of fact we can represent generalized functions via some solutions of the heat equation as follows.

Proposition 3.1 [10]. Let \( u \in S'(\mathbb{R}^n) \). Then its Gauss transform \( Gu(x, t) \) in (3.2) is a \( C^\infty \)-solution of heat equation satisfying

(i) There exist positive constants \( C, M, \) and \( N \) such that
\[
|Gu(x, t)| \leq Ct^{-M} \left( 1 + |x| \right)^N \quad \text{in} \; \mathbb{R}^n \times (0, \delta);
\]
(3.3)
(ii) $G(u, t) \rightarrow u$ as $t \rightarrow 0^+$ in the following sense: for every $\varphi \in S$,

$$
\langle u, \varphi \rangle = \lim_{t \to 0^+} \int G(u, t)\varphi(x) \, dx.
$$

Conversely, every $C^\infty$-solution $U(x, t)$ of heat equation satisfying the growth condition (3.3) can be expressed as $U(x, t) = G(u, t)$ for some $u \in S'$.

Similarly we can represent Fourier hyperfunctions as initial values of solutions of heat equation as a special case of the results in [9]. In this case, estimate (3.3) is replaced by the following: For every $\epsilon > 0$ there exists a positive constant $C_\epsilon$ such that

$$
|G(u, t)| \leq C_\epsilon \exp(\epsilon(|x| + 1/t))
$$

in $\mathbb{R}^n \times (0, \delta)$.

Convolving $E_t(x)E_s(y)$ in each side of the inequalities (1.3') and (1.4') we have the following stabilities of quadratic-additive type and d'Alembert–Cauchy type functional equations for smooth functions $f$:

$$
\|f(x + y, t + s) + f(x - y, t + s) - 2f(x, t) - 2f(y, s)\|_{L^\infty} \leq \epsilon,
$$

(3.5)

$$
\|f(x + y, t + s) + f(x - y, t + s) - 2f(x, t)f(y, s)\|_{L^\infty} \leq \epsilon
$$

(3.6)

for $x, y \in \mathbb{R}^n, t, s > 0$.

Thus we first consider the stabilities of quadratic-additive type functional equation and d'Alembert–Cauchy type functional equation.

**Lemma 1** (Quadratic-additive type). Let $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality

$$
\|f(x + y, t + s) + f(x - y, t + s) - 2f(x, t) - 2f(y, s)\|_{L^\infty} \leq \epsilon.
$$

Then there exists a unique function $g(x, t)$ satisfying the quadratic-additive functional equation

$$
g(x + y, t + s) + g(x - y, t + s) - 2g(x, t) - 2g(y, s) = 0
$$

(3.8)

such that

$$
\|f(x, t) - g(x, t)\|_{L^\infty} \leq \frac{7}{6} \epsilon.
$$

(3.9)

**Proof.** Define an operator $T$ by

$$(Tf)(x, y, t, s) = f(x + y, t + s) + f(x - y, t + s) - 2f(x, t) - 2f(y, s)$$

and let

$$F(x, t) = f(x, t) - f(0, t).
$$

(3.10)

Then we have

$$|(TF)(x, y, t, s)| \leq 2\epsilon.
$$

(3.11)

Putting $y = x, s = t$ in (3.11) and dividing the result by 4 we have

$$\left|\frac{1}{4}F(2x, 2t) - F(x, t)\right| \leq \frac{\epsilon}{2}.$$
Making use of the induction argument and triangle inequality it follows that
\[
\left| 4^{-n} F(2^n x, 2^n t) - F(x, t) \right| \leq \frac{2}{3} \epsilon. \tag{3.12}
\]
On the other hand, putting \(x = y = 0\) and \(s = t\) in (3.7), dividing the result by 2, using the induction argument and the triangle inequality we have
\[
\left| 2^{-n} f(0, 2^n t) - f(0, t) \right| \leq \frac{\epsilon}{2}. \tag{3.13}
\]
Now we set
\[
g_n(x, t) = 4^{-n} F(2^n x, 2^n t) + 2^{-n} f(0, 2^n t).
\]
Then from inequalities (3.12) and (3.13) it is easy to see that \(g_n(x, t)\) is a uniform Cauchy sequence and hence
\[
g(x, t) = \lim_{n \to \infty} g_n(x, t)
\]
exists. Now it follows from inequality (3.12) and (3.13) that
\[
\left| f(x, t) - g_n(x, t) \right| \leq \left| F(x, t) - 4^{-n} F(2^n x, 2^n t) \right| + \left| f(0, t) - 2^{-n} f(0, 2^n t) \right|
\leq \frac{7}{6} \epsilon.
\]
Letting \(n \to \infty\) we get inequality (3.9). Now in view of (3.7) and (3.11) we have
\[
\left| (Tg_n)(x, y, t, s) \right| = \left| 4^{-n} (TF)(2^n x, 2^n y, 2^n t, 2^n s) + 2^{-n} (Tf)(0, 0, 2^n t, 2^n s) \right|
\leq 4^{-n} 2 \epsilon + 2^{-n} \epsilon.
\]
Letting \(n \to \infty\) we get (3.8).

Finally we prove the uniqueness. Let \(G(x, t) = g(x, t) - g(0, t)\). Then \(G(x, t)\) also satisfies Eq. (3.8). Replacing \(g\) in (3.8) by \(G\) and putting \(y = 0\) we have
\[
G(x, t + s) = G(x, t).
\]
Thus \(G(x, t)\) is independent of \(t > 0\) and we may write \(G_0(x) = G(x, 1) = G(x, t)\). Also since \(G\) satisfies Eq. (3.8), \(G_0(x)\) satisfies the quadratic functional equation
\[
G_0(x + y) + G_0(x - y) = 2G_0(x) + 2G_0(y) \tag{3.14}
\]
for all \(x, y \in \mathbb{R}^n\). Consequently, \(G_0\) has the property \(G_0(rx) = r^2 G_0(x)\) and that
\[
G(rx, t) = r^2 G(x, t) \tag{3.15}
\]
for any rational number \(r\).  

Now suppose that \(h(x, t)\) satisfies (3.8) and (3.9) and let \(H(x, t) = h(x, t) - h(0, t)\). Then by property (3.15) and the triangle inequality we have
\[
r^2 \left| G(x, t) - H(x, t) \right| = \left| G(rx, t) - H(rx, t) \right|
\leq \left| g(rx, t) - h(rx, t) \right| + \left| g(0, t) - h(0, t) \right| \leq \frac{28}{6} \epsilon.
\]
Letting $r \to \infty$ we must have $G = H$ and that
\[
|g(x, t) - h(x, t)| = |g(0, t) - h(0, t)| = \frac{1}{k}|g(0, kt) - h(0, kt)| \leq \frac{14}{6k} \epsilon
\] (3.16)
for any positive integer $k$. Letting $k \to \infty$ in (3.16) we conclude that $g = h$. This completes the proof. □

**Lemma 2** (d’Alembert–Cauchy type). Let $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality
\[
\left\| f(x + y, t + s) + f(x - y, t + s) - 2f(x, t)f(y, s) \right\|_{L\infty} \leq \epsilon.
\] (3.17)
Then either
\[
\left\| f(x, t) \right\|_{L\infty} \leq \frac{1}{2}(1 + \sqrt{1 + 2\epsilon})
\] (3.18)
or $f$ satisfies the d’Alembert–Cauchy type functional equation
\[
f(x + y, t + s) + f(x - y, t + s) - 2f(x, t)f(y, s) = 0.
\] (3.19)

**Proof.** First we will show that all the bounded solutions of inequality (3.17) satisfy inequality (3.18). Assume that $f$ is bounded. Put $x = y = 0$ in (3.17) to get
\[
|f(0, t + s) - f(0, t)f(0, s)| \leq \frac{\epsilon}{2}.
\] (3.20)
It is known [8, p. 102] that every bounded solution of inequality (3.20) satisfies the inequality
\[
|f(0, t)| \leq \frac{1}{2}(1 + \sqrt{1 + 2\epsilon}).
\] (3.21)
Assume that there is a bounded solution $f$ of the equation (3.17) satisfying
\[
|f(x_0, t_0)| > \beta = \frac{1}{2}(1 + \sqrt{1 + 2\epsilon})
\] for some $x_0 \in \mathbb{R}^n$, $t_0 > 0$. Putting $x = y = x_0$, $t = s = t_0$ in (3.17) it follows from the triangle inequality and (3.21) that
\[
|f(2x_0, 2t_0)| \geq \frac{2}{\epsilon} |f(x_0, t_0)|^2 - \beta - \epsilon \geq 2(\beta + p)^2 - \beta - \epsilon
\]
\[
\geq \beta + 4p\beta + p^2 \geq \beta + 4p
\]
for some $p > 0$. By the induction argument we conclude that
\[
|f(2^n x_0, 2^n t_0)| \to \infty
\] as $n \to \infty$, which contradicts the assumption that $f$ is bounded.
Now following the similar method as in [8, p. 133] we prove that every unbounded solution $f$ of inequality (3.17) satisfies the functional equation (3.19). Indeed, choose a sequence $(x_n, t_n) \in \mathbb{R}^n \times (0, \infty)$ such that
\[
|f(x_n, t_n)| \to \infty
\]
as \( n \to \infty \) and put \( y = x_n, s = t_n \) in (3.17); we can write
\[
f(x, t) = \lim_{n \to \infty} \frac{f(x + x_n, t + t_n) + f(x - x_n, t + t_n)}{2 f(x_n, t_n)}.
\]
(3.22)

It follows from equality (3.22) that
\[
2 f(x, t) f(y, s) = \lim_{n \to \infty} \frac{1}{2(f(x_n, t_n))^2} \left[ f(x + x_n, t + s + t_n) f(x_n, t_n)
+ f(x + x_n, t + s + t_n) f(y + x_n, s + t_n)
+ f(x - x_n, t + s + t_n) f(x_n, t_n)
+ f(x - x_n, t + s + t_n) f(y - x_n, s + t_n) \right].
\]

Now making use of inequality (3.17) twice we can write
\[
2 f(x, t) f(y, s) = \lim_{n \to \infty} \frac{1}{2(f(x_n, t_n))^2} \left[ f(x + y + x_n, t + s + t_n) f(x_n, t_n)
+ f(x + y - x_n, t + s + t_n) f(x_n, t_n)
+ f(x - y + x_n, t + s + t_n) f(x_n, t_n)
+ f(x - y - x_n, t + s + t_n) f(x_n, t_n) + \epsilon_n \right],
\]
where \( |\epsilon_n| \leq 8 \epsilon \).

Thus it follows from (3.22) that \( f \) satisfies Eq. (3.19). This completes the proof. \( \Box \)

Now we state and prove the stability of the quadratic functional equation and the d’Alembert equation in the spaces of distributions and hyperfunctions.

**Theorem 3.1.** Let \( u \) be a tempered distribution or Fourier hyperfunction satisfying
\[
\| u \circ A + u \circ B - 2 u \circ P_1 - 2 u \circ P_2 \| \leq \epsilon.
\]
(3.23)

Then there exists a unique quadratic form
\[
q(x) = \sum_{1 \leq j \leq k \leq n} a_{jk} x_j x_k
\]
such that
\[
\| u - q(x) \| \leq \frac{7}{6} \epsilon.
\]

**Proof.** Convolving in each side of (3.23) the tensor product \( E_\xi(x) E_\eta(y) \) of \( n \)-dimensional heat kernels as a function of \( x, y \) we have in view of (3.1),
\[
\left[ u \circ A \ast \left( E_\xi(x) E_\eta(y) \right) \right](\xi, \eta) = \left[ u \circ A, E_\xi(\xi - x) E_\eta(\eta - y) \right]
= \left< u_x, \int E_\xi(\xi - x + y) E_\eta(\eta - y) \, dy \right>
\]
\begin{align*}
&= \left< u_x, \int E_t(\xi + \eta - x) E_s(y) \, dy \right>
&= \left< u_x, (E_t * E_s)(\xi + \eta - x) \right> = \left< u_x, E_{t+s}(\xi + \eta - x) \right> = Gu(\xi + \eta, t + s).
\end{align*}

Similarly we have
\begin{align*}
l[(u \circ B) * (E_t(x)E_s(y))] (\xi, \eta) &= Gu(\xi - \eta, t + s),
l[(u \circ P_1) * (E_t(x)E_s(y))] (\xi, \eta) &= Gu(\xi, t),
l[(u \circ P_2) * (E_t(x)E_s(y))] (\xi, \eta) &= Gu(\eta, s),
\end{align*}

where \(Gu(\xi, t)\) is the Gauss transform of \(u\).

Thus inequality (3.23) is converted to
\[\|Gu(x + y, t + s) + Gu(x - y, t + s) - 2Gu(x, t) - 2Gu(y, s)\|_{L^\infty} \leq \epsilon.\] (3.24)

By Lemma 1, there exists a unique function \(g\) satisfying
\[g(x + y, t + s) + g(x - y, t + s) - 2g(x, t) - 2g(y, s) = 0\] (3.25)
such that
\[\|Gu(x, t) - g(x, t)\|_{L^\infty} \leq \frac{7}{6} \epsilon.\] (3.26)

Now since the Gauss transform \(Gu\) is a smooth function, \(g\) is a continuous function as we see in the proof of Lemma 1. Thus the solution \(g(x, t)\) of (3.25) has the form [4, Theorem 3.4]
\[g(x, t) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j + bt.\]

Letting \(t \to 0^+\) in (3.26), it follows that
\[\|u - \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j\| \leq \frac{7}{6} \epsilon.\]

This completes the proof. \(\square\)

**Remark.** The norm inequality \(\|u - q(x)\| \leq (7/6)\epsilon\) implies that \(u - q(x)\) belongs to \((L^1)' = L^\infty\). Thus all the solutions \(u\) in \(S'\) or \(F'\) can be written in the form
\[u = q(x) + \mu,\]
where \(q(x)\) is a quadratic function and \(\mu\) is a bounded measureable function such that
\[\|\mu\|_{L^\infty} \leq \frac{7}{6} \epsilon.\]

**Theorem 3.2.** Let \(u\) be a tempered distribution or Fourier hyperfunction satisfying
\[\|u \circ A + u \circ B - 2u \otimes u\| \leq \epsilon.\] (3.27)

Then \(u\) is a bounded measurable function
\[\|u\| \leq \frac{1}{2} (1 + \sqrt{1 + 2\epsilon}).\] (3.28)
Proof. As in the proof of Theorem 3.1, convolving in (3.27) the tensor product \( E_t(x)E_s(y) \) of heat kernels, inequality (3.27) is converted to the stability of the d’Alembert–Cauchy type functional inequality

\[
\left| Gu(x + y, t + s) + Gu(x - y, t + s) - 2Gu(x, t)Gu(y, s) \right| \leq \epsilon,
\]
where \( Gu(x, t) \) is the Gauss transform of \( u \). By Lemma 2, we have either

\[
\| Gu(x, t) \|_{L_\infty} \leq \frac{1}{2} (1 + \sqrt{1 + 2\epsilon}) \quad \text{(3.29)}
\]
or else \( Gu \) satisfies the d’Alembert–Cauchy type equation

\[
Gu(x + y, t + s) + Gu(x - y, t + s) - 2Gu(x, t)Gu(y, s) = 0. \quad \text{(3.30)}
\]
Letting \( t \to 0^+ \) in (3.29) we get (3.28). Now if \( Gu \) satisfies (3.30), given the smoothness of the Gauss transform \( Gu \), it is known [4] that

\[
Gu(x, t) = \frac{1}{2} e^{bt}(e^{iax} + e^{-iaw}). \quad \text{(3.31)}
\]
Taking the growth conditions (3.3) or (3.4) of \( Gu(x, t) \) into account, \( a \) should be a real vector. Letting \( t \to 0^+ \) we have

\[
u = \cos(a \cdot x),
\]
which also satisfies (3.28). This completes the proof. \( \square \)

Remark. If we consider the inequality (3.27) in a bigger space of generalized functions, for example, the dual space \( \left( S^{1/2} \right)' \) of the Gelfand–Shilov space [6] the solutions \( u \) of the inequality (3.27) are either bounded functions satisfying (3.28) or the trigonometric function

\[
u = \cos(c \cdot x), \quad c \in \mathbb{C}^n.
\]
For more spaces of generalized functions we refer to [5–7,9–11].

References