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Proof of the Conjecture of A. Grothendieck on the Fuglede–Kadison Determinant

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Soit M une algèbre de von Neumann munie d'une trace semi-finie fidèle τ. Pour les éléments τ-compacts x de M, A. Grothendieck introduit dans un "Séminaire Bourbaki," de 1955 une fonction Δ_{1+x} généralisant le déterminant de Fuglede–Kadison, et conjecture l'inégalité

$$\Delta_{1+|x+y|} \leqslant \Delta_{1+|x|}\Delta_{1+|y|}.$$

Dans cet article, nous démontrons cette inégalité. En corollaire, nous obtenons une démonstration directe des inégalités de Clarkson.

Let M be a von Neumann algebra with a faithful semifinite trace τ. For τ-compact elements $x \in M$, Grothendieck introduced in "Séminaire Bourbaki," 1955 a function Δ_{1+x} generalizing the Fuglede–Kadison determinant, and conjectured that

$$\Delta_{1+|x+y|} \leqslant \Delta_{1+|x|}\Delta_{1+|y|}.$$

In this paper, the inequality is proved. As a corollary, a direct proof of the Clarkson inequalities is obtained.

INTRODUCTION

For any trace class operator x in a separable Hilbert space, the Fredholm determinant $\det(1+x)$ makes sense, and we have

$$\det(1+x) = \prod_{n \geqslant 1} (1+\lambda_n(x))$$

where $(\lambda_1(x), \lambda_2(x),...)$ is a listing of all nonzero eigenvalues of x, counted up to algebraic multiplicity (cf. [6]). The fundamental inequality

$$\det(1+|x+y|) \leqslant \det(1+|x|)\det(1+|y|) \qquad (1)$$

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was proven first by Grothendieck (unpublished; see [7]) by inspecting the terms of the classical expansion of the Fredholm determinant $\det(1+h)$ for a trace class operator h. Twenty years after Grothendieck, this inequality was rediscovered by Seiler and Simon in their study of the Yukawa$_2$ quantum field theory (cf. [13, 14]). Several alternative proofs have been found by Rot'feld [12] and Lieb [8].

In [5] Fuglede and Kadison define a determinant function Δ for every II$_1$-factor M and prove that

$$\Delta(xy) = \Delta(x)\,\Delta(y), \qquad \forall x, y \in M.$$

It is natural to ask whether or not inequality (1) remains true in this setting. In fact, the situation is quite different from the classical one, because $\Delta(x)$ ignore the phase of $x \in M$ (i.e., $\Delta(x) = \Delta(|x|)$) and there is no continuous analog for the notion of n-exterior power ($n \in \mathbb{N}$) of an operator. However, (1) remains true and not only for the Fuglede–Kadison determinant, but also for its natural generalization to the case of an arbitrary von Neumann algebra M with a faithful semifinite trace τ.

To be more precise, let $\mu_s(x)$ ($s \geqslant 0$) be the sth singular value of $x \in M$ (cf. [4, 10]) and define the continuous product of the "t-first singular values" of x by

$$\Delta_x(t) = \exp \int_0^t \log \mu_s(x)\, ds.$$

When τ is finite, $\Delta_x(\tau(1))$ is nothing but the Fuglede–Kadison determinant $\Delta(x)$. For τ-compact elements $x, y \in M$, we prove that

$$\Delta_{1+|x+y|} \leqslant \Delta_{1+|x|}\Delta_{1+|y|}. \tag{2}$$

This inequality was conjectured by Grothendieck [7].

The paper is organized as follows: in Section 1, we make the necessary preliminary definitions, and prove the inequality

$$\Delta_{(1+x)(1+y)} \leqslant \Delta_{1+x}\Delta_{1+y}$$

for τ-compact elements x, $y \in M$. This inequality replaces the multiplicativity of the Fuglede–Kadison determinant. In Section 2, we prove the main inequality (2). In Section 3, we give a direct proof of Clarkson's inequalities

$$\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2(\|x\|_p^p + \|y\|_p^p)^{p'/p}$$
$$(1 < p \leqslant 2 \text{ and } p' = p/(p-1)),$$

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2^{p-1}(\|x\|_p^p + \|y\|_p^p) \qquad (2 \leqslant p < \infty)$$

for $x, y \in M$, where $\|x\|_p = \tau(|x|^p)^{1/p}$. The first one has been recently obtained by Zsidó [16], using the more sophisticated interpolation techniques, and the second one goes back to Dixmier [2]. They imply uniform convexity for the L^p-spaces considered by Dixmier [2] and Segal [15].

In Section 4, we give some other applications of the fundamental inequality. We use the usual terminology of von Neumann algebras as in [3].

1. The Determinant Function

Let M be a von Neumann algebra with a faithful semifinite trace τ. Call $x \in M$ *finite rank* (relative to τ) if $\tau(\operatorname{supp}(x^*)) < \infty$, and *compact* if it is a norm limit of finite rank elements. The compact elements are easily seen to be a (two-sided) ideal $C_\infty = C_\infty(M, \tau)$ and the finite rank elements are the smallest ideal whose norm closure is C_∞ (cf. [3, p. 14, Ex. 2]). The *trace ideal* $C_1 = C_1(M, \tau)$ is defined as the set of all $x \in M$ such that

$$\|x\|_1 = \tau(|x|) < \infty.$$

A basic tool in the investigation of the analytical properties of completely continuous operators in a Hilbert space is the notion of "nth singular value." Let us now recall the natural generalization of this notion to our framework.

1.1. Definition. Let $x \in M$ and $t \geqslant 0$. We call "tth *singular value*" of x the number

$$\mu_t(x) = \inf\{\|xe\| \mid e = \textit{projection in } M \textit{ with } \tau(1-e) \leqslant t\}.$$

We have

$$\text{For } x \in M,\ t \mapsto \mu_t(x) \text{ is decreasing and } \mu_0(x) = \|x\|. \tag{1.1.1}$$

$$\text{For } t \geqslant 0,\ x \mapsto 0,\ x \mapsto \mu_t(x) \text{ is increasing on } M_+. \tag{1.1.2}$$

$$\mu_t(x) = \mu_t(x^*) = \mu_t(|x|) \qquad (t \geqslant 0;\ x, y \in M). \tag{1.1.3}$$

Proofs may be found in [4]. Moreover, we have

$$\text{If } \tau(1) = \infty, \text{ then } \mu_s(1 + x) \geqslant 1\ (s \geqslant 0) \text{ for any } x \in C_\infty. \tag{1.1.4}$$

In fact, we have $\|(1 + x)e\| \geqslant 1$ for every infinite projection $e \in M$, because if not exe would be invertible in M_e and hence $C_\infty(M_e, \tau_e) = M_e$ for some infinite projection $e \in M$, a fact which is absurd.

1.2. DEFINITION. Let M be a von Neumann algebra with a faithful semifinite trace τ. Call *determinant function* associated with $x \in M$ the function $\Delta_x: \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Delta_x(t) = \exp \int_0^t \log \mu_s(x)\, ds \qquad (t \geqslant 0).$$

As we have $\log \mu_s(x) \leqslant \log \|x\|$ for $s \geqslant 0$, $\Delta_x(t)$ makes sense (of course, $\int_0^t \log \mu_s(x)\, ds$ may be understood as a lower integral).

1.3. *Remark.* If $\tau(1) < \infty$, $\Delta_x(\tau(1)) = \Delta(x)$, where Δ is the analytical extension (in the terminology of [5]) of the Fuglede–Kadison determinant (see, for example, [4, 2.2.2]).

1.4. LEMMA. *We have*

(i) $\Delta_{1+x} = \Delta_{1+x^*} \leqslant \Delta_{1+|x|}$ $(x \in M)$.

(ii) $\Delta_{xy}(t) \leqslant \|x\|^t \Delta_y(t)$ $(t \geqslant 0; x, y \in M)$.

(iii) $x \mapsto \Delta_{1+x}(t)$ *is increasing on* M_+ *for each* $t \geqslant 0$.

Proof. (i) Let $x \in M$ and $s \geqslant 0$. For any projection e in M, we have

$$\|(1+x)\, e\| \leqslant 1 + \|xe\|$$

so that

$$\mu_s(1+x) \leqslant 1 + \mu_s(x) = 1 + \mu_s(|x|) = \mu_s(1+|x|).$$

The result follows immediately.

(ii) We have $\mu_s(xy) \leqslant \|x\|\, \mu_s(y)$ $(s \geqslant 0)$ by [4, Proposition 1.6(iv)] and the result follows.

(iii) We have $\mu_s(1+x) = 1 + \mu_s(x)$ and we get the result by (1.1.2). ∎

Using the inequality

$$\log(1 + \mu_s(|x|)) \leqslant \mu_s(|x|),$$

we deduce from 1.4(i) and [4, Proposition 1.11] that

$$\Delta_{1+x} \leqslant \exp(\|x\|_1) \qquad \text{for} \quad x \in C_1.$$

1.5. *Remark.* If $\tau(1) = \infty$, $\Delta_{1+x}(t)$ has a limit $(t \to \infty)$ for any $x \in C_\infty$ by virtue of (1.1.4). Put

$$\Delta(1+x) = \lim_{t\to\infty} \Delta_{1+x}(t).$$

Then Δ is a finite positive function on $1+C_1$, and it is easy to see that

$$\Delta(1+|x|)=\exp\tau(\log(1+|x|)) \qquad \text{for} \quad x\in C_1.$$

However, $\Delta(1+x)$ does not generally coïncide with $\exp\tau$ ($\log|1+x|$), so that it is not in any way a "generalized Fredholm determinant."

The main result of this section is the following, which is a natural generalization of [4, Theorem 2.3].

1.6. THEOREM. *Let M be a von Neumann algebra with a faithful semifinite trace, and $x, y\in C_\infty$. Then, we have*

$$\Delta_{(1+x)(1+y)}(t)\leqslant\Delta_{1+x}(t)\,\Delta_{1+y}(t) \qquad \textit{for} \quad t>0.$$

Proof. If M is finite, the result follows immediately from [4, Theorem 2.3]. Assume now that M is infinite and put $|(1+x)(1+y)|^2=1+h$, where $h\in C_\infty$. Fix $t>0$. Using (1.1.4) and (1.1.1), we may assume w.l.o.g. that $\mu_s(1+h)>1$ for $s<t$. Then, it is almost clear (and we shall come back to this point) that there exists two finite projections p, q in M such that

$$\Delta_{(1+x)(1+y)}(t)=\Delta_{q(1+x)(1+y)p}(t).$$

As $q(1+x)$ and $(1+y)p$ are τ-compact elements in M, we get

$$\Delta_{(1+x)(1+y)}(t)\leqslant\Delta_{q(1+x)}(t)\,\Delta_{(1+y)p}(t)$$

by [4, Theorem 2.3]. Using 1.4(ii), we get the result.

Let us now indicate how to find p and q. Let $h=\int_{-1}^{\infty}\lambda\,de_\lambda$ be the spectral decomposition of h. Using [4, Propsition 1.3] and (1.1.4), we get

$$\mu_t(1+h)=1+\mu_0,$$

where $\mu_0=\min\{\mu\geqslant 0\,|\,\tau(1-e_\mu)\leqslant t\}$. Assume first that $\mu_0>0$ and put $p=1-e_{\mu_0^-}$. Then

$$\tau(p)=\lim_{\mu\to\mu_0^-}\tau(1-e_\mu)\geqslant t$$

and we get

$$\mu_s(1+h)=\mu_s(p(1+h)p) \qquad \text{for} \quad s<t$$

by [4, Proposition 1.5]. Then

$$\begin{aligned}\mu_s((1+x)(1+y))&=\mu_s(1+h)^{1/2}\\&=\mu_s(|(1+x)(1+y)p|^2)^{1/2}\\&=\mu_s((1+x)(1+y)p) \qquad (s<t).\end{aligned}$$

But now $k = |p(1+y)^*(1+x)^*|^2$ is compact and there exists as in [4, Lemma .13] a finite projection q in M such that $\mu_s(k) = \mu_s(qkq)$ for $s < t$. It follows that

$$\mu_s((1+x)(1+y)) = \mu_s(k)^{1/2} = \mu_s(|p(1+y)^*(1+x)^*q|^2)^{1/2}$$
$$= \mu_s(q(1+x)(1+y)p) \quad \text{for} \quad s < t.$$

If $\mu_0 = 0$, we put $p = 1 - e_0$ and choose q as before. We have $\tau(p) = t$ and $\mu_s((1+x)(1+y)) = \mu_s(q(1+x)(1+y)p)$ for $s < t$. The proof of Theorem 1.6 is then complete. ∎

1.7 COROLLARY. *Let M be a von Neumann algebra with a faithful semifinite trace, and $x, y \in C_\infty$. Let $w \in 1 + C_\infty$ with $\|w\| \leqslant 1$. Then, we have*

$$\Delta_{(1+x)w(1+y)}(t) \leqslant \Delta_{1+x}(t)\,\Delta_{1+y}(t) \qquad \textit{for} \quad t > 0.$$

We are now in position to prove the main inequality.

2. PROOF OF THE MAIN INEQUALITY

2.1. THEOREM. *Let M be a von Neumann algebra with a faithful semifinite trace. Let $x, y \in C_\infty$. Then, we have*

$$\Delta_{1+|x+y|} \leqslant \Delta_{1+|x|}\Delta_{1+|y|}.$$

The proof is based on Theorem 1.6, combined with the following technical lemma which replaces the wrong inequality $1 + x + y \leqslant 1 + |x| + |y|$.

2.2. LEMMA. *Let M be a von Neumann algebra and $x, y \in M$. Then, there exists an element $w \in M$, $\|w\| \leqslant 1$, such that*

$$1 + x + y = (1 + |x^*| + |y^*|)^{1/2}\, w(1 + |x| + |y|)^{1/2}.$$

Proof. Let $(e_{ij})_{1 \leqslant i,j \leqslant 3}$ be a system of matrix units for $M_3(\mathbb{C})$ and put

$$a = 1 \otimes e_{11} + |x|^{1/2} u^* \otimes e_{21} + |y|^{1/2} v^* \otimes e_{31} \in M \otimes M_3(\mathbb{C})$$
$$b = 1 \otimes e_{11} + |x|^{12} \otimes e_{21} + |y|^{1/2} \otimes e_{31} \in M \otimes M_3(\mathbb{C}),$$

where u and v are the phases of x and y. We get by direct calculation

$$|a| = (1 + u|x|u^* + v|y|v^*)^{1/2} \otimes e_{11} = (1 + |x^*| + |y^*|)^{1/2} \otimes e_{11}$$

and

$$|b| = (1 + |x| + |y|)^{1/2} \otimes e_{11}.$$

Let U (resp. V) be phase of a (resp. b). We get

$$\begin{aligned} a^*b &= |a|\, U^*V\,|b| \\ &= [(1+|x^*|+|y^*|)^{1/2}\otimes e_{11}][w\otimes e_{11}][(1+|x|+|y|)^{1/2}\otimes e_{11}], \end{aligned}$$

where $w\in M$ and $\|w\|\leqslant 1$. But $a^*b=(1+x+y)\otimes e_{11}$, so that the lemma is proved. ∎

Proof of Theorem 2.1. Let $x, y\in C_\infty$ and $t>0$.

Step 1. Let us first show that we have

$$\Delta_{1+x+y}(t)\leqslant \Delta_{1+x}(t)\,\Delta_{1+y}(t)$$

for positive $x, y\in C_\infty$. We have

$$1+x+y=(1+x)^{1/2}\,[1+|y^{1/2}(1+x)^{-1/2}|^2](1+x)^{1/2}$$

and Theorem 1.6 implies

$$\begin{aligned} \Delta_{1+x+y}(t) &\leqslant \Delta_{(1+x)^{1/2}}(t)\,\Delta_{1+|y^{1/2}(1+x)^{-1/2}|^2}(t)\,\Delta_{(1+x)^{1/2}}(t) \\ &= \Delta_{1+x}(t)\,\Delta_{1+|y^{1/2}(1+x)^{-1/2}|^2}(t). \end{aligned}$$

But $\Delta_{1+|z|^2}(t)=\Delta_{1+|z^*|^2}(t)$, and hence

$$\Delta_{1+x+y}(t)\leqslant \Delta_{1+x}(t)\,\Delta_{1+y^{1/2}(1+x)^{-1}y^{1/2}}(t).$$

But $(1+x)^{-1}\leqslant 1$, and we get the result by 1.4(iii).

Step 2. Let us now show that we have

$$\Delta_{1+x+y}(t)\leqslant \Delta_{1+|x|}(t)\,\Delta_{1+|y|}(t) \qquad \text{for} \quad x, y\in C_\infty.$$

By Lemma 2.2, there exists a contraction $w\in M$ such that

$$1+x+y=(1+|x^*|+|y^*|)^{1/2}\,w(1+|x|+|y|)^{1/2}.$$

By Corollary 1.7, we get

$$\Delta_{1+x+y}(t)\leqslant \Delta_{1+|x^*|+|y^*|}(t)^{1/2}\,\Delta_{1+|x|+|y|}(t)^{1/2}.$$

By step 1, we get

$$\begin{aligned} \Delta_{1+|x^*|+|y^*|}(t) &\leqslant \Delta_{1+|x^*|}(t)\,\Delta_{1+|y^*|}(t) \\ &= \Delta_{1+|x|}(t)\,\Delta_{1+|y|}(t) \end{aligned}$$

and

$$\Delta_{1+|x|+|y|}(t)\leqslant \Delta_{1+|x|}(t)\,\Delta_{1+|y|}(t),$$

so that finally

$$\Delta_{1+x+y}(t) \leqslant \Delta_{1+|x|}(t)\, \Delta_{1+|y|}(t).$$

End of the Proof. Let $x+y=U|x+y|$ be the polar decomposition of $x+y$. By step 2, we have

$$\Delta_{1+|x+y|}(t) = \Delta_{1+U^*x+U^*y}(t) \leqslant \Delta_{1+|U^*x|}(t)\, \Delta_{1+|U^*y|}(t).$$

But $\Delta_{1+|U^*x|}(t) = \Delta_{1+(x^*UU^*x)^{1/2}}(t) \leqslant \Delta_{1+|x|}(t)$ by [11, Proposition 1.3.8] and 1.4(iii), so that finally

$$\Delta_{1+|x+y|}(t) \leqslant \Delta_{1+|x|}(t)\, \Delta_{1+|y|}(t). \quad ∎$$

From Theorem 2.1, we shall now deduce many inequalities.

3. Minkowski and Clarkson's Noncommutative Inequalities

Let M be a von Neumann algebra with a faithful semifinite trace τ. For $x \in M$ and $p > 0$, set

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

If $p \geqslant 1$, we have the well-known inequality of Minkowski

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p.$$

Let us first prove an inequality which replaces Minkowski's when $0 < p < 1$.

3.1. Proposition. *Let $x, y \in M$. Then, we have*

$$\tau(|x+y|^p) \leqslant \tau(|x|^p) + \tau(|y|^p) \qquad \textit{for} \quad 0 < p \leqslant 1.$$

Note that 3.1 implies that $\delta(x,y) = \|x-y\|_p^p$ is a metric on $C_p = \{x \in M \mid \|x\|_p < \infty\}$ for $p \leqslant 1$.

The proof of 3.1 is based on the following technical lemma:

3.2. Lemma. *Let φ be a positive function on $\mathbb{R}_+$.*
Assume that φ is bounded and set

$$\pi_t(r) = \exp \int_0^t \log(1 + r\varphi(s))\, ds \qquad (t, r > 0).$$

Then, we have for each p $(0 < p < 1)$

$$\int_0^t \varphi^p(s)\, ds = \frac{p \sin(\pi p)}{\pi} \int_0^\infty \frac{\log \pi_t(r)}{r^{p+1}}\, dr.$$

Proof. We have, for a fixed $t > 0$

$$\frac{d}{dr}\pi_t(r) = \pi_t(r)\int_0^t \frac{\varphi(s)}{1 + r\varphi(s)}\,ds,$$

and hence

$$\int_0^\infty \frac{d(\log \pi_t(r))}{r^p} = \int_0^\infty \frac{1}{r^p}\left(\int_0^t \frac{\varphi(s)}{1 + r\varphi(s)}\,ds\right) dr.$$

But

$$\int_0^t \left[\int_0^\infty \frac{\varphi(s)}{r^p(1 + r\varphi(s))}\,dr\right] ds = \int_0^t \varphi(s)^p \left[\int_0^\infty \frac{du}{u^p(1+u)}\right] ds$$

$$= \frac{\pi}{\sin p\pi}\int_0^t \varphi(s)^p\,ds < \infty,$$

so that we get by the Lebesgue–Fubini theorem

$$\int_0^\infty \frac{d(\log \pi_t(r))}{r^p} = \frac{\pi}{\sin p\pi}\int_0^t \varphi(s)^p\,ds.$$

On the other hand, we have

$$\int_0^R \frac{d(\log \pi_t(r))}{r^p} = \frac{\log \pi_t(R)}{R^p} + p\int_0^R \frac{\log \pi_t(r)}{r^{p+1}}\,dr.$$

As the left-hand side has a limit for $R \to \infty$, the two terms of the right-hand side must also have a limit, and the first one goes to zero. Henceforth, we have

$$\int_0^\infty \frac{\log \pi_t(r)}{r^{p+1}}\,dr = \frac{\pi}{p\sin(p\pi)}\int_0^t \varphi(s)^p\,ds. \quad \blacksquare$$

Proof of Proposition 3.1. This proposition has nontrivial content only if x and y are compact. The case $p = 1$ is well known (see, for example, [4]), so that we shall assume that $0 < p < 1$. By Theorem 2.1, we get

$$\log \Delta_{1+r|x+y|}(t) \leqslant \log \Delta_{1+r|x|}(t) + \log \Delta_{1+r|y|}(t) \quad (t, r > 0)$$

and hence

$$\int_0^t \mu_s(|x+y|)^p\,ds \leqslant \int_0^t \mu_s(|x|)^p\,ds + \int_0^t \mu_s(|y|)^p\,ds$$

by Lemma 3.2. By letting $t \to \infty$ and using [4, Proposition 1.6(ii)], we get the result. $\blacksquare$

The following lemma, that we shall need to prove Clarkson's inequalities in full generality, contains the analog for $p \geqslant 1$ (and x, y nonnegative) of Proposition 3.1.

3.3. LEMMA. *Let x, y be positive elements in M. Then, we have*

$$2^{1-p}\|x+y\|_p^p \leqslant \|x\|_p^p + \|y\|_p^p \leqslant \|x+y\|_p^p \qquad \text{for} \quad p \geqslant 1.$$

Proof. We may assume w.l.o.g. that M acts on a Hilbert space H and that $\|x\|_p$, $\|y\|_p$ and $\|x+y\|_p < \infty$. Let us first check the inequality on the left. Recall first the well-known inequality for positive numbers λ, ρ

$$2^{1-p}(\lambda+\rho)^p \leqslant \lambda^p + \rho^p \qquad (p \geqslant 1).$$

Using [4 Proposition 4.3], we get

$$\int_0^t \mu_s(x+y)^p \, ds \leqslant \int_0^t (\mu_s(x) + \mu_s(y))^p \, ds,$$

and the previous inequality implies

$$2^{1-p} \int_0^t \mu_s(x+y)^p \, ds \leqslant \int_0^t \mu_s(x)^p \, ds + \int_0^t \mu_s(y)^p \, ds.$$

The result follows by letting $t \to \infty$.

To prove the inequality on the right, we proceed essentially as in [9, Lemma 2.6]. We have $x, y \leqslant x+y$, so that there exists elements u, $v \in M$, $\|u\|$, $\|v\| \leqslant 1$, such that

$$x^{1/2} = u(x+y)^{1/2}, \qquad y^{1/2} = v(x+y)^{1/2}$$

(cf. [3, Proposition 10, p. 11]).

Hence, we have

$$x = u(x+y)\,u^*, \qquad y = v(x+y)\,v^*.$$

We claim that $\tau(x^p) \leqslant \tau(u(x+y)^p\, u^*)$. In fact, let t be a positive number and let e be a projection in M with $\tau(1-e) \leqslant t$. For $\xi \in e(H)$, $\|\xi\| = 1$, put $\eta = u^*\xi$. We have

$$(x\xi \mid \xi)^p = ((x+y)\,\eta \mid \eta)^p = \left(\int_0^\infty \lambda d(e_\lambda(\eta) \mid \eta)\right)^p,$$

where $x+y = \int_0^\infty \lambda de_\lambda$ is the spectral decomposition of $x+y$. But

$$\left(\int_0^\infty \lambda d(e_\lambda(\eta) \mid \eta)\right)^p \leqslant \|\eta\|^{p-1} \int_0^\infty \lambda^p d(e_\lambda(\eta) \mid \eta)$$

by Jensen's inequality (note that $t \mapsto t^p$ is convex for $p \geqslant 1$). As $\|\eta\| \leqslant 1$, we get

$$(x\xi \mid \xi)^p \leqslant (u(x+y)^p u^* \xi \mid \xi) \quad \text{for any } \xi \in e(H) \text{ with } \|\xi\| = 1.$$

Hence, we have by [4, Remark 1.4.1]

$$\mu_t(x)^p \leqslant \mu_t(u(x+y)^p u^*)$$

and hence

$$\tau(x^p) \leqslant \tau(u(x+y)^p u^*).$$

Similarly, we get

$$\tau(y^p) \leqslant \tau(v(x+y)^p v^*),$$

and hence

$$\tau(x^p) + \tau(y^p) \leqslant \tau((x+y)^p (u^*u + v^*v)).$$

Now, using cyclicity of the trace together with the obvious equality

$$x + y = (x+y)^{1/2} (u^*u + v^*v)(x+y)^{1/2},$$

we get

$$\begin{aligned} \tau(x^p) + \tau(y^p) &\leqslant \tau((x+y)^{p-1} (x+y)^{1/2} (u^*u + v^*v)(x+y)^{1/2}) \\ &= \tau((x+y)^p). \end{aligned}$$

The inequality on the right is then proved. ∎

We are now in position to prove the noncommutative analogs of the Clarkson inequalities [1].

3.4. THEOREM. *Let $x, y \in M$. We have*

(i) $\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2(\|x\|_p^p + \|y\|_p^p)^{p'/p}$ *for* $1 < p \leqslant 2$ *and* $p' = p/p-1$,

(ii) $\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2^{p-1}(\|x\|_p^p + \|y\|_p^p)$ *for* $2 \leqslant p < \infty$.

These inequalities are due to McCarthy [9] when M is the algebra of all bounded operators in a Hilbert space. The proof of (ii) for general semifinite von Neumann algebras goes back to Dixmier [2], and (i) has been recently obtained by Zsido [16]. Our proof, based on Proposition 3.1, is new and doesn't use the more sophisticated interpolation techniques.

Proof of (i). We may assume that M acts in a Hilbert space H and then choose elements $\xi_\alpha \in H$, $\|\xi_\alpha\| = 1$ $(\alpha \in I)$ such that

$$\tau(u) = \sum_\alpha (u\xi_\alpha | \xi_\alpha)$$

for each positive element u in M. Set

$$a = x^*x + y^*y, \qquad b = x^*y + y^*x$$

The same computation as in [9, Theorem 2.7(ii), p. 261–262] gives us $\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2\{\sum_\alpha 2^{-1}[((a+b)^{p/2}\xi_\alpha|\xi_\alpha) + ((a-b)^{p/2}\xi_\alpha|\xi_\alpha)]\}^{p'/p}$. But the function $t \mapsto t^{p/2}$ is operator concave on $\mathbb{R}_+$ (cf. [11]), so that we get

$$\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2\left[\sum_\alpha (a^{p/2}\xi_\alpha | \xi_\alpha)\right]^{p'/p}$$

and hence

$$\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2\{\tau[(x^*x + y^*y)^{p/2}]\}^{p'/p}.$$

Using Proposition 3.1, we get then

$$\|x+y\|_p^{p'} + \|x-y\|_p^{p'} \leqslant 2(\|x\|_p^p + \|y\|_p^p)^{p'/p}.$$

Proof of (ii). Set $q = p/2$. We have

$$\begin{aligned}
\|x+y\|_p^p + \|x-y\|_p^p &= \||x+y|^2\|_q^q + \||x-y|^2\|_q^q \\
&\leqslant \||x+y|^2 + |x-y|^2\|_q^q \qquad \text{(use 3.3 with } q = p/2 \geqslant 1) \\
&= 2^q \||x|^2 + |y|^2\|_q^q \\
&\leqslant 2^q 2^{q-1}(\||x|^2\|_q^q + \||y|^2\|_q^q) \\
&= 2^{p-1}(\|x\|_p^p + \|y\|_p^p). \quad \blacksquare
\end{aligned}$$

4. Other Applications

4.1.

From the fundamental inequality 2.1, we can derive many other inequalities by using [4, Corollary 4.2]. More precisely, we get

$$\int_0^a g[1 + \mu_s(x+y)]\, ds \leqslant \int_0^a g[(1 + \mu_s(x))(1 + \mu_s(y))]\, ds$$

$$(a > 0; x, y \in C_\infty)$$

for any nondecreasing continuous function $g: [0, +\infty[\to \mathbb{R}$ such that $t \mapsto g(\exp(t))$ is convex on $[-\infty, +\infty[$. Taking $g(t) = t^p (p > 0)$ and using Hölder's inequality for functions, we get for a von Neumann algebra M with finite trace τ and $x, y \in M$

$$\| 1 + |x + y| \|_p \leqslant \| 1 + |x| \|_q \, \| 1 + |y| \|_r \quad \text{if } 1/p = 1/q + 1/r.$$

4.2

Let M be a von Neumann algebra with a faithful semifinite trace τ. From Theorem 2.1, we may deduce the continuity of the function $x \mapsto \Delta(1 + x) = \exp \tau(\log(1 + x))$ on the positive part of C_1. More precisely, let x, y be positive elements in C_1 and $t > 0$. We have

$$\Delta_{1+x}(t) = \Delta_{1+|x-y+y|}(t) \leqslant \Delta_{1+y}(t)\, \Delta_{1+|x-y|}(t) \qquad \text{(by Theorem 2.1)}$$

and hence

$$\Delta(1 + x) \leqslant \Delta(1 + y)\, \Delta(1 + |x - y|)$$

by letting $t \to \infty$. Then

$$\Delta(1 + x) - \Delta(1 + y) \leqslant \Delta(1 + y)[\Delta(1 + |x - y|) - 1].$$

Using the inequality

$$e^u - 1 \leqslant u e^u \qquad \text{for} \quad u \geqslant 0,$$

we get

$$\begin{aligned} \Delta(1 + x) - \Delta(1 + y) &\leqslant \Delta(1 + y)\, \Delta(1 + |x - y|) \int_0^\infty \log(1 + \mu_s(|x - y|))\, ds \\ &\leqslant \exp(\| y \|_1) \exp(\| x - y \|_1) \| x - y \|_1 \\ &\leqslant \exp(\| x \|_1)\, \| x - y \|_1 \qquad \text{(by Lemma 3.3)} \\ &\leqslant \exp(\| x \|_1 + \| y \|_1) \| x - y \|_1 . \end{aligned}$$

By symmetry

$$|\Delta(1 + x) - \Delta(1 + y)| \leqslant \exp(\| x \|_1 + \| y \|_1) \| x - y \|_1$$

so that $x \mapsto \Delta(1 + x)$ is locally Lipschitz on the positive part of C_1.

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