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Strong Nilpotence of Solvable Ideals in Quadratic Jordan Algebras

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An ideal K is called strongly nilpotent in a Jordan algebra J if there is an n such that any product of elements of J having n or more factors from K vanishes. Jacobson has shown that any solvable ideal K is strongly nilpotent in any finite-dimensional linear Jordan algebra J (equivalently, K generates a nilpotent ideal in the universal multiplication envelope $\mathscr{U}(J)$). In this paper we extend this result to quadratic Jordan algebras and weaken the hypotheses to finiteness of K alone: if K is a solvable finitely generated quadratic Jordan algebra, then K generates a nilpotent ideal in $\mathscr{U}(J)$ (and hence is strongly nilpotent in J) for any algebra $J \rhd K$, as long as either (i) the scalars Φ form a noetherian ring or (ii) J/K is also finitely generated. Previously only the case $J = K$ was known: the Albert–Zhevlakov theorem asserts that a finitely generated quadratic Jordan algebra K is solvable iff $\mathscr{U}(K)$ is nilpotent.

Throughout we are concerned with ideals K in quadratic Jordan algebras J over an arbitrary ring of scalars Φ. Since the ideals in J are the same as those in the *unital hull* $\hat{J} = \Phi 1 + J$, J will always denote a *unital* Jordan algebra, while K will denote a not-necessarily-unital Jordan algebra. We will usually be concerned with K which is finitely generated *as algebra*; note that this is a much stronger condition than K being finitely generated *as ideal* in J.

A unital Jordan algebra J is equipped with a product $U_x y$ quadratic in x and linear in y and a unit element 1 such that

$$\begin{aligned} &(\text{QJ1}) && U_1 = Id \\ &(\text{QJ2}) && V_{x,y} U_x = U_x V_{y,x} \qquad (0.1) \\ &(\text{QJ3}) && U_{U(x)y} = U_x U_y U_x \end{aligned}$$

hold strictly. The axiomatization for a non-unital algebra is more involved:

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K is equipped with products $U_x y$ and x^2 quadratic in x and linear in y, such that [5, p. 273]

$$\begin{aligned}
&(\text{CJ1}) && V_{x,x} = V_{x^2} \\
&(\text{CJ2}) && U_x V_x = V_x U_x \\
&(\text{CJ3}) && U_x(x^2) = (x^2)^2 \\
&(\text{CJ4}) && (x^3)^2 = (x^2)^3 \qquad (x^3 = U_x x) \\
&(\text{CJ5}) && U_{x^2} = U_x^2 \\
&(\text{CJ6}) && U_{x^3} = U_x^3
\end{aligned} \tag{0.2}$$

hold strictly. Here

$$V_{x,y}(z) = \{xyz\} = U_{x,z}\,y, \qquad U_{x,z} = U_{x+z} - U_x - U_z,$$
$$V_x(y) = x \circ y, \qquad x \circ y = (x+y)^2 - x^2 - y^2.$$

The archetypal example of a Jordan algebra is the algebra A^q obtained from an associative algebra A via $x^2 = xx$, $U_x y = xyx$ (so $x \circ y = xy + yx$, $\{xyz\} = xyz + zyx$); a Jordan algebra is *special* if it is isomorphic to a subalgebra of some A^q.

1. Strong Nilpotence

In this section we introduce the notion of strong nilpotence of an ideal and relate it to ordinary nilpotence and solvability. An *ideal* $K \lhd J$ is a subspace such that a product falls in K as soon as one factor does: $U_K J$ and $U_J K$ (hence also K^2, $K \circ J$, $\{JJK\}$) are contained in K. We are interested in conditions under which K acts nilpotently on a larger algebra J.

An algebra K is *solvable* if some derived algebra vanishes, $D^n(K) = 0$, where $D^0(K) = K$, $D^{k+1}(K) = D(D^k(K))$ for $D(L) = U_L L$. These derived algebras are again ideals in J in K is, since in general if B, C are ideals so is their quadratic product $U_B C$ (spanned by all $U_b c$ for $b \in B$, $c \in C$).

An algebra K is *nilpotent* if some power vanishes, $K^n = 0$, where K^n is spanned by all products of n or more factors from K (counting $U_x y$ as having two factors x). Nilpotence is a stronger condition than solvability: we always have

$$D(K) = K^3, \qquad D^k(K) \subset K^{3^k} \tag{1.1}$$

but in general we do not have $D^2(K) \supset K^n$ for any n unless K is finitely generated (see 4.4).

The powers K^n of an ideal need not be ideals; to remedy this we define the nth *power* $K^{(n,J)}$ *of* K *in* J to be the space spanned by all products of elements of J having n or more factors from K. By definition the $K^{(n,J)}$'s form a decreasing sequence of ideals in J. We say K is *strongly nilpotent in* J if some $K^{(n,J)}=0$. Since

$$K^n \subset K^{(n,J)}, \tag{1.2}$$

we see that strong nilpotence implies nilpotence.

We can relate strong nilpotence to multiplications of K on J. The *multiplication algebra* $\mathscr{M}(J)$ is the subalgebra of $\mathrm{End}_\Phi(J)$ generated by all multiplication operators U_a, $U_{a,b}$, $V_{a,b}$, V_a $(a, b \in J)$. By $\mathscr{M}_n(K;J)$ we denote the ideal in the multiplication algebra $\mathscr{M}(J)$ spanned by operators with n or more factors from K; here if $x, y \in K$, $a \in J$, then U_x and $U_{x,y}$ and $V_{x,y}$ count as two factors, while V_x and $V_{x,a}$ and $U_{x,a}$ count as one factor from K. Since J is unital, we can write

$$K^{(n,J)} = \mathscr{M}_n(K;J)J + \mathscr{M}_{n-1}(K;J)K \tag{1.3}$$

(compare [3, p. 475]); the second term can be omitted when $\frac{1}{2} \in \Phi$ since then $K = \frac{1}{2}V_K 1$. By (1.3) strong nilpotence of K in J is equivalent to the vanishing of some $\mathscr{M}_n(K;J)$ $(K^{(n,J)}=0 \Rightarrow \mathscr{M}_n(K;J)=0 \Rightarrow K^{(n+1,J)}=0)$. Moreover, vanishing of some $\mathscr{M}_n(K;J)$ is equivalent to ordinary associative nilpotence of the *multiplication ideal of* K *on* J

$$\mathscr{M}(K;J) = \mathscr{M}_1(K;J) = \mathscr{M}(J)\{U_K + V_{K,J}\}\mathscr{M}(J) \tag{1.4}$$

(as in (2.3), (2.4), $V_K = V_{K,1}$ and $V_{K,K} = V_K V_K - U_{K,K}$ and $U_{K,J} = V_K V_J - V_{K,J}$ are generated by $V_{K,J}$ and U_K, so these are the only multiplications involving K that we need) since

$$\mathscr{M}(K;J)^m \subset \mathscr{M}_m(K;J), \qquad \mathscr{M}_{2n}(K;J) \subset \mathscr{M}(K;J)^n. \tag{1.5}$$

Indeed, $\mathscr{M}(K;J)^m \subset \mathscr{M}_m(K;J)$ is clear, and $\mathscr{M}_{2n}(K;J) \subset \mathscr{M}(K;J)^n$ holds since any spanning monomial from $\mathscr{M}_{2n}(K;J)$ contains i factors U_K and j factors $V_{K,J}$ for $2i+j \geqslant 2n$, hence $2(i+j) \geqslant 2n$ and $i+j \geqslant n$, therefore, has at least n factors from $\mathscr{M}(J)\{U_K + V_{K,J}\}\mathscr{M}(J) = \mathscr{M}(K;J)$.

Thus just as nilpotence of K is equivalent to nilpotence of its multiplication algebra, so strong nilpotence of K in J is equivalent to nilpotence of its multiplication ideal.

1.6 PROPOSITION. *An ideal* K *is strongly nilpotent in* J *iff its multiplication ideal* $\mathscr{M}(K;J)$ *is nilpotent in the multiplication algebra* $\mathscr{M}(J)$. ∎

In view of (1.1), (1.2) we have

strong nilpotence $\Rightarrow$ nilpotence $\Rightarrow$ solvability.

Our main goal is the converse of this in the finitely generated case,

1.7 STRONG NILPOTENCE THEOREM. *A solvable finitely generated quadratic Jordan algebra K is strongly nilpotent in any extension $J \rhd K$ (equivalently, its multiplication ideal $\mathcal{M}(K; J)$ is nilpotent) such that either* (i) *Φ is noetherian or* (ii) *J/K is finitely generated.* □

It was Zhevlakov [6] who first showed that results previously obtained for finite-dimensional algebras could be extended to finitely generated algebras. (We will see in 4.9 that K is automatically finitely spanned over Φ when it is solvable and finitely generated, so it is not far from being finite-dimensional in this case). Note also that when K is finitely generated the condition that J/K be finitely generated too is equivalent to the condition that J itself be finitely generated.

1.8 EXAMPLE. The Strong Nilpotence Theorem (indeed, even the Albert–Zhevlakov theorem 5.3) fails if K is not finitely generated: over an arbitrary Φ we can construct a Jordan algebra K with

(i) $J = \hat{K}$ has J/K spanned by one element;

(ii) K is solvable, $D^2(K) = 0$ (even Penico-solvable, $P_J^2(K) = 0$);

(iii) J and K are special;

(iv) K is not nilpotent, $V_{K,K}^n K \neq 0$ for all n.

(Note that we cannot strengthen (ii) to (ii′) $D(K) = 0$ since then $K^3 = D(K) = 0$ and K would be nilpotent.) Let $A = \bigwedge(X)$ be the unital exterior algebra on a space X, and $K \subset M_2(A)^q$ the Jordan subalgebra of all $\begin{pmatrix} x & a \\ 0 & 0 \end{pmatrix}$ for $x \in X$, $a \in A$. Here the Jordan products are

$$\begin{pmatrix} x & a \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix},$$

$$U\left(\begin{pmatrix} x & a \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} y & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xya \\ 0 & 0 \end{pmatrix},$$

$$\left\{\begin{pmatrix} x & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z & c \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 & xyc + zya \\ 0 & 0 \end{pmatrix}$$

since $x^2 = xyx = 0$ in $A = \bigwedge(X)$, so K is indeed a Jordan subalgebra; moreover $P(K) \subset \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ is trivial, $P^2(K) = 0$. If $X = \bigoplus \Phi x_i$ is an infinite-dimensional free module over Φ, then for $k_i = \begin{pmatrix} x_i & 0 \\ 0 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we have $V_{k_1,k_2} \cdots V_{k_{2n-1},k_{2n}} k = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ for $a = x_1 \wedge x_2 \wedge \cdots \wedge x_{2n-1} \wedge x_{2n} \neq 0$. ∎

1.9 *Remark.* It is not clear to what extent conditons (i) or (ii) are necessary for 1.7. As we remarked before, by 4.9, K will always be finitely spanned. Let us show that strong nilpotence holds at least when $K = \Phi k$ is spanned by one element, no matter what J and Φ are (so (i) and (ii) are superfluous in this case). Here $k \circ a = t(a)k$, $k^2 = \varepsilon k$, $k^{2^m} = \varepsilon^{2^m - 1}k$, $2U_k a = k \circ (k \circ a) - k^2 \circ a = \varepsilon t(a)k$, and $V_{k,a_1} \cdots V_{k,a_n} k = V_{k,a_1} \cdots V_{k,a_{n-1}}(2U_k a_n) = \varepsilon^n t(a_1) \cdots t(a_n)k$. But solvability of K forces nilpotence of k, some $\varepsilon^n k = k^m = 0$, hence $V^n_{K,J} K = U^n_K J = 0$, $V^{n+1}_{K,J} = U^n_K = 0$, and $\mathcal{M}(K; J)^{2n} = 0$ (we will see in Lemma 3.1 that $\mathcal{M}(K; J)^m \subset \{U_K + V_{K,J}\}^m \mathcal{M}(J)$, and by (0.1) (QJ2) this is contained in $\{\sum U^i_K V^{m-i}_{K,J}\} \mathcal{M}(J)$). ∎

1.10 *Remark.* In many situations, nilpotence of the multiplication ideal $\mathcal{M}(K; J) = \mathcal{M}(J)\{U_K + V_{K,J}\} \mathcal{M}(J)$ is equivalent to nilpotence of the ideal $\mathcal{I}(K; J) = \mathcal{M}(J)\{U_K + V_K\} \mathcal{M}(J)$ generated by all pure multiplications $\mathcal{M}_J(K)$ of K on J. We always have $\mathcal{M}(K; J) \supset \mathcal{I}(K; J)$, and if $\frac{1}{2} \in \Phi$, we have equality (since $2V_{x,a} = 2V_x V_a - 2U_{x,a} = [V_x, V_a] + V_{x \circ a}$ from (2.4), (2.3)). The proof of 3.5 will show that nilpotence of $\mathcal{I}(K; J)$ implies that of $\mathcal{M}(K; J)$ if either (i) $K/D(K)$ is finitely generated over noetherian Φ or (ii) J/K is finitely generated.

However, in infinitely generated characteristic 2 situations nilpotence of $\mathcal{I}(K; J)$ need not imply that of $\mathcal{M}(K; J)$, as the following example shows.

1.11 Example. Over any Φ of characteristic 2 there exists a Jordan algebra $\hat{J}$ with trivial ideal K, $U_K = V_K = \mathcal{M}_{\hat{J}}(K) = \mathcal{I}(K; \hat{J}) = 0$, but $V^n_{K,\hat{J}} \neq 0$ for all n, so $\mathcal{M}(K; \hat{J})$ is not nilpotent. We construct $\hat{J}$ as the unital hull of the algebra J built on the exterior algebra $\bigwedge(L \oplus M) = L \oplus M \oplus N$ $(N = \oplus_{k \geq 2} \Lambda^k(L \oplus M))$ with Jordan products defined by

$$(l + m + n)^2 = 0,$$

$$U(l + m + n)(l' + m' + n') = l \wedge m \wedge n' + l \wedge m' \wedge n + l' \wedge m \wedge n.$$

It is easy to check that J satisfies the Jordan axioms of (0.2): since $V_x = x^2 = 0$ identically in J, the axioms (CJ2)–(CJ4) are trivial, and the others reduce to (CJ1′) $V_{x,x} = 0$, (CJ5′) $U_x U_x = 0$, (CJ6′) $U_x U_x U_x = 0$, which follow from characteristic 2 and alternation of the product in N (for $x = l + m + n$, $x' = l' + m' + n'$ we have in (CJ1′) $V_{x,x} x' = U_{x,x'} x = (l \wedge m' + l' \wedge m) \wedge n + (l \wedge m \wedge n' + l' \wedge m \wedge n) + l \wedge (m \wedge n' + m' \wedge n) = 2(l \wedge m \wedge n' + l \wedge m' \wedge n + l' \wedge m \wedge n) = 0$, in (CJ5′) $U_x(U_x x') =$

$l \wedge m \wedge (U_x x')$ (as $U_x x' \in N) = l \wedge m \wedge (l \wedge m \wedge n' + l \wedge m' \wedge n + l' \wedge m \wedge n) = 0$, hence in (CJ6′) $U_x U_x U_x = 0$ too). Here $K = M \oplus N$ is a trivial ideal, $K^2 = 0$ and $J^2 \subset N \subset K$, and $\mathscr{M}_J(K) = 0$ since $U_K = V_K = 0$. If $L = \oplus \Phi l_i$, $M = \oplus \Phi m_i$ are infinite-dimensional free Φ-modules, then $V_{l_1, m_1} \cdots V_{l_r, m_r}(l_{r+1} \wedge m_{r+1}) = U_{l_1, m_1} \cdots U_{l_r, m_r}(l_{r+1} \wedge m_{r+1}) = l_1 \wedge m_1 \wedge l_2 \wedge m_2 \cdots \wedge l_r \wedge m_r \wedge l_{r+1} \wedge m_{r+1} \neq 0$ in $\bigwedge(L \oplus M)$, so $V^r_{J,K} K \neq 0$ for any r, and $\mathscr{M}(K; \hat{J})$ is not nilpotent. ∎

1.12 *Remark.* The above Example 1.11 is not special since if $J \subset A^q$ is special of characteristic 2 and $K \lhd J$ is an ideal with $\mathscr{M}_J(K) = 0$ ($U_K = V_K = 0$), then $V^2_{K,J} = 0$ and $\mathscr{M}(K; J)^2 = 0$. Indeed, in terms of the associative product in A, $V_K = 0$ means $k \circ x = kx + xk = [k, x]$ vanishes for all $k \in K$, $x \in J$, so k commutes with J, hence $k[x^2, z] = [xkx, z] \in [K, J] = 0$ since K is an ideal in J, whence $k[x \circ y, z] = 0$ by linearization. Thus $V_{k', z} V_{k, y} x = k'k V_z V_y x = k'kz \circ (x \circ y) = k'k[x \circ y, z] = 0$, and $V_{K,J} V_{K,J} J = 0$. ∎

This does not seem to carry over by induction to show $\mathscr{M}(K; J)^n = 0 \Rightarrow \mathscr{M}(K; J)^{2n} = 0$: any induction involves the action of K and J on a bimodule derived from J, and so involves representations. We now turn to this more general concept.

2. Universal Nilpotence

We want to strengthen the Strong Nilpotence Theorem 1.7 to say that $\mathscr{M}(K; J)$ acts nilpotently not only on J but on any larger algebra $E \supset J$; here we place no finiteness restrictions on E, and K need not remain an ideal in E. This leads us to consider multiplication representations and their abstract versions, quadratic representations and specializations.

If J is a subalgebra of a Jordan algebra E and M a subspace invariant under outer multiplications by J (i.e., under U_J and V_J), then the restrictions $\mu_x = U_x \mid M$, $\nu_x = V_x \mid M$, $\nu_{x,y} = V_{x,y} \mid M$ to M of the multiplication operators by J afford a *multiplication representation* (μ, ν) *of* J *on* M. If J is unital and the unit of J acts as unit on M, then we have a *unital* representation [4, Theorem 4, p. 282] satisfying

$$\text{(UQ1)} \qquad \mu_1 = I$$

$$\text{(UQ2)} \qquad \mu_{U(x)y, x} = \nu_{x,y} \mu_x = \mu_x \nu_{y,x} \tag{2.1}$$

$$\text{(UQ3)} \qquad \mu_{U(x)y} = \mu_x \mu_y \mu_x$$

as in (0.1); for a general multiplication representation one derives from (0.2) that

$$
\begin{aligned}
&\text{(CS1)} && \nu_{x,x} = \nu_{x^2} \\
&\text{(CS2)} && \nu_{x,x^2} = \nu_{x^2,x} = \nu_{x^3} \\
&\text{(CS3)} && \mu_{x^2,y} = \mu_{x,y}\nu_x - \nu_y\mu_x = \nu_x\mu_{x,y} - \mu_x\nu_y \\
&\text{(CS4)} && \mu_{x^3,y} = \mu_{x,y}\nu_{x^2} - \nu_{y,x}\mu_x = \nu_{x^2}\mu_{x,y} - \mu_x\nu_{x,y} \\
&\text{(CS5)} && \mu_{x^2} = \mu_x^2 \\
&\text{(CS6)} && \mu_{x^3} = \mu_x^3
\end{aligned}
\tag{2.2}
$$

hold strictly. Abstractly, any pair of maps (μ, ν) from J to $\mathrm{End}_\Phi(M)$ satisfying (2.2) is called a *quadratic representation of J on M* (and M is called a *J-bimodule*) [4, Proposition 15, p. 298]. Any such abstract representation is realized as a concrete multiplication representation via the *split null extension*

$$E = J \oplus M, \qquad U_{x \oplus m}\, y \oplus n = U_x y \oplus \{\mu_x(n) + \nu_{x,y}(m)\}$$

[4, Theorem 17, p. 300]. More abstractly yet, any pair of maps (μ, ν) from J to any associative algebra A satisfying (2.2) is called a *quadratic specialization of J in A* (quadratic representations being the special case $A = \mathrm{End}_\Phi(M)$). Since any associative algebra A can be realized as an algebra of linear transformations on a space M, these three concepts are equivalent formulations of the behavior of Jordan multiplications by J.

To rearrange operators in quadratic representations we will make use of the following consequences of the defining identities (2.2):

$$\nu_x = \nu_{x,1} = \nu_{1,x} = \mu_{x,1}, \qquad 2\mu_x = \nu_x^2 - \nu_{x^2}, \tag{2.3}$$

$$\nu_{x,y} = \nu_x \nu_y - \mu_{x,y}, \tag{2.4}$$

$$\nu_{x,y} + \nu_{y,x} = \nu_{x \circ y}, \tag{2.5}$$

$$\nu_{x,y}\mu_z + \mu_z \nu_{y,x} = \mu_{\{xyz\},z}, \tag{2.6}$$

$$\mu_{\{xyz\}} + \mu_{U(x)U(y)z,z} = \mu_x\mu_y\mu_z + \mu_z\mu_y\mu_x + \nu_{x,y}\mu_z\nu_{y,x}, \tag{2.7}$$

$$\nu_{x,U(y)z} = \nu_{x,y}\nu_{z,y} - \mu_{x,z}\mu_y, \; \nu_{U(y)z,x} = \nu_{y,z}\nu_{y,x} - \mu_y\mu_{z,x}, \tag{2.8}$$

$$[\mu_x, \mu_y] - \mu_{x \circ y} - \mu_{U(x)y,y} = \mu_y \nu_{x^2} - \mu_{x \circ y,y}\nu_x \tag{2.9}$$

$$[\mu_y, \mu_x] - \mu_{x \circ y} - \mu_{U(x)y,y} = \nu_{x^2}\mu_y - \nu_x\mu_{x \circ y,y}, \tag{2.9*}$$

$$[\mu_x, \nu_{y,z}] = -\nu_{x \circ y, x \circ z} + \nu_{y,U(x)z + x^2 \circ z} + \nu_{U(x)y,z} + \{\nu_{x \circ y,z} - \nu_{y,x \circ z}\}\nu_x, \tag{2.10}$$

$$\mu_x\mu_y\mu_z - \mu_z\mu_y\mu_x = \mu_{\{xyz\}} + \mu_{U(x)U(y)z,z} - \mu_{\{xyz\},z}\nu_{y,x} + \mu_z\nu_{y,U(x)y}. \tag{2.11}$$

These can be found in [2; 3, (10), p. 469); 4] (2.11) comes from (2.7), (2.8), (2.6).

There is a universal gadget for quadratic specializations, the *universal multiplication envelope* [4, p. 289] consisting of an associative algebra $\mathscr{U}(J)$ and a *universal quadratic specialization* (u, v) through which all other quadratic specializations factor. Thus the properties of $\mathscr{U}(J)$ are the "universal" properties satisfied by all quadratic specializations of J. The condition that $\mathscr{M}(K; J)$ act nilpotently on any M inside any E is precisely the condition that the *universal multiplication ideal*

$$\mathscr{U}(K; J) = \mathscr{U}(J)\{u(K) + v(K, J)\}\, \mathscr{U}(J)$$

be nilpotent in $\mathscr{U}(J)$. This ideal is slightly larger than the ideal

$$\mathscr{P}(K; J) = \mathscr{U}(J)\{u(K) + v(K)\}\, \mathscr{U}(J)$$

generated by all pure multiplications by K (this is the universal analogue of $\mathscr{I}(K; J)$ of (1.10), and again we have $\mathscr{P}(K; J) = \mathscr{U}(K; J)$ if $\frac{1}{2} \in \Phi$). Our main goal in this paper is to prove the

2.12 Universal Nilpotence Theorem. *A solvable finitely generated Jordan algebra K generates a nilpotent ideal in the universal multiplication envelope $\mathscr{U}(J)$ of any extension $J \rhd K$ such that either* (i) *Φ is noetherian or* (ii) *J/K is finitely generated: the universal multiplication ideal $\mathscr{U}(K; J)$ is nilpotent in $\mathscr{U}(J)$.* □

Since the *regular multiplication representation* of J on itself factors through the universal one via an associative homomorphism $\mathscr{U}(J) \to \mathscr{M}(J)$ sending $\mathscr{U}(K; J) \to \mathscr{M}(K; J)$, nilpotence of $\mathscr{U}(K; J)$ will imply that of $\mathscr{M}(K; J)$, so Universal Nilpotence 2.12 will imply Strong Nilpotence 1.7. Moreover, we recover Jacobson's result for linear Jordan algebras: if J is finite-dimensional over a field Φ, then it and any ideal $K \lhd J$ are finitely spanned, hence finitely generated as algebras. The same works whenever Φ is noetherian,

2.13 Corollary. *If J is finitely spanned over a noetherian ring Φ, then any solvable ideal $K \lhd J$ generates a nilpotent ideal in the universal multiplication envelope.* ∎

Example 1.8 shows that if a solvable K is not finitely generated, it need not act nilpotently on itself. Now we show that even if K act trivially on itself, it need not act nilpotently on all bimodules.

2.14 Example. The Universal Nilpotence Theorem 2.12 fails if K is not finitely generated (even if K is trivial and Φ is a field and J/K is finitely generated): we can construct an example $K \lhd J$ over an arbitrary Φ such that

(i) $J = \hat{K}$ has J/K spanned by one element,

(ii) K is trivial, $K^2 = D(K) = 0$ (hence $P_J(K) = 0$),

(iii) J and K are special,

(iv) J has a representation with $\mu_K = 0$ but $v^n_{K,K} \neq 0$ for all n, so $\mathscr{U}(K;J)$ is not nilpotent.

Indeed, let $K = \oplus \Phi z_i$ be the trivial Jordan algebra on an infinite-dimensional free module, (μ, v) the quadratic specialization of K in $A = \bigwedge(K)$ given by $\mu_z = 0$, $v_z = z$. Any time that $\mu = 0$ conditions (2.2) (CS3)–(CS6) hold, and (CS1)–(CS2) reduce to $v_{x,x} = v_{x^2}$, $v_{x,x^2} = v_{x^2,x} = v_{x^3}$; when in addition K itself is trivial ($K^2 = U_K K = 0$), these conditions further reduce to the alternating condition $v_x v_x = 0$. Since our v alternates by $x \wedge x = 0$ in $\bigwedge(K)$, we see (μ, v) is indeed a specialization with $\mu_K = 0$ but $v^n_{K,K} = \{v_K v_K\}^n = v_K^{2n} \neq 0$ since $v_{z_1} v_{z_2} \cdots v_{z_n} = z_1 \wedge z_2 \wedge \cdots \wedge z_n \neq 0$ in $\bigwedge(K)$. Then $\mathscr{U}(K;J)$ cannot be nilpotent either since $\mathscr{U}(K;J)^n = 0 \Rightarrow v^n_{K,K} = 0 \Rightarrow v^n_{K,K} = 0$ for all quadratic specializations (μ, v). ∎

3. The First Step

In this section we will establish the first step towards Universal Nilpotence, that a sufficiently high power of $\mathscr{U}(K;J)$ falls into $\mathscr{U}(D(K);J)$; in the following section we will iterate this to get the final result. We begin by showing that we can commute multiplication operators past $v_{K,L}$ and u_K.

3.1 Commutation Lemma. *If K, L are outer ideals in J, then in $\mathscr{U} = \mathscr{U}(J)$ we have*

(i) *the ideal in $\mathscr{U}$ generated by $v_{K,L}$ is $v_{K,L}\mathscr{U} = \mathscr{U} v_{K,L}$,*

(ii) *the ideal in $\mathscr{U}$ generated by u_K is $u_K \mathscr{U} = \mathscr{U} u_K$,*

(iii) *the multiplication ideal of K is $\mathscr{U}(K;J) = \{v_{K,J} + u_K\}\mathscr{U} = \mathscr{U}\{v_{K,J} + u_K\}$,*

(iv) *if S is a unital set of generators for J modulo L, then*

$$v_{K,J}\mathscr{U} = v_{K,S \cup L}\mathscr{U},$$

(v) $\{v_{K_1,L_1}\mathscr{U} + u_{K_2} u_{L_2} \mathscr{U}\}^{m_1+m_2-1} \subset v^{m_1}_{K_1,L_1}\mathscr{U} + (u_{K_2} u_{L_2})^{m_2}\mathscr{U}$,

(vi) $\{v_{K,L}\mathscr{U} + u_K \mathscr{U}\}^m \subset v^m_{K,L}\mathscr{U} + u_K \mathscr{U}$.

Proof. (i) Since $v_{K,L}\mathscr{U}$ is a right ideal and $\mathscr{U} v_{K,L}$ a left ideal, it suffices to prove they coincide, and by symmetry it suffices if $\mathscr{U} v_{K,L} \subset v_{K,L}\mathscr{U}$; but this follows from (2.10), $u_a v_{k,l} = v_{k,l} u_a + \{v_{a\circ k,l} - v_{k,a\circ l}\} v_a - v_{a\circ k, a\circ l} + v_{k,U(a)l + a^2\circ l} + v_{U(a)k,l}$, where $a\circ k$, $U(a)k \in K$ and $a\circ l$, $a^2\circ l \in L$ by outerness if $a\in J$, $k\in K$, $l\in L$. (ii) As above it suffices to show $\mathscr{U} u_K \subset u_K \mathscr{U}$; but this follows from (2.9), $u_a u_k = u_k u_a + u_{a\circ k} + u_{U(a)k,k} - u_{a\circ k,k} v_a + u_k v_{a^2}$ for $a\in J$, $k\in K$. (iii) The right sides coincide by (i), (ii), and they coincide with $\mathscr{U}(K;J)$ since (as in (1.4)) $v_K = v_{K,1}$, $u_{K,J} = v_K v_J - v_{K,J}$, $v_{J,K} = v_{J\circ K,1} - v_{K,J}$ are generated by $v_{K,J}$ by (2.3), (2.4), (2.5). (iv) The set $\{a\in J \mid v_{K,a} \subset v_{K,S+L}\mathscr{U}\}$ is a subalgebra (it is a linear space containing $1\in S$ and closed under the product $U_a b$ since $v_{k,U(a)b} = v_{k,a} v_{b,a} - u_{k,b} u_a = v_{k,a} v_{b,a} + v_{k,b} u_a - v_{k,1} v_b u_a$ by (2.8)), which contains S and L by definition, hence contains all of J. (v) $(I_1 + I_2)^{m_1+m_2-1} \subset I_1^{m_1} + I_2^{m_2}$ for any ideals I_1, I_2 since in a product of $m_1 + m_2 - 1 > (m_1 - 1) + (m_2 - 1)$ factors from $I_1 \cup I_2$ there must be at least m_1 factors from I_1 or at least m_2 from I_2; furthermore $\{v_{K,L}\mathscr{U}\}^m = v_{K,L}^m \mathscr{U}$ by (i), $\{u_K u_L \mathscr{U}\}^m = (u_K u_L)^m \mathscr{U}$ by (ii). (vi) is the special case $m_1 = m$, $m_2 = 1$ of (v). ∎

Instead of using straightening arguments as in [3], we derive all our nilpotence results from alternating arguments: we exhibit multiplication monomials as alternating functions of their arguments modulo higher ideals, so in finitely spanned situations a suitably long product must fall into the higher ideal.

3.2 Alternating Lemma. *If K, L are outer ideals in J then so is $U_K L$, and for $k_i \in K$, $l_i \in L$ we have in $\mathscr{U} = \mathscr{U}(J)$*

(i) $v_{k_1,l_1} \cdots v_{k_r,l_r}$ *is an alternating function of the k's modulo the ideal $\{v_{U(K)L,L} + u_K u_L\}\mathscr{U}$ and of the l's modulo the ideal $\{v_{K,U(L)K} + u_K u_L\}\mathscr{U}$;*

(ii) *if $L \lhd J$ is an ideal, then $u_{k_1} u_{l_1} \cdots u_{l_{r-1}} u_{k_r}$ is an alternating symmetric function of the k's modulo the ideal $\mathscr{U}(U_K L; J)$.*

Proof. $U_K L$ is outer by (2.9): $U_a(U_k l) = U_k(U_a l) + U_{a\circ k} l + U_{U(a)k,k} l + U_k(a^2\circ l) - U_{a\circ k,k}(a\circ l)$ for $a\circ k$, $U_a k \in K$ and $U_a l, a^2\circ l$, $a\circ l \in L$ by outerness of K, L $(a\in J,\ k\in K,\ l\in L)$. (i) follows from (2.8): $v_{k,l} v_{k,l'} = v_{U(k)l,l'} + u_k u_{l,l'}$, $v_{k,l} v_{k',l} = v_{k,U(l)k'} + u_{k,k'} u_l$ and the fact that $v_{U(K)L,L}\mathscr{U}$, $v_{K,U(L)K}\mathscr{U}$, $u_K u_L \mathscr{U} = (u_K \mathscr{U})(u_L \mathscr{U})$ are ideals by the Commutation Lemma 3.1(i, ii). (ii) is symmetric in the k's by (2.11), $u_k u_l u_{k'} - u_{k'} u_l u_k = u_{\{klk'\}} + u_{U(k)U(l)k',k} - u_{\{klk'\},k'} v_{l,k} - u_{k'} v_{l,U(k)l}$ for $U_k l$, $\{klk'\}$, $U_k U_l k' \in U_K L$ (where $U_l k' \in L$ requires $L \lhd J$), and falls into $\mathscr{U}(U_K L; J)$ if some k is repeated by (0.1)(QJ3), $u_k u_l u_k = u_{U(k)l}$. ∎

The two finiteness conditions we are interested in are when $K/D(K)$ (Φ noetherian) and J/K are finitely generated. Both are subsumed under the case when J/L is finitely generated for some $\tilde{K} \supset L \supset K$ (usually $L = \tilde{K}$ or $L = K$). Clearly $J/\tilde{K}$ is finitely generated if J/K already is; we now construct $\tilde{K}$ and show $J/\tilde{K}$ is finitely spanned when $K/D(K)$ is finitely generated and Φ

is noetherian. First we make a simple observation that $K/D(K)$ is finitely spanned as soon as it is finitely generated.

3.3 LEMMA. *If $K/D(K)$ is generated by n elements $\{k_i\}$, then it is spanned by $\bar{n}=\frac{1}{2}n(n+3)$ elements $\{k_i, k_i^2, k_i \circ k_j\ (i<j)\}$.*

Proof. K is spanned modulo $D(K)=K^3$ (see (1.1)) by all monomials in the generators of degree <3, namely, the n elements k_i, the n elements k_i^2, and the $\frac{1}{2}n(n-1)$ elements $k_i \circ k_j$ $(i<j)$, where $n+n+\frac{1}{2}n(n-1)=\frac{1}{2}n(n+3)$. ∎

Next we construct $\tilde{K}=\mathrm{Int}(K, D(K))$ as a special case of an *inner transformer* $\mathrm{Int}(K, L)$, and show that $J/\tilde{K}$ is finitely spanned.

3.4 INNER TRANSFORMER LEMMA. *If K, L are outer ideals in J, then so is the set*

$$\mathrm{Int}(K, L)=\{z \in J \mid U_K z \subset L\}$$

of elements transforming K into L from the inside. In particular, the inner annihilator $\mathrm{Inann}(K)=\mathrm{Int}(K, 0)$ *and the closure* $\tilde{K}=\mathrm{Int}(K, D(K)) \supset K$ *are outer ideals when K is.*

If $K \supset L$ are ideals in J with K/L spanned by n elements over a neotherian ring Φ, then $J/\mathrm{Int}(K, L)$ is also finitely spanned by $m(K, L, J, n)$ elements ($m \leqslant \frac{1}{2}n^2(n+1)$ if Φ is a field).

In particular, if $K/D(K)$ is generated by n elements over a noetherian Φ, then $J/\tilde{K}$ is finitely spanned by $m(K, J, n)$ elements ($m \leqslant (1/16)n^2(n+1)$ $(n+2)(n+3)^2$ if Φ is a field).

Proof. Clearly $Z=\mathrm{Int}(K, L)$ is a linear space, and it is outer-invariant since by (2.9*) for $a \in J$, $z \in Z$, $k \in K$, we have $U_k(U_a z)=U_a(U_k z)+U_{a \circ k} z+U_{U(a)k,k} z+V_{a^2} U_k z-V_a(U_{a \circ k,k} z) \in L$ since $k, a \circ k$, $U_a k \in K$ and $U_K z \subset L$ and L is invariant under U_a, V_{a^2}, V_a. As particular cases we may take $L=0$ and $L=D(K)=U_K K$ (which is outer by (3.2)).

If $K \supset L$ are ideals with K/L finitely spanned, then by passing to $\bar{J}=J/L$ it suffices to assume $L=0$ with K finitely spanned, and to show that $J/\mathrm{Inann}(K)$ is finitely spanned ($\mathrm{Int}(K, L)$ is the preimage in J of $\mathrm{Int}(\bar{K}, \bar{L})=\mathrm{Int}(\bar{K}, \bar{O})=\mathrm{Inann}(\bar{K})$ in $\bar{J}$, so $J/\mathrm{Int}(K, L) \cong \bar{J}/\mathrm{Inann}(\bar{K})$ as linear spaces). But if K is spanned by $k_1, \ldots, k_n$ over Φ, we have imbeddings

$$J/\mathrm{Inann}(K) \rightarrow \mathrm{Quad}_\Phi(K) \rightarrow \mathrm{Symm}(M_n(K)),$$

where $\mathrm{Quad}_\Phi(K)$ denotes the space of quadratic maps from K to itself and $\mathrm{Symm}(M_n(K))$ the symmetric $n \times n$ matrices with entries in K. Here $\mathrm{Inann}(K)$ is precisely the kernel of the map $J \rightarrow \mathrm{Quad}_\Phi(K)$ given by $a \rightarrow q_a$

(where $q_a(k) = U_k a$ is linear in a and quadratic in k, and maps K into itself when K is an inner ideal $U_K J \subset K$). The map $\mathrm{Quad}_\Phi(K) \to \mathrm{Symm}(M_n(K))$ is given by $q \to (q_{ij})$, where $q_{ii} = q(k_i)$, $q_{ij} = q(k_i, k_j) = q(k_i + k_j) - q(k_i) - q(k_j)$; it is an imbedding since if $q(k_i) = q(k_i, k_j) = 0$ on a spanning set $\{k_i\}$ for K, then q vanishes identically on K. Since K (and hence $\mathrm{Symm}(M_n(K))$ too) is finitely spanned over Φ, the same is true for the imbedded subspace $J/\mathrm{Inann}(K)$ when Φ is noetherian. In general we have no estimate for the rank $m(K, O, J, n)$ of this subspace, but when Φ is a field we have $m = \dim_\Phi(J/\mathrm{Inann}(K)) \leqslant \dim_\Phi(\mathrm{Symm}(M_n(K))) = \frac{1}{2}n(n+1)\dim_\Phi(K) = \frac{1}{2}n^2(n+1)$.

In particular, for $L = D(K)$, $\mathrm{Int}(K, L) = \tilde{K}$ we see $J/\tilde{K}$ is finitely spanned by $m(K, J, n) = m(K, D(K), J, \bar{n})$ elements if $K/D(K)$ is generated by n elements (then by (3.3) it is spanned by $\bar{n} = \frac{1}{2}n(n+3)$ elements). If Φ is a field we have $m \leqslant \frac{1}{2}\bar{n}^2(\bar{n}+1) = \frac{1}{2}\{\frac{1}{2}n(n+3)\}^2\frac{1}{2}\{n^2+3n+2\} = (1/16)n^2(n+3)^2(n+1)(n+2)$. ∎

We are finally ready to shove $\mathscr{U}(K; J)$ into $\mathscr{U}(D(K); J)$.

3.5 THEOREM. *There are universal constants $f(n, m)$ such that whenever K is an ideal in J with $K/D(K)$ generated by n elements and $J/\tilde{K}$ is unitally generated by m elements (e.g., if* (i) *Φ is noetherian, $m = m(K, J, n)$ depending only on n when Φ is a field, or* (ii) *J/K is generated by m elements), then*

$$\mathscr{U}(K; J)^{f(n,m)} \subset \mathscr{U}(D(K); J).$$

Proof. By (3.4) the hypotheses are satisfied in both cases (i), (ii). By 3.1(iv) we have $v_{K,J}\mathscr{U} = v_{K,S\cup L}\mathscr{U}$ for $S = \{1, x_1, ..., x_m\}$ a unital generating set for J modulo an outer ideal $L \supset K$ (e.g., $L = \mathrm{Int}(K, D(K)) = \tilde{K}$, using (3.4)). By induction $\{v_{K,J}\mathscr{U}\}^r = v^r_{K,S\cup L}\mathscr{U}$ (if true for r, then $\{v_{K,J}\mathscr{U}\}^{r+1} = v^r_{K,S\cup L}\mathscr{U} \cdot v_{K,J}\mathscr{U} = v^r_{K,S\cup L} \cdot v_{K,S\cup L}\mathscr{U} = v^{r+1}_{K,S\cup L}\mathscr{U}$ by 3.1(i) and the case $r = 1$). By 3.2(i), $v_{k_1,a_1} \cdots v_{k_r,a_r}$ is an alternating function of the a's in $S \cup L$ modulo $\{v_{K,U(J)K} + u_K\}\mathscr{U} \subset \{v_{K,K} + u_K\}\mathscr{U}$, hence for $r > m+1$ either some a_i falls in L or some $a_i \in S$ appears twice, so in either case the product falls in $\{v_{K,L} + u_K\}\mathscr{U}$ since $L \supset K$; thus $v^{m+2}_{K,J} \subset \{v_{K,L} + u_K\}\mathscr{U}$. Using 3.1(vi) twice we see $(\{v_{K,J} + u_K\}\mathscr{U})^{(m+2)r} \subset (\{v_{K,L} + u_K\}\mathscr{U})^r \subset \{v^r_{K,L} + u_K\}\mathscr{U}$, so by 3.1(iii)

(3.6) If $L \supset K$ are outer ideals in J and J/L is unitally generated by m elements, then

$$\mathscr{U}(K; J)^{(m+2)r} \subset \{v^r_{K,L} + u_K\}\mathscr{U}.$$

Assume now that $L \subset \tilde{K}$. If $K/D(K)$ is generated by n elements, then by

3.3 it is spanned by a set T of $\bar{n} = \frac{1}{2}n(n+3)$ elements, and K is spanned by $T \cup D(K)$. By 3.2(i), $V_{k_1,l_1} \cdots V_{k_r,l_r}$ is an alternating function of the $k_i \in T \cup D(K)$ modulo $\{v_{U(K)L,L} + u_K u_L\}\mathscr{U} \subset \{v_{D(K),J} + u_K\}\mathscr{U}$ (BY DEFINITION OF $L \subset \tilde{K} = \text{Int}(K, D(K))$, which vanishes into $v_{D(K),J}\mathscr{U}$ if some $k_i \in D(K)$, and if $r > \bar{n}$, then either some k_i falls into $D(K)$ or some $k_i \in T$ appears twice; hence in either case the product falls into $\{v_{D(K),J} + u_K\}\mathscr{U}$. Thus $v_{K,L}^{\bar{n}+1} \subset u_K\mathscr{U} + \mathscr{U}(D(K); J)$, and $\mathscr{U}(K; J)^{(m+2)(\bar{n}+1)s} \subset (\{v_{K,L}^{\bar{n}+1} + u_K\}\mathscr{U})^s$ (by (3.6)) $\subset (u_K\mathscr{U} + \mathscr{U}(D(K); J))^s$, hence by 3.1(ii)

$$\mathscr{U}(K; J)^{(m+2)(\bar{n}+1)s} \subset u_K^s\mathscr{U} + \mathscr{U}(D(K); J). \qquad (*)$$

By 3.2(ii) with $L = K \lhd J$ we have $u_{k_1}u_{l_1} \cdots u_{l_{r-1}}u_{k_r}$ an alternating function of the $k_i \in K$ modulo $\mathscr{U}(U_K L; J) = \mathscr{U}(D(K); J)$; since K is spanned mod $D(K)$ by $\bar{n}$ elements k_i as in (3.3) we see u_K is spanned mod $\mathscr{U}(D(K); J)$ by the $\bar{\bar{n}} = \frac{1}{2}\bar{n}(\bar{n}+1)$ elements $u_{k_{ij}}$ $(i \leqslant j)$ for $k_{ii} = k_i$, $k_{ij} = k_i + k_j$ $(K = \sum \Phi k_i + \text{D}(K) \Rightarrow u_K = \sum \Phi u_{k_i} + \sum \Phi u_{k_i,k_j} + \sum u_{k_i,D(K)} + u_{D(K)} \subset \sum \Phi u_{k_{ii}} + \sum \Phi(u_{k_{ij}} - u_{k_{ii}} - u_{k_{jj}}) + \mathscr{U}(D(K); J))$, hence if $r > \bar{\bar{n}}$, some $u_{k_{ij}}$ is repeated and the product falls in $\mathscr{U}(D(K); J)$: $u_K^{2r-1} \subset \mathscr{U}(D(K); J)$. If $s = 2r - 1 > 2\bar{\bar{n}} - 1$ (i.e., $s \geqslant 2\bar{\bar{n}} = \bar{n}(\bar{n}+1)$), then $r > \bar{\bar{n}}$ and $u_K^s \subset \mathscr{U}(D(K); J)$, so by $(*)$

$$\mathscr{U}(K; J)^{(m+2)(\bar{n}+1)\bar{n}(\bar{n}+1)} \subset \mathscr{U}(D(K); J). \qquad (**)$$

Thus we may take $f(n, m) = \bar{n}(\bar{n}+1)^2(m+2) = \frac{1}{2}n(n+3)\{\frac{1}{2}(n^2 + 3n + 2)\}^2(m+2) = (1/8)\, n(n+1)^2(n+2)^2(n+3)(m+2)$. ∎

3.7 *Remark.* Example 2.14 shows that Theorem 3.5 fails if $K/D(K)$ is not finitely generated: there $D(K) = 0$, J/K is unitally generated by 0 elements, Φ is arbitrary, yet $\mathscr{U}(K; J)^f \not\subset \mathscr{U}(D(K); J)$ since $v_{K,K}^f \neq 0$ for any f but $\mathscr{U}(D(K); J) = 0$. ∎

3.8 *Remark.* Statement (3.6) holds for any *subspace* $L \supset K$ with $L \supset \{LKJ\}$ such that J/L is unitally generated by m elements since even though $v_{K,L}\mathscr{U}$ need not be an ideal in this case we still have $v_{K,J}v_{K,L} \subset \{v_{K,L} + u_K\}\mathscr{U}$ (by linearized (2.8) when $\{JKL\} \subset L$), and we can use $\mathscr{U} \cdot \mathscr{U}(K; J)^{m+2} = \mathscr{U}(K; J)^{m+2}$ to pass from r to $r+1$. ∎

3.9 *Remark.* At this point we could deduce the Universal Nilpotence Theorem 2.12 from the Albert–Zhevlakov Theorem 5.3: we have $\mathscr{U}(K; J)^{f(n,m)} \subset \mathscr{U}(D(K); J) \subset \{v_{K,K} + u_K\}\mathscr{U}$ (since $v_{D(K),J} \subset v_{K,K}$ by $v_{U(k)k',a} = -v_{U(k)a,k'} + v_{k,\{akk'\}}$), hence $\mathscr{U}(K; J)^{fr} \subset \{v_{K,K} + u_K\}^r\mathscr{U}$ (by 3.1) $\subset \mathscr{U}_J(K)^r\mathscr{U}$, which vanishes for sufficiently large r since $\mathscr{U}_J(K)$ is a homomorphic image of $\mathscr{U}(K)$ (the universal quadratic specialization of J in $\mathscr{U}$ restricts to one of K in $\mathscr{U}_J(K)$, which factors through $\mathscr{U}(K)$), and $\mathscr{U}(K)$ is nilpotent for solvable finitely generated K by Albert–Zhevlakov. ∎

3.10 *Remark.* If $\frac{1}{2} \in \Phi$, then $\mathscr{U}(K;J) = \mathscr{U}\{v_K + u_K\}\mathscr{U} = \mathscr{P}(K;J)$ just as in (1.10). In general, the hypothesis that J/K be finitely generated is necessary in characteristic 2 to get $\mathscr{U}(K;J)$ nilpotent modulo $\mathscr{P}(K;J)$ as in 3.9: Example 1.9 has $\mathscr{I}(K;J) = 0$ but $\mathscr{M}(K;J)$ not nilpotent, hence $\mathscr{M}(K;J)^r \not\subset \mathscr{I}(K;J)$ for any r and therefore $\mathscr{U}(K;J)^r \not\subset \mathscr{P}(K;J)$. ∎

Rather than appeal to the Albert–Zhevlakov Theorem (whose proof in [3] involved the Zhevlakov Straightening Argument and a digression into Penico-solvability), we will give a self-contained proof of Universal Nilpotence by iteration of 3.5.

4. The Final Step

So far we have shown that under certain hypotheses on K a power of the multiplication ideal $\mathscr{U}(K;J)$ falls into the multiplication ideal $\mathscr{U}(D(K);J)$ of the derived ideal. If these hypotheses are inherited by all the derived ideals, we can iterate the procedure: taking $f = f(n_1, m_1) \cdots f(n_k, m_k)$ in 3.5 we get

4.1 Theorem. *There are universal constants $f(n_1, m_1, ..., n_k, m_k)$ such that whenever K is an ideal in J such that for $i = 1, ..., k$ we have $D^{i-1}(K)/D^i(K)$ generated by n_i elements and $J/\mathrm{Int}(D^{i-1}(K), D^i(K))$ unitally generated by m_i elements (e.g., if either* (i) *Φ is noetherian, $m_i = m(D^{i-1}(K), J, n_i)$or* (ii) *each $J/D^{i-1}(K)$ is unitally generated by m_i elements), then*

$$\mathscr{U}(K;J)^{f(n_1, m_1, ..., n_k, m_k)} \subset \mathscr{U}(D^k(K);J). \quad ∎$$

It remains to find useful conditions on K which guarantee that all $D^{i-1}(K)/D^i(K)$ and $J/\mathrm{Int}(D^{i-1}(K), D^i(K))$ remain finitely generated. Our applications will be to solvable ideals $D^k(K) = 0$; if $K/D(K)$, $D(K)/D^2(K)$,..., $D^{k-1}(K)/D^k(K)$ are all finitely generated, then so is $K/D^k(K) = K$, so a natural condition is that K itself be finitely generated. Note that if in addition we assume that J/K is finitely generated by m elements, then automatically J (and all homomorphic images $J/\mathrm{Int}(D^{i-1}(K), D^i(K))$) is generated by $n + m$ elements.

The only difficulty involved with the hypothesis that K is finitely generated is showing it is inherited by derived ideals. The key is the following special case of the Albert–Zhevlakov Theorem.

4.2 Lemma. *If $K/D(K)$ is generated by n elements, then for $d = 2f(n, 0)(n + 2) + 4$ we have*

$$K^d \subset D^2(K).$$

Proof. Set $J = \hat{K}$ where K is generated by $k_1, \dots, k_n$ modulo $D(K)$. Then J/K is unitally generated by $m = 0$ elements, so by 3.5 we have

$$\mathscr{M}(K; J)^{f(n,0)} \subset \mathscr{M}(D(K); J). \qquad (*)$$

It suffices to consider a monomial $p = M_1 \cdots M_r q$ of degree $\geqslant d$ in K, where each M_i is a U_{z_i}, U_{z_i, w_i}, or V_{z_i} (z_i, w_i either a k_j or an element of $D(K)$ since these operators generate $\mathscr{M}_J(K)$), and q has degree 3 or 4, with $2f(n,0)(n+2)+4 = d \leqslant \text{degree}(p) \leqslant 2r+4$ (since $\text{degree}(M_i) \leqslant 2$, $\text{degree}(q) \leqslant 4$). But then $f(n,0)(n+2) \leqslant r$, so $p \in \mathscr{M}(K)^r q \subset \mathscr{M}(K; J)^{f(n,0)(n+2)} D(K)$ ($q \in K^3 = D(K)$ by (1.1)) $\subset \mathscr{M}(D(K); J)^{n+2} D(K)$ (by (*)) $\subset \{V_{D(K),D(K)} + u_{D(K)}\} \mathscr{M}(J)\, D(K)$ (by 3.6 with L, K replaced by $D(K)$ since $J/D(K) = (\Phi 1 + K)/D(K)$ is unitally generated by $m = n$ elements) $\subset U_{D(K)} D(K) = D^2(K)$. ∎

4.3 Finite Generation Theorem. *There are universal constants $d_k(n)$ such that whenever a Jordan algebra K is generated by n elements, then $D^k(K)$ is generated by $\leqslant d_k(n)$ elements.*

Proof [3, Proposition 9, p. 481]. It suffices to construct $d(n) = d_1(n)$ since then by induction $d_{k+1}(n) = d(d_k(n))$ works for $D^{k+1}(K) = D(D^k(K))$. It also suffices to construct d for the free Jordan algebra on n generators (the polynomials with zero constant term inside the free unital Jordan algebra J on $x_1, \dots, x_n$) since any algebra with n generators is a homomorphic image of this free K. We will show we can take $d(n)$ to be the number of Jordan monomials $s(x_1, \dots, x_n)$ with $3 \leqslant \text{degree}(s) < d = 2f(n,0)(n+2)+4$. Let S be the set of all such monomials. To see that S generates $D(K) = K^3$, it suffices to generate all monomials p of degree $\geqslant 3$; if $\text{degree}(p) < d$, then p is already one of the generators while if it has degree $\geqslant d$, then by Lemma 4.2 it lies in $D^2(K) = U_{D(K)} D(K)$, hence is a sum $p = \sum U_{p_i} q_i + \sum U_{p_i, r_i} q_i$ for $p_i, q_i, r_i \in D(K)$ monomials of lower degrees (this property of degrees requires freeness of J), and by induction the p_i, q_i, r_i are generated by S, hence p is too. ∎

Now we are able to extend 5.3 to higher derived ideals, yielding a result slightly stronger than Universal Nilpotence 2.12 and Strong Nilpotence 1.7.

4.4 Universal Constants of Nilpotency Theorem. *There are universal constants $f(n,m,k)$ and $g(n,m,k)$ such that if $K \lhd J$ is an ideal generated as algebra by n elements, where either* (i) *Φ is noetherian ($m = (K, J)$) or* (ii) *J/K is unitally generated by m elements, then*

(1) $K^{(g(n,m,k),J)} \subset D^k(K)$,

(2) $\mathscr{U}(K; J)^{f(n,m,k)} \subset \mathscr{U}(D^k(K); J)$.

If Φ is a field in case (i), *then $f(n, K, J, k)$ and $g(n, K, J, k)$ depend only on n, k, not on K, J.*

Proof. It suffices to construct the $f(n, m, k)$ satisfying (2) since then we may take $g(n, m, k) = 2f(n, m, k) + 1$ in (1): by (1.1), (1.3) $K^g = K^{2f+1} \subset \mathscr{U}_{2f}(K; J)J \subset \mathscr{U}(K; J)^f J \subset \mathscr{U}(D^k(K): J)J$ (by (2) since $\mathscr{U}$ is a homomorphic image of $\mathscr{U}) \subset D^k(K)$ (since $D^k(K) \lhd J$ is an ideal).

We can construct $f(n, m, k) = f(n_1, m_1) \cdots f(n_k, m_k)$ as in Theorem 4.1: we have $D^{i-1}(K)$ (hence also $D^{i-1}(K)/D^i(K)$) generated by $n_i = d_{i-1}(n)$ elements by 4.3, and $J/\mathrm{Int}(D^{i-1}(K), D^i(K))$ by m_i elements (in case (i) $m_i = m(D^{i-1}(K), J, n_i)$ as in 3.4, and in case (ii) we noted we can take $m_i = m + n$ since if J/K has m generators and K has n, then J and all homomorphic images have $m + n$ generators). ∎

4.5 *Remark.* Theorem 4.4 does not seem to follow from Universal Nilpotence 2.12. It is not clear how to construct a universal pair (J_u, K_u) for K_u generated *as algebra* by n elements (it is easy for K_u generated *as ideal* by n elements) and J_u/K_u by m elements. One could try to construct (J_u, K_u) inside a direct product: every pair (J, K) is a homomorphic image of the free algebra $J[x_1, ..., x_m, z_1, ..., z_n]$, and we can form the product $J_s = \prod J_\sigma$, $K_s = \prod K_\sigma$ over all pairs (J_σ, J_σ) with J_σ a quotient of J; if J_u denotes the subalgebra of J_s generated by the $x_{is} = \prod x_{i\sigma}$, and K_u that generated by the $z_{js} = \prod z_{j\sigma}$, then any pair (J, K) is a homomorphic image of (J_u, K_u), but there is no reason that K_u should be an ideal in J_u. ∎

4.6 *Remark.* Theorem 4.4(1, 2) do follow from Theorem 2.12 in the special case $J = \hat{K}$: if K is generated by n elements, then

$$K^{f(n,k)} \subset D^k(K), \qquad \mathscr{U}(K)^{g(n,k)} \subset \mathscr{U}(D^k(K); K)$$

[3, Proposition 8, p. 479] since these refer only to K and hence there is a universal object for this situation (and a bound for the universal object gives a universal bound for all objects). Namely, let K be the free (non-unital) Jordan algebra on n generators; for each k the factor algebra $\bar{K} = K/D^k(K)$ is solvable of index k and finitely generated, hence by 2.12 (or even the weaker Albert–Zhevlakov Theorem 5.3) $\bar{K}$ and $\mathscr{U}(\bar{K})$ are nilpotent: $\bar{K}^f = \mathscr{U}(\bar{K})^g = 0$ for some f, g, hence $K^f \subset D^k(K)$ and $\mathscr{U}(K)^g \subset \mathscr{U}(D^k(K); J)$ (using $\mathscr{U}(J/L) = \mathscr{U}(J)/\mathscr{U}(L; J)$), and since any algebra with n generators is an image of K we see $f = f(n, k)$ and $g = g(n, k)$ are universal bounds. ∎

4.7 *Remark.* We can now see that the apparent generalization from finite-dimensionality to finite generation of the ideal K in the Nilpotence Theorems 1.7, 2.12, 4.4 is largely illusory:

4.8 Proposition. *If a Jordan algebra K is solvable, then it is finitely generated over Φ iff it is finitely spanned over Φ.*

Proof. A finitely spanned algebra is always finitely generated. Suppose conversely that $K = \Phi[x_1, \ldots, x_n]$ is finitely generated and $D^d(K) = 0$. Considering the chain $K \supset D(K) \supset D^2(K) \supset \cdots \supset D^d(K) = 0$, it suffices to show each $D^k(K)/D^{k+1}(K)$ is finitely spanned. But here $L = D^k(K)$ remains finitely generated by Finite Generation 4.3, say $L = \Phi[y_1, \ldots, y_m]$, so $D^k(K)/D^{k+1}(K) = L/D(L)$ is finitely spanned by the $y_i, y_i^2, y_i \circ y_j$ by (3.3). ∎

However, it *is* a surprising generalization that finite-dimensionality of J is unnecessary in the noetherian case—that no hypotheses on J are needed, and universal nilpotence results solely from the solvability and finiteness of K.

5. Consequences

We can recover the results of [3] on nilpotence and Penico-solvability. An ideal $K \lhd J$ is *Penico-solvable* in J if some $P^n(K) = 0$, where $P^0(K) = K$ and $P^{k+1}(K) = P(P^k(K))$ for $P(L) = P_J(L) = U_L J$. K is *intrinsically Penico-solvable* if it is Penico-solvable in its unital hull $J = \hat{K}$. Since

$$\begin{array}{ccccc} & & K^{2^n} & & \\ & \subset & & \subset & \\ D^n(K) \subset P^n_{\hat{K}}(K) & & & & K^{(2^n, J)} \\ & \subset & & \subset & \\ & & P^n_J(K) & & \end{array} \tag{5.1}$$

we see

$$\text{Nilpotence} \Rightarrow \text{intrinsic Penico-solability} \Rightarrow \text{solvability}. \tag{5.2}$$

In the finitely generated case the converse holds.

5.3 Albert–Zhevlakov Theorem [3, p. 479]. *The following are equivalent for a finitely generated Jordan algebra K:*

(i) *K is solvable,*

(ii) *K is intrinsically Penico-solvable,*

(iii) *K is nilpotent,*

(iv) *the multiplication algebra $\mathscr{M}(K)$ is nilpotent.*

(v) *the universal multiplication algebra $\mathscr{U}(K)$ is nilpotent.*

Proof. Always (v) ⇒ (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) in view of (5.2), and (i) ⇒ (v) by applying the Universal Nilpotence Theorem 2.12 to the unital hull $J = \hat{K}$. ∎

5.4 *Remark.* This result shows that local nilpotence and local solvability coincide, so the locally nilpotent radical Loc (J) coincides with the locally solvable radical Locs (J). The latter is easier to construct: a maximal locally solvable ideal $L \lhd J$ always exists, and the only hard part is showing that J/L contains no more locally solvable ideals K/L. This is equivalent to the fact that K/L and L locally solvable imply K locally solvable, i.e., if K_0 is a finitely generated subalgebra modulo L, then it is solvable: $D^n(K_0) = L_0 \subset L \Rightarrow$ some $D^{n+m}(K_0) = D^m(L_0) = 0$.(The corresponding relation $K^{n+m} = (K^n)^m$ for nilpotence is false.) Here it suffices if L_0 is fintely generated, so the only nontrivial step in the construction of the locally solvable radical is to verify that $D(K_0)$ is finitely generated if K_0 is (Finite Generation Theorem 4.3); this seems to require Lemma 4.2 (i.e., 3.5 for $m = 0$ plus 3.6, where 3.5 for $m = 0$ seems to require the same steps and just as much work as 3.5 for general m). It would be desirable to have a more direct and elementary proof of 4.3. ∎

Using 3.6 we can obtain the equivalence of solvability and Penico-solvability for ideals $K \lhd J$ as long as J (but not necessarily K itself) is finitely generated.

5.5 THEOREM [3, Proposition 5, p. 473]. *If J is generated by n elements, then*

$$P_J^{k(n+3)}(K) \subset D^k(K) \tag{5.6}$$

for any ideal $K \lhd J$. In particular, an algebra K is solvable iff it is intrisically Penico-solvable iff it is Penico-solvable in any finitely generated extension $J \rhd K$.

Proof. We have the general fact (compare [3, (20), p. 473])

$$P_J^{r+1}(L) \subset D(L) + V_{P^r(L),J} \cdots V_{P^1(L),J} P(L) \quad (L \lhd J) \tag{5.7}$$

since, for $r = 1$, $P^2(L) = U_{P(L)} J$ is spanned by $U_{U(z)a} J = U_z U_a U_z J \subset U_L L = D(L)$ and $U_{U(z)a, U(z')a'} J \subset V_{P(L),J} P(L)$ for $z \in L$, $a \in J$, and if (5.7) holds for r, then it holds for $r+1$ by $P^{r+2}(J) = P^2(P^r(L)) \subset D(P^r(L)) + V_{P(P^r(L)),J} P(P^r(L))$ (by the case $r = 1$) $\subset D(L) + V_{P^{r+1}(L),J} \{D(L) + V_{P^r(L),J} \cdots V_{P^1(L),J} P(L)\}$ (by the induction hypothesis) $\subset D(L) + V_{P^{r+1}(L),J} \cdots V_{P^1(L),J} P(L)$.

For (5.6) we induct on k, starting with $k = 0$ ($P^0(K) = D^0(K) = K$). If the result is true for k, then it is true for $k+1$ since $P^{(k+1)(n+3)}(K) = P^{k(n+3)}(P^{n+3}(K)) \subset D^k(P^{n+3}(K))$ (by the induction hypothesis on k) $\subset D^k(D(K)) + V_{K,J}^{n+2} K)$ (by (5.7) for $r = n+2$) $\subset D^k(D(K) + \{V_{K,K} + U_K\} K)$ (by 3.6 for $L = K$, $m = n$ noting the ideal K is invariant under multiplications) $= D^k(D(K)) = D^{k+1}(K)$. ∎

5.8 PENICO'S THEOREM [3, p. 473]. *If J is finitely generated, then an ideal $K \lhd J$ is solvable iff it is Penico-solvable.* ∎

5.9 *Remark.* We can further simplify the proof of Universal Constants 4.1 if we assume K is Penico-solvable in J rather than merely solvable: There are universal constants $f(n_1,..., n_k)$ such that whenever K is an ideal in J such that K and all Penico-derived ideals $P_J^{i-1}(K)$ ($i = 1,..., k$) are generated as algebras by n_i elements, then

$$\mathscr{U}(K; J)^{f(n_1,...,n_k)} \subset \mathscr{U}(P_J^k(K); J).$$

Proof. If $K = P^0(K)$ is generated by n_1 elements, then by 3.3 it is spanned by $\bar{n}_1$ elements mod $P(K)$ (even mod $D(K)$); thus, by 3.2(i), $v_{K,J}^{\bar{n}_1+1} \subset \{v_{P(K),J} + u_K\}\mathscr{U}$. By 3.1(iii, vi), $\mathscr{U}(K; J)^{(\bar{n}_1+1)r} \subset \{v_{K,J} + u_K\}^{(\bar{n}_1+1)r}\mathscr{U} \subset \{v_{P(K),J} + u_K\}^r\mathscr{U} \subset \{v_{P(K),J} + u_K^r\}\mathscr{U}$. But $u_K^r \subset \mathscr{U}(P(K), J)$ for $r > \bar{\bar{n}}_1$ by 3.2(ii) (namely, $u_K^r = u_K u_J \cdots u_J u_K$ is a symmetric alternating function of the K's modulo $\mathscr{U}(U_K J; J) = \mathscr{U}(P(K); J)$). Hence if we set $f(n_1) = (\bar{n}_1 + 1)(\bar{\bar{n}}_1 + 1)$, we have $\mathscr{U}(K; J)^{f(n_1)} \subset \mathscr{U}(P(K); J)$; this is the case $k = 1$, and the general case follows by induction. ∎

5.10 COROLLARY. *If K is Penico-solvable and all $P_J^k(K)$ are finitely generated, then $\mathscr{U}(K; J)$ is nilpotent and K is strongly nilpotent in J.* ∎

Note that 2.12(ii) implies 5.10 when J/K is finitely generated; to pass from 5.10 to 2.12(ii) we would need to know that the $P^k(K)$ inherit finite generation. At present no inheritance result like 4.3 is known for the Penico-derived ideals:

If J and K are finitely generated, are all $P_J^k(K)$ finitely generated too? (5.11)

It would suffice if $P_J^k(K)/D^k(K)$ are finitely generated (since all $D^k(K)$ are by 4.3). Also the case $n = 1$ would suffice. This holds if Φ is noetherian since $P_J(K)/D(K) \subset K/D(K)$ and the latter is finitely spanned by 3.3. The result also holds if $J = \hat{K}$ by [3, Proposition 9, p. 481].

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