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ON CLASSIFYING PROPER KNOTS IN OPEN 3-MANIFOLDS

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We examine proper embeddings of the real line into open 3-manifolds and their proper isotopy classes, i.e., proper knots and their equivalence classes. In particular, for proper knots running between distinct ends of an open 3-manifold M, we give conditions on the structure of the ends of M under which proper homotopy implies proper isotopy. To prove this result, geometric techniques are employed which enable one to properly isotope a proper knot that is wild in the neighbourhood of an end to one that is tame.

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proper embedding, open 3-manifold, proper isotopy, proper knot, wild knot

1. Introduction

In this paper, we give an isotopy classification of proper knots in open 3-manifolds in the case that the proper knots run between two distinct ends of the manifold where one of these ends has the structure $S^2\times[0,\infty)$ and the other is a "ladder" end (as explained in Section 2; the definition of ladder ends includes ends with the structure $N^2\times[0,\infty)$ as a special case where N^2 is any closed, connected surface).

Recall that a continuous map $f: X\to Y$ is *proper* if for all compact $K\subset Y, f^{-1}(K)$ is compact in X. A *proper knot* is a smooth, proper embedding $f:\mathbb{R}\to M$ where M is a smooth, open 3-manifold. We define two proper knots to be *equivalent* or *properly isotopic* if they can be connected by a smooth isotopy $H:\mathbb{R}\times I\to M$ which

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is itself a proper map from $\mathbb{R}\times I$ to M. Other definitions of equivalence are possible, e.g., proper concordance, PL proper isotopy, etc. These are interesting but will not be dealt with in this paper.

The main result of this paper, Theorem 4.2, shows that for proper knots running between an end with the structure $S^2\times[0,\infty)$ and a ladder end, proper homotopy of knots implies proper isotopy, i.e., the isotopy classification reduces to a proper homotopy problem. However, it is not clear, in general, how to reduce a proper homotopy problem to a group theory problem.

A problem which motivated this research is the isotopy classification of proper knots in $N^2\times\mathbb{R}$ where N^2 is a connected, closed surface. In case $N^2=S^2$, Corollary 3.4 (a special case of the main theorem) and Proposition 2.2 show that there are exactly four equivalence classes of proper knots in $S^2\times\mathbb{R}$. In case $N^2\neq S^2$, Theorem 4.2 does not apply and the isotopy classification is unknown. Lacking suitable algebraic invariants in a nonambient setting, it is not known if there are only finitely many equivalence classes of proper knots in $N^2\times\mathbb{R}$, $N^2\neq S^2$. Indeed, it is not even known if there are any nontrivial proper knots that run between the two ends of $N^2\times\mathbb{R}$.

This paper continues the research initiated in [7] on proper knots in open 3-manifolds. In [7], a geometric technique of "combing out" along a vector field is introduced to construct proper isotopies (nonambient in general). This technique is briefly described and applied in Proposition 2.2 below. In a different direction, the problem of proper embeddings of planes in noncompact manifolds is studied in [4-6].

One of the distinguishing characteristics of this version of proper knot theory is that, in general, it is a nonambient theory. Artin and Fox in [2] give an example of a wild arc in $\mathbb{R}^3$ that has a nonsimply connected complement. By putting this arc in S^3 and deleting its endpoints, it can be thought of as a proper knot running between the two ends of $S^2\times\mathbb{R}$ with a nonsimply connected complement in $S^2\times\mathbb{R}$. By Corollary 3.4 of this paper, this proper knot is equivalent to the trivial proper knot which follows $\{p\}\times\mathbb{R}$ (some $p\in S^2$). Hence any proper isotopy realizing this equivalence cannot be covered by an ambient isotopy of $S^2\times\mathbb{R}$.

The proof of Theorem 4.2 relies, in part, on a geometric technique—a "lasso"—which enables one to interchange over- and undercrossings in knot diagrams. In addition, Theorem 4.2 employs the technique, introduced in [7], of combing out along vector fields. Theorem 4.2 is in marked contrast to the case of proper knots which send both ends of $\mathbb{R}^1$ to the same end of an open 3-manifold. Examples in [7] show that in this case, proper homotopy does not imply proper isotopy in general.

2. Preliminaries

The following notation will be used throughout: The closed unit interval is denoted by I. The n-disc (open n-disc) of radius r and centred at $0\in\mathbb{R}^n$ is denoted by $D^n(r)$

($OD^n(r)$). As usual, $\bar{U}$ denotes the closure of U in M and $f|A$ is the restriction of $f: X \to Y$ to $A \subset X$.

Simple examples of equivalences between proper knots are generated by the following elementary lemma:

Lemma 2.1. *Let $f: \mathbb{R} \to M$ be a proper knot and let $G: M \times I \to M$ be a smooth ambient isotopy of M. Then the map $H: \mathbb{R} \times I \to M$, defined by $H(t, \mu) = G(f(t), \mu)$, is an equivalence of proper knots.*

Let N be a noncompact manifold. We now recall the definition of the set of ends of N (see, e.g., [1, 3]). Let $\{K_i \mid i = 1, 2, \ldots\}$ be an exhaustion of N by compact sets, i.e., $\forall i$, K_i is compact, $K_i \subset \operatorname{int} K_{i+1}$ and $N = \bigcup_i K_i$. Now form sequences $U_1 \supset U_2 \supset U_3 \supset \cdots$ where each U_i is chosen to be a path component of $N - K_i$ and each U_i has noncompact closure, i.e., $\{U_i\}$ is a nested sequence of nonempty, open, connected subsets of N such that $\forall i$, U_i has compact frontier, $\bar{U}_i$ is noncompact and $\bigcap_i U_i = \emptyset$. Suppose that $\{V_j\}$ is another such sequence generated as the path components of the complements of another compact exhaustion of N. Then we say that $\{U_i\}$ and $\{V_j\}$ are *equivalent* if they are cofinal, i.e., $\forall i$, $\exists j$ such that $V_j \subset U_i$ and $\forall m$, $\exists n$ such that $U_n \subset V_m$. An equivalence class of such sequences is called an *end* of N. The set of ends of N is denoted by $e(N)$.

Let M be a smooth, open 3-manifold. Then $\Gamma \in e(M)$ is called a *collared end* if $\exists \{V_i\} \in \Gamma$ and $\exists j$ such that $\bar{V}_j$ is diffeomorphic to $W \times [0, \infty)$ where W is a smooth, closed, connected surface. We shall also refer to Γ as a *W end.* An end $\Lambda \in e(M)$ is called a *ladder end* if $\exists \{U_i\} \in \Lambda$ and $\exists j$ such that $\bar{U}_j$ is a smooth submanifold of M and $\bar{U}_j$ admits a smooth, proper Morse function $m: \bar{U}_j \to \mathbb{R}$ satisfying

(i) $m(\bar{U}_j) = [0, \infty)$,

(ii) 0 is a regular value of m such that $m^{-1}(0) = \partial \bar{U}_j$,

(iii) the critical points of m are all of index 1.

An example of a manifold with a single, ladder end is provided by the interior of the solid, semi-infinite ladder $T \# T \# T \# \cdots$ where T is the solid torus and $\#$ denotes disc sum along the boundary. Note that a collared end is a special case of a ladder end whose associated Morse function has no critical points. In this case the "collar" is $m^{-1}(0) \times [0, \infty)$.

Let $g: A \to B$ be a proper map between manifolds and let $\{U_i\} \in \Gamma_A \in e(A)$ and $\{V_j\} \in \Gamma_B \in e(B)$. Then g *sends Γ_A to Γ_B* if $\forall j$, $\exists i$ such that $g(U_i) \subset V_j$. It is easy to show that g induces a well-defined map, denoted by $\hat{g}: e(A) \to e(B)$, where $\forall \Gamma \in e(A)$, $\hat{g}$ sends Γ to $g(\Gamma)$. Denote the two ends of $\mathbb{R}$ by $\pm\infty$, i.e., $e(\mathbb{R}) = \{+\infty, -\infty\}$. Given a proper knot $f_0: \mathbb{R} \to M$ and two ends $\Gamma_1, \Gamma_2 \in e(M)$, we say that f_0 *runs between Γ_1 and Γ_2* if $\hat{f}_0(\{+\infty, -\infty\}) = \{\Gamma_1, \Gamma_2\}$. If, furthermore, $\hat{f}_0(-\infty) = \Gamma_1$ and $\hat{f}_0(+\infty) = \Gamma_2$, then f_0 is said to *run from Γ_1 to Γ_2*. Note that in this case, if $H: \mathbb{R} \times I \to M$ is an equivalence between f_0 and another proper knot $f_1: \mathbb{R} \to M$, then f_1 also runs from Γ_1 to Γ_2.

Combing out. There is a geometric technique which is used to construct equivalences (nonambient in general) between proper knots. Given two equivalent proper knots $f, g: \mathbb{R} \to M$, the idea is to use a suitable vector field V on M (usually a gradient vector field) such that g is parallel to trajectories of V. The flow of V is then used to produce a proper isotopy which pushes f to g. This construction, known as "combing out" the knot f, is explained in detail in [7]. Another way of viewing this in the case where M has only 0- and 1-handles is the following: The cores of the 0- and 1-handles form a one-dimensional complex and f and g can be properly isotoped to avoid this complex. M minus the complex has a product structure and this can be used to "straighten" f and g so that they run along fibres out to the ends of M. After performing this procedure, the equivalence between f and g is easily constructed. To illustrate combing out in a case useful for the proof of the main theorem, we prove the following proposition:

Proposition 2.2. *Let M^2 be a smooth, closed surface and let $f, g: \mathbb{R} \to M^2 \times \mathbb{R}$ be two proper knots. Denote the two ends of $M^2 \times \mathbb{R}$ by $M^2 \times \{+\infty\}$ and $M^2 \times \{-\infty\}$.*

(a) *Suppose that f runs from $M^2 \times \{-\infty\}$ to $M^2 \times \{+\infty\}$ and that for some $r \in \mathbb{R}$, f meets $M^2 \times \{r\}$ transversely in a single point $(p, r) \in M^2 \times \mathbb{R}$. Then f is equivalent to a proper knot $\bar{f}: t \mapsto (p, t)$.*

(b) *If f and g both run from $M^2 \times \{+\infty\}$ to $M^2 \times \{+\infty\}$ (or both run from $M^2 \times \{-\infty\}$ to $M^2 \times \{-\infty\}$), then f and g are equivalent (i.e., if they both "stay in the same end", then they are equivalent).*

Proof. (a) By lemma 2.1, we may assume $r = 0$. We may further assume that for some $\varepsilon > 0$, $f(t) = (p, t)$ $\forall t \in [-\varepsilon, \varepsilon]$. Now construct a combing out vector field V on $M^2 \times \mathbb{R}$ which is parallel to lines $\{q\} \times \mathbb{R}$ $(q \in M^2)$ given by $V(q, t) = (0, t) \in T_q M^2 \times T_t \mathbb{R}$ $(t \in \mathbb{R})$. Then use the flow of V to push f to $\bar{f}$, i.e., $f([-\varepsilon, \varepsilon])$ gets pushed

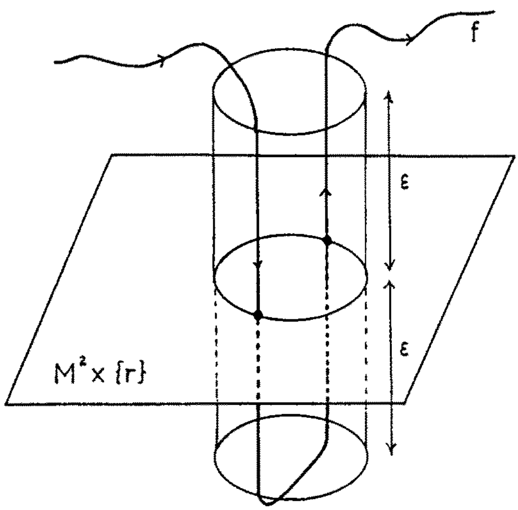

Fig. 1.

to $\bar{f}(\mathbb{R})$ and $f(\mathbb{R}-[-\varepsilon,\varepsilon])$ gets pushed to the ends of $M^2\times\mathbb{R}$. For more details on the reparametrization involved, see [7].

(b) Suppose that f and g both run from $M^2\times\{+\infty\}$ to $M^2\times\{+\infty\}$. We may assume that f and g are both bounded away from $M^2\times\{0\}$ and that $f|D^1=g|D^1$. Let $h:D^2\to M^2$ be a chart for M^2. Then by a piping move, we may isotope $f|D^1(4\varepsilon)$ (some small $\varepsilon>0$) such that it hits $M^2\times\{0\}$ transversely at the two points $h(S^0)\times\{0\}$ and that $f(D^1(3\varepsilon)-\mathrm{OD}^1(\varepsilon))=h(S^0)\times[-\varepsilon,\varepsilon]$ (see Fig. 1). It can be assumed that $g|D^1(4\varepsilon)$ was isotoped simultaneously with f in the same fashion. The flow of a combing out vector field parallel to lines $\{q\}\times\mathbb{R}$, $q\in M^2$, which vanishes on $M^2\times(-\infty,0]$ and whose flow pushes f and g in $M^2\times[0,\infty)$ to $h(S^0)\times[0,\infty)$ then gives rise to the desired equivalence. □

3. Lasso constructions

In the following, M will always denote a smooth, connected, open 3-manifold with at least two ends Γ_1, $\Gamma_2\in e(M)$. Γ_1 will always be assumed to be an S^2 end, i.e., a collared end such that $W=S^2$ (see the definition in Section 2). Denote the associated collar by $E_1=W\times[0,\infty)=S^2\times[0,\infty)\subset M$ and identify Γ_1 with $S^2\times\{\infty\}$.

Interchanging under- and overcrossings. Suppose that $f:\mathbb{R}\to M$ is a proper knot that runs from Γ_2 to Γ_1 and that near Γ_1, f follows a collar line, i.e., for some t, $r\in\mathbb{R}$ and for some $p\in S^2$, $f([t,\infty))=\{p\}\times[r,\infty)\subset E_1$. Let $\theta:U\to\mathbb{R}^3$ be any chart of M where $f(\mathbb{R})\cap U$ consists of a finite number of embedded open arcs. If we have a regular projection (see, e.g, [8]) of $\theta(f(\mathbb{R})\cap U)$, then under- or overcrossings of f in U (i.e., the embedded arcs) may occur relative to this projection. The claim is then:

Lemma 3.1. *Any under- (over-) crossing of f in U can be changed into an over- (under-) crossing by a smooth, compactly supported isotopy of f.*

Proof. The general scheme is to pull an under- or overcrossing arc into a tubular neighbourhood that follows f from the crossing point X out to the S^2 end (see Fig. 2(i)). Flip this around the back of S^2 (Fig. 2(ii)) and shrink back to the crossing point (Fig. 2(iii)). Note that for arcs being pushed out in the $S^2\times\{\infty\}$ direction along f, there are two possible choices, i.e., the under- or overcrossing arc could be chosen. One choice will always head back to the crossing X (Fig. 2(iv)) and the other heads out to $S^2\times\{\infty\}$ without ever encountering X (relative to the regular projection of f in U). The latter choice is the one that we make here. □

The dashed arc depicted in Fig. 2(i)–(iii) is called a *lasso*. If, relative to some projection, it is formed by stretching an under- (over-) crossing piece of arc, then it is called an *under- (over-) crossing lasso.*

The following "folk" theorem of classical knot theory is an easy consequence of Lemma 3.1 (see [13]):

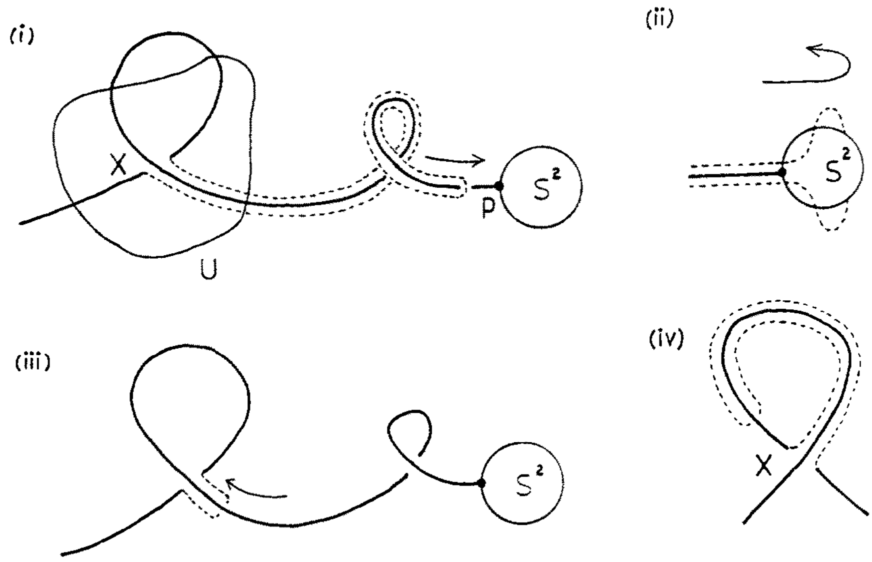

Fig. 2.

Theorem 3.2 ("Light bulb"). *Let $p \in S^2$ be a basepoint. Let $f: I \to S^2 \times I$ be any tamely embedded arc connecting $\{p\} \times \{0\}$ to $\{p\} \times \{1\}$ such that $f((0, 1)) \subset S^2 \times (0, 1)$. Then f is equivalent to $\{p\} \times I$ by an ambient isotopy that fixes $S^2 \times \{0, 1\}$.*

The uses of lassos include unlinking tangles of arcs and isotoping proper knots so that they meet a given surface transversely in a single point. These uses are illustrated in the next three propositions which form the principal technical results of this section.

Proposition 3.3. *Let $f: \mathbb{R}^1 \to M$ be a proper knot such that $f(t)$ runs from Γ_2 to Γ_1. Then there is a smooth, proper isotopy of f that moves f in E_1 onto a collar line $\{p\} \times [r, \infty)$ (for some $p \in S^2$, $r \geq 0$). Furthermore, this isotopy fixes f in $M - E_1$.*

Proof. Since f is proper, we may assume f meets $\Sigma^0 = S^2 \times \{0\}$ transversely in a finite number of points. Let $p_0 = f(t_0)$ be the first point and $p_0' = f(t_0')$ ($t_0 \leq t_0'$) be the last point of intersection of f with Σ^0, i.e., $f(t) \notin E_1$ if $t < t_0$ and $f(t) \in E_1$ if $t \geq t_0'$. Now consider another sphere $\Sigma^1 = S^2 \times \{r_1\}$, $r_1 > 0$. Again we may assume that f meets Σ^1 transversely in a finite number of points: let $p_1 = f(t_1)$, respectively $p_1' = f(t_1')$ ($t_1 \leq t_1'$) be the first, respectively the last, points of intersection of f with Σ^1, i.e., $f(t) \notin S^2 \times [r_1, \infty)$ if $t < t_1$ and $f(t) \in S^2 \times [r_1, \infty)$ if $t \geq t_1'$. Furthermore, since $f((-\infty, t_0'])$ is bounded away from $S^2 \times \{\infty\}$, $r_1 > 0$ can be chosen large enough so that $t_0' < t_1$. Hence $t_0 \leq t_0' < t_1 \leq t_1'$ (see Fig. 3).

Now form two proper knots f_0 and f_1 by smoothly joining on collar lines $\{p_0'\} \times [0, \infty)$ and $\{p_1'\} \times [r_1, \infty)$ to $f((-\infty, t_0'])$ and $f((-\infty, t_1'])$ respectively (f may have to be perturbed slightly to avoid self-intersections in f_0 and f_1).

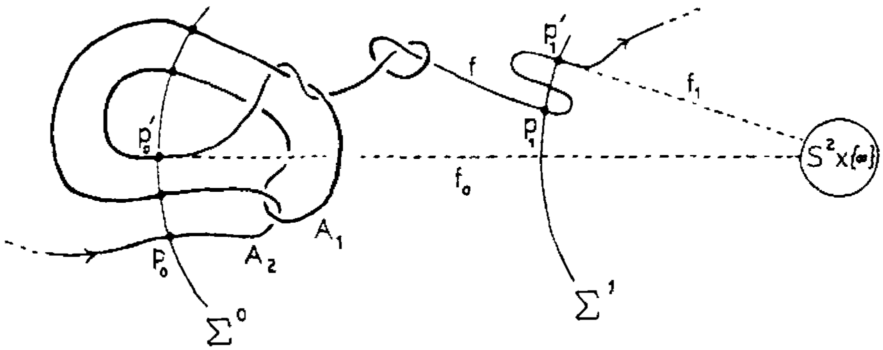

Fig. 3.

We claim that f_1 is equivalent to f_0 by a smooth ambient isotopy that moves f_1 onto f_0 and leaves $M-E_1$ fixed. To see this, first consider the arcs $A_1, \ldots, A_k$ formed by $f([t_0, t_0'])\cap E_1$ (in Fig. 3 we have shown two such arcs, A_1 and A_2). Considering E_1 as a subset of $\mathbb{R}^3$, we may assume that we have a regular projection of f_0 and f_1 in E_1. Then whenever one of the A_j overcrosses $f_1([t_0', t_1'])$ $(=f([t_0', t_1']))$, turn it into an undercrossing by sending an overcrossing lasso around $S^2\times\{\infty\}$ in accordance with the scheme given in Lemma 3.1. Note that the isotopy realizing the lasso fixes $M-E_1$ as the lasso is pushed out in the $S^2\times\{\infty\}$ direction and hence must stay in E_1. Now deform any overcrossings of the A_j with $\{p_0'\}\times[0,\infty)$ to undercrossings (we do not have to use lassos to accomplish this). Then $f_1([t_0',\infty))$ and $p_0'\times[0,\infty)$ lie in a subset of $\mathbb{R}^3$ separated from $\bigcup_i A_i$ by a plane of constant level. Hence, in a manner similar to the Light bulb theorem, $f_1([t_0',\infty))$ can now be unknotted and isotoped to $\{p_0'\}\times[0,\infty)$ by an isotopy that is fixed below this plane, thus avoiding $\bigcup_i A_i$. In restoring the overcrossings of the A_j with $\{p_0'\}\times[0,\infty)$ by using undercrossing lassos, we complete the proof of the claim.

Summarizing the above process: Given the two concentric spheres Σ^0 and Σ^1 and given the proper knot f_0 where $f_0(t)=f(t)$ $\forall t\in(-\infty, t_0']$, then by an ambient isotopy that leaves $M-S^2\times[0,\infty)$ and $f|(-\infty, t_0]$ fixed, we can move f_0 onto the proper knot f_1 where now $f_1(t)=f(t)$ $\forall t\in(-\infty, t_1']\supset(-\infty, t_0']$. This isotopy of f_0 is proper and can be covered ambiently.

We now repeat all of the above, this time using Σ^1 and $\Sigma^2=S^2\times\{r_2\}$ for an appropriately chosen $r_2>r_1$ in place of Σ^0 and Σ^1. This would give an ambient isotopy of M that leaves $M-S^2\times[r_1,\infty)$ and $f|(-\infty, t_1]$ fixed, and moves proper knot f_1 onto proper knot f_2 where $f_2(t)=f_1(t)$ $\forall t\in(-\infty, t_2']\supset(-\infty, t_1']$. This isotopy of f_1 is proper, etc. In this way, we can produce a nested sequence of spheres Σ^0, Σ^1, $\Sigma^2, \ldots$ at collar levels $r_0=0$, r_1, $r_2, \ldots$, where $r_j\to\infty$ as $j\to\infty$, and a sequence of proper knots $f_0, f_1, f_2, \ldots$ and points $p_j=f(t_j)$, $p_j'=f(t_j')$ where

$$t_j\leq t_j'<t_{j+1}\leq t_{j+1}'$$

and $p_j, p_j'\in\Sigma^j$, $j=0, 1, 2, \ldots$ with the following properties: $\forall j$,

(i) f meets Σ^j transversely,

(ii) $f_j(t) = f(t)\ \forall t \leq t_j'$ (and so t_j, $t_j' \to \infty$ because f is proper),

(iii) $f_j((t_j', \infty)) = \{p_j'\} \times (r_j, \infty)\ \forall t > t_j'$,

(iv) f_j is properly isotopic to f_{j+1} via an ambient isotopy of M, say $F^j : M \times I \to M$, which leaves $M - S^2 \times [r_j, \infty)$ and $f|(-\infty, t_j]$ fixed.

Let $G^j = F^j(f_j(\cdot), \cdot) : \mathbb{R} \times I \to M$, i.e., G^j is the proper isotopy connecting f_j and f_{j+1}. We wish to produce a map that has the effect of G^0 followed by G^1 followed by G^2 etc. So smoothly concatenate the G^j in the usual way to produce $G : \mathbb{R} \times [0, 1) \to M$. That is, we have an infinite sequence $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ where $\mu_j \to 1$ as $j \to \infty$ and nondecreasing, onto functions $\theta^j : I_j = [\mu_j, \mu_{j+1}] \to [0, 1]$ satisfying

$$G(\cdot, \mu) = G^j(\cdot, \theta^j(\mu)) \quad \forall \mu \in I_j.$$

Define $H : \mathbb{R} \times I \to M$ by

$$H(\cdot, \mu) = \begin{cases} G(\cdot, \mu) & \text{if } \mu \in [0, 1), \\ f(\cdot) & \text{if } \mu = 1. \end{cases}$$

Then the isotopy we seek is $H(\cdot, 1 - \mu)$ which moves f to f_0 as μ goes from 0 to 1.

We now show that H is a proper map. Note that by (iv) above, given any compact set $K \subset M$, there exists $n > 0$ such that $\forall \mu \geq \mu_n$,

$$[H(\cdot, \mu)]^{-1}(K) = f^{-1}(K).$$

Hence $H^{-1}(K) = A \cup B$ where $A = f^{-1}(K) \times [\mu_n, 1]$ and $B = (G|\mathbb{R} \times [0, \mu_n])^{-1}(K)$. But f is proper and so A is compact. Also, $G|\mathbb{R} = [0, \mu_n]$ is just the concatenation of the proper maps $G^0, G^1, \ldots, G^{n-1}$ and hence is itself a proper map. Thus B is compact. □

In Proposition 3.3, we constructed f "a piece at a time", i.e., at the jth stage, f was assembled from a straight line segment between the two concentric spheres Σ^j, Σ^{j+1} and the $\{\Sigma^j\}$ converged to $S^2 \times \{\infty\}$. In the case $M = S^2 \times \mathbb{R}$, we could have equally as well directed this "building up" process out to $S^2 \times \{-\infty\}$ using concentric spheres. So if we start with an arc $\{p\} \times (-\infty, \infty)$, $p \in S^2$, we can construct any arc we like that runs between $S^2 \times \{-\infty\}$ and $S^2 \times \{\infty\}$ by "building up" out to $S^2 \times \{\infty\}$ followed by "building up" out to $S^2 \times \{-\infty\}$. In other words:

Corollary 3.4. *Up to equivalence, there are exactly two proper knots that run between the two ends of $S^2 \times \mathbb{R}$ and these knots are distinguished by orientation.*

We remark that Proposition 2.2(b) together with Corollary 3.4 gives a classification of proper knots in $S^2 \times \mathbb{R}$ up to equivalence, i.e., we have four possible equivalence classes given by the following representatives: Any open arc running

(i) from $S^2 \times \{-\infty\}$ to $S^2 \times \{+\infty\}$,

(ii) from $S^2 \times \{+\infty\}$ to $S^2 \times \{-\infty\}$,

(iii) from $S^2 \times \{+\infty\}$ to $S^2 \times \{+\infty\}$,

(iv) from $S^2 \times \{-\infty\}$ to $S^2 \times \{-\infty\}$.

Proposition 3.3 shows that we may "straighten" a proper knot near $\Gamma_1 =$ an S^2 end. We now examine conditions under which it is possible to achieve the same result near Γ_2 when Γ_2 is not necessarily an S^2 end.

Proposition 3.5. *Let Γ_2 be a collared end determined by $E_2 = T_2 \times [0, \infty) \subset M$ where T_2 is a smooth, closed surface and $E_2 \cap E_1 = \emptyset$. Let $f: \mathbb{R} \to M$ be a proper knot that runs from Γ_2 to Γ_1 and suppose that f follows a collar line in E_1. Then by a smooth, compactly supported, ambient isotopy of M, f can be moved so that it intersects $T_2 \times \{r_0\}$ (any $r_0 > 0$) transversely at a single point.*

Proof. We may assume that f is transverse to $T_2 \times \{r_0\}$. Recall that $f(t) \to T_2 \times \{\infty\}$ as $t \to -\infty$. As f is proper, $\exists t_0$, t_0', $t_0' \leq t_0$, such that $\forall t > t_0$, $f(t) \notin T_2 \times [r_0, \infty)$ and $\forall t \leq t_0'$, $f(t) \in T_2 \times [r_0, \infty)$. Now consider the arcs $A_1, \ldots, A_k$ formed by $f([t_0', t_0]) \cap T_2 \times [r_0, \infty)$. These are wholly contained in $T_2 \times [r_0, r_1]$ and do not meet $T_2 \times \{r_1\}$ if r_1 $(> r_0)$ is large enough (see Fig. 4). Again, we may assume that f is transverse to $T_2 \times \{r_1\}$. By perturbing f in $T_2 = [r_0, r_1]$ and using the projection $\pi: T_2 \times [r_0, r_1] \to T_2 \times \{r_0\}$ defined by the collar lines, we may produce a regular projection of f in $T_2 \times [r_0, r_1]$ onto $T_2 \times \{r_0\}$. By sending lassos around the S^2 end, then relative to π, any undercrossing of $f((-\infty, t_0'])$ with the A_j can be turned into an overcrossing, i.e., we are modifying $f((-\infty, t_0'])$ in such a way that it always sits "above" the A_j with respect to the projection π. The proof is then completed when we push the A_j down, via collar lines, into

$$T_2 \times [r_0 - 2\varepsilon, r_0 - \varepsilon] \quad (\text{some } \varepsilon > 0)$$

leaving $f(t_0')$ as the single transverse crossing point. $\square$

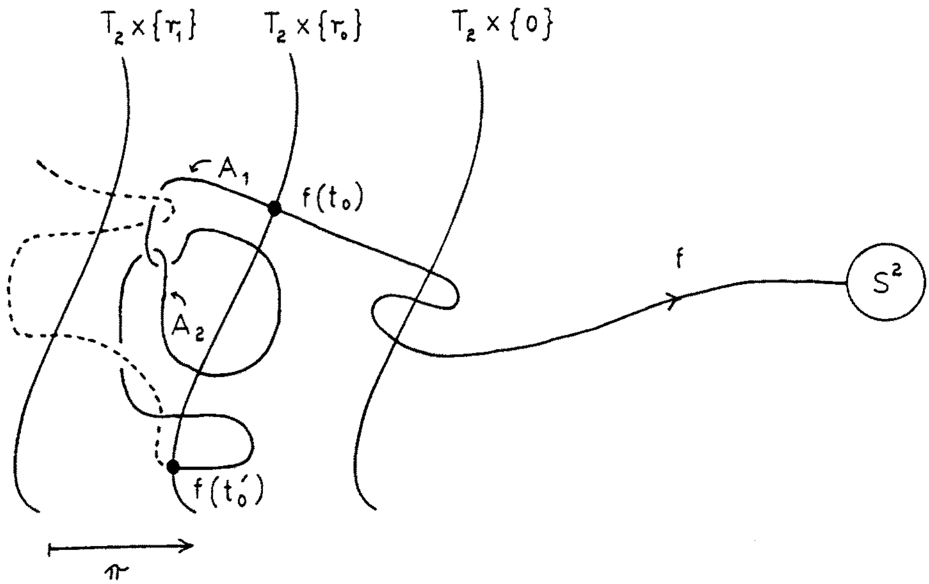

Fig. 4.

Corollary 3.6. *Let f be as in Proposition 3.5. Then f is equivalent to a proper knot which follows a collar line $\{\tau\}\times[r_0,\infty)$, some $\tau\in T_2$, $r_0>0$. The equivalence fixes f in $M-E_2$.*

Proof. By Proposition 3.5, we may assume that f has a single, transverse intersection point with $T_2\times\{r_0\}$. A combing out vector field which is parallel to the collar lines and zero on $M-T_2\times[r_0,\infty]$ is then used to generate the required isotopy (this is similar to Proposition 2.2(a)). □

Corollary 3.6 may, by using Proposition 3.5, be extended to the case where Γ_2 is a ladder end. Referring to the notation used in the definition of a ladder end in Section 2, there is an open set $U=U_j\in\{U_i\}\in\Gamma_2$ and a smooth Morse function $m:\bar U\to\mathbb{R}$ satisfying conditions (i)-(iii) of the definition. Denote the submanifold $\partial\bar U$ by W_0. $\bar U$ has a smooth vector field V which is a gradient vector field of m (see [12]) and the trajectories of V are all defined on $[0,\infty)$, i.e., V has flow $\chi:\bar U\times[0,\infty)\to\bar U$. The next proposition is then:

Proposition 3.7. *Let $f:\mathbb{R}\to M$ be a proper knot that runs from the ladder end Γ_2 to the S^2 end Γ_1. Then f is equivalent to a proper knot $\bar f:\mathbb{R}\to M$ which follows a flow line of V in the ladder end, say $\chi(u,[t_0,\infty))$ (some $u\in U$, $t_0>0$), and $\bar f$ also follows a collar line in the S^2 end (i.e., f can be "straightened" at both ends).*

Proof. We may assume that m has only one critical point per critical level. Now construct an auxiliary vector field V_1 on $\bar U$ in the following way: Let $c>0$ be the lowest critical value of m (we assume that there is at least one critical point or else we just have the previous case where Γ_2 was a collar end). Let $V_1|m^{-1}([\frac{1}{2}c,\infty))=V|m^{-1}([\frac{1}{2}c,\infty))$ and let V_1 be parallel to V in $m^{-1}((0,\frac{1}{2}c))$ but arrange for $V_1(p)\to 0$ as $p\to W_0=m^{-1}(0)$, i.e., $V_1|W_0\equiv 0$ and V_1 is nonzero off W_0 and critical points of m. In addition, $V_1|U$ has all its trajectories defined on all of $\mathbb{R}$ and these run along the same track and in the same direction as trajectories of V. Hence V_1 is a complete vector field with flow $\chi_1:\bar U\times\mathbb{R}\to\bar U$. Define a *critical trajectory* of V_1 to be a trajectory that runs between two critical points of m or between W_0 and a critical point of m.

By Proposition 3.3, it can be assumed that f follows a collar line in E_1.

The first step is to perturb f so that it is transverse to W_0, say $f\mapsto f_0$. Then take a tubular neighbourhood of f_0 and find a parallel translate of f_0 in the tubular neighbourhood which avoids all critical points and critical trajectories of V_1, say $f_0\mapsto f_1$ (see [7] for details), i.e., in a ladder end, f_0 can be pushed off the critical trajectories of V_1. This allows us to comb out f_1 parallel to trajectories of V once we have arranged f_1 to be transverse in a single point to some noncritical surface of m. This is what we shall do next.

Now $\exists t_0$, t_0', $t_0'\le t_0$ such that $\forall t>t_0$, $f_1(t)\notin\bar U$ and $\forall t\le t_0'$, $f(t)\in\bar U$. Choose a regular value of m, $b>0$ (say), large enough so that $f_1([t_0',t_0])$ does not meet $m^{-1}(b)=W_1$ and so that f_1 is transverse to W_1. Notice that $W_1\times[0,\infty)\cong\chi_1(W_1,(-\infty,0])\subset\bar U$ where $\forall a\ge 0$, $W_1\times\{a\}$ corresponds to $\chi_1(W_1,-a)$. Also, since

f_1 avoids all critical trajectories and critical points of V_1 and since the only trajectories of V_1 starting in $W_1 \times (0, \infty)$ that fail to pass through W_1 ($\cong W_1 \times \{0\}$) are critical trajectories, it follows that $f_1(\mathbb{R}) \cap m^{-1}([0, b]) \subset W_1 \times [0, \infty)$. We now use arguments similar to those in Proposition 3.5 to properly isotope f_1 in $W_1 \times [0, \infty)$ to a proper knot $f_2 : \mathbb{R} \to M$ which hits W_1 transversely in a single point. To see this, consider the set $S = f_1(\mathbb{R}) \cap W_1 \times [0, \infty)$ which consists of arcs $A_1, A_2, \ldots, A_m$ starting and finishing at $W_1 \times \{0\}$ and (open) arcs $B_1, B_2, \ldots, B_n$ starting and finishing at $W_1 \times \{\infty\}$ together with an arc C which starts at $p \in W_1 \times \{0\}$ and finishes at $W_1 \times \{\infty\}$ ($\cong W_0$). Let $\pi : W_1 \times [0, \infty) \to W_1 = W_1 \times \{0\}$ be projection onto W_1. We may assume that we have a regular projection of S onto W_1 relative to π. By using lassos, arrange for the A_i to undercross both C and the B_j. Then push the A_i down through $W_1 \times \{0\}$ so they lie in $W_1 \times [-\varepsilon, 0)$ (some $\varepsilon > 0$). The new proper knot that is formed in this way is the required f_2 which hits W_1 transversely in the single point $p \in W_1$ and p lies on a noncritical trajectory of V_1.

It can be assumed that f_2 also misses all critical points and critical trajectories in $m^{-1}([b, \infty))$. So to complete the proof of the proposition, comb out f_2 from W_1 parallel to trajectories of V using another vector field V_2. V_2 is constructed from V in a way that is similar to V_1 except that this time, we arrange for V_2 to be zero on all of $m^{-1}([0, b])$. $\square$

4. Theorems

We need a lemma about level preserving maps for use in the main theorem. A fuller discussion of the ideas employed here may be found in [9].

Lemma 4.1. *Let $F^1 : \mathbb{R} \times I \to M$ be a smooth proper map such that*

(a) *there is an $\varepsilon \in (0, \frac{1}{2})$ such that for each $\mu \in [0, \varepsilon] \cup [1-\varepsilon, 1]$, $F^1(\cdot, \mu)$ is a smooth embedding,*

(b) *there is a $\beta > 0$ such that for each $\mu \in I$, $F^1(t, \mu)$ is a smooth embedding when restricted to values of t where $|t| \geq \beta$.*

Then we may perturb $F^1 \times \mathrm{id}_I : \mathbb{R} \times I \to M \times I$ to a smooth, level preserving map $F^{1*} : \mathbb{R} \times I \to M \times I$ where

(i) *F^{1*} is an immersion whose singularities consist of a finite number of transverse double points,*

(ii) *$F^2 = \pi_M \circ F^{1*} : \mathbb{R} \times I \to M$ is a proper map and $\forall \mu \in I$, $F^2(\cdot, \mu)$ is an immersion with at most one singularity and that is a double point,*

(iii) *F^2 agrees with F^1 on $\{(t, \mu) \mid \mu \in [0, \delta] \cup [1-\delta, 1]$ or $|t| \geq \gamma\}$ where $0 < \delta < \varepsilon$ and $\gamma > \beta$.*

Proof. Let A be the rectangle $\{(t, \mu) \mid \mu \in [\varepsilon, 1-\varepsilon]$ and $|t| \leq \beta\}$, Then by hypotheses (a) and (b), $F = F^1 \times \mathrm{id}_I$ is already an embedding on $\mathbb{R} \times I - A$. Since $\dim(M \times I) = 4 = 2 \cdot \dim(\mathbb{R} \times I)$, then using standard arguments, we need only alter F on a rectangle

B, where $A \subset \operatorname{int} B \subset B \subset \mathbb{R} \times (0, 1)$, so that F is in general position with itself (see, e.g., [10]). So say we produce a map $G: \mathbb{R} \times I \to M \times I$ which is an immersion with transverse double points. We may assume that G agrees with F on $\mathbb{R} \times I - B$ and that the singular set of G is contained in B. G is also a proper map and hence, since B is compact, G can only have a finite number of singularities. Furthermore, by composing G with a suitable diffeomorphism of $M \times I$ which is close to the identity and is supported in a neighbourhood of the double point set, one may arrange, if necessary, that distinct double points of G lie on distinct levels of $M \times I$.

The next task is to change G so that it is level preserving. So now write

$$G(t, \mu) = (G_1(t, \mu), G_2(t, \mu)).$$

Because we can choose $G_2: \mathbb{R} \times I \to I$ to be arbitrarily close to $\pi_2: \mathbb{R} \times I \to I$ (where $\pi_2(t, \mu) = \mu$) in the Whitney C^∞ topology, we may choose the map $\eta: \mathbb{R} \times I \to \mathbb{R} \times I$ given by $\eta(t, \mu) = (t, G_2(t, \mu))$ to be arbitrarily close to $\mathrm{id}_{\mathbb{R} \times I}$ and hence we may assume that η is a C^∞ diffeomorphism (see, e.g., [11]), equal to $\mathrm{id}_{\mathbb{R} \times I}$ on $\mathbb{R} \times I - B$. Let $\eta^{-1}(t, \lambda) = (t, H(t, \lambda))$ and set $F^{1*} = G \circ \eta^{-1}$. We claim F^{1*} is now level preserving:

$$\begin{aligned} F^{1*}(t, \lambda) &= G(t, H(t, \lambda)) \\ &= (G_1(t, H(t, \lambda)), G_2(t, H(t, \lambda))), \end{aligned}$$

but

$$\begin{aligned} \eta \circ \eta^{-1}(t, \lambda) = (t, \lambda) &\Rightarrow \eta(t, H(t, \lambda)) = (t, G_2(t, H(t, \lambda))) = (t, \lambda) \\ &\Rightarrow G_2(t, H(t, \lambda)) = \lambda. \end{aligned}$$

So $F^{1*}(t, \lambda) = (G_1(t, H(t, \lambda)), \lambda)$. Note that as F^{1*} is just a reparameterization of $G(\mathbb{R} \times I)$; then F^{1*} still only has a finite number of transverse double points as singularities and distinct double points of F^{1*} occur on distinct levels of $M \times I$. Furthermore $\pi_M \circ F^{1*}$ agrees with F^1 as in (iii) above. It follows that the map $F^{1*}: \mathbb{R} \times I \to M \times I$ satisfies all the conclusions of the lemma. $\square$

Having established the necessary machinery, the main result of this paper can now be stated and proved:

Theorem 4.2. *Let Γ_1, Γ_2 be distinct ends of a smooth, open 3-manifold M such that Γ_1 is an S^2 collar end and Γ_2 is a ladder end. Let $f, g: \mathbb{R} \to M$ be any two proper knots that run between Γ_1 and Γ_2 such that f and g can be connected by a smooth, proper homotopy. Then f and g can be connected by a smooth, proper isotopy (i.e., they are equivalent).*

Proof. Let U and $m: \bar{U} \to \mathbb{R}$ be the open submanifold and Morse function associated to the ladder end Γ_2 and let V be an associated gradient vector field of m.

One may suppose that f and g both run from Γ_2 to Γ_1.

We are given a smooth, proper homotopy $H^0:\mathbb{R}\times I\to M$ connecting f to g. Hence, by Proposition 3.7, there is a smooth, proper homotopy $H^1:\mathbb{R}\times I\to M$ connecting $\bar{f}$ to f to g to $\bar{g}$ where $\bar{f}$ and $\bar{g}$ are the end straightened versions of f and g. Call a trajectory of V critical if it runs from $\partial\bar{U}$ to a critical point of m or if it runs between critical points. Now find t_1, $t_2\in\mathbb{R}$ and a regular value of m, $b>0$ (say), such that $\bar{f}(t_1)$, $\bar{g}(t_2)\in U$ and $m(\bar{f}(t_1))$, $m(\bar{g}(t_2))<b$ and $\forall t<t_1$, $\forall t'<t_2$, $\bar{f}(t)$ and $\bar{g}(t')$ follow noncritical trajectories of V. These trajectories will pass through $m^{-1}(b)$ transversely in single points. Also find $r\geqslant 0$ such that $\bar{f}$ and $\bar{g}$ follow collar lines in $S^2\times[r,\infty)\subset E_1$.

The first step is to change H^1 so that the whole homotopy is "end straightened". The idea is to replace the ends of H^1 by collar lines in E_1 and by noncritical trajectories of V in U (we may assume $E_1\cap U=\emptyset$). This is easy to accomplish in E_1 but requires a bit more care in U. So, choose $c>0$ large enough so that $\forall t\leqslant -c$, $H^1(t,I)\subset m^{-1}((b,\infty))=X$ and $\forall t\geqslant c$, $H^1(t,I)\subset S^2\times[r,\infty)$. Denote $H^1(-c,\cdot)$ and $H^1(c,\cdot)$ by $\alpha(\cdot):I\to X$ and $\gamma(\cdot):I\to E_1$ respectively. What we would like is to have $\alpha(I)$ missing all critical points and critical trajectories. The following argument shows how to alter H^1 in order to achieve this. Now there exists a smooth homotopy of α, rel$\{0,1\}$, that improves α to an embedding $\alpha':I\to X$. Consider a "pinched" tubular neighbourhood of α' (i.e., the neighbourhood squeezes to points $\alpha'(0)$ and $\alpha'(1)$ at 0 and 1 respectively). Then on $(0,1)$, there is a translate of α' in this neighbourhood which avoids all critical points and critical trajectories of V (see [7]). This provides a smooth isotopy, rel$\{0,1\}$, between α' and $\alpha'':I\to X$ (say) and α'' avoids all critical points and critical trajectories. Hence we may construct a smooth homotopy $A:I\times I\to X$ which deforms α to α'' and then back to α (rel$\{0,1\}$) where $A(\cdot,\frac{1}{2})=\alpha''(\cdot)$. Splice A into H^1 by cutting H^1 along the strut $H^1(-c,\cdot)=\alpha(\cdot)$ and smoothly insert A to produce a new smooth homotopy $H^2:\mathbb{R}\times I\to M$ such that $H^2(-c,\cdot)=H^2(-c-1,\cdot)=\alpha(\cdot)$ and $H^2(-c-\frac{1}{2},\cdot)=\alpha''(\cdot)$. Notice that $H^2(t,0)$ and $H^2(t,1)$ follow the image of $\bar{f}(t)$ and $\bar{g}(t)$ respectively, but for $t\in[-c-1,-c]$, we no longer have embeddings (as A moved α rel$\{0,1\}$). However, $H^2(\cdot,0)$ and $H^2(\cdot,1)$ are clearly homotopic to $\bar{f}$, $\bar{g}$ respectively by smooth, proper homotopies that merely reparametrize $H^2(\mathbb{R},0)$ and $H^2(\mathbb{R},1)$. In this way, we obtain a smooth, proper homotopy $H^3:\mathbb{R}\times I\to M$ connecting $\bar{f}$ to $H^2(\cdot,0)$ to $H^2(\cdot,1)$ to $\bar{g}$ and the strut $H^3(-c-\frac{1}{2},\cdot)=\tilde{\alpha}(\cdot):I\to X$ still misses all critical points and critical trajectories. So H^3 is the required modification of H^1 and $\tilde{\alpha}$ the curve with the required properties. By keeping $H^3(\cdot,\mu)$ stationary near $\mu=0$, 1, it can also be arranged that

$$H^3_\mu(\cdot)=H^3(\cdot,\mu)=\begin{cases}\bar{f}(\cdot) & \forall\mu\in[0,\varepsilon],\\ \bar{g}(\cdot) & \forall\mu\in[1-\varepsilon,1].\end{cases}$$

Now that we have the curves $\tilde{\alpha}(\cdot)=H^3(-c-\frac{1}{2},\cdot)$ and $\gamma(\cdot)=H^3(c,\cdot)$, H^3 can be replaced by an "end straightened" C^0 proper homotopy $H^4:\mathbb{R}\times I\to M$. This is constructed from H^3 by cutting H^3 off at $\tilde{\alpha}$ and γ and joining on noncritical trajectories of V at $\tilde{\alpha}(I)$ and collar lines of E_1 at $\gamma(I)$ in a C^0 fashion to $\tilde{H}^3=H^3|\{(t,\mu)\,|\,\mu\in I,\ t\in[-c-\frac{1}{2},c]\}$. Note that this construction does not alter the homotopy near $\mu = 0$, 1, i.e.,

$$H^4_\mu(\cdot) = H^4(\cdot, \mu) = \begin{cases} \bar{f}(\cdot) & \forall \mu \in [0, \varepsilon], \\ \bar{g}(\cdot) & \forall \mu \in [1-\varepsilon, 1]. \end{cases}$$

By using standard approximation techniques, we may improve H^4 to a C^∞ proper homotopy $H^5 : \mathbb{R} \times I \to M$ such that for some $c' > c + \frac{1}{2}$, $0 < \varepsilon' < \varepsilon$, if $B = \{(t, \mu) \mid \mu \in [0, \varepsilon'] \cup [1-\varepsilon', 1]$ or $|t| \geq c'\}$, then $H^5 | B = H^4 | B$. So H^5 is a smooth, proper homotopy connecting $\bar{f}$ to $\bar{g}$ such that $\forall \mu \in I$, $H^5(\cdot, \mu)$ eventually follows trajectories of V in U and collar lines in E_1.

From the construction of H^5, it can be seen that Lemma 4.1 applies and so we get another smooth, proper homotopy $H^6 : \mathbb{R} \times I \to M$ such that $H^6_0 = H^5_0$ and $H^6_1 = H^5_1$. Furthermore, $H^6 \times \mathrm{id}_I : \mathbb{R} \times I \to M \times I$ has only a finite number of singularities and these are transverse double points. Also, $\forall \mu \in I$, H^6_μ is an immersion which has at most one singularity and that is a double point.

Now examine H^6_μ singular points for each singular level $\mu_0 \in I$. Suppose $H^6_{\mu_0}(t_1) = H^6_{\mu_0}(t_2)$, $t_1 \neq t_2$. Then by part (i) of Lemma 4.1, as μ goes from $\mu_0 - \nu$ to $\mu_0 + \nu$ (some small $\nu > 0$), locally, we would see two arcs approach each other, cross at a single point $H^6_{\mu_0}(t_1)$ and then separate. But that is the same as changing an overcrossing to an undercrossing by passing one arc through the other. As there is only one crossing point at each singular level, $\forall \mu \in I$, H^6_μ can be changed into an embedding by using a lasso at singular levels. In the case of the singular level $\mu = \mu_0$, we stop at level $\mu = \mu_0 - \nu$, send out a lasso to change over the order of the crossing arcs and then use the rest of H^6 to proceed as before.

In this way, we produce a proper isotopy $H^7 : \mathbb{R} \times I \to M$ such that $H^7_0 = H^5_0$ and $H^7_1 = H^5_1$. The required equivalence between f and g comes from the chain of equivalences

$$f \sim \bar{f} = H^5_0 = H^7_0 \sim H^7_1 = H^5_1 = \bar{g} \sim g. \qquad \square$$

The next theorem, Theorem 4.3, actually follows from Theorem 4.2, but it is worth stating as a separate result.

Theorem 4.3. *Let Γ_1, Γ_2 be two distinct collar ends of a smooth, open 3-manifold M and let Γ_1 be an S^2 end. Let $f, g : \mathbb{R} \to M$ be any two proper knots that run between Γ_1 and Γ_2 such that f and g can be connected by a smooth, proper homotopy. Then f and g can be connected by a smooth proper isotopy.*

The proof is similar to that of Theorem 4.2 but more direct as there are no 1-handles in Γ_2 to consider. In fact, Proposition 3.7 is not needed for the proof and when H^4 is formed from $\tilde{H}^3$, we just join collar lines on at both ends.

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