New Methods for Linear Inequalities

Yair Censor
Department of Mathematics
University of Haifa, Mt. Carmel, Haifa 31999, Israel

and

Tommy Elfving
National Defence Research Institute
Box 1165, S-581 11 Linköping, Sweden

Submitted by Richard A. Brualdi

ABSTRACT

The iterative method of Cimmino for solving linear equations is generalized to linear inequalities. We also present a Richardson-type iterative method for solving the inequality problem, which includes the generalized Cimmino scheme. Convergence proofs are provided.

1. INTRODUCTION

Cimmino [3] devised a beautiful iterative scheme for the solution of a finite system of linear equations in the Euclidean n-dimensional space $\mathbb{R}^n$. Cimmino's method starts with an arbitrary point in $\mathbb{R}^n$ as an initial approximation, and then calculates at each step the centroid of a system of masses placed at the reflections of the previous iterate with respect to the hyperplanes defined by the system of equations. This centroid is taken as the new iterate.

Our purpose in this paper is twofold. First we derive from Cimmino's method a new iterative algorithm for solving a system of *linear inequalities*. The idea is to calculate at each step the centroid of a subsystem of masses placed at the reflections of the previous iterate with respect to the bounding hyperplanes of only the *violated* half spaces defined by the system of inequalities.

Secondly, we show how to modify a Richardson-type iterative least-squares algorithm in order to obtain a new algorithmic scheme for computing a solution of a system of linear inequalities. We prove that the sequence of iterates generated by this scheme converges to a solution of the system of

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inequalities, provided that there exists one, from any initial approximation. Finally, we show that the Cimmino-like algorithm is actually a special case of the algorithmic scheme, and therefore also converges.

The problem of solving systems of linear inequalities arise in numerous fields, e.g., in linear programming [4,12], or in image reconstruction from projections [2,7]. In addition to Cimmino [3] the reader may consult Householder [8, p. 119] or Gastinel [5, p. 160]. Kammerer and Nashed [10, 11] generalized Cimmino's method to integral equations of the first kind. Votruba [16] examined Cimmino's method in the setting of generalized inverses.

2. A CIMMINO-LIKE ALGORITHM FOR LINEAR INEQUALITIES

We consider the system of linear inequalities

$$\langle a_i, x\rangle \leqslant b_i, \qquad i \in \mathcal{P}, \tag{1}$$

where $\langle a_i, x\rangle$ is the Euclidean inner product of a_i and x in $\mathbb{R}^n$, $b_i \in \mathbb{R}$, and $\mathcal{P} \triangleq \{1,2,\dots,p\}$. To avoid triviality we assume that $p \geqslant 2$.

For each $i \in \mathcal{P}$ define the closed half space $L_i \triangleq \{x \in \mathbb{R}^n | \langle a_i, x\rangle \leqslant b_i\}$ and its bounding hyperplane $H_i \triangleq \{x \in \mathbb{R}^n | \langle a_i, x\rangle = b_i\}$. Define $L \triangleq \bigcap_{i\in\mathcal{P}} L_i$, and assume throughout that $L \neq \varnothing$, i.e., we assume that there exists a solution to (1). The task of solving (1) will be referred to as the *linear feasibility problem*.

We would like to emphasize that the assumption $L \neq \varnothing$, i.e., that the linear feasibility problem is indeed feasible, is made throughout the whole paper. The question of how the algorithms presented here behave when there is no solution to the system (1) is a critical one, since with many problems it cannot be said *a priori* whether a solution exists. However, we will not consider the issue in this paper, but leave it open for further investigation.

Let $\{\hat{m}_i\}_{i\in\mathcal{P}}$ be a given set of positive real numbers called masses. We work hereafter with the *normalized masses* $\{m_i\}_{i\in\mathcal{P}}$ obtained by

$$m_i \triangleq \frac{\hat{m}_i}{\sum_{i=1}^{p} \hat{m}_i}, \qquad i \in \mathcal{P}, \tag{2}$$

for which $\sum_{i=1}^{p} m_i = 1$ and $0 < m_i < 1$ for all $i \in \mathcal{P}$.

For $x^k \in \mathbb{R}^n$ and $i \in \mathcal{I}$ define

$$c_i^k \triangleq \min\left\{0, \frac{b_i - \langle a_i, x^k \rangle}{\|a_i\|^2}\right\}, \tag{3}$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. Accordingly, if $x^k \notin L_i$, then

$$c_i^k = \frac{b_i - \langle a_i, x^k \rangle}{\|a_i\|^2} < 0,$$

and $c_i^k = 0$ otherwise.

Define $I_k \triangleq \{i | c_i^k < 0\}$, i.e., the set of indices of $\mathcal{I}$ for which x^k violates the half space L_i in the sense that $x^k \notin L_i$. Next define

$$\mu_k \triangleq \begin{cases} \sum_{i \in I_k} m_i & \text{if} \quad |I_k| \geq 2, \\ 1 & \text{if} \quad |I_k| = 1, \end{cases} \tag{4}$$

where $|I_k|$ denotes the number of elements in I_k. Observe that μ_k is defined only when $I_k \neq \varnothing$; if $I_k = \varnothing$, then x^k is a solution of (1).

With these definitions and notations at hand we state the Cimmino-like algorithm for solving linear inequalities.

ALGORITHM 1. $x^0 \in \mathbb{R}^n$ is arbitrary; calculate I_k; if $I_k = \varnothing$ then stop. Otherwise,

$$x^{k+1} = x^k + \frac{2}{\mu_k} \sum_{i=1}^{p} m_i c_i^k a_i. \tag{5}$$

Convergence of this algorithm to a solution of the linear feasibility problem will follow from the results presented in the sequel. As a matter of fact, the factor 2 in (5), which ensures the positioning of the masses at the reflection points, may be replaced by a sequence of relaxation parameters.

By replacing each linear equation with a pair of linear inequalities, Cimmino's original method for systems of equations is readily recovered from

Algorithm 1. Cimmino's original algorithm reads:

$$x^{k+1}=x^k+\frac{2}{\mu}\sum_{i=1}^{p} m_i \frac{b_i-\langle a_i, x^k\rangle}{\|a_i\|^2} a_i, \tag{6}$$

where $\mu=\sum_{i=1}^{p} m_i$. It solves the problem

$$\langle a_i, x\rangle=b_i, \qquad i\in\mathcal{P}. \tag{7}$$

3. A MODIFIED LEAST-SQUARES ALGORITHM

In matrix notation, the system (1) takes the form

$$Ax\leqslant b, \tag{8}$$

where A is a $p\times n$ matrix whose ith row is a_i^T. Denote by $\rho(Q)$ the spectral radius of a matrix Q, and by $\mathcal{R}(Q)$ its range. The following theorem presents a well-known iterative method for solving the least-squares problem. The proof may be found, e.g., in [15].

Theorem 1. *Let M be a given positive definite matrix, and define $\|x\|_M \triangleq x^TMx$. The following method:*

$$x^{k+1}=x^k+\alpha A^TM(b-Ax^k), \tag{9}$$

with $0<\alpha<2/\rho(A^TMA)$ and $x^0\in\mathcal{R}(A^T)$, generates the solution of the problem

$$\begin{array}{rl} \text{Min} & \|x\| \\ \text{subject to} & x\in\{x \mid \|b-Ax\|_M \text{ is minimum}\} \end{array}. \tag{10}$$

In order to state our modification of (9) some definitions are needed. Let $\{x^k\}_{k=0}^{\infty}$ be a sequence of iterates; denote by $r^k=b-Ax^k$ the residual vector, and by r_i^k the ith component of r^k. We also introduce a diagonal matrix D^k

defined by

$$D_{ii}^k \triangleq \begin{cases} 1 & \text{if } j\in I_k \quad \left(\text{i.e., if } r_i^k<0\right), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

ALGORITHM 2. Let $M=(m_{ij})$ be a positive definite matrix with nonnegative elements. $x^0\in\mathbb{R}^n$ is arbitrary. Calculate $r^k=b-Ax^k$. If $r^k\geqslant 0$ then stop. Otherwise,

$$x^{k+1}=x^k+\alpha_k A^T M^k(b-Ax^k), \tag{12}$$

where $M^k \triangleq D^k M D^k$ and the parameters $\{\alpha_k\}$ are restricted to $0<\alpha_k<2/\rho(A^T M^k A)$.

THEOREM 2. *Any sequence of iterates* $\{x^k\}_{k=0}^{\infty}$ *generated by Algorithm* 2 *converges to a solution of the system* (1).

4. CONVERGENCE

In proving Theorem 2 we will use a convergence theory developed by Gubin, Polyak, and Raik [6].

DEFINITION 1. A sequence $\{x^k\}_{k=0}^{\infty}$ is called *Fejer-monotone with respect to the set L* if for every $x\in L$

$$\|x^{k+1}-x\|\leqslant\|x^k-x\| \qquad \text{for all} \quad k\geqslant 0. \tag{13}$$

It is easy to check that every Fejer-monotone sequence is bounded.

Denote by $d(x, L_i)$ the Euclidean distance between a point $x\in\mathbb{R}^n$ and a set L_i [i.e., $d(x,L_i)\triangleq\inf_{y\in L_i}\|x-y\|$], and define

$$\phi(x)\triangleq\sup_{i\in\mathcal{I}} d(x,L_i). \tag{14}$$

The fundamental convergence theorem of Gubin et al. [6] is:

THEOREM 3. *Let $L_i \subseteq \mathbb{R}^n$ be convex closed sets for all $i \in \mathcal{P}$, and $L \triangleq \bigcap_{i \in \mathcal{P}} L_i \neq \varnothing$. If for a sequence $\{x^k\}_{k=0}^{\infty}$ the following conditions hold:*

(i) $\{x^k\}_{k=0}^{\infty}$ is Fejer-monotone with respect to L, and
(ii) $\lim_{k \to \infty} \phi(x^k) = 0$,

then $x^k \underset{k \to \infty}{\longrightarrow} x^* \in L$.

Proof. Follows from Lemma 5 and Lemma 6 of Gubin et al. [6]. ∎

Theorem 2 will be proved by establishing the conditions of Theorem 3.

PROPOSITION 1. *Any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 2 is Fejer-monotone with respect to L, provided that $x^k \notin L$ for all $k \geq 0$.*

Proof. Let $x \in L$ (i.e., $b - Ax \geq 0$), and define $e^k \triangleq x^k - x$. Then from (12)

$$e^{k+1} = e^k + \alpha_k A^T d^k, \tag{15}$$

with $d^k \triangleq M^k r^k$. It follows that

$$\|e^{k+1}\|^2 = \|e^k\|^2 + \alpha_k^2 \|A^T d^k\|^2 + 2\alpha_k \langle A^T d^k, e^k \rangle. \tag{16}$$

From $x \in L_i$ we obtain

$$r_i^k \geq -\langle a_i, e^k \rangle; \tag{17}$$

hence,

$$\langle A^T d^k, e^k \rangle = -\sum_{i=1}^{p} d_i^k(-\langle a_i, e^k \rangle) \leq -\sum_{i=1}^{p} d_i^k r_i^k$$

$$= -\langle d^k, r^k \rangle, \tag{18}$$

provided that

$$d_i^k \leqslant 0 \tag{19}$$

for all i and all k.

To see that (19) holds, observe that

$$d_i^k = (M^k r^k)_i = (D^k M D^k r^k)_i = D_{ii}^k \sum_{j \in I_k} m_{ij} r_j^k, \tag{20}$$

where m_{ij} are the entries of M, which are nonnegative by assumption, and that $r_j^k < 0$ whenever $j \in I_k$.

Turning now to the second term on the right-hand side of (16), we decompose the semidefinite matrix M^k as $M^k = W^T W$ and use the inequality

$$\langle Qy, y \rangle \leqslant \rho(Q) \langle y, y \rangle, \tag{21}$$

which holds for any symmetric and positive semidefinite matrix Q (e.g. [14, p. 35]), to obtain

$$\begin{aligned} \| A^T d^k \|^2 &= \langle A^T M^k r^k, A^T M^k r^k \rangle = \langle M^k A A^T M^k r^k, r^k \rangle \\ &= \langle (W A A^T W^T) W r^k, W r^k \rangle, \\ &\leqslant \rho(W A A^T W^T) \langle W r^k, W r^k \rangle \\ &= \rho(A^T M^k A) \langle d^k, r^k \rangle. \end{aligned} \tag{22}$$

Combining (18) and (22) into (16), we get

$$\| e^{k+1} \|^2 \leqslant \| e^k \|^2 + \alpha_k \left[\alpha_k \rho(A^T M^k A) - 2 \right] \langle d^k, r^k \rangle, \tag{23}$$

where $\langle d^k, r^k \rangle = \langle W r^k, W r^k \rangle \geqslant 0$.

Since $0 < \alpha_k < 2/\rho(A^T M^k A)$ for all $k \geqslant 0$ in Algorithm 1, the desired conclusion

$$\| e^{k+1} \| \leqslant \| e^k \|$$

follows. ■

Let the orthogonal projection of $x^k \in \mathbb{R}^n$ onto the half space $L_i \subseteq \mathbb{R}^n$ be denoted by $P_i(x^k)$.

LEMMA 1. *$c_i^k a_i = P_i(x^k) - x^k$ for all $i \in \mathcal{I}$ and all $k \geq 0$.*

Proof. Simple. ■

The next proposition establishes condition (ii) of Theorem 3.

PROPOSITION 2. *Any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 2 has the property*

$$\lim_{k\to\infty} \phi(x^k) = 0. \tag{24}$$

Proof. Fejer-monotonicity implies that the sequence $\{\|x^k - x\|\}_{k=0}^{\infty}$ is monotonically decreasing, thus converging. It follows then from (23) that

$$\lim_{k\to\infty} \langle d^k, r^k \rangle = 0. \tag{25}$$

But $\langle d^k, r^k \rangle = \langle MD^k r^k, D^k r^k \rangle$; thus,

$$\lim_{k\to\infty} D^k r^k = 0 \tag{26}$$

and

$$(D^k r^k)_i = c_i^k \|a_i\|^2$$

imply that

$$d(x^k, L_i) = \|P_i(x^k) - x^k\| = \|c_i^k a_i\|$$

$$= \frac{|(D^k r^k)_i|}{\|a_i\|} \underset{k\to\infty}{\longrightarrow} 0 \tag{27}$$

for every $i \in \mathcal{I}$, and the finiteness of $\mathcal{I}$ ensures that

$$\lim_{k\to\infty} \phi(x^k) = 0.$$ ■

5. THE CIMMINO-LIKE ALGORITHM AS A SPECIAL CASE

Here we show that the Cimmino-like algorithm (Algorithm 1) is a special case of the modified least-squares algorithm (Algorithm 2), thereby providing the desired convergence of the Cimmino-like algorithm.

The connection between the two algorithms is obtained by defining, for Algorithm 2, the $p\times p$ matrix

$$M=(m_{ij})=\left(\delta_{ij}\frac{m_i}{\|a_i\|^2}\right) \tag{28}$$

where δ_{ij} is the Kronecker delta, m_i are the masses of Algorithm 1, and a_i^T are the rows of A.

The following lemma will be used.

LEMMA 2. $\rho(A^TM^kA)\leqslant \sum_{i\in I_k} m_i\leqslant 1.$

Proof. From the equality

$$A^TM^kA=\sum_{i\in I_k} m_{ii}\, a_i\, a_i^T \tag{29}$$

and from the fact that $\rho(Q)=\|Q\|_2$, the l_2-norm of Q, for any symmetric matrix Q (see, e.g., [14, p. 41]), we obtain

$$\rho(A^TM^kA)=\|A^TM^kA\|_2=\|\sum_{i\in I_k} m_{ii}a_ia_i^T\|_2$$

$$\leqslant \sum_{i\in I_k} m_{ii}\|a_ia_i^T\|_2=\sum_{i\in I_k} m_{ii}\rho_i. \tag{30}$$

Here each $\rho_i\triangleq\rho(a_ia_i^T)$ is a simple eigenvalue (since $a_ia_i^T$ has rank one and we have assumed $a_i\neq 0$) which corresponds to the eigenvector $x=a_i$, and

$$\rho_i=\|a_i\|^2. \tag{31}$$

Now (28), (30), and (31) imply the desired result, since $\Sigma_{i\in I_k}m_i\leqslant 1$. ■

To establish our claim that Algorithm 1 may be regarded as a special case of Algorithm 2, we prove that

$$0<\frac{2}{\mu_k}<\frac{2}{\rho(A^TM^kA)} \tag{32}$$

for all $k\geqslant 0$.

Proof. *Case* 1. If $|I_k|=1$, then by Lemma 2

$$\rho^k \triangleq \rho(A^TM^kA)=\sum_{i\in I_k} m_i=m_l<1,$$

where $I_k=\{l\}$; hence

$$\frac{2}{\rho^k}=\frac{2}{m_l}>2=\frac{2}{\mu_k}.$$

Case 2. For $|I_k|>1$ let us consider for a moment instead of (5) the iteration

$$x^{k+1}=x^k+\beta_k\sum_{i=1}^{p} m_i c_i^k a_i, \qquad \beta_k=\frac{\omega}{\mu_k}. \tag{33}$$

From Lemma 2

$$\rho^k\leqslant \sum_{i\in I_k} m_i=\mu_k;$$

hence $2/\rho^k\geqslant 2/\mu_k>\beta_k>0$ as long as $0<\omega<2$. Therefore, Algorithm 1 with (5) replaced by (33) converges for $0<\omega<2$. We now investigate separately the case $\omega=2$.

Lemma 2 guarantees that $2/\rho^k\geqslant 2/\mu_k$. If for some $k\geqslant 0$ this inequality is not strict, then

$$\rho^k=\mu_k=\sum_{i\in I_k} m_i, \tag{34}$$

which is, by (30), equivalent to

$$\| \sum_{i \in I_k} m_{ii} a_i a_i^T \|_2 = \sum_{i \in I_k} m_{ii} \| a_i a_i^T \|_2. \tag{35}$$

Corollary A1, which we defer to the Appendix of the paper, shows that if (35) holds, then the vectors a_i are almost identical for all $i \in I_k$, i.e.,

$$a_i = k_i a \qquad i \in I_k. \tag{36}$$

Therefore, for an iteration index k for which (34) holds, the system (1) splits into

$$\left.\begin{aligned} \langle a, x \rangle &\leq \frac{b_i}{k_i}, \qquad (k_i > 0) \\ \langle a, x \rangle &\geq \frac{b_i}{k_i}, \qquad (k_i < 0) \end{aligned}\right\}, i \in I_k \tag{37a}$$

$$\langle a_i, x \rangle \leq b_i, \qquad i \in \mathcal{P} \setminus I_k. \tag{37b}$$

Put $b_1' = \min_{i \in I_k} \frac{b_i}{k_i}$, $k_i > 0$, and $b_2' = \max_{i \in I_k} \frac{b_i}{k_i}$, $k_i < 0$. Then (37a) is equivalent to

$$\begin{aligned} \langle a, x \rangle &\leq b_1' \\ \langle a, x \rangle &\geq b_2'. \end{aligned} \tag{38}$$

Note that $b_2' \leq b_1'$ since $L \neq \varnothing$. Suppose $|I_k| > 1$. Then $\langle a, x^k \rangle > b_1'$ and $\langle a, x^k \rangle < b_2'$ and hence $b_2' > b_1'$ which is a contradiction. We conclude that only one type of inequality in (38) is present which implies that the new $\hat{I}_k$ of the system is a singleton and case 1 of this proof applies. ∎

APPENDIX

Here we prove a corollary which we used in the proof of (32) above.

COROLLARY A1. *If*

$$\| \sum_{i \in I_k} m_{ii} a_i a_i^T \|_2 = \sum_{i \in I_k} m_{ii} \| a_i a_i^T \|_2,$$

then

$$a_i = k_i a \qquad i \in I_k,$$

where k_i are nonzero numbers.

We need the following two lemmas.

LEMMA A1. *Let A, B be two positive semidefinite and symmetric matrices. Assume the $(A+B)u = \rho_{A+B}u$, $Az = \rho_A z$ and $Bv = \rho_B v$, where the spectral radii ρ_A and ρ_B are simple eigenvalues of A and B. Then,*

$$\|A+B\|_2 = \|A\|_2 + \|B\|_2 \quad \Rightarrow \quad u = k_1 z = k_2 v. \tag{39}$$

Proof. $A_z = \rho_A z$ implies $z^T A z = \rho_A z^T z$, so that $\rho_A = z^T A z / z^T z$. Similarly, $\rho_B = v^T B v / v^T v$ and $\rho_{A+B} = u^T (A+B) u / u^T u$. From $\|A+B\|_2 = \rho(A+B) = \rho(A) + \rho(B)$ we get

$$S \triangleq \frac{u^T A u}{u^T u} - \frac{z^T A z}{z^T z} = \frac{v^T B v}{v^T v} - \frac{u^T B u}{u^T u} \triangleq T.$$

But $S \leqslant 0$ since $\rho_A > \dfrac{u^T A u}{u^T u}$ and, by the same token, $T \geqslant 0$, so that $S = T = 0$, hence

$$\frac{u^T A u}{u^T u} = \frac{z^T A z}{z^T z} = \rho_A.$$

This shows that $Au = \rho_A u$ which proves that $u = k_1 z$ because ρ_A is simple. In a similar manner we get $u = k_2 v$. ■

LEMMA A2. *Assume that $\{A_i\}_{i \in I}$ is a given family of symmetric and positive semidefinite matrices where I is some finite index set. Also let $C = \sum_{i \in I} A_i$, $Cx = \rho_C x$,* and $A_i x_i = \rho_i x_i$ where ρ_i are simple eigenvalues. Then

$$\|C\|_2 = \sum_{i \in I} \|A_i\|_2 \quad \Rightarrow \quad x_i = k_i x \quad \forall i \in I. \tag{40}$$

Proof. Use induction and Lemma A1. ■

Corollary A1 now follows directly from Lemma A2 since $a_i a_i^T a_i = \rho_i a_i$, as in the proof of Lemma 2, and ρ_i are simple eigenvalues, provided $a_i \neq 0$. ■

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