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BOUNDARY CONTROL, WAVE FIELD CONTINUATION AND INVERSE PROBLEMS FOR THE WAVE EQUATION

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Abstract—We consider the problem of the wave field continuation and recovering of coefficients for the wave equation in a bounded domain in $\mathbf{R}^n$, $n > 1$. The inverse data is a response operator mapping Neumann boundary data into Dirichlet ones. The reconstruction procedure is local. This means that, observing boundary response for larger times, we may recover coefficients deeper in the domain. The approach is based upon ideas and results of the boundary control theory, yielding some natural multidimensional analogs of the classical Gel'fand-Levitan-Krein equations.

1. INTRODUCTION

Inverse problems (IP) under consideration are related to the well-posed initial-boundary value problem for the wave equation

$$\rho u_{tt} - \operatorname{div}(\mu \nabla u) + qu = 0 \qquad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$u_{|t<0} \equiv 0, \tag{1.2}$$

$$\mu \frac{\partial u}{\partial \nu}_{|\Gamma \times [0,T]} = f, \tag{1.3}$$

in a bounded domain $\Omega \in \mathbf{R}^n$. Its solution is denoted by $u^f(x,t)$. The problem (1.1)–(1.3) originates a response operator $R : f \to u^f_{|\Gamma \times [0,T]}$ and a control operator $W : f \to u^f(\cdot, T)$. We are interested in the following IP: given a response operator R, recover coefficients in (1.1). More exactly, we suggest a procedure for solving three IP: (i) to recover μ for $\rho = 1$, $q = 0$; (ii) to recover ρ for $\mu = 1, q = 0$ and (iii) to recover q for $\rho = \mu = 1$. An important stage of the procedure is determination of the control operator W via given R. Determination of W appears to be a particular problem which we call the wave field continuation (WFC) problem. Formulation of the IP and WFC problem will be made more precise below.

In recent years, there appeared numerous papers devoted to boundary IP. So we mention only those approaches that are more or less similar to the approach proposed in this paper. Recovery of coefficients in a domain, via a given boundary operator of the type of the response one, is considered in [1–9] (see also literature cited there). Multidimensional IP in time-domain are discussed in [10–15].

An essential property of the proposed procedure is the property of locality (causality). Namely, if we know the response operator for larger T, then we are able to recover coefficients for x lying deeper in Ω. The requirement of the procedure to be local reflects some essential features of the wave propagation governed by Equation (1.1). A. S. Blagovestchensky was the first to put forward the local approach for the classical one-dimensional IP for an inhomogeneous string [16]. Later, [17,18] he elaborated it for the matrix Sturm-Liouville IP and IP in layered media.

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The approach under consideration was put forward by M. I. Belishev [19] and was developed in [20–31]. It is based upon some ideas and results of the boundary control theory [32–34] and may be considered as an extension of the local approach to the multidimensional case. Let us note that locality is also appropriate to approaches considered in [35,36].

Let us discuss briefly the content and results of the article. Previous works [19–31] concern mostly the problem of recovering ρ. In the present paper, we propose a variant of the approach suitable also for recovering μ and q. The work may be conventionally divided into three parts. The first one includes precise formulation of the problems under consideration (Section 2) and a discussion of some necessary direct results. In Section 3, we introduce systems (Ω, ρ, μ, q) available to the approach. They are described in terms of boundary control theory and are called normal systems. Ray coordinates and some of their properties are discussed in Section 4. In this article, we restrict ourselves to the description of the inversion procedure in the boundary layer where ray coordinates are regular. We should like to emphasize that this restriction is made only for simplicity and may be removed by the same considerations as in [27] (see also Subsection 12.1). Some results on the propagation of discontinuities are described in Section 5.

An operator outline of the approach is considered in the second part of the paper. In Section 6, we introduce an operator C^T playing a crucial role in the work. It determines an isometry between the outer space of boundary controls and the inner space of waves. Linear equations of IP are deduced in Section 7. They are due to the relevant problems of boundary control and appear to be some natural multidimensional analogs of the classical equations of Gel'fand-Levitan-Krein. In Section 8, we reconstruct the so called weighted wave fields, i.e., waves multiplied by a function determined by ray geometry. Let us emphasize that the procedure of the weighted wave fields reconstruction may be extended to hyperbolic equations of more general types.

In the third part, we give solutions to the IP described in (i)–(iii). Namely, we recover μ, ρ and q in Sections 9, 10, and 11, respectively. Some concluding remarks are considered in Section 12. The Appendix contains demonstrations of some auxiliary statements.

2. FORMULATION OF PROBLEMS

2.1. Initial-Boundary Value Problem

Let us consider the problem

$$\rho\, u_{tt} - \operatorname{div}(\mu \nabla u) + qu = 0 \qquad \text{in} \quad \mathbf{Q}^T, \tag{2.1}$$

$$u_{|t<0} = 0, \tag{2.2}$$

$$\mu\, \partial_\nu u_{|_{\Sigma^T}} = f, \tag{2.3}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$ with a simple-connected boundary $\Gamma := \partial\Omega \in C^\infty$. Coefficients ρ, μ and q are the functions of the class $C^\infty(\bar{\Omega})$, $\rho, \mu \geq \text{const} > 0$; $\mathbf{Q}^T := \Omega \times (0,T)$; $\Sigma^T := \Gamma \times (0,T)$; $\nu = \nu(\gamma)$ is an outward normal to the boundary at the point $\gamma \in \Gamma$; $\partial_\nu = \partial/\partial\nu$. Function $f = f(\gamma, t)$ is called (boundary) control, $f \in F^T := L_2(\Sigma^T)$.

REMARK 2.1. All functional spaces in the article are real.

The solution $u = u^f(x,t)$, $(x,t) = (x^1, \ldots, x^n, t) \in \mathbf{Q}^T$, for the problem (2.1)–(2.3) is regarded in the sense of $L_2(\mathbf{Q}^T)$. Its definition as well as theorems of existence and uniqueness are given in [33]. Let us describe some properties of u^f necessary for further considerations (see e.g., [33,37]):

(i) The mapping $f \to u^f$ is continuous from F^T into $C([0,T], L_2(\Omega))$.

(ii) For every $f \in F^T$, there exists the trace $u^f_{|_{\Sigma^T}} \in F^T$ and the mapping $f \to u^f_{|_{\Sigma^T}}$ is continuous.

(iii) For $f \in C_0^\infty(\Sigma^T)$, the solution u^f is a classical one; $u^f \in C^\infty(\mathbf{Q}^T)$.

We call (Ω, ρ, μ, q) a system with density ρ, tension μ and stiffness q, these notations correspond to the well-known mechanical interpretation of the problem. The solution u^f represents the wave initiated by the boundary control f and propagating towards the interior of Ω. The value of the function $c := \rho^{-1/2}\mu^{1/2}$ at the point $x \in \Omega$ coincides with the wave velocity at this point.

The space of controls F^T is called the outer space.

2.2. Response Operator

The initial-boundary value problem (2.1)–(2.3) originates the response operator $R^T : F^T \to F^T$,

$$R^T f := u^f_{|_{\Sigma^T}}. \tag{2.4}$$

In view of (ii) of Section 2.1, R^T is a continuous linear operator in F^T. In what follows, operator R^T is taken to be the inverse data (ID) for the IP under consideration.

There are two usual ways of the operator R^T representation:

(i) Let us consider (2.1)–(2.3) with an instantaneous point source on the boundary $f = \delta(\gamma,\gamma')\,\delta(t)$ where $\delta(\cdot,\gamma')$ is a surface δ-function on Γ; $\sup\delta(\cdot,\gamma') = \{\gamma'\}$. Its solution $u^{\delta\delta}(x,\gamma',t)$ has the meaning of the Green's function, i.e.,

$$u^f(x,t) = \int_{\Gamma\times[0,t]} d\Gamma_{\gamma'}dt'\, u^{\delta\delta}(x,\gamma',t-t')\cdot f(\gamma',t'),$$

so that

$$[R^T f](\gamma,t) = \int_{\Gamma\times[0,t]} d\Gamma_{\gamma'}dt'\, r(\gamma,\gamma',t-t')\cdot f(\gamma',t') \tag{2.5}$$

where $r := u^{\delta\delta}_{|_{\Sigma^T}}$.

(ii) Let us consider the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and corresponding eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ of the elliptic boundary value problem

$$L\phi := \left[-\rho^{-1}\cdot\ \mathrm{div}\ (\mu\nabla) + \rho^{-1}q\right]\phi = \lambda\phi, \qquad \text{in}\quad \Omega \tag{2.6}$$

$$\partial_\nu\phi_{|_\Gamma} = 0. \tag{2.7}$$

Functions $\{\phi_k\}_{k=1}^{\infty}$, normalized by

$$\int_\Omega dx\,\rho(x)\,\phi_i(x)\,\phi_k(x) = \delta_{ik}, \qquad (i,k = 1,2,\dots) \tag{2.8}$$

form an orthonormal basis in $H := L_{2,\rho}(\Omega)$. Operator R^T may be represented via the given spectrum $\{\lambda_k\}_{k=1}^{\infty}$ and traces of eigenfunctions on the boundary $\{\phi_{k_{|_\Gamma}}\}_{k=1}^{\infty}$

$$[R^T f](\gamma,t) = \int_{\Gamma\times[0,t]} d\Gamma_{\gamma'}dt' \left\{\sum_{k=1}^{\infty}\frac{\sin\sqrt{\lambda_k}(t-t')}{\sqrt{\lambda_k}}\,\phi_k(\gamma)\,\phi_k(\gamma')\right\} f(\gamma',t'). \tag{2.9}$$

2.3. Reduced Problems

The family of the "reduced" problems analogous to (2.1)–(2.3) is defined as the problems

$$\rho\, u_{tt} - \ \mathrm{div}\ (\mu\,\nabla u) + qu = 0, \qquad \text{in}\quad \mathbf{Q}^\xi \quad (0<\xi\le T), \tag{2.10}$$

$$u_{|_{t<0}} = 0 \tag{2.11}$$

$$\mu\,\partial_\nu u_{|_{\Sigma^\xi}} = f \tag{2.12}$$

where control $f \in F^\xi := L_2(\Sigma^\xi)$. Let $\mathcal{T}^T_{T-\xi} : F^\xi \to F^T$ be a time-delay operator

$$\left[\mathcal{T}^T_{T-\xi} f\right](\gamma,t) = f_{T-\xi}(\gamma,t) := \begin{cases} 0, & 0<t<T-\xi,\\ f(\gamma,t-(T-\xi)), & T-\xi<t<T.\end{cases} \tag{2.13}$$

As coefficients of Equation (2.1) don't depend on t then the time-delay of the control yields the same time-delay of the corresponding wave

$$u^f(x,\xi) = u^{f_{T-\xi}}(x,T), \qquad x\in\Omega. \tag{2.14}$$

In view of the definition of R^T (2.4), equality (2.14) restricted to the boundary yields

$$R^\xi = \left[\mathcal{T}^T_{T-\xi}\right]^*\, R^T\, \mathcal{T}^T_{T-\xi}, \tag{2.15}$$

where R^ξ is a response operator for the reduced problem (2.10)–(2.12).

Subspaces of controls delayed by $T-\xi$

$$F^T_{T-\xi} := \mathcal{T}^T_{T-\xi}F^\xi = \{f\in F^T :\ f=0,\ \text{for}\ t<T-\xi\}\subset F^T \tag{2.16}$$

are invariant with respect to R^T, i.e., $R^T\,F^T_{T-\xi}\subset F^T_{T-\xi}$.

2.4. Eikonal

Operator L (2.6) induces the Riemannien metric on Ω

$$d(x,x') := \inf \int_{l[x,x']} c^{-1}(\zeta)\,|d\zeta|, \tag{2.17}$$

where $c = \rho^{-1/2}\,\mu^{1/2}$ is the wave velocity and infimum is taken over the set of all piecewise smooth curves $l[x,x']$ in $\bar{\Omega}$, connecting x and x'. The distance $d(x,x')$ is equal to the wave propagation time from x to x'.

The function

$$\tau(x) := \inf_{\gamma\in\Gamma} d(x,\gamma) = \mathrm{dist}_d(x,\Gamma) \tag{2.18}$$

is called eikonal.

As Equation (2.1) is hyperbolic, the velocity of the wave propagation is finite. The set

$$\Omega^\xi := \{x \in \Omega : \tau(x) < \xi\} \qquad (0 < \xi \le T) \tag{2.19}$$

coincides with the subdomain of Ω filled with the waves from Γ at $t = \xi$, i.e.,

$$\bar{\Omega}^\xi = \bigcup_{f\in F^\xi} \sup u^f(\cdot,\xi) = \bigcup_{f\in F^T_{T-\xi}} \sup u^f(\cdot,T), \qquad (0 < \zeta \le T) \tag{2.20}$$

(see (2.14) and Figure 1). The time needed for the wave from the boundary to fill the whole Ω is given by $T_* := \max_{x\in\Omega} \tau(x)$.

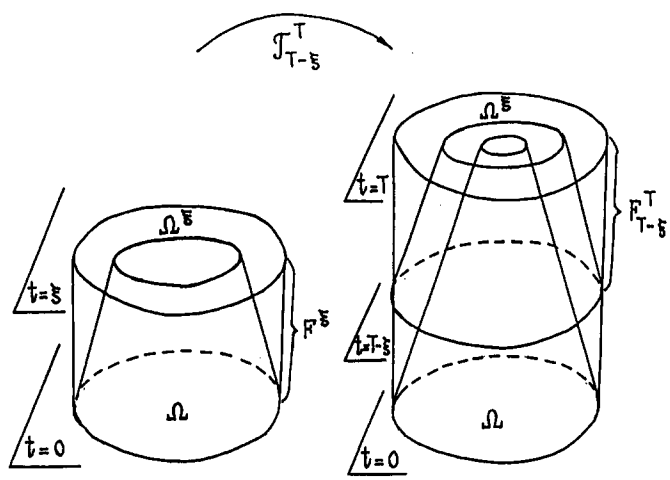

Figure 1. Domains Ω^ξ filled with waves.

2.5. Inverse Problems

Let $H := L_{2,\rho}(\Omega)$ be a Hilbert space with the inner product

$$(y,z)_H = \int_\Omega dx\,\rho(x)\,y(x)\,z(x) \tag{2.21}$$

and let H^T be a subspace of H

$$H^T := \{y \in H : \sup y \in \bar{\Omega}^T\}.$$

In view of the property (i) of Section 2.1 and (2.20), waves $u^f(\cdot,T)$ belong to H^T. We call H^T the inner space for the system (Ω,ρ,μ,q).

The problem (2.1)–(2.3) originates a control operator $W^T : F^T \to H^T$,

$$W^T f := u^f(\cdot, T). \tag{2.22}$$

In view of the property (i) of Section 2.1, operator W^T is a continuous linear operator from F^T to H^T.

REMARK 2.2. It may be obtained by means of the results of [37] that W^T is compact.

In the article we consider the following IP:

(i) Given a response operator R^{2T} to recover W^T and the tension $\mu_{|_{\Omega^T}}$ for the system $(\Omega, 1, \mu, 0)$;

(ii) Given a response operator R^{2T} to recover W^T and the density $\rho_{|_{\Omega^T}}$ for the system $(\Omega, \rho, 1, 0)$;

(iii) Given a response operator R^{2T} to recover W^T and the stiffness $q_{|_{\Omega^T}}$ for the system $(\Omega, 1, 1, q)$.

Let us give heuristic reasons for the time doubling in the formulation of the problems (i)–(iii). The system's response observed on Γ for $0 < t < 2T$ is quite determined by the behavior of the coefficients ρ, μ, q in Ω^T. (In Figure 2, arrows indicate waves reflected from inhomogeneities and moving back to Γ). It is natural to assume that, inversely, the boundary response given only for $0 < t < 2T$, determines the coefficients in Equation (2.1) and wave fields u^f in Ω^T uniquely. The approach to hyperbolic IP proposed in the article is just based upon this idea. Such an approach was called by its author, A. S. Blagovestchensky, a local (causal) one [16,17]. In [19–21,23,25,27] it was extended to a multidimensional case.

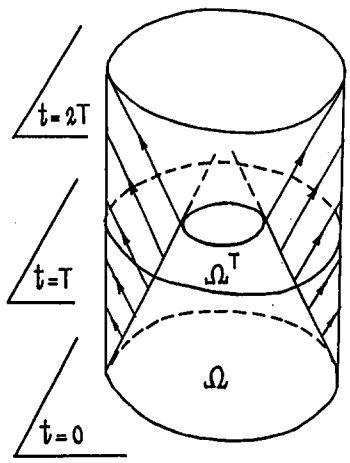

Figure 2. Reflections of waves and time doubling.

Let us remark also that the reconstruction of wave fields in natural (ray) coordinates makes up an important stage of the approach. The procedure of their reconstruction is the same for problems (i)–(iii). So we distinguish the problem of W^T recovering via given R^{2T} as a particular one which we call the problem of the wave field continuation (WFC).

2.6. Boundary Control (BC) Problem

This is the BC problem: for an arbitrary $y \in H^T$ to find a control $f \in F^T$ so that

$$u^f(\cdot, T) = y, \tag{2.23}$$

i.e., to find an appropriate control that transforms the system from the initial zero state to the given final one in the moment T [32,33]. The problem (2.23) is equivalent to the equation

$$W^T f = y, \tag{2.24}$$

so that the BC problem is equivalent to the inversion problem for the control operator W^T.

A class of systems (Ω, ρ, μ, q), subjected to the approach discussed in the article, is introduced in Section 3.

3. NORMAL SYSTEMS

3.1. Solvability of the BC Problem; Controllability

THEOREM 1. *For $T < T_*$*

$$\text{Ker } W^T = \{0\}. \tag{3.1}$$

The proof is given in the Appendix. It is based upon the fact that for $T < T_*$, mes $\{\Omega \backslash \Omega^T\} > 0$, where $\Omega \setminus \Omega^T$ is a subdomain of Ω free of waves for $t = T$.

COROLLARY. *For $T < T_*$ the BC problem may have not more than one solution.*

The range of values of the control operator in the control theory is called the reachable set. The solvability of the BC problem (2.23) is closely related to the character of the imbedding of this set into H^T. Systems satisfying the relation of the type

$$\text{cl}_H\, W^T F^T = H^T, \tag{3.2}$$

are called (approximately) controllable [32]. (Here cl is a symbol for the closure).

REMARK 3.1. The system is called exactly controllable [32] if its reachable set coincides with the whole space, i.e., (2.24) is solvable for every $y \in H^T$. In the considered case of the Neumann BC, the operator W^T is compact (see Remark 2.2). That is why $W^T F^T \neq H^T$ and the system is not exactly controllable. It is interesting to note that exact controllability is valid for the Dirichlet BC in the one-dimensional case and is invalid for $n \geq 2$ [30].

It is quite routine for the BC theory that controllability is equivalent to a certain uniqueness theorem. In the considered case, it comes out to be the Holmgren-John uniqueness theorem concerning the Cauchy problem for (2.1) with data on the time-like surface Σ^T. Let us discuss this equivalence in more detail.

3.2. Dual system; Observability

The problem

$$\rho\, v_{tt} - \text{ div } (\mu\, \nabla v) + qv = 0, \qquad \text{in } \mathbf{Q}^T, \tag{3.3}$$

$$v_{|_{t=T}} = 0, \qquad v_{t|_{t=T}} = y \in H^T, \tag{3.4}$$

$$\mu\, \partial_\nu v_{|_{\Sigma^T}} = 0, \tag{3.5}$$

is called dual to the problem (2.1)–(2.3). Its (generalized) solution $v = v^y(x,t)$ of the class $L_2(\mathbf{Q}^T)$ is considered in [33]. It is shown in this work that there exists the trace $v^y_{|_{\Sigma^T}} \in F^T$, the map $y \to v^y_{|_{\Sigma^T}}$ being continuous from H^T to F^T.

The dual problem (3.3)–(3.5) originates the continuous linear operator $O^T : H^T \to F^T$,

$$O^T y := -v^y_{|_{\Sigma^T}}, \tag{3.6}$$

which is called the observation operator.

LEMMA 1. *For every $T > 0$*

$$O^T = [W^T]^*. \tag{3.7}$$

PROOF. Let u^f and v^y correspond to the smooth $f \in C_0^\infty(\Sigma^T)$ and $y \in C_0^\infty(\Omega^T)$. Then, integrating by parts and taking into account (2.1)–(2.3) and (3.3)–(3.5), we obtain

$$\begin{aligned}
0 &= \int_{\mathbf{Q}^T} dxdt\, [\rho\, u^f_{tt} - \text{ div } (\mu\, \nabla u^f) + q\, u^f] \cdot v^y \\
&= \int_\Omega dx\ \rho[u^f_t\, v^y - u^f\, v^y_t]\Big|_{t=0}^{t=T} - \int_{\Sigma^T} d\Gamma_\gamma dt\, \mu\, [v^y\, \partial_\nu u^f - u^f\, \partial_\nu v^y] \\
&+ \int_{\mathbf{Q}^T} dxdt\, u^f\, [\rho\, v^y_{tt} - \text{ div } (\mu\, \nabla v^y) + q\, v^y] \\
&= -\int_\Omega dx\, \rho(x)\, u^f(x,T)\, y(x) - \int_{\Sigma^T} d\Gamma_\gamma dt\, f(\gamma,t)\, v^y(\gamma,t),
\end{aligned}$$

so that

$$(W^T f, y)_{H^T} = \int_\Omega dx\, \rho(x)\, u^f(x,t)\, y(x)$$
$$= \int_{\Sigma^T} d\Gamma_\gamma dt\, f(\gamma,t)\,[-v^y(\gamma,t)] = (f, O^T y)_{F^T}.$$

These equalities yield Equation (3.7) in view of the density of smooth f and y in F^T and H^T, respectively. ∎

COROLLARY. *The density property (3.2) is equivalent to the relation*

$$\text{Ker } O^T = \{0\}. \tag{3.8}$$

The Corollary is an immediate consequence of the known equality

$$\text{Ker } O^T = H^T \Theta \text{cl}_H\, W^T F^T.$$

In the BC theory, systems satisfying (3.8) are called observable. The corollary concerns the well-known equivalence of controllability and observability [32].

3.3. Connection with the Holmgren-John Theorem

THEOREM 2. *Let coefficients ρ, μ, q be real-analytic functions in Ω and let Γ be a real-analytic manifold. If the function v^y solving (3.3)–(3.5) satisfies the boundary condition*

$$v^y_{|_{\Sigma^T}} = 0,$$

then $H^T \ni y = 0$.

This statement is an adequate version of the Holmgren-John uniqueness theorem concerning the Cauchy problem for the hyperbolic equation (3.3) with data on the time-like surface $\Gamma \times [0,T]$. The proof is given in [38] for $q = 0$ and may be easily extended by means of the results of [39] to $q \neq 0$. It may be noted that the Holmgren-John theorem is used in [38] to obtain controllability for the full state $(u^f(\cdot,T), u^f_t(\cdot,T))$.

Theorem 2 yields (3.8). Indeed for $y \in$ Ker O^T, we have $v^y_{|_{\Sigma^T}} = O^T y = 0$ that implies $y = 0$.

The generalization of Theorem 2 for $\rho, \mu, q \in C^2$ and $\Gamma \in C^2$ was announced by D. L. Russell in [32; Theorem 5.1, p. 685]. Unfortunately, there is no proof of this statement and we cannot prove it at present. As approximate controllability makes the basis of the proposed approach we postulate it in the following definition.

3.4. Class of Normal Systems (Class N)

DEFINITION 1. *A system (Ω, ρ, μ, q) is called normal (belongs to N) if*

$$\text{Ker } O^T = \{0\}, \qquad \text{for all} \quad T > 0. \tag{3.9}$$

We think that this definition is well motivated by the following heuristic arguments. The invalidity of (3.9) means that for some $T > 0$, there exist velocity perturbations $v_{t\,|_{t=T}} = y \in$ Ker O^T initiating waves with abnormal behavior. These waves don't affect Γ, i.e.,

$$v_{|_{\Sigma^T}} = \partial_\nu v^y_{|_{\Sigma^T}} = 0,$$

in spite of the fact that Γ lies in the domain of influence of supp y. For any smooth continuation of ρ, μ, q into $\mathbf{R}^n \setminus \Omega$, these waves would not propagate out of Ω in spite of the absence of any obstacles. This is not the case for systems (Ω, ρ, μ, q) of the class N.

DEFINITION 2. *A system* (Ω, ρ, μ, q) *is called normal if*

$$cl_H\, W^T F^T = H^T \qquad \text{for all} \quad T > 0. \tag{3.10}$$

Definition 2 is equivalent to Definition 1 due to the equivalence of controllability and observability discussed in Section 3.2.

Definition 2 yields obviously an important property of normal systems: Let $(\Omega, \rho, \mu, q) \in N$ and $\{f_k\}_{k=1}^{\infty}$ be a family of controls complete in F^T, i.e.,

$$\mathrm{cl}_{H^T}\ \mathrm{Lin}\ \{f_k\}_{k=1}^{\infty} = F^T, \qquad (T > 0), \tag{3.11}$$

where Lin is a linear span. Then the family of the induced waves is complete in H^T, i.e.,

$$\mathrm{cl}_{H^T}\ \mathrm{Lin}\ \{u^{f_k}(\cdot, T)\}_{k=1}^{\infty} = H^T, \qquad (T > 0). \tag{3.12}$$

Class N was introduced in [25,27]. Let us repeat that the normality of systems (Ω, ρ, μ, q) with $\rho, \mu, q \in C^2(\bar{\Omega})$ is still unknown because, in the literature, there is no proof of D. Russell's results [32].

4. RAY COORDINATES

4.1. Definitions

Let us associate with a point x lying in the neighborhood of Γ the pair $[\gamma, \tau]$, where $\tau = \tau(x)$ and γ is the point on the boundary nearest to x in the metric (2.17): $d(x, \gamma) = \tau(x)$. The pair $[\gamma, \tau]$ is called the ray (semi-geodesic) coordinates of the point $x := x[\gamma, \tau]$. It is well-known (see, e.g., [40]) that the ray coordinates are regular in some layer near the boundary. Let T_r be the largest of those T for which the ray coordinates are regular in Ω^T.

REMARK 4.1. The value T_r may be estimated from below via given *a priori* information on the velocity c and its derivatives up to the second order [22].

An eikonal level surface $\Gamma^\xi := \{x \in \Omega : \tau(x) = \xi\}$ $(0 < \xi \le T)$, is called a (wave) front; $\Gamma^0 := \Gamma$. For $\xi < T_r$ the front Γ^ξ is a smooth $(n-1)$-dimensional manifold. It is the inner component of the boundary Ω^ξ, Ω^ξ being a subdomain of Ω filled with waves at $t = \xi$.

A level curve $l_\gamma := \{x \in \bar{\Omega} : x = x[\gamma, \tau]\ \tau \ge 0\}$, i.e., the geodesic normal to the boundary at the initial point $\gamma \in \Gamma$ is called a ray, τ being the natural parameter along it. For x lying on the ray l_γ the boundary point γ is called the (geodesic) projection of x onto Γ and is denoted by $\gamma = \sigma(x)$.

Rays l_γ are known to be orthogonal to fronts Γ^ξ (see, e.g., [40]).

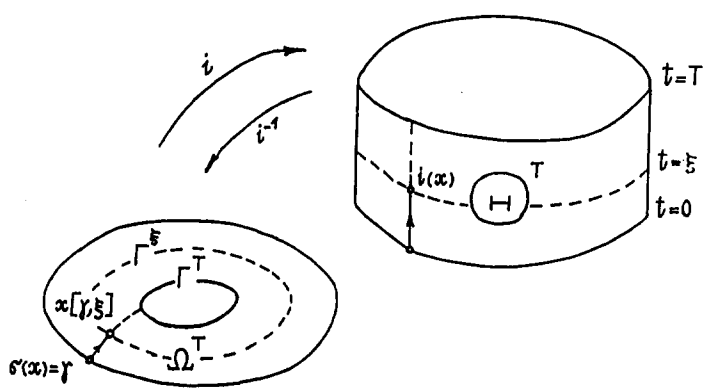

Figure 3. Ray coordinates.

4.2. Correspondence between Cartesian and Ray Coordinates

Let $T < T_r$. Transformation from Cartesian to ray coordinates provides the mapping $i : \Omega^T \to \Theta^T := \Gamma \times [0,T]$,

$$i(x) = [\gamma, \tau]. \tag{4.1}$$

The set Θ^T is the range of ray coordinates. It is convenient for us to consider Θ^T as a copy of $\Gamma \times [0,T]$ different from Σ^T because Θ^T and Σ^T are supports of functions of a different nature. Namely functions on Σ^T are controls while functions on Θ^T appear to be the waves in ray coordinates. The mapping i is a diffeomorphism of Ω^T onto Θ^T.

Let us discuss ray coordinates in more detail. For a fixed $x \in \Omega^T$ let $V_x \subset \Omega^T$ be a (small enough) neighborhood of x. Then the projection $\sigma(V_x) := \{\sigma(x') : x' \in V_x\} \subset \Gamma$ turns out to be a boundary neighborhood of $\gamma = \sigma(x)$. If $(\gamma^1, \dots, \gamma^{n-1})$ are some local coordinates on $\sigma(V_x)$, then the variables $[\gamma^1, \dots, \gamma^{n-1}, \tau]$ make up a regular coordinate system for V_x. In what follows, we denote functions $h(x)$, $x \in \Omega^T$, written in ray coordinates by $h[\gamma, \tau] = h[\gamma^1, \dots, \gamma^{n-1}, \tau] := h(i^{-1}[\gamma, \tau])$.

The Jacobi matrix corresponding to the inverse mapping $i^{-1} : [\gamma^1, \dots, \gamma^{n-1}, \tau] \to (x^1, \dots, x^n)$ is denoted by $G = \{g_{ik}[\gamma, \tau]\}_{i,k=1}^n$ where

$$g_{ik} = \frac{\partial x^i}{\partial \gamma^k}, \quad g_{in} = \frac{\partial x^i}{\partial \tau}, \qquad (i = 1, \dots, n;\ k = 1, \dots, n-1). \tag{4.2}$$

It induces a matrix $M = \{m_{ik}[\gamma, \tau]\}_{i,k=1}^n := G \cdot G^+$ ("+" is the symbol of transposition):

$$\begin{aligned} m_{ik} &= \left(\frac{\partial x}{\partial \gamma^i}, \frac{\partial x}{\partial \gamma^k}\right); \quad m_{kn} = m_{nk} = 0, \qquad (i, k = 1, \dots, n-1), \\ m_{nn} &= \left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \tau}\right) = c^2. \end{aligned} \tag{4.3}$$

Here $(\cdot, \cdot)$ is the standard inner product in $\mathbf{R}^n$ and $c = c[\gamma, \tau]$ is the wave velocity. Equalities $m_{kn} = m_{nk} = 0$ are due to the mentioned orthogonality of rays and wave fronts while the equality $m_{nn} = c^2$ is a consequence of the definition of rays. The matrix M is the matrix of the metric tensor in ray coordinates:

$$(dx)^2 = \sum_{i,k=1}^{n-1} m_{ik}[\gamma, \tau]\, d\gamma^i d\gamma^k + c^2[\gamma, \tau]\, (d\tau)^2.$$

Let us introduce the function

$$J[\gamma, \tau] := c^{-1}[\gamma, \tau] \mid \det G[\gamma, \tau] \mid, \tag{4.4}$$

so that the following relation for elementary volumes in Ω^T is valid

$$dx = dx^1 \cdot \ldots \cdot dx^n = c[\gamma, \tau]\, J[\gamma, \tau]\, d\gamma^1 \cdot \ldots \cdot d\gamma^{n-1} d\tau$$

As the (Lebesgue) measure of an elementary area on Γ in local coordinates $(\gamma^1, \dots, \gamma^{n-1})$ is determined by

$$d\Gamma_\gamma = J[\gamma, 0]\, d\gamma^1 \cdot \ldots \cdot d\gamma^{n-1},$$

then

$$dx = c[\gamma, \tau] \cdot j[\gamma, \tau] \cdot d\Gamma_\gamma d\tau, \tag{4.5}$$

where the function $j[\gamma, \tau] := J[\gamma, \tau]\, J^{-1}[\gamma, 0]$ is called the divergence of the ray field (see, e.g., [40,41]).

REMARK 4.2. Values of j have an invariant geometric meaning of the quotient of elementary surface measures on Γ^τ and Γ, respectively [40,41].

4.3. Inner Product in Ray Coordinates

The mapping i defined in Subsection 4.1 induces a transformation

$$I^T : y(x) \to y[\gamma, \tau] := y(i^{-1}[\gamma, \tau]), \tag{4.6}$$

transferring functions defined on Ω^T into functions defined on Θ^T.

Changing variables x to $[\gamma, \tau]$ in the integral

$$(y, z)_{H^T} = \int_{\Omega^T} dx\, \rho(x)\, y(x)\, z(x),$$

we obtain, in view of (4.5), the representation

$$\begin{aligned}(y, z)_{H^T} &= \int_0^T d\tau \int_\Gamma d\Gamma_\gamma\, \rho[\gamma, \tau]\, c[\gamma, \tau]\, j[\gamma, \tau]\, y[\gamma, \tau]\, z[\gamma, \tau] \\ &= \int_{\Theta^T} d\Gamma_\gamma d\tau\, \beta^2[\gamma, \tau]\, y[\gamma, \tau]\, z[\gamma, \tau],\end{aligned} \tag{4.7}$$

function $\beta = \beta[\gamma, \tau]$,

$$\beta := \sqrt{\rho c j} = \sqrt{\frac{j\mu}{c}} \geq \text{const.} > 0, \tag{4.8}$$

being a smooth function on Θ^T. Let $H^{[T]} := L_{2,\beta^2}(\Theta^T)$ be a Hilbert space with the inner product defined by the last integral in (4.7). Then the transformation $I^T : H^T \to H^{[T]}$ turns out to be an isometry from H^T onto $H^{[T]}$:

$$(y, z)_{H^T} = (I^T y, I^T z)_{H^{[T]}}. \tag{4.9}$$

4.4. Operator L in Ray Coordinates

The change of variables $x \to [\gamma, \tau]$ provides the transformation of the operator L (2.6) in H^T into the operator $\mathcal{L} := I^T L (I^T)^{-1}$ in $H^{[T]}$. In the local coordinate system $[\gamma^1, \ldots, \gamma^{n-1}, \tau]$ the operator $\mathcal{L}$ has the following form:

LEMMA 2. *Operator $\mathcal{L}$ is a second-order differential operator of the form*

$$\mathcal{L} = -\frac{\partial^2}{\partial \tau^2} + \sum_{i,k=1}^{n-1} b_{ik}[\gamma, \tau] \frac{\partial^2}{\partial \gamma^i \partial \gamma^k} + \sum_{k=1}^{n-1} b_k[\gamma, \tau] \frac{\partial}{\partial \gamma^k} + b_n[\gamma, \tau] \frac{\partial}{\partial \tau} + b_0[\gamma, \tau], \tag{4.10}$$

and

$$b_0[\gamma, \tau] = \rho^{-1}[\gamma, \tau]\, q[\gamma, \tau]. \tag{4.11}$$

Let B be the matrix of the main coefficients of $\mathcal{L}$

$$B = \{b_{ik}[\gamma, \tau]\}_{i,k=1}^n; \quad b_{ik} = b_{ki};\ b_{in} = 0,\ b_{nn} = -1, \qquad (i, k = 1, \ldots, n-1) \tag{4.12}$$

Then the following relations are valid

$$B = -c^2 M^{-1}; \qquad \det B[\gamma, \tau] = (-1)^n\, c^{2(n-1)}[\gamma, \tau]\, J^{-2}[\gamma, \tau]. \tag{4.13}$$

The proof of Lemma 2 is given in the Appendix.

The second relation in (4.13) yields in view of the definition of j, (4.4), and Equation (4.8) that

$$c^{2(n-2)}[\gamma, \tau]\, \mu^2[\gamma, \tau] = \beta^4[\gamma, \tau]\, J^2[\gamma, 0]\, |\det B[\gamma, \tau]|, \tag{4.14}$$

this equality being used in IP.

4.5. Differential Equation Along the Ray

In this section, we describe one geometric lemma. It is used below to determine the correspondence between Cartesian and ray coordinates. Let us remember that everywhere in Section 4 we restricted our considerations to the domain Ω^T for $T < T_r$ so that ray coordinates are regular in it.

Let $A = \{a_{ik}[\gamma,\tau]\}_{i,k=1}^n$ be a matrix determined via matrix M

$$\begin{aligned} a_{ik} &= \frac{1}{2}\frac{\partial}{\partial\tau} m_{ik}, \quad (i,k = 1,\dots,n-1); \qquad a_{nn} = \frac{1}{2}\frac{\partial}{\partial\tau} m_{nn} = \frac{1}{2}\frac{\partial}{\partial\tau} c^2; \\ a_{in} &= -a_{ni} = \frac{1}{2}\frac{\partial}{\partial\gamma^i} m_{nn} = \frac{1}{2}\frac{\partial}{\partial\gamma^i} c^2, \qquad (i = 1,\dots,n-1), \end{aligned} \tag{4.15}$$

and let $Z = \{z_{ik}[\gamma,\tau]\}_{i,k=1}^n$ be a matrix of the form

$$Z := A\,M^{-1}. \tag{4.16}$$

LEMMA 3. *The Jacobi matrix $G = G[\gamma,\tau]$ (4.2) is the unique solution of the Cauchy problem*

$$\frac{\partial}{\partial\tau} G = Z\,G, \qquad \tau \in (0,T), \tag{4.17}$$

$$G[\gamma,0] = \{g_{ik}[\gamma,0]\}_{i,k=1}^n, \tag{4.18}$$

the initial data having the form

$$g_{in}[\gamma,0] = -c[\gamma,0]\cos\psi_i(\gamma); \quad g_{ik}[\gamma,0] = \frac{\partial x^i}{\partial\gamma^k}(\gamma), \qquad (i = 1,\dots,n; k = 1,\dots,n-1). \tag{4.19}$$

Here $\psi_i(\gamma)$ is an angle between the outward normal $\nu = \nu(\gamma)$ and the axis x^i and functions $x^i(\gamma) = x^i(\gamma^1,\dots,\gamma^{n-1})$, $(i = 1,\dots,n)$, define the surface Γ in $\mathbf{R}^n$.

The proof of this Lemma is given in the Appendix. It is obvious from (4.15) and (4.16) that the matrix $Z_{|_{\Theta^T}}$ depends smoothly on $[\gamma,\tau]$ so the Cauchy problem (4.17) and (4.18) is well-posed.

Let us explain in advance how this Lemma is used in IP. The inversion procedure proposed below to recover μ (see Section 9) provides $\mu_{|_{\Theta^T}}$, $Z_{|_{\Theta^T}}$ and the initial data (4.18). Solving the Cauchy problem (4.17) and (4.18), we obtain the Jacobi matrix $G_{|_{\Theta^T}}$, its n-th column being $\partial x/\partial\tau = \{\partial x^k/\partial\tau[\gamma,\tau]\}_{k=1}^n$. As $x^k[\gamma,0] = x^k(\gamma)$ are known, the Cartesian coordinates $x^k[\gamma,\tau]$ can be determined by integration:

$$x^k[\gamma,\xi] = x^k[\gamma,0] + \int_0^\xi \frac{\partial x^k}{\partial\tau}[\gamma,\tau]\,d\tau, \qquad (0 \le \xi \le T). \tag{4.20}$$

Afterwards, we recover $\mu_{|_{\Omega^T}}$ in Cartesian coordinates.

5. PROPAGATION OF DISCONTINUITIES

5.1. Dual Problem with Discontinuous Data

Let us consider the initial-boundary value problem (3.3)–(3.5)

$$\rho\, v_{tt} - \operatorname{div}\,(\mu\,\nabla v) + qv = 0, \qquad \text{in} \quad \Theta^T, \tag{5.1}$$

$$v_{|_{t=T}} = 0, \qquad v_{t|_{t=T}} = y(\cdot)\,\chi_\xi(\cdot), \tag{5.2}$$

$$\mu\,\partial_\nu v_{|_{\Sigma^T}} = 0, \tag{5.3}$$

where $y \in C^\infty(\Omega^T)$ and $\chi_\xi = \chi_\xi(x)$ is an indicator function of the subdomain $\Omega^T \setminus \Omega^\xi$ $(0 < \xi < T < T_r)$. The velocity perturbation

$$v_t(x,T) = y(x)\,\chi_\xi(x) = \begin{cases} y(x), & \text{for} \quad \Omega^T \setminus \Omega^\xi, \\ 0, & \text{outside} \quad \Omega^T \setminus \Omega^\xi, \end{cases} \tag{5.4}$$

is located at the final moment $t = T$ in the layer $\Omega^T \setminus \Omega^\xi$ and has a jump discontinuity on the front Γ^ξ for $y_{|\Gamma^\xi} \not\equiv 0$.

The discontinuity of the data (5.4) yields two discontinuities of the solution. One of them moves (in reverse time) towards the boundary and the other moves from it. The latter does not affect the boundary for $0 < t < T$ and is not of interest for our considerations. The former reaches Γ at $t = T - \xi$ and provides the jump of the velocity v_t on it. The value of this jump can be obtain by means of the ray method [40, 41]. Omitting standard calculations let us write out the result

$$v_t^{y\chi}(\gamma, t)_{|t=T-\xi-0} = y[\gamma, \xi]\, \frac{\beta[\gamma, \xi]}{\beta[\gamma, 0]}, \tag{5.5}$$

$v^{y\chi}_{|\Gamma}$ vanishing identically for $T - \xi < t < T$. In view of definition (3.6)

$$-v^{y\chi}_{|\Sigma^T} = O^T(y\chi_\xi),$$

so that the relation (5.5) takes the form

$$-\left\{\frac{\partial}{\partial\tau} O^T(y\chi_\xi)\right\}(\gamma, T - \xi - 0) = y[\gamma, \xi]\, \frac{\beta[\gamma, \xi]}{\beta[\gamma, 0]}, \qquad (\gamma \in \Gamma,\ 0 < \xi < T). \tag{5.6}$$

5.2. Initial-Boundary Value Problem with Instantaneous Control

Let us consider the problem (2.1)–(2.3) with $f := 1(\gamma)\,\delta(t)$, where $1(\gamma)$ is a function of Γ identically equal to 1. From the physical point of view, the solution of this problem describes the effect of the instantaneous boundary impulse with constant amplitude. Its solution $u^{1\delta}$ is a wave propagating from Γ towards the interior of Γ, $u^{1\delta}$ having a jump discontinuity on its wave front. It can be obtained by means of the ray method that

$$u^{1\delta}_{|\Sigma^{2T}} = \beta^{-2}\,[\gamma, 0]\,\theta(t) + \ldots,$$

where $\theta(t)$ is the Heaviside function and points ... denote the term tending to 0 for t tending to $+0$. So

$$\{R^{2T}[\,1(\cdot)\,\delta(\cdot)]\}\,(\gamma, +0) = \beta^{-2}\,[\gamma, 0], \qquad (\gamma \in \Gamma), \tag{5.7}$$

this relation being used to determine $\beta[\gamma, 0]$.

6. ISOMETRY OF OUTER AND INNER SPACES

6.1. Operator C^T

In view of the definition (2.2) and Equation (3.7) of Lemma 1, we have for arbitrary $f, g \in F^T$:

$$(u^f(\cdot\,, T), u^g(\cdot\,, T))_{H^T} = (W^T f, W^T g)_{H^T} = (C^T f, g)_{F^T}, \tag{6.1}$$

where

$$C^T := O^T\, W^T. \tag{6.2}$$

The operator C^T introduced in [23] is a nonnegative operator in F^T, Equation (3.1) yielding

$$\operatorname{Ker} C^T = \{0\}, \qquad (0 < T < T_*). \tag{6.3}$$

The crucial role of this operator in solving IP is due to the remarkable possibility to represent it explicitly via given response operator R^{2T} [23,27,29]. Let $S^T : F^T \to F^{2T}$ be an operator of the odd continuation with respect to $t = T$

$$[S^T f](\gamma, t) := \begin{cases} f(\gamma, t), & t \in (0, T), \\ -\,f(\gamma, 2T - t), & t \in (T, 2T), \end{cases} \tag{6.4}$$

and let $Y^\xi : F^\xi \to F^\xi$, $(0 < \xi \le 2T)$, be an integral operator

$$[Y^\xi f](\gamma, t) := \int_0^t f(\gamma, t')\, dt'. \tag{6.5}$$

THEOREM 3. *For any $T > 0$*

$$C^T = -\frac{1}{2}[S^T]^* Y^{2T} R^{2T} S^T. \tag{6.6}$$

REMARK 6.1. Using the representation (2.5) for the response operator R^{2T}, we can represent C^T in the form of an integral operator with a distribution kernel

$$[C^T f](\gamma, t) = \int_{\Sigma^T} d\Gamma_{\gamma'} dt' \left[\frac{1}{2}\int_{|t-t'|}^{2T-t-t'} r(\gamma, \gamma', \eta)\, d\eta\right] f(\gamma', t').$$

6.2. Demonstration of Theorem 3

For arbitrary $f, g \in C_0^\infty(\Sigma^T) \subset F^T$, let us consider the function

$$w^{fg}(s,t) := (u^f(\cdot, s),\, u^g(\cdot, t))_{H^T}, \qquad (0 \le s, t \le T).$$

Integrating by parts and taking into account (2.1)–(2.3), we obtain

$$\begin{aligned}
\left(\frac{\partial}{\partial t^2} - \frac{\partial}{\partial s^2}\right) w^{fg}(s,t) &= \int_\Omega dx\, \rho(x)\, [u^f(x,s)\, u^g_{tt}(x,t) - u^f_{ss}(x,s)\, u^g(x,t)] \\
&= \int_\Omega dx\, [u^f(x,s)\ \mathrm{div}\ (\mu\, \nabla u^g(x,t)) - \ \mathrm{div}\ (\mu \nabla u^f(x,s)) u^g(x,t)] \\
&= \int_\Gamma d\Gamma_\gamma\, [u^f(\gamma,s)\, \mu(\gamma)\, \partial_\nu u^g(\gamma,t) - \mu(\gamma)\, \partial_\nu u^f(\gamma,s)\, u^g(\gamma,t)] \\
&= \int_\Gamma d\Gamma_\gamma\, \left\{[R^T f](\gamma,s)\, g(\gamma,t) - f(\gamma,s)\, [R^T g](\gamma,t)\right\}.
\end{aligned}$$

As the waves u^f and u^g satisfy initial conditions (2.2) the comparison of the first and the last terms of these equalities show that w^{fg} appears to be a solution of the following initial-boundary value problem

$$\left(\frac{\partial}{\partial t^2} - \frac{\partial}{\partial s^2}\right) w^{fg}(s,t) = \int_\Gamma d\Gamma_\gamma\, \left\{[R^T f](\gamma,s)\, g(\gamma,t) - f(\gamma,s)\, [R^T g](\gamma,t)\right\}, \text{ for } s,t > 0, \tag{6.7}$$

$$w^{fg}_{|t<0} = 0, \qquad w^{fg}_{|s=0} = 0. \tag{6.8}$$

It follows from the definition of w^{fg} that the values of this function for $0 < s,\ t < T$ are completely determined by the behavior of f and g for $0 < s,\ t < T$. It is convenient to continue f and g onto the time interval $(0, 2T)$ by means of the operator S^T (6.4). This yields the continuation of the right-hand side of (6.7) onto $0 < s,\ t < 2T$ so that the initial-boundary value problem (6.7)–(6.9) may be solved by means of the D'Alambert formula:

$$w^{fg}(s,t) = \frac{1}{2}\iint_{\Pi^{st}} d\eta\, d\xi \int_\Gamma \left\{[R^{2T} S^T f](\gamma,\xi)\, [S^T g](\gamma,\eta) - [S^T f](\gamma,\xi)\, [R^{2T} S^T g](\gamma,\eta)\right\}, \tag{6.9}$$

where Π^{st} is the domain bounded by characteristics of Equation (6.7) (see Figure 4).

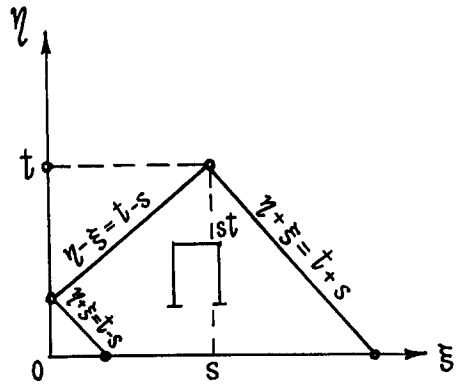

Figure 4. Domain of integration.

For $s = t = T$, Formula (6.9) takes the form

$$w^{st}(T,T) = \frac{1}{2}\int_0^T d\eta \int_\eta^{2T-\eta} d\xi \int_\Gamma d\Gamma_\gamma \left\{ [R^{2T}S^T f](\gamma,\xi)[S^T g(\gamma,\eta)] - [S^T f](\gamma,\xi)[R^{2T}S^T g](\gamma,\eta) \right\}.$$

The oddness of the function $[S^T f](\cdot,\xi)$ with respect to $\xi = T$ is responsible for vanishing of the integral over ξ in the second term of the right-hand side in the preceding equality. So

$$\begin{aligned} w^{fg}(T,T) &= \frac{1}{2}\int_{\sum^T} d\Gamma_\gamma d\eta g(\gamma,\eta) \int_\eta^{2T-\eta} d\xi [R^{2T}S^T f](\gamma,\xi) \\ &= \frac{1}{2}\int_{\sum^T} d\Gamma_\gamma d\eta g(\gamma,\eta)[Y^{2T}R^{2T}S^T f](\gamma,\xi)\Big|_{\xi=\eta}^{\xi=2T-\eta}. \end{aligned} \tag{6.10}$$

Let $[S^T]^* : F^{2T} \to F^T$ be an operator conjugate to S^T. It may be easily shown that

$$[S^T]^* h(\gamma,\eta) = [h(\gamma,\eta) - h(\gamma,2T-\eta)]_{|\eta\in(0,T)}.$$

Then the relation (6.10) may be written as

$$w^{fg}(T,T) = (g, -\frac{1}{2}[S^T]^* Y^{2T}R^{2T}S^T f)_{F^T}.$$

On the other hand, we have in view of the definition of C^T (6.1)

$$w^{fg}(T,T) = (g, C^T f)_{F^T}.$$

Comparing the last two relations and taking into account the arbitrariness of f and g, we obtain Equation (6.6). ∎

6.3. Space Φ^T

Let $T < T_*$ and let us provide F^T with a new inner product

$$(f,g)_{\Phi^T} := (C^T f, g)_{F^T} = (W^T f, W^T g)_{H^T}, \tag{6.11}$$

this bilinear form being non-degenerate in view of the property (6.3). Let Φ^T be a completion of F^T according to the C^T-form (6.11), this completion being called the "outer" space as well. It contains a certain class of distributions including those of the form $f = \alpha(\gamma)\,\delta(t)$ for $\alpha \in C(\Gamma)$ (see Section 5.2). Unfortunately, we don't know a precise analytic characterization of the space Φ^T for dimensions $n > 1$.

In view of the definition (6.11), the operator $W^T : \Phi^T \to H^T$, $D(W^T) = F^T$ obviously has to be isometric. Let $\mathcal{W}^T$ be its continuous extension onto Φ^T.

The operator C^T can also be continuously extended from F^T onto Φ^T, this extension being equal to $O^T\mathcal{W}^T$. We denote this extension by C^T as well. The extended C^T also has a trivial kernel (6.3). For the outer space Φ^T, let us introduce subspaces Φ^ξ formed by time-delayed controls (2.16)

$$\Phi^\xi := cl_{\Phi^T} F^T_{T-\xi} = cl_{\Phi^T} \mathcal{T}^T_{T-\xi} F^\xi, \quad (0 < \xi < T). \tag{6.12}$$

For the inner space H^T, let us also consider subspaces of functions localised in the filled subdomains Ω^ξ

$$H^\xi := y \in H^T : \text{ supp } y \subset \bar{\Omega}^\xi, \quad (0 < \xi < T). \tag{6.13}$$

PROPOSITION 1. *Let $(\Omega, \rho, \mu, q) \in N$. Then $\mathcal{W}^T$ is an isometry of Φ^T onto H^T and*

$$\mathcal{W}^T \Phi^\xi = H^\xi, \quad (0 \le \xi \le T). \tag{6.14}$$

The validity of (6.14) is an immediate consequence of the definition of $\mathcal{W}^T$ and the baisc property of normal systems (3.10) (where T is replaced by $\xi \in [0, T]$). From the point of view of the BC problem, the C^T-completion of F^T corresponds to the minimal extension of the class of controls to provide solvability to the problem (2.23) for every $y \in H^T$. Its unique (generalized) solution is given by

$$f = [\mathcal{W}^T]^{-1} y \in \Phi^T. \tag{6.15}$$

6.4. Projectors $\mathcal{P}^\xi$ and P^ξ. Wave Bases

Let $\mathcal{P}^\xi$ be an (ortho) projector of Φ^T onto Φ^ξ and P^ξ be a projector of H^T onto H^ξ. In view of (6.13), P^ξ cuts functions onto the subdomain Ω^ξ:

$$[P^\xi y](x) = \begin{cases} y(x), & x \in \Omega^\xi, \\ 0, & x \in \Omega^T \setminus \Omega^\xi. \end{cases} \tag{6.16}$$

Using the definition of the projectors $\mathcal{P}^\xi$, P^ξ, we may replace Proposition 1 by the equivalent

PROPOSITION 2. *Let $(\Omega, \rho, \mu, q) \in N$. Then*

$$\mathcal{W}^T \mathcal{P}^\xi = P^\xi \mathcal{W}^T, \quad (0 \le \xi \le T). \tag{6.17}$$

To construct $\mathcal{P}^\xi$ let us take some complete system of time-delayed controls $\{f_k\}_{k=1}^\infty \subset F^T_{T-\xi} \subset \Phi^\xi$,

$$cl_{F^T} \text{ Lin } \{f_k\}_{k=1}^\infty = F^T_{T-\xi},$$

and orthogonalize it with respect to the C^T-form (6.11). The resulting system $\{h_k^\xi\}_{k=1}^\infty$ forms an orthonormal basis in Φ^ξ so that

$$\mathcal{P}^\xi = \sum_{k=1}^\infty (\cdot, h_k^\xi)_{\Phi^T} h_k^\xi. \tag{6.18}$$

REMARK 6.2. It is necessary to point out that this procedure is quite practicable for IP under consideration. Indeed, to carry it out, it is sufficient to know C^T, this operator was obtained via the given R^{2T} (see (6.6)).

Let (Ω, ρ, μ, q) be a normal system. Then in view of (6.14) the waves $\{W^T h_k^\xi\}_{k=1}^\infty$ form an orthonormal (wave) basis in H^ξ so that

$$P^\xi = \sum_{k=1}^\infty (\cdot, \mathcal{W}^T h_k^\xi)_{H^T} \mathcal{W}^T h_k^\xi. \tag{6.19}$$

6.5. Operator L in Wave Basis

Let $(\Omega, \rho, \mu, q) \in N$. Let us construct a smooth wave basis in H^T using the procedure described in Subsection 6.4. To this end, we take some complete system of controls in F^T so that $\{f_k\}_{k=1}^{\infty} \subset C_o^{\infty}(\sum^T)$ and C^T-orthogonalize it. Then the resulting orthonormal basis consists of controls $\{h_k\}_{k=1}^{\infty} \subset C_o^{\infty}(\sum^T)$. In view of the smoothness property (iii) of Section 2.1, the initiated waves $\{\mathcal{W}^T h_k\}_{k=1}^{\infty}$ are smooth in Ω and satisfy the Neumann boundary condition (2.7). So the wave basis $\{\mathcal{W}^T h_k\}_{k=1}^{\infty}$ belongs to $D(L)$. Equation (2.1) yields for an arbitrary $h \in C_o^{\infty}(\sum^T)$:

$$L\mathcal{W}^T h = -\frac{\partial^2}{\partial t^2}\mathcal{W}^T h = \mathcal{W}^T\left[-\frac{\partial^2}{\partial t^2}h\right], \tag{6.20}$$

the last equality being an obvious consequence of the independence on t of ρ, μ, q (see also (2.11)).

Substituting $h = h_k$, we obtain

$$L\mathcal{W}^T h_k = \mathcal{W}^T\left[-\frac{\partial^2}{\partial t^2}h_k\right], \quad (k = 1, 2, \dots). \tag{6.21}$$

This formula is used below in solving IP.

7. EQUATIONS OF THE INVERSE PROBLEM

7.1. BC Problem of a Particular Type

Let $T < T_*$ and let $a = a(x) \in C^2(\bar{\Omega})$ be quasi-harmonic, i.e., $[\operatorname{div}(\mu\nabla) - q]a = 0$ in Ω. Let us also consider the BC problem for $a^T := P^T a \in H^T$:

$$u^f(\cdot, T) = a^T. \tag{7.1}$$

The unique solution of this problem is given by formula (6.15):

$$f = q_a^T := [\mathcal{W}^T]^{-1} a^T \in \Phi^T. \tag{7.2}$$

Let us emphasize that the control q_a^T may be also characterized in terms of IP [23,25,27,29].

THEOREM 4. *The control q_a^T is the unique solution for the following equation*

$$C^T f = \kappa^T a_{|\Gamma} - [R^T]^*(\kappa^T \mu \partial_\nu a_{|\Gamma}), \tag{7.3}$$

in the outer space Φ^T. Here $\kappa^T = \kappa^T(t) := T - t$; $a_{|\Gamma}, \partial_\nu a_{|\Gamma}$ are traces on Γ and $[R^T]^*$ is an operator conjugate to R^T in F^T.

PROOF. In view of the definition of C^T (6.2) and Equations (2.1)–(2.3), we have for an arbitrary $f \in C_o^{\infty}(\Sigma^T)$

$$\begin{aligned}
(C^T q_a^T, f)_{F^T} &= (\mathcal{W}^T q_a^T, \mathcal{W}^T g)_{H^T} = (a^T, u^f(\cdot, T))_{H^T} = (a, u^f(\cdot, T))_H \\
&= \int_\Omega dx\, \rho(x)\, a(x)\, u^f(x, T) = \int_\Omega dx\, \rho(x)\, a(x) \int_0^T dt (T - t)\, u^f{}_{tt}(x, t) \\
&= \int_0^T dt (T - t) \int_\Omega dx\, a(x)[\operatorname{div}(\mu(x)\nabla u^f(x, t)) - q(x)\, u^f(x, t)] \\
&= \int_0^T dt (T - t) \int_\Omega dx\, u^f(x, t)[\operatorname{div}(\mu(x)\nabla a(x)) - q(x)\, a(x)] \\
&+ \int_0^T dt (T - t) \int_\Gamma d\Gamma_\gamma [a(\gamma)\, \mu(\gamma)\, \partial_\nu u^f(\gamma, t) - u^f(\gamma, t)\, \mu(\gamma)\, \partial_\nu a(\gamma)] \\
&= \int_{\sum^T} d\Gamma_\gamma\, dt \left\{(T - t)\, a(\gamma)\, f(\gamma, t) - [R^T f](\gamma, t)\, (T - t)\, \mu(\gamma)\, \partial_\nu a(\gamma)\right\} \\
&= (\kappa^T a_{|\Gamma} - [R^T]^*(\kappa^T \mu \partial_\nu a_{|\Gamma}), \quad f)_{F^T}.
\end{aligned}$$

Taking into account the density of $f \in C_o^\infty(\Sigma^T)$ in F^T and comparing the first and the last terms, we obtain (7.3). ∎

Equation (7.3) is a natural multidimensional analog of the classical integral equations for one-dimensional IP. Namely, for IP (ii) (inhomogeneous string), this equation may be simply transformed into the Krein one. For IP (iii) (the Sturm-Liouville problem), it may be transformed into the Gel'fand-Levitan equation.

Let us call Equation (7.3) the Inverse Problem Equation (IPE).

7.2. Reduced IPE

Let $a(x)$ be the same as in Section 7.1. and let $a^\xi := P^\xi a^T \in H^\xi (0 < \xi < T)$. It is convenient to denote the unique solution of the reduced BC problem

$$u^f(\cdot, \xi) = a^\xi, \tag{7.4}$$

by $f = \bar{q}_a^\xi$. Then due to Theorem 4 (where T is replaced by ξ), the control $\bar{q}_a^\xi$ gives the unique solution for the reduced IPE:

$$C^\xi f = \kappa^\xi a_{|\Gamma} - [R^\xi]^*(\kappa^\xi \mu \partial_\nu a_{|\Gamma}). \tag{7.5}$$

Using the time-delayed control $q_a^\xi := \mathcal{T}_{T-\xi}^T \bar{q}_a^\xi \in \Phi^\xi \subset \Phi^T$ we can rewrite (7.4) in the form

$$a^\xi = \mathcal{W}^\xi \bar{q}_a^\xi = \mathcal{W}^T \mathcal{T}_{T-\xi}^T \bar{q}_a^\xi = \mathcal{W}^T q_a^\xi. \tag{7.6}$$

On the other hand, relations (6.17) and (7.2) yield

$$a^\xi = P^\xi a^T = P^\xi \mathcal{W}^T q_a^T = \mathcal{W}^T \mathcal{P}^\xi q_a^T. \tag{7.7}$$

Applying the operator $[\mathcal{W}^T]^{-1}$ to both sides of (7.6) and (7.7) and comparing the results, we obtain

$$\mathcal{P}^\xi q_a^T = q_a^\xi = \mathcal{T}_{T-\xi}^T \bar{q}_a^\xi, \quad (0 < \xi < T). \tag{7.8}$$

This formula provides us the way to determine the projections of q_a^T onto Φ^ξ by solving reduced IPE.

Finally let us deduce one useful representation for reduced operators

$$C^\xi = [\mathcal{T}_{T-\xi}^T]^* C^T \mathcal{T}_{T-\xi}^T, \quad (0 \le \xi \le T). \tag{7.9}$$

Indeed, taking into account the definition of C^T (6.2) and Equation (2.14) we have for arbitrary $f, g \in F^\xi$

$$(C^\xi f, g)_{F^\xi} = (u^f(\cdot, \xi), u^g(\cdot, \xi))_{H^\xi} = (u^{f_{T-\xi}}(\cdot, T), u^{g_{T-\xi}}(\cdot, T))_{H^T} = ([\mathcal{T}_{T-\xi}^T]^* C^T \mathcal{T}_{T-\xi}^T f, g)_{F^\xi},$$

these equalities yield (7.9).

8. WAVE FIELDS IN RAY COORDINATES

Throughout this section, we suppose $(\Omega, \rho, \mu, q) \in N$ and $T < T_r$.

8.1. Weighted Waves; Space $\tilde{H}^{[T]}$

Let $f \in \Phi^T$ and

$$u^f([\gamma, \tau; T] := I^T u^f(\cdot, T)[\gamma, \tau] = (I^T \mathcal{W}^T f)[\gamma, \tau] = (\mathcal{W}^{[T]} f)[\gamma, \tau), \tag{8.1}$$

be the initiated wave in ray coordinates (4.6). In view of (4.9) and (6.11), I^T and $\mathcal{W}^T$ are isometries of the relevant spaces so the operator $\mathcal{W}^{[T]} := I^T \mathcal{W}^T$ is an isometry of Φ^T onto H^T.

DEFINITION. *Let $\beta_o[\gamma,\tau] = \beta^{-1}[\gamma,0]\beta[\gamma,\tau]$, where β is defined by (4.8). Then the function*

$$\tilde{u}^f[\gamma,\tau;T] := \beta_o[\gamma,\tau]\, u^f[\gamma,\tau;T], \quad ([\gamma,\tau] \in \Theta^T).$$

is called the weighted wave corresponding to the control f.

We introduce this notion because weighted waves take meaning for more general hyperbolic equations and systems and the inversion procedure discussed below starts in their reconstruction.

Let us introduce the Hilbert space $\tilde{H}^{[T]} := L_{2,\beta^2[\cdot,0]}(\Theta^T)$, the isometry of $H^{[T]}$ onto $\tilde{H}^{[T]}$ being provided by $\hat{\beta}_o$, where $\hat{\beta}_o$ is an operator of multiplication by $\beta_o[\gamma,\tau]$. Then the operator $\tilde{\mathcal{W}}^{[T]} := \hat{\beta}_o \mathcal{W}^{[T]}$ is isometric from Φ^T onto $\tilde{H}^{[T]}$. The definition of weighted waves may be rewritten in the following form

$$\tilde{u}^f = (\hat{\beta}_o I^T \mathcal{W}^T) f = (\hat{\beta}_o \mathcal{W}^{[T]}) f = \tilde{\mathcal{W}}^{[T]} f, \tag{8.2}$$

so that weighted waves belong to $\tilde{H}^{[T]}$. It is convenient to illustrate these considerations by the diagram shown in Figure 5.

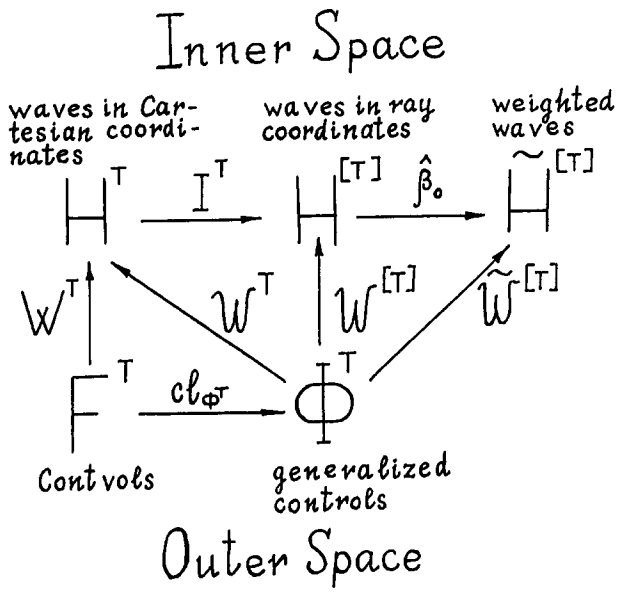

Figure 5. Spaces and operators.

8.2. Principal Relations for WFC

The isometry $\hat{\beta}_o I^T : H^T \to \tilde{H}^{[T]}$ induces the operator $\tilde{\mathcal{L}}$ in $\tilde{H}^{[T]}$

$$\tilde{\mathcal{L}} := [\hat{\beta}_o I^T]\, L [\hat{\beta}_o I^T]^{-1} = \hat{\beta}_o\, \mathcal{L}\, \hat{\beta}_0^{-1}. \tag{8.3}$$

The WFC-procedure is based upon the following result

THEOREM 5. *For an arbitrary $f \in C_o^\infty(\sum^T)$, the following relations*

$$\tilde{u}^f[\gamma,\xi;T] = -\left\{\frac{\partial}{\partial t} C^T \mathcal{P}_\perp^\xi f\right\}(\gamma, T-\xi-0), \tag{8.4}$$

$$(\tilde{\mathcal{L}}\tilde{u}^f)[\gamma,\xi;T] = -\left\{\frac{\partial}{\partial t} C^T \mathcal{P}_\perp^\xi \left[-\frac{\partial^2}{\partial t^2}\right] f\right\}(\gamma, T-\xi-0), \tag{8.5}$$

are valid. Here $[\gamma,\xi] \in \Theta^T$, $\mathcal{P}_\perp^\xi$ is a projector from Φ^T onto $\Phi^T \Theta \Phi^\xi$; $\mathcal{P}_\perp^\xi := E^T - \mathcal{P}^\xi$, E^T being the identical operator in Φ^T.

PROOF. Let us consider the dual problem (5.1)–(5.3) with $y = u^f(\cdot,T) = \mathcal{W}^T f \in H^T$. Then, due to the intertwining property (6.17),

$$u^f(\cdot,T)\chi_\xi(\cdot) = P_\perp^\xi\, u^f(\cdot,T) = P_\perp^\xi \mathcal{W}^T f = \mathcal{W}^T \mathcal{P}_\perp^\xi f, \tag{8.6}$$

where $P_\perp^\xi$ is a projector from H^T onto $H^T \ominus H^\xi$. So the relation (5.6) takes the form

$$-\left\{\frac{\partial}{\partial t} O^T \mathcal{W}^T \mathcal{P}_\perp^\xi f\right\}(\gamma, T-\xi-0) = \beta_o[\gamma,\xi]\, u^f[\gamma,\xi;T]. \tag{8.7}$$

The operator $O^T\mathcal{W}^T$ coincides with the extended C^T (see Section 6.3) while the right-hand side of (8.7) is the weighted wave $\tilde{u}^f$. So (8.7) provides (8.4). At last (8.5) follows from (8.4) in view of Equation (6.20). ∎

COROLLARY 1. *Let $\{h_k\}_{k=1}^\infty$, $h_k \in C_o^\infty(\sum^T)$, be a basis of Φ^T constructed in Section 6.5. Let $\{\tilde{u}_k\}_{k=1}^\infty := \tilde{u}_k = \tilde{u}_k[\gamma,\xi] := (\tilde{\mathcal{W}}^{[T]} h_k)[\gamma,\xi]$ be the weighted wave basis in $\tilde{H}^{[T]}$ corresponding to $\{h_k\}_{k=1}^\infty$. Then for every $k = 1,2,\ldots$ and $[\gamma,\xi] \in \Theta^T$*

$$\tilde{u}_k[\gamma,\xi] = -\left\{\frac{\partial}{\partial t} C^T \mathcal{P}_\perp^\xi h_k\right\}(\gamma, T-\xi-0), \tag{8.8}$$

$$(\tilde{\mathcal{L}}\tilde{u}_k)[\gamma,\xi] = -\left\{\frac{\partial}{\partial t} C^T \mathcal{P}_\perp^\xi \left[-\frac{\partial^2}{\partial t^2} h_k\right]\right\}(\gamma, T-\xi-0). \tag{8.9}$$

COROLLARY 2. *Let $q_a^T \in \Phi^T$ (7.2) be a solution of the BC problem (7.1). Then*

$$\beta_o[\gamma,\xi]\, a[\gamma,\xi] = -\left\{\frac{\partial}{\partial t} C^T [q_a^T - q_a^\xi]\right\}(\gamma, T-\xi-0), \tag{8.10}$$

for every $[\gamma,\xi] \in \Theta^T$.

PROOF. Equations (8.8) and (8.9) coincide with (8.4) and (8.5) for $f = h_k$. To prove (8.10) let us remark that due to (7.8)

$$\mathcal{P}_\perp^\xi q_a^T = (E^T - \mathcal{P}^\xi)\, q_a^T = q_a^T - q_a^\xi = q_a^T - \mathcal{T}_{T-\xi}^T\, \bar{q}_a^\xi.$$

So (8.10) follows directly from (8.4) taking into account that $\tilde{\mathcal{W}}^{[T]} q_a^T = \hat{\beta}_o a$. ∎

8.3. Reconstruction of Weighted Wave Fields

Just now we are entering upon solving IP. The reconstruction of the operator $\tilde{\mathcal{W}}^{[T]}$ via the given R^{2T} is described in this section; this procedure is the same for all IP under consideration.

STEP 1. Using Equation (6.6) we find the operator C^T via the given R^{2T}.

STEP 2. By means of C^T-orthogonalization of some complete in $F_{T-\xi}^T$ and F^T systems, we construct projectors $\mathcal{P}^\xi, (0 \le \xi \le T)$, (see Section 6.4) and an (orthonormal) basis $\{h_k\}_{k=1}^\infty \subset C_o^\infty(\sum^T)$ in Φ^T.

STEP 3. Using Equation (8.8), we determine the weighted wave basis $\{\tilde{u}_k\}_{k=1}^\infty \in \tilde{H}^{[T]}$ and then extend $\tilde{\mathcal{W}}^{[T]}$ onto the whole Φ^T by linearity and continuity.

So the operator $\tilde{\mathcal{W}}^{[T]}$ is reconstructed.

8.4. WFC in Ray Coordinates

Having determined $\tilde{\mathcal{W}}^{[T]}$ we reconstruct now the control operator $\mathcal{W}^{[T]}$. This operator provides WFC in ray coordinates (see Figure 5).

To this end the function $\beta_o[\gamma,\xi]$ is needed. It may be found by means of different methods for different cases.

STEP 4. Step 4 is devoted to recovering β_o. The detailed description is given in Sections 9–11 for IP items (i)–(iii), respectively.

STEP 5. Via the given β_o and $\tilde{\mathcal{W}}^{[T]}$, we construct $\mathcal{W}^{[T]} = \hat{\beta}_o^{-1}\tilde{\mathcal{W}}^{[T]} : \Phi^T \to H^{[T]}$, this operator giving WFC in ray coordinates $[\gamma,\xi]$.

8.5. Reconstruction of Coefficients of Operator $\mathcal{L}$

To determine $\mathcal{W}^T$ via $\mathcal{W}^{[T]}$ one has to come back from ray coordinates to Cartesian ones. For IP (i)–(iii) the inverse transformation $(I^T)^{-1}$ is obtained by different methods. In particular in two cases we use the coefficients of $\mathcal{L}$. To find them we need the following auxiliary result.

Let $\{h_k\}_{k=1}^{\infty}$ be a basis in Φ^T constructed previously (Step 2) and let β_o be already known from Step 4. Then, multiplying (8.8) and (8.9) by $\beta_o^{-1}[\gamma,\xi]$ and using the relation $\mathcal{L} = \hat{\beta}_o^{-1}\tilde{\mathcal{L}}\hat{\beta}_o$ (see (8.3)), we obtain

$$u_k[\gamma,\xi;T] = -\beta_o^{-1}[\gamma,\xi]\left\{\frac{\partial}{\partial t}C^T\mathcal{P}_\perp^\xi h_k\right\}(\gamma,T-\xi-0), \tag{8.11}$$

$$(\mathcal{L}u_k)[\gamma,\xi;T] = \beta_o^{-1}[\gamma,\xi]\left\{\frac{\partial}{\partial t}C^T\mathcal{P}_\perp^\xi\left[\frac{\partial^2}{\partial t^2}h_k\right]\right\}(\gamma,T-\xi-0). \tag{8.12}$$

Here $\{u_k\}_{k=1}^{\infty} = \hat{\beta}_o^{-1}\{\tilde{u}_k\}_{k=1}^{\infty} = \{\mathcal{W}^{[T]}h_k\}_{k=1}^{\infty}$ is the wave basis in $H^{[T]}$ corresponding to the basis $\{h_k\}_{k=1}^{\infty}$.

As the right-hand sides of (8.11) and (8.12) are already determined, these formulas may be used to reconstruct the smooth wave basis $\{u_k\}_{k=1}^{\infty}$ and its image $\{\mathcal{L}u_k\}_{k=1}^{\infty}$. To find the coefficients of $\mathcal{L}$, we use relations

$$\left\{-\frac{\partial^2}{\partial\tau^2} + \sum_{i,k=1}^{n-1} b_{ik}[\gamma,\tau]\frac{\partial^2}{\partial\gamma^i\partial\gamma^k} + \sum_{k=1}^{n-1} b_k[\gamma,\tau]\frac{\partial}{\partial\tau^k} + b_n[\gamma,\tau]\frac{\partial}{\partial\tau} + b_o[\gamma,\tau]\right\}u_p = \mathcal{L}u_p, \tag{8.13}$$

$(p = 1,2,\dots)$ considered as an algebraic system for b_{ik}, b_k, b_n, b_o.

We summarize our considerations of Section 8.5 in

STEP 5′. Using (8.11) and (8.12), we determine the wave basis $\{u_k\}_{k=1}^{\infty}$ in $H^{[T]}$ and its image $\{\mathcal{L}u_k\}_{k=1}^{\infty}$. Coefficients b_{ik}, b_k, b_n, b_o, $(i,k = 1,\dots,n-1)$, are then obtained by solving (8.13).

9. INVERSE PROBLEM FOR THE SYSTEM $(\Omega,1,\mu,0) \in N$, $T < T_r$

9.1. Reconstruction of $\beta_o|_{\Theta^T}$

The determination of $\beta_o[\gamma,\tau]$ consists of the omitted Step 4 of the inversion procedure described in Section 8. It is based on the quasi-harmonicity of the function $1(x) \equiv 1$, $x \in \Omega$, in the considered case, i.e., $L1 = 0$ for $L = -\operatorname{div}(\mu\nabla)$. Due to this fact, β_o may be determined by means of IPE (7.3) and (7.5). For $a = 1(\cdot)$, these equations take the form

$$C^T f = \kappa^T 1; \quad C^\xi f = \kappa^\xi 1, \quad (0 < \xi < T). \tag{9.1}$$

Let e^T, $\bar{e}^\xi$ be their (unique) solutions, $e^T \in \Phi^T$, $e^\xi := \mathcal{T}_{T-\xi}^T\bar{e}^\xi \in \Phi^\xi$. Substituting them into Equation (8.10), we obtain

$$\beta_o[\gamma,\xi] = -\left\{\frac{\partial}{\partial t}C^T(e^T - e^\xi)\right\}(\gamma,T-\xi-0), \tag{9.2}$$

for every $[\gamma,\xi] \in \Theta^T$. So in this case, we proceed as follows:

STEP 4. By means of (7.9), we construct C^ξ, $(0 < \xi < T)$, via the given C^T. Solving Equations (9.1) with these C^T, C^ξ, we find e^T, e^ξ. Then, using (9.2), we determine $\beta_0[\gamma,\xi]$.

9.2. Tension in Ray Coordinates

Having already carried out Step 5′, we possess the matrix $B = \{b_{ik}[\gamma,\tau]\}_{i,k=1}^n$, for every $[\gamma,\xi] \in \Theta^T$ (see (4.12) and Section 8.5). Taking into account the relation $c^2 = \rho^{-1}\mu = \mu$ (see Section 2.1) we obtain in view of (4.14)

$$\mu^n[\gamma,\tau] = \beta_o^4[\gamma,\tau]\,\beta^4[\gamma,0]\,J^2[\gamma,0]|\det B[\gamma,\tau]|, \tag{9.3}$$

the function $J[\gamma,0]$ being known for the chosen local coordinates on Γ (see Section 4.2).

STEP 6. Using the formula (5.7), we obtain $\beta[\gamma,0]$. Then we determine $\mu[\gamma,\tau]$ by means of Equation (9.3).

The completive part of the procedure is purely geometric. Having determined the velocity c in ray coordinates, we determine the mapping $i^{-1} : \Theta^T \to \Omega^T$.

9.3. Returning to Cartesian Coordinates

The matrix B and the coefficient $\mu = c^2$ being known, we are able to construct the matrix Z determining the evolution of G along the ray (4.17). For this end, we use formulas (4.13) and (4.15) and the definition of Z (4.16). Taking into account that the function

$$c[\gamma, 0] = \beta^2[\gamma, 0], \quad (\gamma \in \Gamma), \tag{9.4}$$

is already known, the matrix Z and the initial data for the Cauchy problem (4.17) and (4.18) are available.

STEP 7. We construct $c[\gamma, 0]_{|\Gamma}$ using (9.4). Then we find $M_{|\Theta^T}$, $A_{|\Theta^T}$ and $Z_{|\Theta^T}$ using (4.13), (4.15) and (4.16), respectively, via the known $B_{|\Theta^T}$ and $\mu_{|\Theta^T}$.

STEP 8. Solving the Cauchy problem (4.17) and (4.18) we determine the Jacobi matrix $G_{|\Theta^T}$. Integration of its last column yields $\{x^k[\gamma, \tau]\}_{k=1}^n$, $[\gamma, \tau] \in \Theta^T$ (see (4.20)).

The mapping $i^{-1} : \Theta^T \to \Omega^T$ in Equation (4.1),

$$i^{-1}([\gamma, \xi]) = x = \{x^1[\gamma, \xi], \ldots, x^n[\gamma, \xi]\}, \tag{9.5}$$

determines returning from ray coordinates to Cartesian ones and induces the operator $(I^T)^{-1} : H^{[T]} \to H^T$

$$\{(I^T)^{-1} h[\cdot, \cdot]\}(x) = h[i(x)]. \tag{9.6}$$

The following Step concludes the procedure.

STEP 9. The operator $(I^T)^{-1}$ is determined by means of (9.5) and (9.6). Using it, we recover the tension $\mu_{|\Omega^T} = (I^T)^{-1}\mu_{|\Theta^T}$ and the extended control operator $\mathcal{W}^T : \Phi^T \to H^T$, $\mathcal{W}^T = (I^T)^{-1}\mathcal{W}^{[T]}$.

The IP (i) is thus solved.

10. INVERSE PROBLEM FOR THE SYSTEM $(\Omega, \rho, 1, 0) \in N$, $(T < T_r)$

10.1. Reconstruction of $\beta_{o|_{\Theta^T}}$

As in Section 9, we begin with the description of β_o-reconstruction in ray coordinates (Step 4 of the procedure in Section 8). This procedure is quite the same as in Section 9.1 because the function $1(x)$ remains to be quasi-harmonic for the operator $L = -\rho^{-1}\Delta$, where $\Delta =$ div (grad) is the Laplace operator.

10.2. Returning to Cartesian Coordinates

The method of the determination of coordinates functions $x^k[\gamma, \tau]$, $[\gamma, \tau] \in \Theta^T$, was put forward in [27] and is based upon the fact that the coordinate functions $x^k = x^k(\cdot)$ are quasi-harmonic, i.e., $Lx^k = 0$ in Ω. So IPE (7.3) and (7.5) may be used to find these functions in ray coordinates.

For $a = x^k$, $k = 1, \ldots, n$, Equations (7.3) and (7.5) take the form

$$C^T f = \kappa^T x^k_{|\Gamma} - [R^T]^*(\kappa^T \cos \Psi_{k|\Gamma}), \; C^\xi f = \kappa^\xi x^k_{|\Gamma} - [R^\xi]^*(\kappa^\xi \cos \Psi_{k|\Gamma}), \quad (0 < \xi < T), \tag{10.1}$$

$\Psi_k(\gamma)$ being an angle between the outward normal $\nu = \nu(\gamma)$ and x^k-axis (see also (4.19)).

Let q_k^T, $\bar{q}_k^\xi$ be the (unique) solutions of these equations, $q_k^\xi = T^T_{T-\xi}\bar{q}_k^\xi : q_k^T \in \Phi^T$, $q_k^\xi \in \Phi^\xi$. Substituting them into (8.10), we obtain

$$x^k[\gamma, \xi] = -\beta_o^{-1}[\gamma, \xi]\left\{\frac{\partial}{\partial t} C^T[q_k^T - q_k^\xi]\right\}(\gamma, T - \xi - 0), \tag{10.2}$$

for every $[\gamma, \xi] \in \Theta^T$, $k = 1, \ldots, n$.

Remember previous steps and then make use of Equation (10.2).

STEPS 1–5′. Steps 1–5′ are carried out in just the same way as in Section 8 and Subsection 9.1. We note only that Step 5′ is not necessary in this case.

STEP 6. Solving IPE (10.1), we obtain q_k^T, q_k^ξ. Substituting them into (10.2), we determine $x^k_{|\Theta^T}$ in ray coordinates.

STEP 7. Using the coordinate functions, we reconstruct the mapping i^{-1} (see (9.5)) and the operator $(I^T)^{-1}$ (see (9.6)). By means of $(I^T)^{-1}$, we determine the control operator $\mathcal{W}^T : \Phi^T \to H^T$, $\mathcal{W}^T = (I^T)^{-1}\mathcal{W}^{[T]}$. So the WFC problem is solved.

10.3. Recovery of the Density $\rho_{|_{\Omega^T}}$

To recover ρ, let us remark that, in view of Equation (4.3),

$$\sum_{k=1}^{\infty}\left(\frac{\partial x^k}{\partial \tau}[\gamma,\xi]\right)^2 = \left(\frac{\partial x}{\partial \tau},\frac{\partial x}{\partial \tau}\right) = c^2[\gamma,\xi] = \rho^{-1}[\gamma,\xi], \tag{10.3}$$

for every $[\gamma,\xi] \in \Theta^T$. The last equality in (10.3) is valid for systems $(\Omega,\rho,1,0)$ due to the fact that $c^2 = \rho^{-1}\mu$ (see Section 2.4).

STEP 8. Using (10.3), we find $\rho_{|_{\Theta^T}}$ and then recover $\rho_{|_{\Omega^T}} = (I^T)^{-1}\rho_{|_{\Theta^T}}$.

The IP (ii) is thus solved.

11. INVERSE PROBLEM FOR THE SYSTEM $(\Omega,1,1q) \in N$, $T < T_r$

11.1. Reconstruction of $\beta_{o|_{\Theta^T}}$

The reconstruction of β_o (Step 4) for the equation $u_{tt} - \Delta u + qu = 0$ is based upon the fact that $c \equiv 1$. So the ray field consists of the straight lines normal to Γ and all properties of this field including the divergence j and $\beta_0 = j^{1/2}$ (see (4.8)) are available for given Ω. Moreover, the mapping i^{-1} (9.5) and the operator $(I^T)^{-1}$ (9.6) are already known *a priori*.

11.2. Inversion Procedure

STEPS 1-5′. Steps 1–5′ are carried out in just the same way as in Section 8, $\beta_{0|_{\Theta^T}}$ being available for the given Ω.

STEP 6. Using the known operator $(I^T)^{-1}$ and alredy found $\mathcal{W}^{[T]}$ (see Step 5), we determine the control operator in Cartesian coordinates $\mathcal{W}^T : \Phi^T \to H^T$, $\mathcal{W}^T = (I^T)^{-1}\mathcal{W}^{[T]}$.

So the WFC problem is solved.

STEP 7. Equation (4.11) in the considered case takes the form $b_{0|_{\Theta^T}} = q_{|_{\Theta^T}}$, providing the stiffness of q in ray coordinates. Returning to Cartesian coordinates, we recover $q_{|_{\Theta^T}} = (I^T)^{-1}q_{|_{\Theta^T}}$.

The IP (iii) is thus solved

12. CONCLUDING REMARKS

12.1. $T > T_r$

An attempt to extend this approach to larger times meets several difficulties as there appear singularities of the ray field and the mapping i fails to be one–to–one. Nevertheless, the procedures of Sections 8–11 may be generalized to $T > T_r$. We mention here only the main ideas of this generalization (for details see [27]).

(i) Let $\omega \in \Omega$ be the set of points connected with Γ by more than one minimal geodesic. This set is called "cut locus" in geometry (see e.g. [42]). As it may be shown, $\mathrm{mes}_{\mathbb{R}^n}\omega = 0$, so ray coordinates turn out to be regular on the subset $\Omega \setminus \omega$ of the full measure.

(ii) Ray method for discontinuities propagation remains valid for $\Omega \setminus \omega$.

These observations enable us to extend the procedure of WFC and coefficients recovering onto $\Omega \setminus \omega$.

12.2. Other Generalizations

The approach is valid also for other types of boundary controls, for example for Dirichlet control $u_{|_{\Sigma^T}} = f$ with the response operator $R^T : f \to \partial_\nu u^f_{|_{\Sigma^T}}$. Attempts to extend the approach onto systems with several velocities are under consideration now.

If the Holmgren-John theorem is valid for Equation (2.1) with finitely smooth coefficients (see [32]) then the method is applicable to such cases.

12.3. Numerical Testing

The approach was tested numerically for the one-dimensional IP for an inhomogeneous string [26,28]. The numerical experiments have shown its efficiency giving possibility to recover non-monotone densities ρ without using any regularization algorithm. An attempt to calculate the two-dimensional case for the equation $\rho u_{tt} - \Delta u = 0$ was not successful. The difficulty, which was not overcome until now, is due to the fact that the series of spectral data of the type of (2.9) converges too slowly. So it is impossible to calculate the elements of the Gram matrix $(C^T f_i, f_k)_{F^T}$ necessary to solve IPE. However, we hope that the use of more efficient computers (we used $PC \setminus AT$) would enable us to overcome this difficulty.

References

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APPENDIX

A1. Sketch of Proof of Theorem 1

Let u^f_- be a continuation of u^f onto the whole time–axis

$$u^f_-(x,t) = \begin{cases} u^f(\gamma,t), & t \in (-\infty, T), \\ -u^f(\gamma, 2T-t), & t \in [T,\infty), \end{cases}$$

u^f being a solution of (2.1)–(2.3). Let f_- be an analogous continuation of f

$$f_- = \begin{cases} f(\gamma,t), & t \in (-\infty, T), \\ -f(\gamma, 2T-t), & t \in [T,\infty). \end{cases}$$

Functions u^f_- and f_- are odd with respect to $t = T$ and for $|\, t - T \,| > T$, $u^f_-(\cdot,t) = 0$, $f_-(\cdot,t) = 0$ in view of (2.2). It may be shown that

$$\left[\rho\frac{\partial^2}{\partial t^2} - \operatorname{div}\,(\mu\nabla) + q\right] u^f_-(x,t) = -2\rho(x)u^f(x,T)\delta'(t-T) \tag{A1}$$

$$\mu\partial_\nu u^f_-|_{\Gamma\times\mathbf{R}} = f_-, \tag{A2}$$

where δ' is the derivative of the Dirac δ–function. It appears in (A1) due to a possible jump of $u^f_-(\cdot,t)$ at $t = T$.

For $T < T_*$, let $g \in \operatorname{Ker} W^T \subset F^T$. We'll show that $g = 0$. For $g \in \operatorname{Ker} W^T$, we have $u^g(\cdot,T) = W^T g = 0$, so that (A1) and (A2) take the form

$$\left[\rho\frac{\partial^2}{\partial t^2} - \operatorname{div}\,(\mu\nabla) + q\right] u^g_- = 0, \quad \text{in} \quad \Omega\times\mathbf{R}, \tag{A3}$$

$$\mu\partial_\mu|_{\Sigma^T} u^g_- = g_-. \tag{A4}$$

As the solution u^g vanishes identically in the unfilled subdomain $\Omega \setminus \bar{\Omega}^T$ for $0 \le T \le T_*$, the same is true for its odd continuation

$$u^g_-(x,t) = 0 \quad \text{in} \quad \{\Omega/\bar{\Omega}^T\} \times \mathbf{R}. \tag{A5}$$

The Fourier transforms with respect to t

$$u^g_-(\cdot,\omega) = F[u^g_-(\cdot,t)] := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} u^g_-(\cdot,t)\,dt, g_-(\cdot,\omega) := F[g_-(\cdot,t)]$$

satisfy, in view of (A3) and (A4), the following equations

$$[\,\mathrm{div}\,(\mu\nabla) - q + \omega^2\rho]\, u^g_-(x,\omega) = 0, \quad \text{for} \quad x \in \Omega, \tag{A6}$$

$$\mu\partial_\nu\, u^g_-(\omega)_{|\Gamma} = g_{-(\omega)}, \tag{A7}$$

while $(A5)$ transforms into

$$u^g_-(x,\omega) = 0, \quad \text{for} \quad x \in \Omega \setminus \bar{\Omega}^T. \tag{A8}$$

In view of $(A6)$ and $(A8)$, u^g_- appears to be a solution of the homogeneous elliptic equation vanishing in the subdomain $\Omega \setminus \bar{\Omega}^T$ of a positive measure. Due to the known uniqueness theorem [43],

$$u^g_-(x,\omega) = 0 \quad \text{in} \quad \Omega,$$

so that

$$g_-(\cdot,\omega) = \partial_\nu\, u^g_-(\cdot,\omega)_{|\Gamma} = 0.$$

By means of the inverse Fourier transform, we obtain

$$g_{-|\Gamma\times\mathbf{R}} = 0, \quad g = g_{-|\Sigma^T} = 0.$$

So the assumption $g \in$ Ker W^T yields $g = 0$. ∎

For detailed proof see [30].

A2. Proof of Lemma 2

Let $z^k = z^k(x^1,\ldots,x^n)$, $k = 1,\ldots,n$, be a smooth non-degenerate change of variables $x \to z$. Then

$$\frac{\partial}{\partial x^i} = \sum_{k=1}^{n} \frac{\partial z^k}{\partial x^i}\frac{\partial}{\partial z^k};$$

$$\begin{aligned} -\rho^{-1}\,\mathrm{div}\,(\mu\nabla u) + \rho^{-1}q = -\rho^{-1}\mu \sum_{i,k=1}^{n} (\nabla_x z^i, \nabla_x z^k)\frac{\partial^2}{\partial z^i \partial z^k} \\ - \rho^{-1}\sum_{k=1}^{n}\{\mu\Delta_x z^k + (\nabla_x\mu, \nabla_x z^k)\}\frac{\partial}{\partial z^k} + \rho^{-1}q. \end{aligned} \tag{A9}$$

Let G be the Jacobi matrix for the inverse change of variables $z \to x$

$$G = \{g_{ik}\}_{i,k=1}^n; \quad g_{ik} = \frac{\partial x^i}{\partial z^k}.$$

The matrix $\left\{\frac{\partial z^i}{\partial x^k}\right\}_{i,k=1}^n$ is the inverse to G so that

$$\{(\nabla_x z^i, \nabla_x z^k)\}_{i,k=1}^n = [G^{-1}]^+ G^{-1}. \tag{A10}$$

Let

$$B = \{b_{ik}\}_{i,k=1}^n, \quad b_{ik} = -\rho^{-1}\mu(\nabla_x z^i, \nabla_x z^k), \tag{A11}$$

be the matrix of the main coefficients of the operator (A9). Comparing (A10) and (A11) and taking into account that $\rho^{-1}\mu = c^2$, we obtain

$$B = -c^2[G^{-1}]^+ G^{-1}. \tag{A12}$$

In view of the definition of matrix M (4.3), Equation (A12) yields the first of Equations (4.13)

$$B = -c^2 M^{-1}.$$

For $z = [\gamma^1,\ldots,\gamma^{n-1},\tau]$ being the ray coordinates, we find by means of the definition of J (see Section 4.2) and Equation (A12)

$$J^2c^2 = [\,\det G]^2 = |\det c^2 B^{-1}| = c^{2n}|\det B|^{-1},$$

the second Equation (4.13)

$$|\det B| = c^{2(n-1)}J^{-2}$$

being the immediate consequence of the preceding equations. ∎

A3. Proof of Lemma 3

The change of variables $x \to [\gamma, \tau]$ is non-degenerate so the vector components of the Jacobi matrix G, i.e., vectors $\partial x/\partial\gamma^1, \dots, \partial x/\partial\gamma^{n-1}$, $\partial x/\partial\tau$ form a basis in $\mathbf{R}^n$. The matrix M of the metric tensor (4.3) is just the Gram matrix of this basis.

To find coefficients of the expansion of the second order derivatives $\partial^2 x/\partial\tau\partial\gamma^1, \dots, \partial^2 x/\partial\tau\partial\gamma^{n-1}$, $\partial^2 x/\partial\tau^2$ over this basis, we need some auxiliary relations

$$\begin{aligned} &\left(\frac{\partial^2 x}{\partial\tau^2}, \frac{\partial x}{\partial\tau}\right) = \frac{1}{2}\frac{\partial m_{nn}}{\partial\tau} := a_{nn}; \left(\frac{\partial^2 x}{\partial\tau^2}, \frac{\partial x}{\partial\gamma^i}\right) = -\frac{1}{2}\frac{\partial m_{nn}}{\partial\gamma^i} := a_{ni}; \\ &\left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\tau}\right) = \frac{1}{2}\frac{\partial m_{nn}}{\partial\gamma^i} := a_{in}; \left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) = -\frac{1}{2}\frac{\partial m_{ik}}{\partial\tau} = a_{ik} \quad (i,k = 1,\dots,n-1). \end{aligned} \tag{A13}$$

Let us prove for example the last of Equations (A13); the others can be obtained similarly. We have

$$\left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) = \frac{\partial}{\partial\tau}\left(\frac{\partial x}{\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) - \left(\frac{\partial x}{\partial\gamma^i}, \frac{\partial^2 x}{\partial\tau\partial\gamma^k}\right). \tag{A14}$$

On the other hand, in view of Equations (4.3),

$$\left(\frac{\partial x}{\partial\tau}, \frac{\partial x}{\partial\gamma^k}\right) = m_{kn} = 0,$$

so that

$$\left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) = \frac{\partial}{\partial\gamma^i}\left(\frac{\partial x}{\partial\tau}, \frac{\partial x}{\partial\gamma^k}\right) - \left(\frac{\partial x}{\partial\tau}, \frac{\partial^2 x}{\partial\gamma^i\partial\gamma^k}\right) = -\left(\frac{\partial x}{\partial\tau}, \frac{\partial^2 x}{\partial\gamma^i\partial\gamma^k}\right). \tag{A15}$$

The equality (A15) yields

$$\left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) = \left(\frac{\partial x}{\partial\gamma^i}, \frac{\partial^2 x}{\partial\tau\partial\gamma^k}\right),$$

so (A14) takes the form

$$\left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right) = \frac{\partial}{\partial\tau}m_{ik} - \left(\frac{\partial^2 x}{\partial\tau\partial\gamma^i}, \frac{\partial x}{\partial\gamma^k}\right), \quad (i,k = 1,\dots,n-1)$$

these equations being equivalent to those under consideration.

Rewriting (A13) in matrix form

$$\left[\frac{\partial}{\partial\tau}G\right]G^+ = A,$$

we obtain

$$\frac{\partial}{\partial\tau}G = A[G^+]^{-1} = A[G^{-1}]^+ G^{-1}G = AM^{-1}G = ZG;$$

this formula coincides with (4.16). The validity of the initial data (4.19) follows obviously from the definition of the Jacobi matrix G. ∎