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MULTIRATE DIGITAL ADAPTIVE CONTROL

M. DE LA SEN
Departamento de Fisica, Facultad de Ciencias, Universidad del País Vasco,
Leioa (Vizcaya), Spain

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Abstract—A design method is developed which involves the use of several adaptive controllers working in parallel at different sampling rates while maintaining the asymptotic stability of the overall adaptive scheme, as well as the boundedness of all the signals within the system. The use of adaptive sampling is considered as an additional loop in a hierarchical organization of the scheme. This strategy allows the designer to take advantage of the suitable properties of such schemes which are well known in classical multirate control designs. These advantages arise from the need for alleviating control computer throughput requirements of accommodating sensor information available at mulitiple rates and for compensating excitations of the fast modes of the plant in the presence of high-frequency disturbances. As a direct consequence of the involved methodology, the transient behaviour of the adaptive system becomes in some cases greatly improved.

1. INTRODUCTION

Nonperiodic sampling has been found to be a powerful tool for certain design types in control problems; some instances follow.

- Adaptation of the sampling interval to the variations of some signals within the system to improve sampling efficiencies and transient behaviours. Some extensions have been made to the adaptative control context by using the tracking or regulation error as the signal to be adapted.
- Compensation of discrete systems to known variations in the parameters of the continuous plant while maintaining the nominal controllers in the control loop.
- Improvement of the transmission of measuring and/or rounding errors towards the results when analyzing from an algebraic point of view some properties of dynamic systems such as controllability, observability and identifiability.

For the above and related topics, a list of references is given in [3b], where an input–output modeling for nonperiodic systems is developed. Such a modeling leads to results which are structurally similar to those associated with the use of the z-transform for periodic sampling systems. However, it is based upon the use of the Cayley–Hamilton theorem from linear algebra.

The study of the transients in deterministic adaptive control has merited some attention recently[1–4,6,18,20]. In [3a,d] suboptimization techniques which involve the use of quadratic criteria for optimal regulation are given as an approach to solve this problem. By developing and optimizing an approximate linear model for the overall adaptive scheme, the standard parameter-adaptive algorithms can be reapplied on a finite-time horizon. This process is made prior to the generation of the current input and involves the use of updated values of the free-algorithm parameters obtained from the optimization procedure. Another approach for this problem which consists of the use of adaptive sampling, as mentioned above, has been taken in [3c].

The problem has also received attention from an optimal stochastic adaptive control context[8,9,15] by stating dual and nondual adaptive controllers and introducing the concepts of caution and probing in the design philosophy. In this context, some practical rules have been given for the choice of the forgetting factors in both the constant and time-varying parameter cases.

The objective of this paper is to study the use of multirate (constant and adaptive) sampling in adaptive control involving the use of several adaptive controllers. Multirate sampling is useful in classical control since the slower controllers alleviate the drawbacks derived from the presence of unmodeled dynamics, which leads to undesirable modes within the system while the faster sampling controllers improve the stability and damping characteristics of the plant response[5]. In addition, multirate digital control alleviates control computer throughput requirements and is valid to accommodate sensor information at multiple rates. Two cases are considered, namely the deterministic case and the case of additive disturbances with known upper bounds for their magnitudes. The paper is organized as follows. In section 2, an adaptive scheme consisting of several controllers working in parallel at different rates is given. A dead-beat adaptation scheme[14] is proposed for the case when disturbances are present. In section 3, a method which involves the use of adaptive variations of the sampling period of each adaptive controller with respect to their nominal values is given. The resulting schemes are useful for improving the transient system behaviour. In section 4, the boundary of the dead-beat zone used for adaptation under the presence of disturbances is related to the upper bound of the disturbances, which is determined by using *a priori* knowledge on bounds for the magnitudes of the plant parameters. Section 5 presents simulation results for the different theoretical aspects discussed in the paper and, finally, conclusions end the paper. The notation used throughout the paper is very simple. Some minimal notation variations are used in each section depending on the use of continuous or discrete system models.

2. DISCRETE ADAPTIVE ALGORITHMS AND MULTIRATE CONTROL

2.1 *Model reference direct adaptive control*

Consider a SISO discrete linear time-invariant plant described by

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})e(k), \qquad d > 0 \tag{1}$$

was considered with $A(q^{-1})$ and $B(q^{-1})$ being polynomials defined by

$$\begin{aligned} A(q^{-1}) &= 1 + a_1q^{-1} + \cdots + a_{n_A}q^{-n_A}, \\ B(q^{-1}) &= b_0 + b_1q^{-1} + \cdots + b_{n_B}q^{-n_B}, \end{aligned} \tag{2}$$

where q^{-1} is the backward shift operator, d represents the plant time delay, $e(k)$ and $y(k)$ are the plant input and output sequences, respectively. It is assumed that the zeroes of $B(z^{-1})$ are all inside the unit circle.

Both tracking and regulation objectives are achieved if the input is generated in such a way that the following equation holds:

$$C_r(q^{-1})\epsilon(k + d) = C_r(q^{-1})(y(k + d) - y^M(k + d)) = 0, \qquad \text{all } k \geq 0, \tag{3}$$

where $y^M(\cdot)$ is a bounded reference sequence (being identically zero in the regulation case), $\epsilon(\cdot)$ is the tracking/regulation error, and $C_r(q^{-1})$ is a monic asymptotically stable polynomial of arbitrary degree n_{c_r}, defined by

$$C_r(q^{-1}) = 1 + c_1q^{-1} + \cdots + c_{n_{c_R}}q^{-n_{c_R}}. \tag{4}$$

It is a well known[10] fact that a unique polynomial identity

$$C_r(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d}R(q^{-1}) \tag{5}$$

holds if

$$S(q^{-1}) = 1 + s_1q^{-1} + \cdots + s_{n_s}q^{-n_s}, \qquad n_s = d - 1; \tag{6}$$

$$R(q^{-1}) = r_0 + r_1q^{-1} + \cdots + r_{n_R}q^{-n_R}, \qquad n_R = \max\,(n_A - 1, n_{c_r} - d). \tag{7}$$

Using (5)–(7), (3) results in the adaptive control case (for types of parameterizations, see, e.g. [6,11,12]), equivalent to

$$C_r(q^{-1})\epsilon(k + d) = \theta^T\phi(k) - \hat{\theta}^T(k)\phi(k) = \tilde{\theta}^T(k)\phi(k) \tag{8}$$

with

$$\begin{aligned} \theta &= [b_0, \theta_0^T]^T, \qquad \phi(k) = [e(k), \phi_0^T(k)]^T, \\ \theta_0 &= [b_0 s_1 + b_1, b_0 s_2 + b_1 s_1 + b_2, \ldots, \\ &\quad b_{n_B} s_{d-1}, r_0, r_1, \ldots, r_{n_R}]^T, \\ \phi_0(k) &= [e(k - 1), e(k - 2), \ldots, \\ &\quad e(k - d - n_B + 1), y(k), y(k - 1), \ldots, y(k - n_R)]^T, \end{aligned} \tag{9}$$

$\hat{\theta}(k)$ being updated according to any adaptive algorithm.

2.2 *Multirate adaptive control*

The advantages of multirate adaptive control in classical control (sect. 2.1) invite us to develop multirate adaptive control schemes for cases of not-completely-known systems or systems of slowly time-varying parameters.

The following assumptions are made for the theoretical analysis of this paper.

Assumption 1. All the elements of the chosen set of sampling intervals are integer multiples of those which are smaller.

Assumption 2. The input to the plant is obtained as the sum of the partial inputs (generated by each controller) available at the current sampling point.

Assumption 3. The discrete plant is parameterized according to the fastest sampling rate.

Assumption 4. All the zeroes of the discrete transfer function of the plant (for the taken parameterization) are assumed to be inside the unit complex circle. Furthermore, upper bounds for the degrees of the numerator and denominator polynomials of the plant transfer function as well as the plant delay are known.

Assumption 5. The reference sequence must be bounded and defined for all sampling points.

Assumption 6. Under bounded output additive noise, this one is assumed to be bounded and of known upper bound for its magnitude. Also, the system is assumed asymptotically stable. ∎

These assumptions are not more restrictive than the usual ones in adaptive control. For instance, Assumptions 4 and 5 are necessary to prove asymptotic convergence with a bounded input–output sequence. Assumptions 1 and 3 allow the construction of a difference equation of constant parameters which exactly describes the input–output sequence from all the sampling points for the fastest sampler. Assumption 6 will be invoked in a scheme of sect. 2.3 which involves the use of a dead-beat zone for the case of additive disturbances.

Thus, the plant is described by the equation

$$A(q^{-1})y(k_i) = q^{-d}B(q^{-1})e(k_i); \qquad k_i = 0, 1, \ldots, i = 1, 2, \ldots, m. \tag{10}$$

The meaning of the different polynomials and magnitudes is the same as in sect. 2.1. The ith index of the positive integer k_i indicates that one is dealing with the k_iT_i sampling point of the ith controller, $T_1 < T_2 < \cdots < T_m$ where $T_i = \lambda_{ij}T_j$ ($\lambda_{ij} \geq 1$ being fixed integers $1 \leq j \leq i \leq m$) is the set of sampling periods. In this way, if t_l is a sampling point of the jth then $t_l = k_jT_j$, $j = 1, 2, \ldots, i$; some $i \leq m$ and some set of positive integers $k_1, k_2, \ldots, k_i$ with $k_l = \lambda_{lj}T_j$. If it is desired to specify a sampling order between controllers, the modified notation k_{ij}, instead of k_j, will denote that $k_{ij}T_j$ is the last sampling point of the jth controller prior or equal to the sampling point k_iT_i of the ith controller for all $j \leq i$. This notation is not cumbersome and allows the interpretation of the equations without ambiguity.

2.3 *Adaptation algorithms and control law*

The natural generalization of the adaptive algorithms of De la Sen[3c] leads in the multirate context (see Fig. 1) to

$$\hat{\theta}_i(k_i) = \hat{\theta}_i(k_i - 1) + \frac{F_i(k_i)\phi(k_i - d)[C_r(q^{-1})y(k_i) - \hat{\theta}_{0i}^T(k_i - 1)\phi(k_i - d)]}{c_i(k_i) + \phi^T(k_i - d)F_i(k_i)\phi(k_i - d)},$$

$$F_i(k_i + 1) = \frac{1}{\lambda_i(k_i)}\left[F_i(k_i) - \frac{F_i(k_i)\phi(k_i - d)\phi^T(k_i - d)F_i(k_i)}{c_i(k_i) + \phi^T(k_i - d)F_i(k_i)\phi(k_i - d)}\right], \tag{11}$$

$$\|\hat{\theta}_i(0)\| < \infty, \qquad F_i(0) = F_i^T(0), \qquad \|F_i(0)\| < \infty,$$

with $0 < \lambda_i(k_i) \leqslant 1$, $0 < c_i(k_i) < \infty$, all $k_i \geqslant 0$, $i = 1, 2, \ldots, m$, where

$$\phi(k_i) = [e(k_i),\ \phi_0^T(k_i)]^T,$$

$$\phi_0(k_i) = [e(k_i - 1),\ e(k_i - 2), \ldots, e(k_i - d - n_B + 1),\ y(k_i),\ y(k_i - 1), \ldots, y(k_i - n_R)]^T,$$

$$e(k_1) = u_1(k_1) + \sum_{j=2}^{m} u_j(k_{1j}), \qquad \text{all integer } k_1 \geqslant 0, \tag{12}$$

$$u_i(k_i) = \frac{1}{\hat{b}_{0i}(k_i)}\left[C_r(q^{-1})y^M(k_i + d) - \sum_{\substack{j=1\\ j\neq i}}^{m} \hat{b}_{0j}(k_{ij})u_j(k_{ij}) - \hat{\theta}_{0i}^T(k_i)\phi_0(k_i)\right], \qquad \text{all integer } k_{ij} \geqslant 0,\ i, j = 1, 2, \ldots, m$$

for the nominal parameter vector being defined as in (9) according to Assumption 3. The regulation and tracking objectives for this problem are stated similarly to (3).

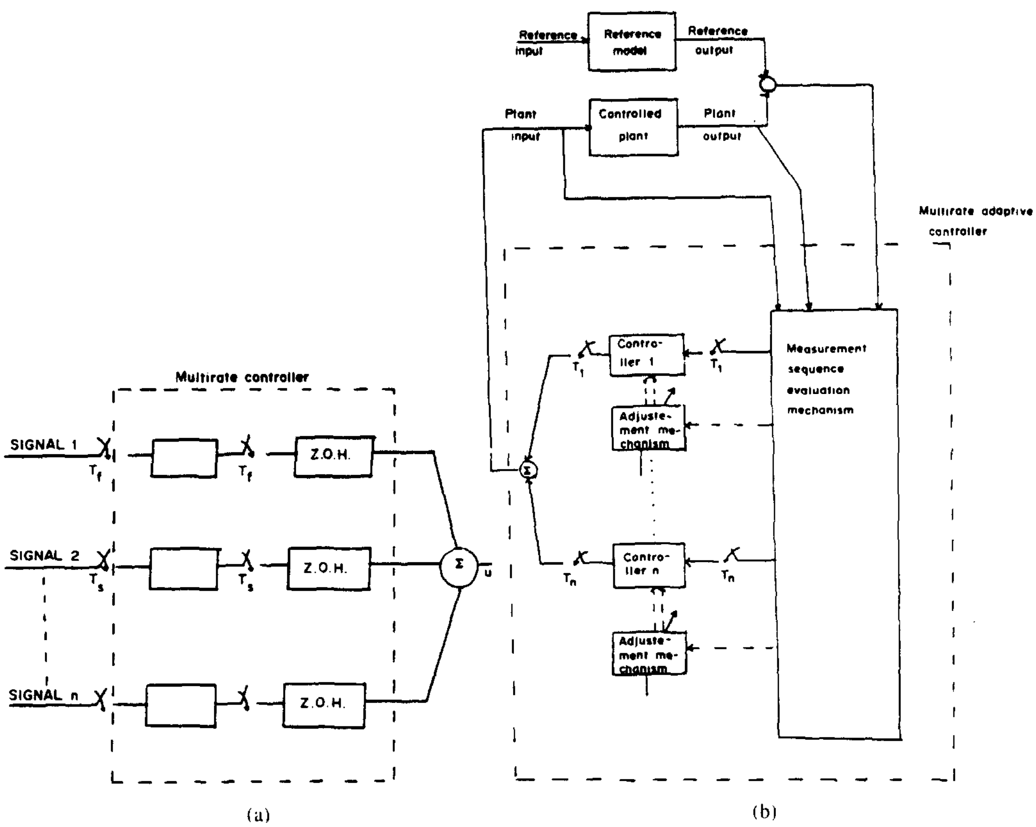


Fig. 1. (a) Multirate digital controller, (b) multirate model reference digital adaptive control ($T_s = lT_i$).

Remarks 2.1

1. In the input generation eqn (12), division by zero must be avoided by appropriate local variations of the free algorithm parameters when necessary. In some of the existing adaptive control schemes such a problem is overcome by using different parameterizations which enter directly the reference sequence in the measurement vector.

2. Note that the computation of the input sequence from eqns (12) can lead to the simultaneous solution of a set of linear equations. This occurs at sampling points in which several controllers must modify their parameters since then each partial input $u_{(\cdot)}$ supplied for one controller depends on some of the others at the same time. However, the linear systems of equations remains compatible and so can be solved provided a determinant involving the leading parameters of the inputs is nonzero. This is not a very restrictive condition in the transient since these parameters can be varied with the use of the free parameters of the algorithms. As time increases solvability can be lost (for instance if several of the leading parameters tend to the same values). The problem can be then overcome by deleting from the scheme some of the adaptive controllers. In practice computational problems inherent in the solution of systems of equations can arise when more than three controllers are present. Approach 2 for multirate adaptive sampling in the next section does not lead to the solution of sets of equations for computation of the inputs since, in general, multirate controllers are not synchronous. This represents an additional advantage to the use of adaptive sampling.

3. In order that the computation be admissible from an applicability point of view, we must have a bounded adaptation matrix and adaptive controller parameters $\lambda_i(k_{ij}) = 1$ all $k_{ij} > k, i, j = 1, 2, \ldots, m$, some finite k, or they must be chosen according to the maintenance of a bounded or constant trace of the adaptation matrix[3c].

4. Note that all the adaptive controllers involve the use of the plant delay d (which indicates d times the smaller sampling period of delay in the continuous plant). Otherwise, the adaptation errors in the controllers would not be available. Note also that sample index k_1 defines any sampling point. This is crucial to understand some of the proofs of convergence in the Appendix. ■

THEOREM 2.1

Under Assumptions 1–5, the system (10) subject to the adaptive controller (11) fulfills the regulation and tracking objectives; i.e. $\lim_{k\to\infty} \epsilon(k) \triangleq 0$ with a bounded input–output sequence.

A proof is given in Appendix. ■

The number of parameters to be updated by the jth adaptive controller, $j = 2, 3, \ldots, m$, can be reduced arbitrarily while respecting the control law in (12). This does not affect the stability proof of Theorem 2.1 since it is directly related to the use of a complete measurement vector [i.e., that which involves the use of all the necessary vector components given in (12)] and the parameter vector for the first adaptive controller. This strategy allows the designer to diminish the overall number of parameters to be adapted, and is directly related to the classical philosophy that is usual in multirate control about the reduction of computational effort. This idea is also useful from a filtering point of view since not all the controllers need the same filtering characteristics and thus the same complexity. Finally, note that the multirate strategy implies that samples occur within the updating process of some adaptive controllers. This idea has been studied for the case of using one adaptive controller (see, for instance, [12]).

2.4 *Multirate adaptive control with a dead-beat adaptation zone*

The use of a dead-beat zone for parameter adaptation has been proposed in Peterson and Narendra[14] for the case of unknown additive disturbances in continuous adaptive systems. The philosophy involved was the following. If the adaptation error (which is subject to an unmeasurable noisy component) lies outside a certain domain around zero, the adaptation takes place, else the adaptive controller maintains its parameters constant. Two conditions which were imposed on the problem were that the noise is bounded and that an upper bound on its magnitude is known. The dead-beat zone for adaptation was defined according to such a bound. The determination of an appropriate bound will be given in section 5. Assume that output additive

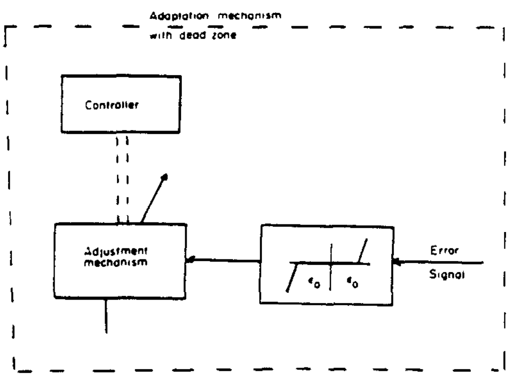


Fig. 2. Adaptation algorithm with dead zone.

noise $v_{(\cdot)}$ is present. Thus, instead of (10), one has

$$A(q^{-1})y(k_i) = q^{-d}B(q^{-1})e(k_i) + v(k_i), \qquad k_i \geq 0. \tag{13}$$

2.5 *Adaptive algorithms and control law*

The adaptive algorithms (11) are modified according to

$$\begin{aligned} \hat{\theta}(k_j) &= \hat{\theta}(k_j - 1) \qquad \text{if } |\epsilon(k_j)| \leq \epsilon_0, \\ F(k_j) &= F(k_j - 1) \qquad \text{if } |\epsilon(k_j)| \leq \epsilon_0, \end{aligned} \tag{14}$$

$\hat{\theta}(k_j)$ is updated as in (11) if $|\epsilon(k_j)| > \epsilon_0$ (ϵ_0 is a known upper bound for the magnitude of the tracking/regulation error), all integers $k_j > 0$, $j = 1, 2, \ldots, m$.

The control law is obtained as in (12).

The main stability result of this adaptive scheme is stated as follows.

THEOREM 2.2

The plant (13) subject to the adaptive algorithm (11)–(12), with the variants of (14), generates an input–output sequence which is bounded under Assumptions 1–6.

Proof is given in Appendix. ■

Remarks 2.2

1. In this case the filtered adaptation error $C_r(q^{-1})\epsilon(k_j + d)$, $j = 1, 2, \ldots, m$ is a disturbed tracking error due to the influences of the measurement output. It deviates from the noise-free tracking error in a bounded signal.

2. All the considerations in section 2.2 about the reduction of parameters in the adaptive controllers remain valid in this design.

The philosophy of this scheme is shown in Fig. 2. ■

3. ADAPTIVE SAMPLING FOR MULTIRATE ADAPTIVE CONTROL

Discrete industrial systems usually result from the discretization of continuous processes and the use of discrete controllers. The advantages which derive from the use of discrete controllers, such as their compatibility with current computer technology or their possibilities for achieving specifications which are prohibitive for continuous controllers, make their use of great practical interest. In this context, the sampling period may be considered as an external parameter for the continuous plant which can be used as an additional design tool. This consideration was taken into account in adaptive control for improving the transient response by adapting the sampling period to the tracking or regulation error derivatives[3c]. The plant was first modelled by means of a time-varying difference equation[3b]. As a consequence of the use of this model, the input sequence was measured at the sampling points while the output sequence was measured at the so-called induced sampling points associated with each current modeling interval (i.e. the time interval which is the union of the last n_A sampling intervals). These (fictitious) sampling points are successively defined by means of multiple integers of the mean sampling period on each current modeling interval. The choice of the admissible domains for sampling adaptation must be made carefully since the discrete input–output delay (which derives from the continuous plant delay) can vary when variations of the sampling intervals occur around their nominal values.

The main variations of the schemes to be presented now with respect to those given in section 2 consist of the following.

1. The sampling periods of each controller suffer intentional local variations around their nominal values. Such variations are obtained by adaptation of the sampling intervals to the variations of the tracking/regulation errors. The objective which is pursued with this design is to improve the adaptation transients by means of the use of an adaptive sampling controller for each plant-adaptive controller.

2. The discretized plant is described by a time-varying model. This arises from the use of nonperiodic sampling.

3.1 *Continuous and discrete models*

The continuous reference model is assumed to be explicit,† stable and given by the input–output description

$$\mathcal{A}^M(p)y^M(t) = \mathcal{B}^M(p)e^M(t - \sigma_M), \qquad \sigma_M \geq 0; \qquad \mathcal{A}^M_{(\cdot)}, \mathcal{B}^M_{(\cdot)}: \text{polynomials} \tag{15}$$

and the continuous plant is given by

$$\mathcal{A}(p)y(t) = \mathcal{B}(p)e(t - \sigma), \qquad \sigma \geq 0; \qquad \mathcal{A}(\cdot), \mathcal{B}: \text{polynomials}, \tag{16}$$

where $\{y^M(\cdot)\}$, $\{y(\cdot)\}$, $\{e^M(\cdot)\}$ and $\{e(\cdot)\}$ are the output and input sequence, σ_M and σ are the constant delays and p is the time-derivative operator (i.e. $p \triangleq \mathrm{d}/\mathrm{d}t$). It is assumed that $\mathcal{B}(s)$ has all its zeroes in Re $(s) < 0$ and that $\{e^M(t)\}$ is a bounded sequence. Since (15) is stable, these assumptions are directly connected with Assumptions 4 and 5 in section 2. In addition, the following hypothesis is made.

Assumption 7. Models (15) and (16) correspond to strictly proper transfer functions. ■

Remark 3.1

Assumption 7 implies $d \geq 1$. Otherwise d would be zero. Thus, the same hypotheses as in (10) are held for the adaptive sampling case. ■

Discretizing (16) by a set of sampler and zero-order holds, each one being related to a controller within a multirate parallel distribution, one obtains the finite time-varying difference equation[3b] related to the lowest sampling period:

$$A[q'^{-1}(k_1)]y(t_{n_A+k_1}) = q^{-d(k_1)}B[q^{-1}(k_1)]e(t_{n_A+k_1}), \qquad \text{all } k_1 \geq 0, \tag{17}$$

where the $t_{(\cdot)}$ are the corresponding sampling points. Equation (17) is subjected to the initial conditions

$$\begin{aligned} &y(t_0),\ y(t_1),\ \ldots,\ y(t_{n_A-1}),\\ &e(t_0),\ e(t_1),\ \ldots,\ e(t_{n_A-1}), \end{aligned} \tag{18}$$

where $q^{-1}(\cdot)$ and $q'^{-1}(\cdot)$ are backward time shifts which act, respectively, on the real and induced sampling instants. The sampling points are obtained from the sampling law Ω (see Table 1 for particular sampling criteria), while the induced ones are supplied by the "induced"

†This assumption is unnecessary and the reference model could be also defined by an arbitrary bounded function defined for all time (implicit model) or by a discrete model of constant or time-varying parameters.

Table 1. Adaptive sampling control laws obtained in [7]

No.	Sampling control law T_i	Approximate sampling control laws e_i $\left(\frac{1}{T_{i-1}}[e_i - e_{i-1}];\ T_i = \right)$	Sampling law parameters
1	$\frac{T_{max}}{C\dot{e}_i^2 + 1}$	$\frac{T_{max} - T_{i-1}}{C[e_i - e_{i-1}]^2 + 1}$	$C = \frac{2}{3AB^2}$ $B = 1/T_{max}$
2	$\frac{C}{\lvert\dot{e}_i\rvert}$	$\frac{CT_{i-1}}{\lvert e_i - e_{i-1}\rvert}$	$C = (AB)^{1/2}$
3	$\frac{C}{\lvert\dot{e}_i\rvert^{2/3}}$	$\frac{CT_{i-1}^{2/3}}{\lvert e_i - e_{i-1}\rvert^{2/3}}$	$C = \frac{(3AB)^{1/2}}{2}$
4	$T_{max} - C\lvert\dot{e}_i\rvert$	$T_{max} - \frac{C\lvert e_i - e_{i-1}\rvert}{T_{i-1}}$	$C = 1/2AB^2$ $B = 1/T_{max}$
5	$\frac{T_{max}}{C\lvert\dot{e}_i\rvert + 1}$	$\frac{T_{max}T_{i-1}}{C\lvert e_i - e_{i-1}\rvert + T_{i-1}}$	$C = 1/AB^2$ $B = 1/T_{max}$
6	$\frac{C}{\lvert\dot{e}_i\rvert}$	$\frac{CT_{i-1}}{\lvert e_i - e_{i-1}\rvert}$	$C = AB$
7	$\frac{C}{\lvert\dot{e}_i\rvert^{1/2}}$	$\frac{C(T_{i-1})^{1/2}}{\lvert e_i - e_{i-1}\rvert^{1/2}}$	$C = (2/3AB)^{1/2}$

sampling criterion $\Omega'(\Omega)$, which is dependent on the current modeling interval $[k_1, k_1 + n_A]$, and defined by $\Omega'(\Omega) = t'_{k_1+1}(l) + l\bar{T}_{n_A+k_1,k_1}$ for all integers $k_1 \geq 0$, $l \in [0, n_A - 1]$ with

$$\bar{T}_{n_A+k_1} = \frac{1}{n_A} \sum_{j=k_1}^{k_1+n_A-1} (T_j)$$

being the mean sampling interval of $[t_{k_1}, t_{n_A+k_1}]$. The discrete time delay is

$$d(k) = \begin{cases} 1 \text{ if } \sigma = 0 \text{ (if the plant is assumed to be strictly proper),} \\ \text{min integer} \left\{ z \geq 2, \sum_{i=2}^{z} T_{n_A+k_1-i} \geq \sigma \right\} \text{ if } \sigma > 0. \end{cases} \tag{19}$$

3.2 *Adaptive sampling laws*

For design purposes, one establishes an admissibility domain $D \triangleq [T_1^*, -\Delta\bar{T}_1, T_1^* + \Delta\bar{T}_1]$ around the lowest nominal sampling period T_1^*. D must satisfy the following requirements.

1. It must be placed within the stability domain.
2. $\Delta\bar{T}_1$ must be sufficiently small to maintain $d(k)$ constant. Otherwise, the number of parameters to be adapted suffer real-time variations.
3. In addition, $\Delta\bar{T}_1$ must be as small as necessary in order that the controlled plant does not suffer great variations in their parameters due to the sampling process. Also, it is interesting that it must be sufficiently significant to achieve the pursued purposes for improving the transient behaviour. A "trade off" between these two situations must be found by using *a priori* knowledge about the system.

In order to maintain the delay constant and each induced sampling instant between the preceding and the following related sampling points, the following result is useful.

THEOREM 3.1

Assume that $T_1(k_1) \in D$ and that d is the delay corresponding to the lowest nominal sampling period T_1^*. Then the following holds.

(i) A necessary condition for the discrete delay to be constant is that

$$0 < \Delta \bar{T}_1 \leq \frac{T_1^*}{d - 1}. \tag{20}$$

(ii) If there exist known lower and higher positive real bounds $\sigma_{\min} > (d - 1)T_1^*$, $\sigma_{\max} < dT_1^*$ for the continuous-time delay σ, then a sufficient condition for the discrete delay to be constant is that

$$0 < \Delta \bar{T}_1 \leq \min\left(\frac{\sigma_{\min}}{d - 1} - T_1^*, T_1^* - \frac{\sigma_{\max}}{d}\right). \tag{21}$$

(iii) The induced sampling points are placed between the preceding and the following sampling points related to its associated sampling instants (i.e. $t_{k_1+i-1} \leq t'_{k_1+1}(k_1) \leq t_{k_1+i+1}$, all $i = 1, 2, \ldots, n_A - 1$) if

$$0 < \Delta \bar{T}_1 \leq \frac{T_1^*}{2n_A + 1}. \tag{22}$$

General criteria for adaptive sampling are shown in Table 1. ■

When multirate sampling has to be used, some modifications of the adaptive sampling laws must be introduced since one deals with several sampling controllers as a natural consequence of the problem at hand. Two approaches are given in the following.

Approach 1. The sampling periods of the multirate scheme are adapted individually but subsequently a (weighted or not) least-squares approximation is implemented to find a value of the nominal lowest sampling interval. Subsequently, this modified value is used together with the nominal factors λ_{ij}, $i, j = 1, 2, \ldots, m$ (which relate each sampling interval to the others) in order to determine the final sampling intervals of the overall multirate sampling scheme. Under the above considerations, the adaptation scheme results in

$$\begin{aligned}\Delta T_i(k_i + d) &= \lambda_{i1}\Delta T(k_{1i} + d) = \lambda_{ij}\Delta T_j(k_{ij} + d) \\ &= \lambda_{i1}\left(\frac{\sum_{j=1}^{m} \sigma_j \lambda_{j1} \Delta T'_j(k_{1j} + d)}{\sum_{j=1}^{m} \sigma_j \lambda_{j1}^2}\right), \qquad k_i, k_j > 0,\ i, j = 1, 2, \ldots, m,\end{aligned} \tag{23}$$

where $\Delta T_i(k_i + d)$ is the above modification module which is related to the values $\Delta T'_{(\cdot)}(\cdot)$ which are obtained from any of the adaptation sampling laws of Table 1 through the least-squares approximation scheme

$$J(t) = \sum_{j=1}^{m} \sigma_j(\Delta T'_j(k_{1j} + d)) - \lambda_{j1}\Delta T(k_{1j} + d))^2, \tag{24}$$

all t being equal to

$$\sum_{k_{1j}=0}^{p_j} T_j(k_{1j}), \qquad \text{all } j = 1, 2, \ldots, m,$$

and some integer $p_j > 0$ (i.e. a time interval which includes a sampling point of each adaptive sampling controller). The positive scalars σ_j, $j = 1, 2, \ldots, m$ are the weighting factors used in the least-squares approximation by each sampling controller.

Approach 2. The sampling rate of each adaptive controller is adapted independently of the nominal factors λ_{ij}, $i, j = 1, 2, \ldots, m$ which relate each nominal sampling interval to the remaining ones. In this way, the sampling laws of Table 1 are used to generate the set $T_1'(k_1), \ldots, T_m'(k_m)$ at each sampling point of any (or various) controllers. Subsequently, the lowest sampling period is projected into the boundary of D, if it lies outside, while computing a proportionality factor applicable to obtain all the remaining periods. The sampling adaptation process is summarized in the following algorithm.

Step 0. (Initialization). Make $l, k_1, k_2, \ldots, k_m \leftarrow 0$.

Step 1. At the current sampling point t_l, obtain the sampling periods $\hat{T}_l(k_j + d)$ of the set of controllers J_l which act at this instant.

Step 2. Construct the list SP of all the future sampling points as $\text{SP} \equiv \{\hat{t}_{l+1}, \hat{t}_{l+2}, \ldots, \hat{t}_{l+m}\}$.

Step 3. *If* the first controller acts at t_l, *then* compute the correction factor

$$\alpha(k_1) = \begin{cases} \dfrac{\hat{T}_1(k_1 + d)}{T_1^* - \Delta\bar{T}_1} & \text{if } \hat{T}_1(k_1 + d) < T_1^* - \Delta\bar{T}_1, \\[2ex] \dfrac{\hat{T}_1(k_1 + d)}{T_1^* + \Delta\bar{T}_1} & \text{if } \hat{T}_1(k_1 + d) > T_1^* + \Delta\bar{T}_1, \\[2ex] 1 & \text{if } \hat{T}_1(k_1) \in D; \end{cases} \tag{25}$$

else go to Step 4.

Step 4. Modify the list SP as

$$\hat{t}_l(k_j + 1) \leftarrow \frac{\hat{t}_l(k_j + 1)}{\alpha(k_l)}, \qquad j = 1, 2, \ldots, m. \tag{26}$$

Step 5. Order the list SP according to

$$\text{SP} = \{\hat{t}_{l+1}, \hat{t}_{l+2}, \ldots, \hat{t}_{l+m} | \hat{t}_{l+i} \leq \hat{t}_{l+i+1}, \qquad \text{all } i = 1, 2, \ldots, m - 1\}$$

with

$$\hat{t}_{l+i} = \left\{ \min_{1 \leq j \leq m} \left[\frac{\hat{t}_l(k_j)}{\alpha(k_1)} \,\middle|\, \frac{\hat{t}_l(k_j + 1)}{\alpha(k_1)} > t_{l+i+1}, \qquad t_l' = 0 \right] \right\}.$$

Step 6. Obtain the next sampling point at $t_{l+1} \leftarrow \hat{t}_{l+1}$. Make $l \leftarrow l + 1$, $k_j \leftarrow k_j + 1$ for all $j \in J_l$.

Step 7. End.

Remarks 3.2

1. Because of its structure, Approach 1 maintains the synchronism of the overall multirate scheme since there exist sampling points which correspond to all the adaptive sampling controllers. However, the sampling adaptation takes place only at the synchronism points. From a sampling adaptation point of view, this circumstance makes Approach 1 less efficient than the step-by-step adaptive Approach 2.

2. In Approach 2, the periodic synchronism of all the sampling controllers becomes lost because the factors λ_{ij}, $i, j = 1, 2, \ldots, m$ which relate each nominal sampling interval to the remaining ones are not used for the on-line obtention of the sampling intervals.

3. In the above algorithm for implementation of Approach 2, the initial sampling periods obtained in Step 1 are not necessarily subjected to constraints. At this design level, the most important question is to achieve adaptive sampling laws that have easy implementation. For this reason, Step 3 implements a projection of the current computed values of the sampling periods within the admissibility domain D. With the given organization, the algorithm is not either excessively computing-time consuming or demanding of great memory requirements.

3.3 *Multirate adaptive controllers with local adaptive sampling control laws*

The last step of the design pursued in this section is to combine the adaptive sampling design with the use of multirate adaptive controllers.

The polynomial factorization (5) now becomes

$$C_r[q'^{-1}(k_1)] = 1 + \sum_{i=1}^{n_{c_r}} c_i(k_1)q'^{-i}(k_1) = A[q'^{-1}(k_1)]S[q'^{-1}(k_1)] + q'^{-d}R[q'^{-1}(k_1)], \qquad k_1 = 0, 1, \ldots, \tag{27}$$

where the coefficients of the $C_r[q'^{-1}(k_1)]$ polynomial may be chosen as those of $C_r(q^{-1})$ with $S[q'^{-1}(k_1)]$ and $R[q'^{-1}(k_1)]$ being time-varying polynomials defined in the same way as those in section 2. Thus, in this case†

$$C_r[q'^{-1}(k_1 + d - n_A)]y(t_{k_1+d}) = S[q'^{-1}(k_1 + d - n_A)]B[q^{-1}(k_1 + d - n_A)] \times e(t_{k_1+d}) + R[q'(k_1 + d - n_A)]y(t_{k_1+d}) = \theta^T(k_1)\phi(k_1). \tag{28}$$

The (asymptotic) control and regulation objectives are expressed as

$$C_r[q'^{-1}(k_i + k - n_A)]y^M(k_i + d) = \hat{\theta}^T(k_i)\phi(k_i), \qquad k_i \geq 0, i = 1, 2, \ldots, m. \tag{29}$$

The time-varying parameter vector is explicited by

$$\theta(k_1) = [b_0(k_1), \theta_0^T(k_1)]^T = \Bigg[b_0(k_1), \sum_{i=0}^{n_B}\sum_{j=0}^{d-1} b_i(k_1)s_j(k_1) \times \{\delta(1 - z_{ij}(k_1 + d - n_A)), \ldots, \delta(d + n_B - 1 - z_{ij}(k_1 + d - n_A))\}, \times r_0(k_1), \ldots, r_{n_R}(k_1)\Bigg], \tag{30}$$

with

$$\delta(j) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_{ij}(k_1 + d - n_A) = \min_{0 \leq z \leq n_A} \text{integer}\left\{z, \sum_{l=1}^{z} T_1(k_1 + d - l) \geq \sum_{l=1}^{j} T_1(k_1 + d - l) - i\,\overline{T}_1(k_1 + d, k_1 + d - n_A)\right\}, \tag{31}$$

where this slightly modified notation, with respect to that of section 3.2, is used for $T_{(\cdot)}(\cdot)$ and $\overline{T}_{(\cdot)}(\cdot)$ to specify the controller which is referred to [the first one in (31)]. The adaptive controller parameter vectors $\hat{\theta}_i(k_i)$, the associate adaptation gain matrices $F_i(k_i)$ and the input to the plant are computed as in (11) and (12), and the measurement vector involves the use of induced sampling points for the output sequence. Thus, one has

$$\phi(k_i) = [e(k_i), \phi_0^T(k_i)]^T = [e(k_i), e(k_i - 1), \ldots, e(k_i - d - n_B + 1), \times y(k_i'), y(k_i' - 1), \ldots, y(k_i' - n_R)]^T, \tag{32}$$

with the superscript primes denoting the induced sampling points within each current modeling interval (i.e. $k_i' - l$ stands for $t'_{k_i}(n_A - 1)$), $l \in [0, n_R]$ with $t'_{k_i}(n_A) = t_{k_i}$, all integer $k_i \geq 0$, $i = 1, 2, \ldots, m$. In spite of the use of the induced sampling, the application of the results in Theorem 3.1(iii) help to maintain a weak computation effort. Since the last sampling point of each modeling interval and its related induced value are coincident, the control law (12) is physically implementable if the measurement vector (32) is used.

The multirate adaptive sampling adaptive control scheme is shown in Fig. 3.

The following stability result is useful.

†In previous papers, a more involved notation was used for polynomials, measurements and plant and adaptive controller parameters by forming an index to design each current modeling interval. Since no ambiguity results from deleting such an index, it is not used in this paper.

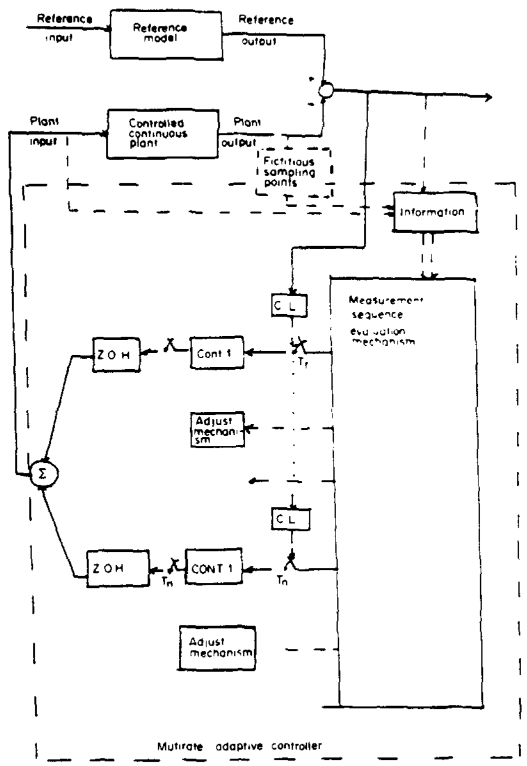


Fig. 3. Multirate adaptive control with adaptive sampling.

THEOREM 3.2

Under Assumptions 1–5 and Assumption 7, the time-varying discrete plant (17)–(18), subject to the adaptation scheme (11)–(12) and Theorem 3.1(ii), fulfills asymptotically the regulation and tracking objectives if $\lim_{k_i \to \infty} T_i(k_i) = T_i$, $i = 1, 2, \ldots, m$.

Proof is given in Appendix. ■

Remarks 3.3

1. It is evident that the asymptotic condition of constant sampling in Theorem 3.2 implies that the discrete plant is asymptotically parameterized by a constant parameter vector.

2. A natural extension of the use of a dead-beat zone by the adaptive controller when disturbances are present may be easily made by using the tools of section 2.3 and the extensions of (16) to the adaptive scheme (11)–(12). Stability is preserved under similar conditions to those given in section 2. (See [14] for the continuous standard case). ■

4. DEAD-BEAT ERROR BOUNDS FOR ADAPTATION

In sections 2 and 3, adaptive schemes which involve the use of dead-beat zones for adaptation have been proposed when additive output disturbances are present. In such a case, instead of the difference equation

$$A(q^{-1})y(k_1 + d) = B(q^{-1})e(k_1), \qquad \text{all } k_1 \geqslant 0 \tag{33}$$

for the discrete plant, one has

$$A(q^{-1})\hat{y}(k_1 + d) = B(q^{-1})\hat{e}(k_1) + v(k_1 + d), \qquad \text{all } k_1 \geqslant 0, \tag{34}$$

where $\hat{y}(\cdot)$ and $\hat{e}(\cdot)$ are measured variables which are influenced through the difference eqn (34) and the control law (12) by previous output disturbances $v(\cdot)$. From (34), the input fulfills

$$\hat{e}(k_1) = \frac{A(q^{-1})}{B(q^{-1})}\hat{y}(k_1 + d - 1) - \frac{1}{B(q^{-1})}v(k_1 + d), \qquad \text{all } k_1 \geqslant 0. \tag{35}$$

On the other hand, from (12), assuming that the true parameter vector θ is known, one has

$$\hat{e}(k_1) = \sum_{i=1}^{m} \frac{1}{\hat{b}_{0i}(k_i)} \left\{ C_r(q^{-1})y^M(k_i + d)\phi_0(k_i) - \sum_{\substack{j=1 \\ j\neq i}}^{m} b_{0j} \times [C_r(q^{-1})y^M(k_{ij} + d) - \theta_0^T\hat{\phi}_0(k_{ij})] \right\}, \quad \text{all } k_1 \geqslant 0, \quad (36)$$

where $\hat{\phi}_{(\cdot)}$ denotes the disturbed measurement vector.

Let $\hat{v}(\cdot)$ denote the accumulated output (and regulation/tracking error) disturbance due to the transmission of $v(\cdot)$ through $\hat{e}(\cdot)$ and $\hat{y}(\cdot)$. From (33) and (34) one has

$$A(q^{-1})\hat{v}(k_1 + d) = B(q^{-1})[\hat{e}(k_1) - e(k_1)] + v(k_1 + d), \quad \text{all } k_1 \geqslant 0. \quad (37)$$

Also, from (34) and (36), it is clear that

$$\begin{aligned} \hat{e}(k_1) - e(k_1) &= \sum_{i=1}^{m} \left\{ \sum_{j=1}^{m} \theta_0^T[\hat{\phi}_0(k_{ij}) - \phi_0(k_{ij})] - \frac{1}{b_0(k_i)} \theta_0^T \times [\hat{\phi}_0(k_i) - \phi_0(k_i)] \right\} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\sum_{l=1}^{d+n_B'-1} \theta^T[\hat{e}(k_{ij} - l) - e(k_{ij} - l)] + \sum_{l=d+n_B}^{d+n_B+n_R} \theta_{0j}^{T(l)} \right. \\ &\quad \times [\hat{y}(k_{ij} - l + d + n_B) - y(k_{ij} - l + d + n_B)], \quad \text{all } k_1 \geqslant 0, \end{aligned} \quad (38)$$

where the superscript ls denote the components of the parameter vector. Thus, (38) may be expressed as

$$\mathcal{A}(q^{-1})[\hat{e}(k_1) - e(k_1)] = \mathcal{B}(q^{-1})[\hat{y}(k_1) - y(k_1)], \quad \text{all } k_1 \geqslant 0, \quad (39)$$

where $\mathcal{A}(q^{-1})$ and $\mathcal{B}(q^{-1})$ are polynomials defined directly from (38). The q^{-1} delay operator is related to each output sample irrespective of the controller which is acting. From (39) into (37), one has as transfer function for the noise transmission

$$\hat{v}(k_1) = \frac{\mathcal{A}(q^{-1})}{\mathcal{A}(q^{-1})A(q^{-1}) - q^{-d}\mathcal{B}(q^{-1})B(q^{-1})} v(k_1), \quad \text{all } k_1 \geqslant 0. \quad (40)$$

For the parallel case of adaptive sampling treated in section 3, the q' (modeling interval dependent) operator must be introduced in the ARMA model associated with (40).

Now it is possible to determine the bound ϵ_0 for the adaptation algorithms with dead zone from the knowledge of a bound v_0 such that $|v(k_1)| < v_0$, all $k_1 \geqslant 0$, as follows. From (40), it follows immediately that

$$|\hat{v}(k_1)| \leqslant \frac{\overline{\mathcal{A}} v_0}{\overline{\mathcal{A}}\overline{A} - \underline{\mathcal{B}}\underline{B}}, \quad \text{all } k_1 \geqslant 0, \quad (41)$$

where $\overline{\mathcal{A}}$ and A are positive constants being the sum of known upper bounds for the absolute corresponding values of the $\mathcal{A}(q^{-1})$ and $A(q^{-1})$ polynomials and $\mathcal{B}$ and B are defined in the same way by considering lower bounds for the $\mathcal{B}(q^{-1})$ and $B(q^{-1})$ polynomials.

The point of view adopted for determining such bounds can be more or less pessimistic according to *a priori* knowledge on the plant parameters. That the tracking error can be ensured to be closer to the accumulated noise component and taking a less pessimistic point of view are adopted in determining (41). A very pessimistic computation of the bounds in (41) would translate into greater tracking errors but not in instability. The above development has been made for bounds of the true parameter vector. Since the adaptive algorithms in sections 2 and

3 are such that the adaptive controller parameter remains bounded, the above development is valid for sufficiently accurate absolute bounds of the time evolution of this vector.

5. PRACTICAL EXPERIMENTATION

5.1 *General considerations*

There are both advantages and drawbacks for both choices of the sampling intervals (namely, great or small intervals). Generally speaking, the following are well known in sampling theory.

1. Upper bounds for the allowable sampling period must be taken according to the desired stability. Lower bounds are related to the closed-loop system bandwidth (typically, from 2 to 20 times smaller than the bandwidth).
2. As the sampling period decreases, the system behaviour becomes very close to its equivalent continuous system. In general, the discretization effects greatly increase with the instability degree (and thus with the length of transient behaviour) if certain compensator types, such as a dead-beat controller, are not used. This phenomenon is not general but it is usual.
3. In the presence of unmodeled dynamics[17], the sampling rate must be slower as other considerations pertain in adaptive control problems. This makes the tracking error closer to that registered with a correct modeling. However, in the presence of higher-frequency disturbances no advantage is found by the use of this strategy. However, if the sampling rate grows, the disturbances can be filtered and the stability of the adaptive scheme improved.

Thus, in a discrete adaptive problem in which the sampling period can be manipulated, both advantages and disadvantages can arise from a choice of great or small values. Therefore, multirate sampling may be useful. Furthermore, in certain applications (see aircraft applications[5,16,19]) it is useful to design autopilots having an internal stabilator loop with fast sampling and an external control loop with slow sampling for the design of the closed-loop system bandwidth and gain scheduling for different flight conditions.

Now different simulations are presented for a second-order plant using the methodologies of sections 2 and 3. Two adaptive controllers at different sampling rates are involved. The results obtained are better, under the assumptions of the presence of unmodeled dynamics and output disturbances, than those obtained from the use of an adaptive controller only.

5.2 *Examples*

The plant which is assumed has the transfer function

$$W_p(s) = \frac{5}{s^2 + 6s + 8}\left(\frac{1}{s + 20}\right)$$

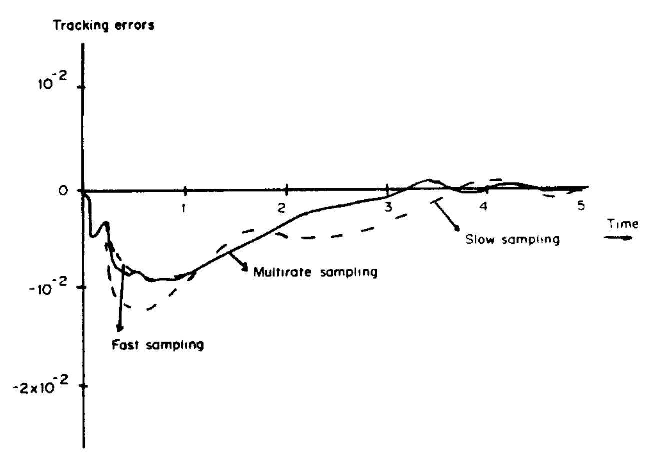


Fig. 4. Tracking errors for the example of Case A.

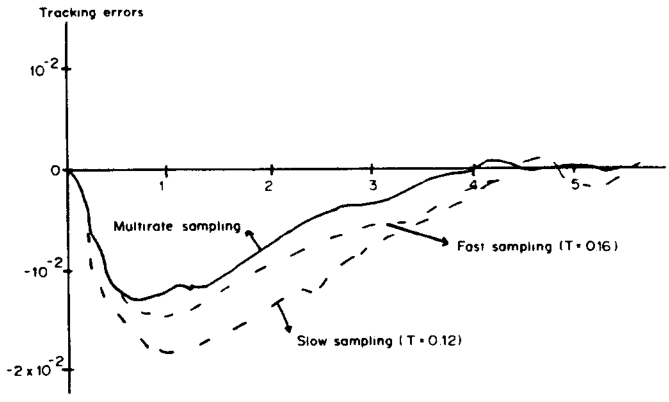


Fig. 5. Tracking errors for the example of Case B.

(the term in parentheses is unmodeled dynamics for the controller design). The reference model is given by

$$W_M(s) = \frac{5}{s^2 + 4s + 3}.$$

The reference input to the reference model is $r(t) = 10 \sin 0.17t$. The output disturbance $v_1(t) = 4 \sin t$, and C (constant of the adaptive sampling laws) is 0.11×10^{-6}. The observation interval for the experiments is [0,9]. The two multirate controllers have as nominal sampling periods $T_1^* = 0.13$ and $T_2^* = 0.26$. The allowed variations for the experiments related to the developments in section 3 are $T_{min}^{(1)} = 0.11$, $T_{max}^{(1)} = 0.15$, $T_{min}^{(2)} = 0.22$ and $T_{max}^{(2)} = 0.30$; all the parameters of the adaptive controllers are initialized to unity; $F(0) = \text{Diag}\,(10^4)$; $\lambda(t) = c(t) = 1$ in the adaptive algorithms. Using the technique developed in section 5 for determining an upper bound for the accumulated tracking/regulation error noise, it is found that $\epsilon_0 = 5$. This bound is computed by assuming that the upper bound 4 is known for the additive noise and that the values of the parameters of the plant vary around 30% of its nominal values. The filter $C_r(q^{-1})$ is taken as unity. For the adaptive sampling laws, their approximated versions in Table 1 are used. The disturbance is assumed to be zero for all $t \geq 3$.

Case A. (two multirate controllers for the plant without output disturbances but with unmodeled dynamics.) In Fig. 4 the tracking errors are shown for the two adaptive controllers working both separately and together. The combined multirate design is shown to be more efficient during the transient in terms of tracking error signal levels.

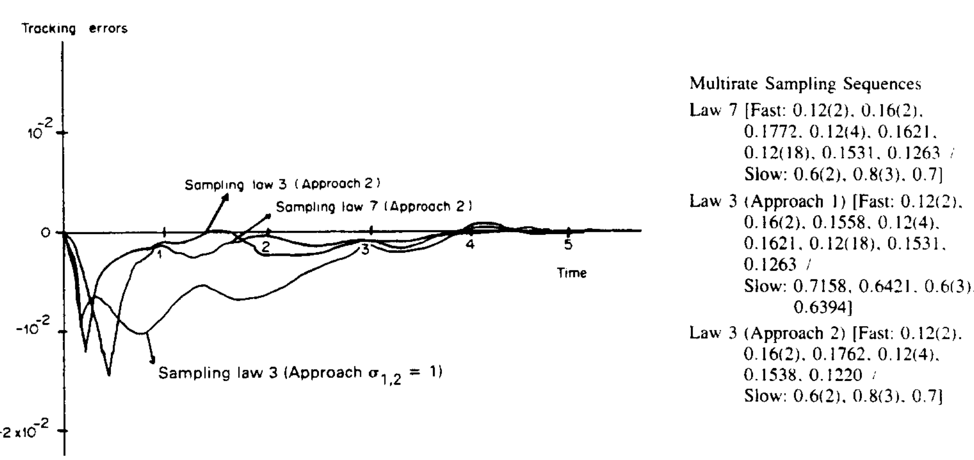
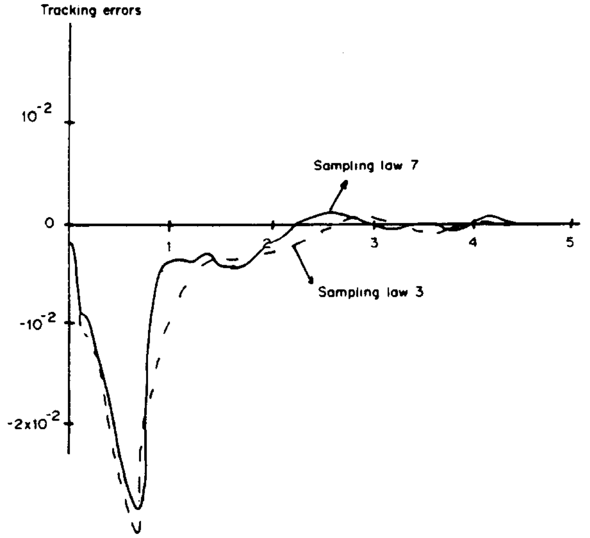


Fig. 6. Tracking errors for the example of Case A, adaptive sampling law 3, by using Approaches 1 and 2 in 3.2. Same as for the sampling law 7 by using Approach 2.



Multirate Sampling Sequence
Law 3 [Fast: 0.12(2), 0.14(2), 0.1751, 0.12(4), 0.1338, 0.1241, 0.16(18), Slow: 0.6(2), 0.8(4)]
Law 7 [Fast: 0.12(2), 0.14(2), 0.1772, 0.12(4), 0.1538, 0.16(20) / Slow: 0.6(2), 0.8(4)]

Fig. 7. Tracking errors for the example of Case B, adaptive sampling laws 3–7 with Approach 2.

Case B. (as above, with bounded output disturbances and unmodeled dynamics). The same conclusions as in Case A are obtained. The tracking errors are shown in Fig. 5. This could be expected from the known advantages of the dead-beat control in the presence of additive disturbances[14].

Case C. (adaptive sampling for Case A). The tracking errors are shown in Fig. 6 for, respectively, the adaptive sampling laws 3 and 7 of Table 1. Note that Approach 2 is more efficient than Approach 1 for a multirate adaptive design.

Case D. (adaptive sampling for Case B). The tracking errors are shown in Fig. 7 for the adaptive sampling laws 3 and 7 of Table 1. T_{min} and T_{max} are selected *a priori* according to the allowed variation for the sampling period around its nominal value.

6. CONCLUSIONS

In this paper multirate sampling design has been applied to a discrete adaptive control algorithm. The cases of a dead-beat adaptation zone for the case of the presence of additive disturbances has been considered. Also, an adaptive sampling approach has been used to improve the transient characteristics of the system behaviour.

Examples have shown that the proposed designs can improve those associated with a single-rate design at fast or slow sampling rates under the presence of bounded disturbances and unmodeled dynamics.

REFERENCES

1. J. Alster and P. R. Belanger, *Automatica* **13**, 627 (1974).
2. R. Bellman, *Adaptive Control: A Guided Tour.* Princeton, New York (1961).
3. M. De La Sen, a. *Proc. IEEE* **72**, 131 (1984).
   b. *Int. J. Systems Sci.* **15**, 315 (1984).
   c. *Int. J. Control* **40**, 639 (1984).
   d. *Proc. IEEE-D* **131**, 146 (1984).
   e. *Int. J. Control* **41**, 1189 (1985).
4. M. De La Sen and M. B. Paz, *Proc. IEEE* **72**, 986 (1984).
5. D. P. Glasson, *Control Systems Magazine* **3**, 2 (1983).
6. G. C. Goodwin, P. J. Ramadge and P. E. Caines, *IEEE. Trans. Autom. Control* **AC-25**, 449 (1980).
7. T. C. Hsia, *IEEE Trans. Autom. Control* **AC-19**, 39 (1974).
8. O. L. R. Jacobs and J. W. Patchell, *Int. J. Control* **16**, 189 (1972).
9. O. L. R. Jacobs and P. Saratchandran, *Automatica* **16**, 89 (1980).
10. V. Kučera, *Discrete Linear Control: The Polynomial Equation Approach.* Marcel Dekker, New York (1979).
11. J. M. Martin-Sanchez, *Proc. IEEE* **64**, 1209 (1976).
12. R. L. Morris and C. P. Neuman, *IEEE Trans. Autom. Control* **AC-26**, 534 (1981).

13. K. S. Narendra and I. H. Khalifa, Report No. 8206, Yale University (1982).
14. B. B. Peterson and K. S. Narendra, *IEEE Trans. Autom. Control* **AC-27** 1161 (1982).
15. J. W. Patchell and O. L. R. Jacobs, *Int. J. Control* **13**, 337 (1971).
16. K. S. Rattan, *Proc. of the 1981 Advanced Flight Control Symposium*, Colorado Springs, 1981.
17. C. E. Rohrs, *Ph.D. Dissertation*, Massachusetts Institute of Technology (1982).
18. A. J. Udink Ten Cate and H. B. Verbruggen, *Int. J. Control* 993 (1978).
19. R. F. Withbeck and G. Hoffmann, *AIAA J. Guidance and Control* **1**, 319 (1978).
20. L. G. Westphal, *IEEE Trans. Autom. Control* **AC-23**, 1075 (1978).

APPENDIX

Proof of Theorem 2.1. The two following intermediate results are needed for the proof.

LEMMA A.1

For the adaptive scheme (10)–(12), the following propositions hold for $i = 1, 2, \ldots, m$.

(i) $V_i(k_i) = \tilde{\theta}_i^T(k_i)F_i^{-1}(k_i+1)\tilde{\theta}_i(k_i) \leq V_i(0) < \infty$ if $V_i(0) < \infty$.
(ii) $\lim_{k_i\to\infty} V_i(k_i) = V_i < \infty$.
(iii) If $\lambda_i(k_i) = 1$ (or, alternatively, is chosen according to the achievement of a constant or bounded trace of the adaptation matrix) and all $k_i \geq k$ (some finite k), then $\|\hat{\theta}_i(k_i)\| < \infty$.
(iv) $\lim_{k_i\to\infty} \hat{\theta}_i(k_i) = \lim_{k_i\to\infty} \hat{\theta}_i(k_i - p) = \hat{\theta}_i$, some finite integer p, where $\tilde{\theta}_i(k_i) = \theta - \hat{\theta}_i(k_i)$, $i = 1, 2, \ldots, m$.

Proof. From standard results it follows that $V_i(k_i)$ is a nonincreasing positive function. This implies propositions (i), (ii). Also, from (10)–(12), one has

$$\lim_{k_i\to\infty}(V_i(k_i+1) - V_i(k_i)) = \lim_{k_i\to\infty}\frac{(\tilde{\theta}_i^T(k_i)\phi(k_i-d))^2}{c_i(k_i) + \phi^T(k_i-d)F_i(k_i)\phi(k_i-d)} = 0 \tag{A.1}$$

Thus, either $\lim_{k_i\to\infty} \tilde{\theta}_i^T(k_i)\phi(k_i-d) = 0$ or

$$\lim_{k_i\to\infty}\frac{|\tilde{\theta}_i^T(k_i-d)|}{c_i(k_i) + \phi^T(k_i-d)F_i(k_i)\phi(k_i-d)} = 0. \tag{A.2}$$

If the first possibility holds, the regulation and tracking objectives (Theorem 2.1) are accomplished. The first possibility implies directly (A.2). Equation (A.2) together with (10) and (11) implies (iv). Proposition (iii) follows from the relations

$$\lambda_{\min}(F_i^{-1}(k_i+1))\|\tilde{\theta}_i(k_i)\|_F^2 \leq V_i(k_i) \leq V_i,$$

and

$$\lambda_{\min}(F_i^{-1}(k_i+d)) \geq \lambda_{\min}(F_i^{-1}(k_i)) > 0$$

if $k_i < \infty$ ($\lambda_{\min}(\cdot)$ and $\|(\cdot)\|_F$ denote, respectively, the minimum eigenvalue of a matrix and the euclidean norm). Q.E.D.

The second intermediate result is a direct extension of a key result stated in Goodwin, Ramadge and Caines[6] for minimum-phase systems.

LEMMA A.2

Under Assumptions A.1–A.5, the measurement vectors $\phi(k_i)$, $0 \leq k_i \leq \infty$, $i = 1, 2, \ldots, m$ in the adaptive scheme (10)–(12) do not grow more than linearly with the associate tracking and regulation errors.

Proof. Note from (12) that†

$$C_i(q^{-1})y^{(j)}(k_i+d) = \hat{\theta}_i'^T(k_i)\phi'(k_i), \qquad k_i \geq 0, j = 1, 2, \ldots, m. \tag{A.3}$$

From the control objectives (7) and the model (10), one has

$$C_i(q^{-1})y(k_1+d) = \theta'^T\phi'(k_1), \qquad \text{all integer } k_1 \geq 0. \tag{A.4}$$

From (A.3) and (A.4) one deduces that

$$C_i(q^{-1})\epsilon(k_1+d) = \tilde{\theta}_i'^T(k_1)\phi'(k_1), \qquad \text{all integer } k_1 \geq 0. \tag{A.5}$$

Using well-known results in Goodwin, Ramadge and Caines[6], one has

$$\|\phi(k_1)\| \leq C_1 + C_2 \max_{0\leq t\leq k_1}(\|\theta'^T\phi'(t)\|), \qquad \text{some constants } 0 \leq C_1 < \infty,\ 0 \leq C_2 < \infty \text{ (linear boundedness condition)} \tag{A.6}$$

†The following equations always stand for vectors $\theta' \triangleq [\theta^T, (n-1) \text{ zeros}]^T$ and $\hat{\theta}_i'(k_i) \triangleq [\hat{\theta}_i^T(k_i), \hat{b}_{0i}(k_{i}), \ldots, \hat{b}_{0mi}(k_i)]^T$ together with their associate measurement vectors [see eqns (12)].

Since, from Lemma A.1, $\|\hat{\theta}_j'(k_j)\|$ is bounded, there exist constants $0 \le C_{ij}' = C_1 + C_2 \max_{0 \le t \le k_j} (|\hat{\theta}_j'^T(t)\phi'(t)|) = C_1 + C_2 \max_{0 \le t \le k_j} (|C_r(q^{-1})y^M(t+d)|) < \infty$, $j = 1, 2, \ldots, m$, such that

$$\|\phi(k_i)\| \le C_1' + C_2 \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\tilde{\theta}_j'^T(t)\phi_j'(t)\|)], \tag{A.7}$$

where $C_1' = \max_{1 \le j \le m} C_{1j}''$ for all $t = k_{ij}T_j$, $j = 1, 2, \ldots, m$ $(k_{ii} = k_i)$ (i.e. t takes the values of all the possible sampling points). Q.E.D.

Proof of Theorem 2.1 follows directly by applying Lemma A.2 to Lemma A.1. (In the case of using some controllers of lower dimensionality, (A.7) remains valid with the remaining components of $\bar{\theta}(\cdot)$ being equal to those of θ.) ∎

Proof of Theorem 2.2. Let z^+ the set of positive integers including zero. One defines the sets

$$U^+ \triangleq \{t \in z^+, |\epsilon(t)| > \epsilon_0; U' \triangleq \{t \in z^+, |\epsilon(t)| \le \epsilon_0\}. \tag{A.8}$$

The adaptation scheme (14) implies that the scalar function $V(\cdot)$, defined as in Lemma A.1, verifies that

$$\begin{aligned} V_i(k_i+1) &= V_i(k_i) \qquad \text{if } k_i \in U^-, \\ V_i(k_i+1) - V_i(k_i) &\le 0 \qquad \text{if } k_i \in U^+ \end{aligned} \tag{A.9}$$

for all integers $0 \le k_i \le \infty$, $i = 1, 2, \ldots, m$. Let $\angle(\cdot)$ be the Lebesgue measure of the set $(\cdot)$. If $\angle(U^-) = \infty$ or $\angle(U^+) = \infty$ (or both are infinite, namely the disturbed adaptation error enters alternately from the adaptation/no adaptation zone to the other), then Lemma A.1 applies *mutatis mutandis*.

From Assumption 6, the disturbed measurement vector is bounded. Thus, instead of (A.7), one has

$$\begin{aligned} \|\phi'(k_i)\| &\le \|\phi'^{\text{fn}}(k_i)\| + \|\tilde{\phi}'(k_i)\| \\ &\le C_1' + C_2 \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\tilde{\theta}_j'^T(t)\phi_j'^{\text{fn}}(t)\|)] \\ &\le C_1'' + C_2' \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\tilde{\theta}_j'^T(t)\phi_j'(t)\|)], \end{aligned} \tag{A.10}$$

where $\tilde{\phi}'(\cdot)$ denotes the noise component of $\phi'(\cdot)$ and superscripts fn denote its noise-free component, with the positive bounded constant C_1'' being defined by

$$C_1'' = C_1' + C_2' \max_{1 \le j \le m} [\max_{0 \le t \le k_j} (\|\tilde{\theta}_j'^T(t)\tilde{\phi}_j'(t)\|)] + \|\tilde{\phi}_i'(k_i)\|]. \tag{A.11}$$

Thus Lemma A.2 is also fulfilled and $\lim_{t \to \infty} \tilde{\theta}'^T(t)\phi'(t) = 0$ if $\angle(U^-) < \infty$ and $\angle(U^+) = \infty$. In all the cases, all the signals within the system are bounded. Q.E.D.

Proof of Theorem 3.2. Since $T_1(k_1)$ tends to a finite limit T_1 as k_1 tends to infinity, there exists a non-zero limit parameter vector $\theta = \lim_{k_1 \to \infty} \theta_1(k_1)$. By continuity of $\theta(k_1)$ around $\bar{T}_1(k_1) = [T_1(k_1), T_1(k_1 - 1), \ldots, T_1(k_1 - n_1 + 1)]^T$, a Lipschitz condition type

$$\|\theta(T_{k_i}) - \theta\| \le C_4\|\bar{T}_{k_i} - \bar{T}_i\|, \qquad \text{some } 0 < C_4 < \infty,\ k_i \ge 0,\ i = 1, 2, \ldots, m \tag{A.12}$$

is fulfilled since $T_1 \in D$ and $T_1(k_1) \in D$, all $k_1 \ge 0$. From (A.12), one has

$$\|\theta(t)\| \le \|\theta\| + C_4\|\bar{T}_{k_1} - \bar{T}_1\| \le \|\theta\| + C_5 = C_6\|\theta\|, \qquad \text{all integral } k_1 \ge 0 \tag{A.13}$$

for some bounded constants C_5 and C_6. The linear boundedness condition[6] is also fulfilled by time-varying systems whose inverses are stable.† Thus (A.6), together with (A.12) and (B.1) and (B.2), result in

$$\begin{aligned} \|\phi(k_i)\| &\le C_1 + C_2 \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\theta'^T(t)\phi'(t)\|)] \\ &\le C_1 + C_2 \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\hat{\theta}_j'^T(t)\phi'(t)\|)] \\ &\quad + C_2C_6 \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\theta'^T\phi'(t)\|)] \\ &\le C_1' + C_2' \max_{1 \le j \le m} [\max_{0 \le t \le k_{ij}} (\|\hat{\theta}_j'^T(t)\phi'(t)\|)], \qquad i = 1, 2, \ldots, m, \end{aligned} \tag{A.14}$$

which is of the same type as (A.7), with bounded constants

$$\begin{aligned} C_1' &= C_1 + C_2(1 + C_6) \max_{1 \le j \le m} [\max_{0 \le t \le \infty} (\|\hat{\theta}_j'^T(t)\phi'(t)\|)], \\ C_2' &= C_2C_6. \quad \text{Q.E.D.} \end{aligned} \tag{A.15}$$

†This arises since such a property is associated with state-space models rather than with transfer function models.