Discrete Mathematics 311 (2011) 1803–1812

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Counting lattice chains and Delannoy paths in higher dimensions

John S. Caughman [a,*], Charles L. Dunn [b], Nancy Ann Neudauer [c], Colin L. Starr [d]

[a] Fariborz Maseeh Department of Mathematics & Statistics, Portland State University, Box 751, Portland, OR 97202, United States
[b] Department of Mathematics, Linfield College, 900 SE Baker Street, McMinnville, OR 97128, United States
[c] Department of Mathematics & Computer Science, Pacific University, Forest Grove, OR 97116, United States
[d] Department of Mathematics, Willamette University, Salem, OR 97301, United States

ARTICLE INFO

Article history:
Received 26 March 2009
Received in revised form 13 April 2011
Accepted 14 April 2011
Available online 6 May 2011

Keywords:
Lattice chain
Delannoy number
Generating function

ABSTRACT

Lattice chains and Delannoy paths represent two different ways to progress through a lattice. We use elementary combinatorial arguments to derive new expressions for the number of chains and the number of Delannoy paths in a lattice of arbitrary finite dimension. Specifically, fix nonnegative integers $n_1, \ldots, n_d$, and let L denote the lattice of points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ that satisfy $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. We prove that the number of chains in L is given by

$$2^{n_d+1} \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d+k-1}{n_d} \prod_{j=1}^{d-1} \binom{n_j+i-1}{n_j},$$

where $k'_{\max} = n_1 + \cdots + n_{d-1} + 1$. We also show that the number of Delannoy paths in L equals

$$\sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d+k-1}{n_d} \prod_{j=1}^{d-1} \binom{n_d+i-1}{n_j}.$$

Setting $n_i = n$ (for all i) in these expressions yields a new proof of a recent result of Duchi and Sulanke [9] relating the total number of chains to the central Delannoy numbers. We also give a novel derivation of the generating functions for these numbers in arbitrary dimension.

1. Introduction

Lattice chains and Delannoy paths are natural combinatorial objects with a rich history, and they have enjoyed a surge of interest in recent decades. Popular expositions of combinatorics, such as those by Comtet [8] and Stanley [16], promote their study and provide several tools for their analysis. Interesting connections have appeared between chains and other topics, including probability theory, ballot numbers, Legendre polynomials, simplicial complexes, and formal languages, to name only a few [3,4,6,11,15]. The interested reader can find more on these topics in the survey by Banderier and Schwer [3], which has over 75 bibliographic references.

The study of chains and paths often reveals an interplay between counting arguments and generating functions. In Stanley [16], a problem involving lattice chains and Delannoy paths in two dimensions is used to illustrate a technique for extracting the diagonal of a generating function. Specifically, the problem is to show that, in an $n \times n$ lattice, the number

* Corresponding author.
E-mail address: caughman@pdx.edu (J.S. Caughman).

0012-365X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.04.024

of chains exceeds the number of Delannoy paths by a factor of 2^{n+1}. Stanley proves this using generating functions, and then poses the question of finding a combinatorial proof of the same result.

This challenge was met by Sulanke [17], who found a bijective correspondence, and later by Caughman et al. [5], who offered an inclusion–exclusion argument. More recently, Duchi and Sulanke [9] generalized Sulanke's result from the central case in two dimensions to the central case in arbitrary dimension, again by means of explicit bijections. Since then, Tărnăuceanu has derived an expression for the central Delannoy numbers in arbitrary dimension [18]. In this article, we generalize all the above results by considering the general (not necessarily central) case in arbitrary dimension. In this general setting, we use counting techniques to derive new formulas, both for chains and for Delannoy paths. The resulting expressions that count these two objects are strikingly similar in form, and upon an appropriate substitution they yield an alternate proof of Duchi and Sulanke's theorem.

We also consider the generating functions for lattice chains and Delannoy paths. It is perhaps not surprising that the coefficients for these satisfy similar recurrence relations. We introduce a concept that describes the recursive structure they have in common, and we exploit this to provide a novel method for deriving any such generating function easily and uniformly for arbitrary dimension. In the special case of chains and Delannoy paths, the generating functions had previously been computed by MacMahon using more specific ad hoc techniques [13, pp. 156–9].

Our results are organized as follows. In Section 2, we fix the notation and describe our results on lattice chains, including k-chains, reducible chains, and the total number of chains. In Section 3, we consider Delannoy paths, first finding the number of k-step paths, and then the total number of paths. Evaluating the expressions for chains and Delannoy paths in the central case, we obtain as a corollary the result of Duchi and Sulanke [9]. In Section 4, we introduce the class of $\mathfrak{a}$-recurrent sequences of functions – a class that includes both the chain numbers and the Delannoy numbers as special cases – and we offer an explicit expression for their generating functions in any dimension.

2. Results on lattice chains

We begin by defining the lattice $L(\mathbf{n})$. Let $\mathbb{N}$ denote the nonnegative integers and $\mathbb{P}$ the positive integers. Fix $d \in \mathbb{P}$ and $\mathbf{n} \in \mathbb{N}^d$, where $\mathbf{n} = (n_1, \ldots, n_d)^T$. Let $L(\mathbf{n})$ denote the lattice of integer points $(a_1, \ldots, a_d)^T \in \mathbb{N}^d$ satisfying $a_i \le n_i$ for $1 \le i \le d$.

Recall that $L(\mathbf{n})$ is partially ordered by the dominance relation, defined as follows. Given $\mathbf{a}, \mathbf{b} \in L(\mathbf{n})$ with $\mathbf{a} = (a_1, \ldots, a_d)^T$ and $\mathbf{b} = (b_1, \ldots, b_d)^T$, we say that $\mathbf{a} \preceq \mathbf{b}$ whenever $a_i \le b_i$ for each $i (1 \le i \le d)$. We write $\mathbf{a} \prec \mathbf{b}$ whenever $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \ne \mathbf{b}$.

Define the *weight* of an element $\mathbf{a} = (a_1, \ldots, a_d)^T \in L(\mathbf{n})$ by $\mathrm{wt}(\mathbf{a}) = a_1 + \cdots + a_d$. We define the *truncation* of $\mathbf{a}$ to be the $(d-1)$-tuple $\mathbf{a}' = (a_1, \ldots, a_{d-1})^T$.

2.1. Counting k-chains and some variations

By a *chain* in $L(\mathbf{n})$, we mean a subset of $L(\mathbf{n})$ that is totally ordered by $\preceq$. A *k-chain* is a chain with k elements. Let $C(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$, and, for each integer k, let $C_k(\mathbf{n})$ denote the set of k-chains in $L(\mathbf{n})$. In this section, we review expressions for $|C_k(\mathbf{n})|$ and $|C(\mathbf{n})|$, and introduce a useful variation of $C_k(\mathbf{n})$.

Expressions for $|C_k(\mathbf{n})|$ have been computed in several places for the special case of a hypercube, where $n_i = 1$ for all i [10,14], and the general case was solved by Chou [7]. Each of these derivations proceeds either by solving an appropriate recurrence or by employing generating functions. More recently, a direct counting argument was given for $|C_k(\mathbf{n})|$ in the general case using the principle of inclusion/exclusion by Caughman et al. [5].

A few basic results on k-chains are summarized in the following lemma.

Lemma 1. *Fix $\mathbf{n} \in \mathbb{N}^d$, where $\mathbf{n} = (n_1, \ldots, n_d)^T$, and, for each $k \in \mathbb{N}$, let $C_k(\mathbf{n})$ denote the set of k-chains in the corresponding lattice $L(\mathbf{n})$, and $\widetilde{C}_k(\mathbf{n})$ the set of chains in $C_k(\mathbf{n})$ that contain the maximum element $\mathbf{n}$. Then the following hold.*

(i) *The maximum length of a chain in $L(\mathbf{n})$ is given by*

$$k_{\max} = \mathrm{wt}(\mathbf{n}) + 1.$$

(ii) [5, *Thm.* 1] *For any integer k $(1 \le k \le k_{\max})$, the number of k-chains in $L(\mathbf{n})$ is given by*

$$|C_k(\mathbf{n})| = \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

(iii) *For any integer k $(1 \le k \le k_{\max})$, the number of k-chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}_k(\mathbf{n})| = \sum_{i=1}^{k} (-1)^{i+1} |C_{k-i}(\mathbf{n})|.$$

Proof. (iii) Note that $|C_0(\mathbf{n})| = 1$ and $|\widetilde{C}_0(\mathbf{n})| = 0$. For $k \ge 1$, each k-chain containing $\mathbf{n}$ corresponds to a unique $(k-1)$-chain that does not contain $\mathbf{n}$ (and conversely). So $|\widetilde{C}_k(\mathbf{n})| = |C_{k-1}(\mathbf{n}) \setminus \widetilde{C}_{k-1}(\mathbf{n})| = |C_{k-1}(\mathbf{n})| - |\widetilde{C}_{k-1}(\mathbf{n})|$. The result now follows by a simple induction. □

2.2. Counting reducible chains

We say that a chain ξ is *reducible* if the truncations of its elements are distinct. Equivalently, a k-chain ξ in $L(\mathbf{n})$ is reducible iff the set ξ', formed by truncating the elements of ξ, remains a k-chain in $L(\mathbf{n}')$. For example, let $\mathbf{n} = (2, 4, 4)^T$, and suppose that ξ_1 and ξ_2 are the following 3-chains:

$$\xi_1 : \begin{pmatrix}0\\3\\1\end{pmatrix} \prec \begin{pmatrix}0\\3\\2\end{pmatrix} \prec \begin{pmatrix}1\\3\\3\end{pmatrix} \quad \text{and} \quad \xi_2 : \begin{pmatrix}0\\2\\1\end{pmatrix} \prec \begin{pmatrix}1\\3\\1\end{pmatrix} \prec \begin{pmatrix}2\\3\\2\end{pmatrix}.$$

Then ξ_1 is *not* reducible, since the first two elements have identical truncations. On the other hand, ξ_2 *is* reducible. Note that ξ_1' has only two distinct elements, while ξ_2' remains a 3-chain:

$$\xi_1' : \begin{pmatrix}0\\3\end{pmatrix} \prec \begin{pmatrix}1\\3\end{pmatrix} \quad \text{and} \quad \xi_2' : \begin{pmatrix}0\\2\end{pmatrix} \prec \begin{pmatrix}1\\3\end{pmatrix} \prec \begin{pmatrix}2\\3\end{pmatrix}.$$

The next result is the analogue of Lemma 1 for reducible chains.

Lemma 2. *With the notation of Lemma 1, let $C^{\text{red}}(\mathbf{n})$ denote the set of reducible chains in $L(\mathbf{n})$, and let $\widetilde{C}^{\text{red}}(\mathbf{n})$ denote the set of reducible chains that contain $\mathbf{n}$. Then the following hold.*

(i) *The maximum length of a reducible chain in $L(\mathbf{n})$ is given by*

$$k'_{\max} = \text{wt}(\mathbf{n}') + 1.$$

(ii) *The number of reducible chains in $L(\mathbf{n})$ is given by*

$$|C^{\text{red}}(\mathbf{n})| = \sum_{k=0}^{k'_{\max}} \binom{n_d + k}{n_d} |C_k(\mathbf{n}')|.$$

(iii) *The number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}^{\text{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} |\widetilde{C}_k(\mathbf{n}')|.$$

Proof. (i) By truncation, every reducible k-chain ξ in $L(\mathbf{n})$ corresponds to a unique k-chain ξ' in $L(\mathbf{n}')$. Therefore, $k \le k'_{\max}$ by Lemma 1(i). Conversely, since any chain in $L(\mathbf{n}')$ extends to a reducible chain in $L(\mathbf{n})$, there are reducible chains in $L(\mathbf{n})$ of length $k'_{\max}$.

(ii) Fix an integer $k (1 \le k \le k'_{\max})$ and let ξ be any reducible k-chain. Truncation gives a unique k-chain ξ' in $L(\mathbf{n}')$, and the dth coordinates of the elements in ξ form a non-decreasing sequence of integers between 0 and n_d (inclusive). Conversely, such a sequence and a k-chain in $L(\mathbf{n}')$ correspond to a unique reducible chain in $L(\mathbf{n})$. The number of such sequences is $\binom{n_d+k}{n_d}$. Multiplying by $|C_k(\mathbf{n}')|$ and summing over k, we obtain the result.

(iii) As in (ii) above, each reducible chain ξ in $\widetilde{C}_k(\mathbf{n})$ corresponds to a unique ξ' in $\widetilde{C}_k(\mathbf{n}')$ and a non-decreasing sequence of integers between 0 and n_d (inclusive), which contains n_d at least once. The number of such sequences is $\binom{n_d+k-1}{n_d}$. Multiplying by $|\widetilde{C}_k(\mathbf{n}')|$ and summing over k, we obtain the result. □

Combining the preceding lemmas, we have the following.

Theorem 3. *With the notation of Lemma 2, the number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}^{\text{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+1} \binom{n_d + k - 1}{n_d} |C_{k-i}(\mathbf{n}')|.$$

Proof. Immediate, by Lemmas 1(iii) and 2(iii). □

Using Lemma 1(ii), we can evaluate $|C_{k-i}(\mathbf{n}')|$ in Theorem 3 to obtain a triple sum. As the next result shows, however, this reduces to a double sum.

Theorem 4. *With the notation of Lemma 2, the number of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$ is given by*

$$|\widetilde{C}^{\text{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d + k - 1}{n_d} \prod_{j=1}^{d-1} \binom{n_j + i - 1}{n_j}.$$

Proof. Consider the expression for $|\widetilde{C}^{\mathrm{red}}(\mathbf{n})|$ given in Theorem 3. Recall that $|C_0(\mathbf{n})| = 1$, and for $i < k$ we can evaluate $|C_{k-i}(\mathbf{n})|$ using Lemma 1(ii) to obtain

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} \left[(-1)^{k+1} + \sum_{i=1}^{k-1} \sum_{r=0}^{k-i-1} (-1)^{r+i+1} \binom{k-i-1}{r} \prod_{j=1}^{d-1} \binom{n_j + k - i - r}{n_j} \right].$$

With the change of variables $r = k - i - t$, this simplifies to

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} (-1)^{k+1} \left[1 + \sum_{i=1}^{k-1} \sum_{t=1}^{k-i} (-1)^t \binom{k-i-1}{t-1} \prod_{j=1}^{d-1} \binom{n_j + t}{n_j} \right].$$

Interchanging the order of summation over i and t, this is equivalent to

$$|\widetilde{C}^{\mathrm{red}}(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d + k - 1}{n_d} (-1)^{k+1} \left[1 + \sum_{t=1}^{k-1} (-1)^t \prod_{j=1}^{d-1} \binom{n_j + t}{n_j} \sum_{i=1}^{k-t} \binom{k-i-1}{t-1} \right].$$

A common binomial identity [1, Thm. 1.8] states that $\sum_{i=1}^{k-t} \binom{k-i-1}{t-1} = \binom{k-1}{t}$. Applying this identity and then substituting $t = i - 1$, the bracketed expression simplifies to give the desired result. $\square$

Remark. The authors would like to thank the anonymous referees for several useful suggestions concerning the above presentation. Indeed, Theorem 4 admits a number of interesting proofs using inclusion/exclusion, and we have chosen the derivation that seems the clearest. The interested reader is invited to experiment on his/her own.

2.3. The total number of chains

Keeping the notation of Lemma 2, we let $\widetilde{C}(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$ that contain $\mathbf{n}$. It is convenient to count $|\widetilde{C}(\mathbf{n})|$ rather than $|C(\mathbf{n})|$ directly. The difference is minimal, however, since removing $\mathbf{n}$ from each chain in $\widetilde{C}(\mathbf{n})$ gives a bijection between $\widetilde{C}(\mathbf{n})$ and $C(\mathbf{n}) \setminus \widetilde{C}(\mathbf{n})$, so that

$$|C(\mathbf{n})| = 2 \cdot |\widetilde{C}(\mathbf{n})|. \tag{1}$$

Let $\mathcal{P}$ denote the power set of $\{0, 1, \ldots, n_d - 1\}$, and recall that $\widetilde{C}^{\mathrm{red}}(\mathbf{n})$ denotes the set of reducible chains in $L(\mathbf{n})$ that contain $\mathbf{n}$. We now establish a bijection ϕ between $\widetilde{C}(\mathbf{n})$ and $\mathcal{P} \times \widetilde{C}^{\mathrm{red}}(\mathbf{n})$.

Roughly speaking, ϕ can be described as follows. A chain ξ that contains $\mathbf{n}$ fails to be reducible if the truncations of its elements are not distinct. The function ϕ removes from ξ any elements whose truncations are repeated by a later element in ξ. Doing so produces a reducible chain ξ^{red}. The dth coordinates of the removed elements are recorded in a set A_ξ. The output of ϕ is the pair $(A_\xi, \xi^{\mathrm{red}})$.

More formally, we have the following.

Definition 5. Suppose that a chain ξ in $\widetilde{C}(\mathbf{n})$ has k elements $\mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})^T$ for each $i (1 \le i \le k)$. We define

$$A_\xi = \{a_{id} \mid \mathbf{a}'_i = \mathbf{a}'_{i+1}\}, \quad \text{and} \quad \xi^{\mathrm{red}} = \xi \setminus \{\mathbf{a}_i \mid \mathbf{a}'_i = \mathbf{a}'_{i+1}\},$$

and we let $\phi(\xi)$ denote the pair $(A_\xi, \xi^{\mathrm{red}})$. $\square$

To illustrate this definition, let $\mathbf{n} = (3, 3, 3)^T$ and suppose that ξ denotes the following 8-chain in $\widetilde{C}(\mathbf{n})$:

$$\begin{array}{l} \mathbf{a}_1 \prec \mathbf{a}_2 \prec \mathbf{a}_3 \prec \mathbf{a}_4 \prec \mathbf{a}_5 \prec \mathbf{a}_6 \prec \mathbf{a}_7 \prec \mathbf{a}_8 \\ \xi : \begin{pmatrix}1\\1\\0\end{pmatrix} \prec \begin{pmatrix}2\\1\\\mathbf{0}\end{pmatrix} \prec \begin{pmatrix}2\\1\\\mathbf{1}\end{pmatrix} \prec \begin{pmatrix}2\\1\\2\end{pmatrix} \prec \begin{pmatrix}2\\2\\2\end{pmatrix} \prec \begin{pmatrix}3\\2\\\mathbf{2}\end{pmatrix} \prec \begin{pmatrix}3\\2\\3\end{pmatrix} \prec \begin{pmatrix}3\\3\\3\end{pmatrix}. \end{array}$$

Notice that $\mathbf{a}'_2 = \mathbf{a}'_3 = \mathbf{a}'_4$ and $\mathbf{a}'_6 = \mathbf{a}'_7$. The reducible chain ξ^{red} is formed by removing $\mathbf{a}_2$ and $\mathbf{a}_3$ (keeping $\mathbf{a}_4$), and removing $\mathbf{a}_6$ (keeping $\mathbf{a}_7$). For each of the elements removed, their last coordinates (third coordinates in this case) are recorded in the set A_ξ. Then ξ^{red} is a reducible 5-chain in $\widetilde{C}^{\mathrm{red}}(\mathbf{n})$, the set A_ξ is a subset of $\{0, 1, 2\}$, and $\phi(\xi)$ denotes the pair $(A_\xi, \xi^{\mathrm{red}})$ below:

$$A_\xi = \{0, 1, 2\} \quad \text{and} \quad \xi^{\mathrm{red}} : \begin{pmatrix}1\\1\\0\end{pmatrix} \prec \begin{pmatrix}2\\1\\2\end{pmatrix} \prec \begin{pmatrix}2\\2\\2\end{pmatrix} \prec \begin{pmatrix}3\\2\\3\end{pmatrix} \prec \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Observe that $\phi(\xi) \in \mathcal{P} \times \widetilde{C}^{\mathrm{red}}(\mathbf{n})$.

Next, we describe how the original chain ξ can be recovered from the pair $(A_\xi, \xi^{\text{red}})$. Given the information above, we simply must reinsert into ξ^{red} the missing elements, one belonging to each member of A_ξ. Each x in A_ξ is the dth coordinate x_d of an element $\mathbf{x}$ that is to be inserted immediately to the left of the first $\mathbf{y}$ in ξ^{red} for which $x_d < y_d$. In our case, 0 and 1 belong left of $(2, 1, 2)^T$, while 2 belongs left of $(3, 2, 3)^T$:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \prec (\mathbf{0}) \prec (\mathbf{1}) \prec \begin{pmatrix}2\\1\\2\end{pmatrix} \prec \begin{pmatrix}2\\2\\2\end{pmatrix} \prec (\mathbf{2}) \prec \begin{pmatrix}3\\2\\3\end{pmatrix} \prec \begin{pmatrix}3\\3\\3\end{pmatrix}.$$

Observe that, for each x in A_ξ, such a $\mathbf{y}$ is guaranteed to exist in ξ^{red} by the fact that every element of A_ξ is strictly less than n_d, while $\mathbf{n}$ belongs to ξ^{red}. Indeed, this motivates our choice to work with $\widetilde{C}(\mathbf{n})$ rather than $C(\mathbf{n})$. To complete the recovery of ξ, note that the remainder of each new element $\mathbf{x}$ is determined by the condition that $\mathbf{x}' = \mathbf{y}'$. In our case, 0 and 1 are topped by $(2, 1)^T$, while 2 is topped by $(3, 2)^T$. Doing so produces the original chain ξ.

The casual reader may skip to Theorem 9 without loss of continuity, as the next three lemmas merely verify that ϕ is well defined, injective, and surjective.

Lemma 6. *With the above notation, ϕ is a function from $\widetilde{C}(\mathbf{n})$ to $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.*

Proof. For $\xi \in \widetilde{C}(\mathbf{n})$, recall that $\phi(\xi) = (A_\xi, \xi^{\text{red}})$. If $x \in A_\xi$, then $x = a_{id}$ for some i, where $\mathbf{a}'_i = \mathbf{a}'_{i+1}$. But $\mathbf{a}_i \prec \mathbf{a}_{i+1}$, so $a_{id} < a_{i+1,d}$. Thus every element of A_ξ is strictly less than n_d, and $A_\xi \in \mathcal{P}$. To show that $\xi^{\text{red}} \in \widetilde{C}^{\text{red}}(\mathbf{n})$, note that $\xi^{\text{red}} \subseteq \xi$, so ξ^{red} is totally ordered by $\prec$. And $\mathbf{n} \in \xi$ since $\xi \in \widetilde{C}(\mathbf{n})$, while $\mathbf{n} \notin \{\mathbf{a}_i | \mathbf{a}'_i = \mathbf{a}'_{i+1}\}$ so $\mathbf{n} \in \xi^{\text{red}}$. It remains to show that ξ^{red} is reducible. Suppose that there were $\mathbf{x} \prec \mathbf{y}$ in ξ^{red} such that $\mathbf{x}' = \mathbf{y}'$. Then $\mathbf{x} = \mathbf{a}_i$ and $\mathbf{y} = \mathbf{a}_j$ for some $i < j$. But $\mathbf{a}'_i \preceq \mathbf{a}'_{i+1} \preceq \mathbf{a}'_j$, so $\mathbf{a}'_i = \mathbf{a}'_{i+1}$, and thus $\mathbf{a}_i \notin \xi^{\text{red}}$, a contradiction. It follows that $\xi^{\text{red}} \in \widetilde{C}^{\text{red}}(\mathbf{n})$. □

Lemma 7. *With the above notation, ϕ is injective.*

Proof. Let ξ_1, ξ_2 be in $\widetilde{C}(\mathbf{n})$ and suppose that $\phi(\xi_1) = \phi(\xi_2)$. Then $\xi_1^{\text{red}} = \xi_2^{\text{red}}$, and, to prove that $\xi_1 = \xi_2$, it remains to show that $\xi_1 \setminus \xi_1^{\text{red}} = \xi_2 \setminus \xi_2^{\text{red}}$. We accomplish this by proving that, for any chain ξ in $\widetilde{C}(\mathbf{n})$, each element $\mathbf{x} \in \xi \setminus \xi^{\text{red}}$ corresponds to a unique element $x_d \in A_\xi$, and that, in fact, $\mathbf{x}$ can be explicitly constructed from the element $x_d \in A_\xi$ and the chain ξ^{red}. Performing this construction for each element of A_ξ then yields the entire set $\xi \setminus \xi^{\text{red}}$. To describe the construction, let $\mathbf{x}$ be any element of $\xi \setminus \xi^{\text{red}}$, and let k denote the length of ξ. Then $\mathbf{x} = \mathbf{a}_i$ for some i, where $\mathbf{a}'_i = \mathbf{a}'_{i+1}$ and $x_d = a_{id} \in A_\xi$. Let $t = \max\{j | , \mathbf{a}'_i = \mathbf{a}'_j\}$. Then $t \geq i + 1$ and $\mathbf{a}'_i = \mathbf{a}'_{i+1} = \cdots = \mathbf{a}'_t$. Also, either $t < k$ and $\mathbf{a}'_t \neq \mathbf{a}'_{t+1}$ or else $t = k$ and $\mathbf{a}_t = \mathbf{n}$. In either case, $\mathbf{a}_t \in \xi^{\text{red}}$, and $\mathbf{a}_i, \ldots, \mathbf{a}_{t-1} \notin \xi^{\text{red}}$. So $\mathbf{a}_t = \min\{\mathbf{y} | \mathbf{y} \in \xi^{\text{red}} \text{ and } \mathbf{x} \prec \mathbf{y}\}$. Observe that, since $\mathbf{x} \prec \mathbf{a}_t$ and $\mathbf{x}' = \mathbf{a}'_t$, it must be the case that $x_d < a_{td}$. It follows that $\mathbf{a}_t = \min\{\mathbf{y} | \mathbf{y} \in \xi^{\text{red}} \text{and} x_d < y_d\}$. Since $\mathbf{x}' = \mathbf{a}'_t$ and has dth coordinate x_d, we have now shown that $\mathbf{x}$ is completely determined by the element x_d in A_ξ and the chain ξ^{red}. It follows that ξ is determined by the pair $(A_\xi, \xi^{\text{red}})$, so ϕ is injective. □

Lemma 8. *With the above notation, ϕ is surjective.*

Proof. To see that ϕ is surjective, we associate a chain in $\widetilde{C}(\mathbf{n})$ with any given pair (A, ζ) in $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$. Suppose that ζ has t elements $\mathbf{b}_1 \prec \cdots \prec \mathbf{b}_t$, where $\mathbf{b}_i = (b_{i1}, \ldots, b_{id})^T$ for each $i (1 \leq i \leq t)$. For each x in A, define $m := \min\{j | x < b_{jd}\}$ and set $\mathbf{b}_x = (b_{m1}, \ldots, b_{m(d-1)}, x)^T$ in $L(\mathbf{n})$. In other words, we define $\mathbf{b}_x$ by putting $\mathbf{b}'_x := \mathbf{b}'_m$ and setting the dth coordinate of $\mathbf{b}_x$ equal to x. Then the chain in $\widetilde{C}(\mathbf{n})$ that we associate with the pair (A, ζ) is simply $\xi_{(A,\zeta)} := \zeta \cup \{\mathbf{b}_x | x \in A\}$. It is easy to check that $\phi(\xi_{(A,\zeta)}) = (A, \zeta)$, as desired. □

The preceding lemmas yield the following.

Theorem 9. *With the above notation, the map ϕ is a bijection between $\widetilde{C}(\mathbf{n})$ and $\mathcal{P} \times \widetilde{C}^{\text{red}}(\mathbf{n})$.*

Combining Theorems 4 and 9, we obtain the following corollary.

Corollary 10. *Fix $\mathbf{n} \in \mathbb{N}^d$ and let $C(\mathbf{n})$ denote the set of chains in $L(\mathbf{n})$. Then*

$$|C(\mathbf{n})| = 2^{n_d+1} \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d + k - 1}{n_d} \prod_{j=1}^{d-1} \binom{n_j + i - 1}{n_j}.$$

Proof. By (1) and Theorem 9, $|C(\mathbf{n})| = 2 \cdot |\widetilde{C}(\mathbf{n})| = 2 \cdot |\mathcal{P}| \cdot |\widetilde{C}^{\text{red}}(\mathbf{n})|$. But $|\mathcal{P}| = 2^{n_d}$, so the result follows by Theorem 4. □

3. Results on Delannoy numbers

The set $D = D(\mathbf{n})$ of (generalized) Delannoy paths contains precisely those chains in $L(\mathbf{n})$ that contain both the origin $\mathbf{0} = (0, \ldots, 0)^T$ and $\mathbf{n} = (n_1, \ldots, n_d)^T$, and whose successive elements differ by at most one in each coordinate. In other words, the elements of $D(\mathbf{n})$ correspond to walks from $\mathbf{0}$ to $\mathbf{n}$ in which only positive steps from the d-dimensional unit hypercube are allowed. This follows Kaparthi and Rao [12]. The cardinalities $|D(\mathbf{n})|$ are referred to as (generalized) Delannoy numbers. For more about generalizations of the Delannoy numbers, we refer the reader to the literature [2,12].

When all the n_i share a common value n, we have $\mathbf{n} = (n, \ldots, n)^T$, and we refer to the cardinalities $|D(\mathbf{n})|$ as the (d-dimensional) *central* Delannoy numbers. Below, we use inclusion/exclusion to find an expression for the general Delannoy numbers which specializes to a useful expression for the central case in Theorem 13.

3.1. *Delannoy paths with k steps*

In contrast with lattice chains, it is common to refer to the size of a Delannoy path by the number of steps it contains, rather than the number of elements it has. In other words, if a chain ξ in $D(\mathbf{n})$ has elements

$$\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k,$$

then we say ξ has k steps. (Notice that ξ has $k+1$ elements, and hence length $k+1$ as a chain.) The set of all k-step Delannoy paths is denoted by $D_k(\mathbf{n})$.

Due to symmetry, the ordering of the dimensions in L is often irrelevant, so we frequently assume that $n_1 \le n_2 \le \cdots \le n_d$. Under this assumption, it is easy to show that the minimum number of steps a Delannoy path can have is n_d, while the maximum is $n_1 + \cdots + n_d$, which equals $k_{\max} - 1$.

Finally, we remark that, since each k-step Delannoy path begins and ends with the points $\mathbf{a}_0 = \mathbf{0}$ and $\mathbf{a}_k = \mathbf{n}$, we can uniquely represent a path $\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$ by the sequence $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$, where each $\mathbf{b}_i = \mathbf{a}_i - \mathbf{a}_{i-1}$. In this representation, the $\mathbf{b}_i$ are nonzero d-tuples of zeros and ones. This observation is the key to the following result.

Theorem 11. *Fix* $\mathbf{n} \in \mathbb{N}^d$ *such that* $n_1 \le n_2 \le \cdots \le n_d$. *Let* $k_{\max} = n_1 + \cdots + n_d + 1$. *Then, for each* $k (n_d \le k \le k_{\max} - 1)$, *the number of* k*-step Delannoy paths in the lattice* $L(\mathbf{n})$ *is given by*

$$|D_k(\mathbf{n})| = \binom{k}{n_d} \sum_{i=0}^{k-n_d} (-1)^i \binom{k-n_d}{i} \prod_{j=1}^{d-1} \binom{k-i}{n_j}.$$

Proof. Each k-step Delannoy path $\mathbf{a}_0 \prec \mathbf{a}_1 \prec \cdots \prec \mathbf{a}_k$ corresponds to a unique sequence $\mathcal{B} = \langle \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \rangle$, where each $\mathbf{b}_i = \mathbf{a}_i - \mathbf{a}_{i-1}$. Each $\mathbf{b}_i$ is a nonzero d-tuple $(b_{i1}, b_{i2}, \ldots, b_{id})^T$ of zeros and ones. By the definition of a Delannoy path, projection of $\mathcal{B}$ onto the j-coordinate must give a sequence $\mathcal{B}_j = \langle b_{1j}, b_{2j}, \ldots, b_{kj} \rangle$ of zeros and ones that contains precisely n_j ones, for each $j (1 \le j \le d)$.

To count the number of such sequences $\mathcal{B}$, we first choose the sequence $\mathcal{B}_d = \langle b_{1d}, b_{2d}, \ldots, b_{kd} \rangle$ of n_d ones and $k - n_d$ zeros. There are $\binom{k}{n_d}$ choices for $\mathcal{B}_d$. Next, we choose sequences $\mathcal{B}_j$ for $1 \le j \le d-1$ so that each $\mathcal{B}_j$ has exactly n_j ones, and so that, when they are combined, the resulting sequence $\mathcal{B}$ has no zero terms. This amounts to ensuring that, for each i where $b_{id} = 0$, there is a j for which $b_{ij} = 1$. This is achieved by inclusion/exclusion, as follows.

Let $\mathcal{Z} = \{i | b_{id} = 0\}$, and, for each $\mathcal{T} \subseteq \mathcal{Z}$, let $s(\mathcal{T})$ be the number of sequences $\mathcal{S} = \langle \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k \rangle$, such that (i)–(iv) hold below.

(i) Each $\mathbf{s}_i (1 \le i \le k)$ is a d-tuple $(s_{i1}, s_{i2}, \ldots, s_{id})^T$ of zeros and ones.
(ii) The d-projection of $\mathcal{S}$ satisfies $\mathcal{S}_d = \mathcal{B}_d$.
(iii) Each j-projection $\mathcal{S}_j = \langle s_{1j}, s_{2j}, \ldots, s_{kj} \rangle$ has precisely n_j ones.
(iv) For each $t \in \mathcal{T}$, the term $\mathbf{s}_t = \mathbf{0}$.

To count $s(\mathcal{T})$, note that (ii) fixes the d-projection of $\mathcal{S}$. To satisfy (iii), we independently form $d-1$ sequences $\mathcal{S}_j$, each with the specified number n_j of ones, and subject to the constraint (iv) that $s_{tj} = 0$ for $t \in \mathcal{T}$. For each j, there are $\binom{k-t}{n_j}$ choices for $\mathcal{S}_j$, where $t = |\mathcal{T}|$. Taken together, we find that $s(\mathcal{T}) = \prod_{j=1}^{d-1} \binom{k-t}{n_j}$.

Now, since $|\mathcal{T}|$ can range from 0 to $k - n_d$, the number of sequences $\mathcal{B}$ that have the specified d-projection $\mathcal{B}_d$ and contain no zero terms is, by inclusion/exclusion,

$$\sum_{\mathcal{T} \subseteq \mathcal{Z}} (-1)^{|\mathcal{T}|} s(\mathcal{T}) = \sum_{t=0}^{k-n_d} (-1)^t \binom{k-n_d}{t} \prod_{j=1}^{d-1} \binom{k-t}{n_j}.$$

Recalling that the number of choices for $\mathcal{B}_d$ was $\binom{k}{n_d}$, we obtain the result. □

3.2. The total number of Delannoy paths

Now that the number of k-step Delannoy paths has been given in Theorem 11, it is a simple matter to find the total number of Delannoy paths.

Theorem 12. *Fix $\mathbf{n} \in \mathbb{N}^d$ such that $n_1 \le n_2 \le \cdots \le n_d$ and let $k'_{\max} = n_1 + \cdots + n_{d-1} + 1$. Then the total number of Delannoy paths in the lattice $L(\mathbf{n})$ is given by*

$$|D(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \binom{n_d+k-1}{n_d} \prod_{j=1}^{d-1} \binom{n_d+i-1}{n_j}.$$

Proof. To find $|D(\mathbf{n})|$, we sum the expression for $|D_k(\mathbf{n})|$ from Theorem 11 over all k from n_d to $k_{\max} - 1$ to obtain

$$|D(\mathbf{n})| = \sum_{k=n_d}^{k_{\max}-1} \binom{k}{n_d} \sum_{i=0}^{k-n_d} (-1)^i \binom{k-n_d}{i} \prod_{j=1}^{d-1} \binom{k-i}{n_j}.$$

Reindexing the outer sum, this simplifies to

$$|D(\mathbf{n})| = \sum_{k=1}^{k'_{\max}} \binom{n_d+k-1}{n_d} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=1}^{d-1} \binom{n_d-1+k-i}{n_j}.$$

Replacing i by $k-i$ reverses the order of the inner sum, and simplification gives the desired result. □

The expressions for the total number of chains from Corollary 10 and the total number of Delannoy paths from Theorem 12 are strikingly similar in form. Specializing to the central case, we have the following.

Theorem 13 ([9]). *Fix $\mathbf{n} \in \mathbb{N}^d$ such that $n_i = n$ for all $i (1 \le i \le d)$. Then*

$$C(\mathbf{n}) = 2^{n+1} D(\mathbf{n}).$$

Proof. Immediate from Corollary 10 and Theorem 12. □

4. Generating functions

Another approach to the study of lattice chains and Delannoy paths is to investigate their generating functions. These functions have been separately computed by MacMahon using techniques that will not be reproduced here, but that can found in his well-known text [13, pp. 156–9]. However, noticing that the coefficients for these sequences satisfy similar recurrence relations, we introduce a concept that describes the recursive structure they share. We exploit this concept to derive an explicit formula for any such generating function easily and uniformly in arbitrary dimension.

4.1. The class of $\mathfrak{a}$-recurrent sequences

The chain numbers in various dimensions satisfy similar recurrence relations. Consider the following.

- When $d = 1$ and $i \in \mathbb{P}$, the number of chains $c_i = |C(i)|$ satisfies

  $$c_i = 2c_{i-1},$$

  where $c_0 = 2$.
- When $d = 2$ and $i, j \in \mathbb{P}$, the number of chains $c_{i,j} = |C(i,j)|$ satisfies

  $$c_{i,j} = 2(c_{i-1,j} + c_{i,j-1}) - 2c_{i-1,j-1},$$

  where $c_{i,0} = c_{0,i} = c_i$, and where $c_{0,0} = c_0$.
- When $d = 3$ and $h, i, j \in \mathbb{P}$, the number of chains $c_{h,i,j} = |C(h,i,j)|$ satisfies

  $$c_{h,i,j} = 2(c_{h-1,i,j} + c_{h,i-1,j} + c_{h,i,j-1}) - 2(c_{h,i-1,j-1} + c_{h-1,i,j-1} + c_{h-1,i-1,j}) + 2c_{h-1,i-1,j-1},$$

  where $c_{i,j,0} = c_{i,0,j} = c_{0,i,j} = c_{i,j}$, and $c_{i,0,0} = c_{0,i,0} = c_{0,0,i} = c_i$, and $c_{0,0,0} = c_0$.

Notice the increasing depth of the recurrence as d increases, and the pattern of alternating 2s as coefficients in these recurrences. Notice also how, in the boundary cases, when one or more of the parameters is zero, the values are determined by the lower-dimensional numbers.

Generalizing to higher dimensions requires some additional notation. For any $d \in \mathbb{P}$, define $\mathcal{B}_d := \{0, 1\}^d \setminus \{\mathbf{0}\}$. Also, let $[d] := \{1, 2, \ldots, d\}$ and recall that the *support* of $\mathbf{n} \in \mathbb{N}^d$ is

$$\mathrm{supp}(\mathbf{n}) := \{j \in [d] \mid n_j \neq 0\}.$$

Observe that for $\mathbf{v} \in \mathcal{B}_d$ we have that $\mathrm{wt}(\mathbf{v}) = |\mathrm{supp}(\mathbf{v})|$, which is nonzero by the definition of $\mathcal{B}_d$.

Suppose that $T \subsetneq [d]$, where $|T| = m$. For $\mathbf{n} \in \mathbb{N}^d$, we define $\mathbf{n}_T$ to be the $(d-m)$-tuple obtained by removing the i-components of $\mathbf{n}$ for each $i \in T$. Note that when $T = \emptyset$ we have $\mathbf{n}_T = \mathbf{n}$.

Using the above notation, we can now describe the recursive structure of the chain numbers. For any dimension d, and for any $\mathbf{n} \in \mathbb{P}^d$, we have

$$|C(\mathbf{n})| = \sum_{\mathbf{v} \in \mathcal{B}_d} 2(-1)^{\mathrm{wt}(\mathbf{v})+1} |C(\mathbf{n} - \mathbf{v})|.$$

Considering the boundary, for nonzero $\mathbf{n} \in \mathbb{N}^d \setminus \mathbb{P}^d$, we have

$$C(\mathbf{n}) = C(\mathbf{n}_T),$$

where $T := [d] \setminus \mathrm{supp}(\mathbf{n})$. And for a base case, we have $|C(\mathbf{0})| = 2$.

We make the following definition.

Definition 14. Let $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$ be a sequence of real numbers. A sequence of functions $\mathcal{F} = \langle F_1, F_2, F_3, \ldots \rangle$, where $F_d : \mathbb{N}^d \to \mathbb{R}$ for all $d \in \mathbb{P}$, is $\mathfrak{a}$-*recurrent* if the following hold.

(i) For all $d \in \mathbb{P}$ and $\mathbf{n} \in \mathbb{P}^d$,

$$F_d(\mathbf{n}) = \sum_{\mathbf{v} \in \mathcal{B}_k} a_{\mathrm{wt}(\mathbf{v})} F_d(\mathbf{n} - \mathbf{v}).$$

(ii) For all $d \in \mathbb{P}$ and nonzero $\mathbf{n} \in \mathbb{N}^d \setminus \mathbb{P}^d$,

$$F_d(\mathbf{n}) = F_{d-|T|}(\mathbf{n}_T),$$

where $T := [d] \setminus \mathrm{supp}(\mathbf{n})$.

(iii) For all $d \in \mathbb{P}$,

$$F_d(\mathbf{0}) = a_0.$$

It is not difficult to see that, for any given sequence $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$ of real numbers, there is a unique sequence of functions which is $\mathfrak{a}$-recurrent.

To illustrate this definition, let $F_d(\mathbf{n}) = |C(\mathbf{n})|$, where $\mathbf{n} \in \mathbb{N}^d$. Then the sequence of chain numbers is $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 2, 2, -2, 2, -2, 2, \ldots \rangle$, where the signs alternate after the first two entries.

Similarly, if we let $F_d(\mathbf{n}) = |D(\mathbf{n})|$, where $\mathbf{n} \in \mathbb{N}^d$, then the sequence of Delannoy numbers is seen to be $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 1, 1, 1, \ldots \rangle$.

4.2. *Generating functions for* $\mathfrak{a}$*-recurrent sequences*

In this final section, we establish a theorem that offers an explicit expression for the generating functions of any $\mathfrak{a}$-recurrent sequence of functions. This theorem generalizes the expressions for the generating functions of chains and Delannoy paths given by MacMahon [13, pp. 156–9].

Theorem 15. *Let* $\mathcal{F} = \langle F_1, F_2, F_3, \ldots \rangle$ *be an* $\mathfrak{a}$*-recurrent sequence of functions with* $\mathfrak{a} = \langle a_0, a_1, a_2, \ldots \rangle$. *For* $d \in \mathbb{P}$, *let*

$$g_d(\mathbf{x}) = g_d(x_1, x_2, \ldots, x_d) = \sum_{(n_1, \ldots, n_d)} F_d(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$$

be the generating function for F_d. *Then*

$$g_d(\mathbf{x}) = a_0 \left(1 - \sum_{\emptyset \neq S \subseteq [d]} a_{|S|} \prod_{i \in S} x_i \right)^{-1}.$$

Proof. We proceed by induction on d. If $d = 1$, then $F_1(n) = a_1 F_1(n-1)$ for $n \in \mathbb{P}$ by Definition 14(i) and $F_1(0) = a_0$ by Definition 14(iii). So

$$\sum_{n=1}^{\infty} F_1(n) x^n = \sum_{n=1}^{\infty} a_1 F_1(n-1) x^n.$$

Thus $g_1(x) - a_0 = x a_1 g_1(x)$, giving that $g_1(x) = a_0(1 - a_1 x)^{-1}$, as desired.

Now, fix an integer $d \geq 2$ and suppose that the statement holds for all $j \in [d-1]$. Using the fact that our sequence of functions is $\mathfrak{a}$-recurrent, we have that

$$\sum_{\mathbf{n}\in\mathbb{P}^d} F_d(\mathbf{n}) x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} = \sum_{\mathbf{n}\in\mathbb{P}^d} \sum_{\mathbf{v}\in\mathcal{B}_d} a_{\mathrm{wt}(\mathbf{v})} F_d(\mathbf{n}-\mathbf{v}) x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}. \tag{2}$$

Consider the left-hand side of (2). By inclusion–exclusion and the definition of $\mathfrak{a}$-recurrent, we have

$$\sum_{\mathbf{n}\in\mathbb{P}^d} F_d(\mathbf{n}) x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} = (-1)^d a_0 + \sum_{S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S). \tag{3}$$

Now, consider the right-hand side of (2). Let $\mathbf{v} \in \mathcal{B}_d$ and let $S_{\mathbf{v}} := \mathrm{supp}(\mathbf{v}) = \{i_1, i_2, \ldots, i_{\mathrm{wt}(\mathbf{v})}\}$. Note that $\mathrm{wt}(\mathbf{v}) \geq 1$. Again by inclusion–exclusion, for this particular $\mathbf{v}$, we have

$$\sum_{\mathbf{n}\in\mathbb{P}^d} a_{\mathrm{wt}(\mathbf{v})} F_d(\mathbf{n}-\mathbf{v}) x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} = a_{\mathrm{wt}(\mathbf{v})} x_{i_1} x_{i_2} \cdots x_{i_{\mathrm{wt}(\mathbf{v})}} \sum_{T\subseteq[d]\setminus S_{\mathbf{v}}} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T). \tag{4}$$

By (2), the expression on the right-hand side of (3) must equal the sum over all $\mathbf{v} \in \mathcal{B}_d$ of the expression on the right-hand side of (4), giving

$$(-1)^d a_0 + \sum_{S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) = \sum_{\mathbf{v}\in\mathcal{B}_d} a_{\mathrm{wt}(\mathbf{v})} x_{i_1} x_{i_2} \cdots x_{i_{\mathrm{wt}(\mathbf{v})}} \sum_{T\subseteq[d]\setminus S_{\mathbf{v}}} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T). \tag{5}$$

Each $\mathbf{v} \in \mathcal{B}_d$ corresponds to a unique nonempty subset $V \subseteq [d]$ and conversely, so (5) can be rewritten as

$$(-1)^d a_0 + \sum_{S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) = \sum_{\emptyset\neq V\subseteq[d]} a_{|V|} \prod_{i\in V} x_i \sum_{T\subseteq[d]\setminus V} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T).$$

Swapping the order of summation on the right yields

$$(-1)^d a_0 + \sum_{S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) = \sum_{T\subsetneq[d]} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T) \sum_{\emptyset\neq V\subseteq[d]\setminus T} a_{|V|} \prod_{i\in V} x_i.$$

Collecting all instances of $g_d(\mathbf{x})$ on the left-hand side, we obtain

$$\begin{aligned} g_d(\mathbf{x}) \left(1 - \sum_{\emptyset\neq V\subseteq[d]} a_{|V|} \prod_{i\in V} x_i\right) &= (-1)^{d+1} a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) \\ &\quad + \sum_{\emptyset\neq T\subsetneq[d]} (-1)^{|T|} g_{d-|T|}(\mathbf{x}_T) \sum_{\emptyset\neq V\subseteq[d]\setminus T} a_{|V|} \prod_{i\in V} x_i. \end{aligned} \tag{6}$$

It remains to show that the right-hand side equals a_0. To this end, observe that the two sums on the right can be combined. Doing so reduces the right-hand side of the above to

$$(-1)^{d+1} a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|} g_{d-|S|}(\mathbf{x}_S) \left(1 - \sum_{\emptyset\neq V\subseteq[d]\setminus S} a_{|V|} \prod_{i\in V} x_i\right).$$

By the induction hypothesis,

$$g_{d-|S|}(\mathbf{x}_S) = a_0 \left(1 - \sum_{\emptyset\neq V\subseteq[d]\setminus S} a_{|V|} \prod_{i\in V} x_i\right)^{-1}.$$

Therefore, substitution into (6) gives us

$$\begin{aligned} g_d(\mathbf{x}) \left(1 - \sum_{\emptyset\neq V\subseteq[d]} a_{|V|} \prod_{i\in V} x_i\right) &= (-1)^{d+1} a_0 - \sum_{\emptyset\neq S\subsetneq[d]} (-1)^{|S|} a_0. \\ &= a_0 \left[(-1)^{d+1} \sum_{i=1}^{d-1} \binom{d}{i} (-1)^i\right]. \end{aligned}$$

The bracketed expression equals 1, by a well-known identity [1, Th. 1.7], and the result now follows. □

To illustrate the above result, we apply it to the chains and Delannoy numbers in any dimension to obtain their generating functions.

Corollary 16 ([13, p. 156]). *Let $g_d^{\mathrm{C}}(x_1, \ldots, x_d)$ be the generating function for the d-dimensional chain numbers. Then*

$$g_d^{\mathrm{C}}(x_1, \ldots, x_d) = \frac{2}{1 + 2 \sum_{\emptyset \neq S \subseteq [d]} (-1)^{|S|} \prod_{i \in S} x_i}.$$

Proof. The sequence of chain numbers is $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 2, 2, -2, 2, \ldots \rangle$. □

Corollary 17 ([13, p. 159]). *Let $g_d^{\mathrm{D}}(x_1, \ldots, x_d)$ be the generating function for the d-dimensional Delannoy numbers. Then*

$$g_d^{\mathrm{D}}(x_1, \ldots, x_d) = \frac{1}{1 - \sum_{\emptyset \neq S \subseteq [d]} \prod_{i \in S} x_i}.$$

Proof. The sequence of Delannoy numbers is $\mathfrak{a}$-recurrent with $\mathfrak{a} = \langle 1, 1, 1, \ldots \rangle$. □

References

[1] I. Anderson, A First Course in Discrete Mathematics, Springer, London, 2001.
[2] J.-M. Autebert, S.R. Schwer, On generalized Delannoy paths, SIAM Journal on Discrete Mathematics 16 (2) (2003) 208–223.
[3] C. Banderier, S. Schwer, Why Delannoy numbers? Journal of Statistical Planning and Inference 135 (1) (2005) 40–54.
[4] J. Bertoin, J. Pitman, Path transformations connecting Brownian motion, excursion and meander, Bulletin des Sciences Mathématiques 118 (2) (1994) 147–166.
[5] J.S. Caughman, C.R. Haithcock, J.J.P. Veerman, A note on lattice chains and Delannoy numbers, Discrete Mathematics 308 (2008) 2623–2628.
[6] G.-S. Cheon, E.-M. Moawwad, Extended symmetric Pascal matrices via hypergeometric functions, Applied Mathematics and Computation 158 (1) (2004) 159–168.
[7] W.-S. Chou, Formulas for counting chains in multisets, Utilitas Mathematica 40 (1991) 3–12.
[8] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.
[9] E. Duchi, R.A. Sulanke, The 2^{n-1} factor for multi-dimensional lattice paths with diagonal steps, Séminaire Lotharingien de Combinatoire 51 (2004) Article B51c.
[10] H.W. Gould, M.E. Mays, Counting chains of subsets, Utilitas Mathematica 31 (1987) 227–232.
[11] G. Hetyei, Central Delannoy numbers and balanced Cohen–Macaulay complexes, Annals of Combinatorics 10 (2006) 443–462.
[12] S. Kaparthi, H.R. Rao, Higher dimensional restricted lattice paths with diagonal steps, Discrete Applied Mathematics 31 (1991) 279–289.
[13] P. MacMahon, Combinatory Analysis, Chelsea, New York, 1955.
[14] R.B. Nelson Jr., H. Schmidt, Chains in power sets, Mathematics Magazine 64 (1991) 23–31.
[15] S.R. Schwer, S-arrangements avec répétitions, Comptes Rendus de l'Academie des Sciences, Serie Generale, la Vie des Sciences 334 (2002) 1–6.
[16] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.
[17] R.A. Sulanke, Counting lattice paths by Narayana polynomials, The Electronic Journal of Combinatorics 7 (1) (2000) 1–9.
[18] M. Tărnăuceanu, The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, European Journal of Combinatorics 30 (2009) 283–287.