A generalized localization theorem and geometric inequalities for convex bodies

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Abstract

In this article, we generalize a localization theorem of Lovász and Simonovits [Random walks in a convex body and an improved volume algorithm, Random Struct. Algorithms 4–4 (1993) 359–412] which is an important tool to prove dimension-free functional inequalities for log-concave measures. In a previous paper [Fradelizi and Guédon, The extreme points of subsets of $s$-concave probabilities and a geometric localization theorem, Discrete Comput. Geom. 31 (2004) 327–335], we proved that the localization may be deduced from a suitable application of Krein–Milman's theorem to a subset of log-concave probabilities satisfying one linear constraint and from the determination of the extreme points of its convex hull. Here, we generalize this result to more constraints, give some necessary conditions satisfied by such extreme points and explain how it may be understood as a generalized localization theorem. Finally, using this new localization theorem, we solve an open question on the comparison of the volume of sections of non-symmetric convex bodies in $\mathbb{R}^d$ by hyperplanes. A surprising feature of the result is

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that the extremal case in this geometric inequality is reached by an unusual convex set that we
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1. Introduction

The purpose of this paper is to develop and extend the localization principle which
happens to be a very powerful tool to prove dimension free functional inequalities
for log-concave measures, such as isoperimetric or concentration inequalities. The pur-
pose of this localization theorem is to reduce an $n$-dimensional inequality to the one-
dimensional case. It has been applied to get isoperimetry on the sphere by Gromov and
Milman [9]. Later on it has been considerably developed by Lovász and Simonovits
[13] who also emphasized its importance as a tool of independent interest and ap-
plied it with Kannan [12] to prove isoperimetric inequalities for log-concave measures
on $\mathbb{R}^n$. It was then popularized under this form and applied in different domains to
get, for example, Kahane–Khintchine type inequalities for negative exponents [10], in-
equalities for the estimate of the growth of the $L^p$ norm of polynomials on convex
bodies [2,6], inequalities for the distribution of zeroes of random analytic functions
[16]. In a previous paper [8], we proved that this reduction is made possible by the
fact that the extreme points of the convex hull of the subset of log-concave probabili-
ties satisfying one linear constraint are Dirac measures and one-dimensional log-affine
probabilities. This constitutes a functional analysis approach to prove inequalities for
log-concave measures. Indeed, by Krein–Milman’s theorem, the supremum of an upper
semi-continuous convex functional on the subset of log-concave probabilities satisfying
one linear constraint will be attained on these extreme points. Therefore it remains to
study this functional on these Dirac measures and on these one-dimensional log-affine
probabilities and this is exactly the principle of the localization theorem of Lovász and
Simonovits [13].

This paper aims at describing some necessary conditions satisfied by the extreme
points of the subset of log-concave probabilities satisfying $p$ linear constraints, $p \geq 2$
(the case $p = 1$ was treated in [8]). There are two cases where we can give a satisfactory
description of these extreme points. For every dimension $n$, when $p = 2$ we prove that
necessarily these extreme points are some Dirac measures, or some one-dimensional
log-concave measures with a density $e^{-V}$ w.r.t. the Lebesgue measure on a segment
$[a, b]$ such that $V$ is the maximum of two affine functions, or some two dimensional
log-affine measures. When $n = 1$, for any number of constraints $p$, we prove that
necessarily these extreme points are some Dirac measures, or some one-dimensional
log-concave measures with a density $e^{-V}$ w.r.t. the Lebesgue measure on a segment
$[a, b]$ such that $V$ is the maximum of $p$ affine functions. The precise statements are
given in Theorem 1 and Corollary 1.
The proofs of these results are not only based on the bisection method as in [13,8]. In fact, the main new idea is to introduce the notion of “degree of freedom” of a log-concave function $e^{-V}$. Roughly speaking, this is the largest integer $k$ such that there is a $k$-dimensional cube around $e^{-V}$ in the set of log-concave functions (this approach is usual in convex optimization, but however the set of log-concave probabilities is not a convex set). Then, simple linear algebra shows that the density of an extreme point of the subset of log-concave probabilities satisfying $p$ linear constraints cannot have degree of freedom greater or equal than $p + 2$. Hence our set of extreme points is included in the set of probabilities whose densities have degree of freedom less or equal than $p + 1$. Therefore we study and try to describe in part 2 the set of convex functions $V$ on $\mathbb{R}^n$ such that $e^{-V}$ has a fixed degree of freedom. This description is complete in the case $n = 1$. However, using the work of Johansen [11] (for $n = 2$) and Bronshtein [5] (for $n \geq 3$) we prove exhibit a deep difference between the one-dimensional case and the $n$-dimensional case ($n \geq 2$).

Another purpose of this paper is to get optimal bounds on the ratio between the volumes of some sections of non-symmetric convex bodies in $\mathbb{R}^n$ by slabs and hyperplanes. There are (at least) three natural and classical candidates for generalizing the notion of central section by a hyperplane $H$ of a symmetric convex body to the non-symmetric setting: the hyperplane parallel to $H$ passing through the center of gravity, the hyperplane parallel to $H$ separating the convex body into two parts of equal volume and the hyperplane parallel to $H$ of maximal volume among its translates. It is natural to compare the volumes of these three sections and to investigate the extremal cases in the relevant inequalities. This has partially been done in [14,7]. Here, we prove a sharp upper bound for the ratio of the volume of the hyperplane section passing through the center of gravity with the parallel hyperplane section separating the convex body into two parts of equal volume. The classical tools of convex geometry developed in [14,15,7] appear to be inoperant. This is why we develop a strategy using a constrained optimization problem to which we apply the results obtained in Theorem 1. Moreover, we manage to identify the asymptotic extremal case in this geometric inequality. It is the union of a cone in a direction $u$ whose basis is a convex body in $u^\perp$ and a truncature of a cone in the direction $-u$ with the same basis and we get equality when the dimension $n$ goes to infinity. This concludes the comparison between the volumes of these three parallel hyperplane sections.

The paper is organized as follows. We introduce and study the notion of degree of freedom of a log-concave function in part 2. The description of the extreme points of the set of log-concave probabilities satisfying several linear constraints is given in part 3. The last part is devoted to the geometric application of this generalized localization theorem.

Notations and well known facts: Let $n$ be a positive integer, $K$ a compact convex set in $\mathbb{R}^n$. We denote by $P(K)$ the set of probabilities in $\mathbb{R}^n$ supported by $K$.

A measure $\mu$ on $\mathbb{R}^n$ is said to be log-concave if for every $\lambda \in [0, 1]$, for every compact sets $A, B \subset \mathbb{R}^n$,

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^{\lambda} \mu(B)^{1-\lambda}.$$
We denote by \( L(K) \) the set of log-concave measures supported by \( K \). A function \( f : \mathbb{R}^n \to \mathbb{R}^+ \) is log-concave if the inequality
\[
 f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}
\]
holds true for every \( x \) and \( y \) in \( \mathbb{R}^n \) and every \( \lambda \in [0, 1] \). Moreover, we say that \( f \) is log-affine if \( \log f \) is affine on its support. The link between log-concave measures and log-concave functions is completely understood since the work of Borell [4]. Indeed, if \( \mu \in L(K) \), then denoting by \( S \) its convex support and \( \text{aff}(S) \) the affine subspace generated by \( S \), \( \mu \) has a density \( e^{-V} \) with respect to the Lebesgue measure \( \text{aff}(S) \) where \( V \) is a convex function defined on \( S \) taking values in \( \mathbb{R} \cup \{+\infty\} \) (and we can extend it by \( +\infty \) on \( \mathbb{R}^n \setminus S \)). Moreover, if \( V \) is any convex function defined on a convex subset \( S \) of \( K \), then the measure of density \( e^{-V} \) (and 0 outside \( S \)) with respect to the Lebesgue measure on \( \text{aff}(S) \) is log-concave. It is clear that \( \mu \) and \( e^{-V} \) have the same support.

If \( S \) is a convex set in \( \mathbb{R}^n \), we say that \( S \) is a \( d \)-dimensional set if the dimension of \( \text{aff}(S) \) is equal to \( d \).

2. Degree of freedom of log-concave functions

We start with the definition of this notion of degree of freedom.

**Definition.** Let \( V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a convex function, denote by \( D \) the domain of \( V \) (i.e. \( D = \{x : V(x) < +\infty\} \)) and by \( S \) the closure of \( D \). The set \( S \) is the support of the log-concave function \( e^{-V} : \mathbb{R}^n \to \mathbb{R}^+ \). We define the degree of freedom of \( e^{-V} \) by the largest integer \( k \) such that:

- there exist \( z > 0 \) and linearly independent continuous functions \( V_1, \ldots, V_k : D \to \mathbb{R} \) such that for every \( (\varepsilon_1, \ldots, \varepsilon_k) \in [-z, z]^k \), the function \( e^{-V} + \sum_{i=1}^k \varepsilon_i V_i \) remains a log-concave function.

Let us start with some simple observations. Let \( k, z \) and \( V_1, \ldots, V_k \) satisfying the above condition. Then by choosing \( \varepsilon_i = z \) or \( -z \) and all the other one equal 0, we get that for all \( i \) the functions \( e^{-V} + zV_i \) and \( e^{-V} - zV_i \) are log-concave, hence non-negative on \( D \). This gives that \( |V_i(x)| \leq e^{-V(x)}/z \). Therefore, there exist \( W_i : D \to \mathbb{R} \) such that \( V_i = W_i e^{-V} \). We have \( |W_i(x)| \leq 1/z \) on \( D \), hence the functions \( W_i \) are continuous and bounded on \( D \), thus we may extend them as continuous bounded functions on \( S \). Therefore we have proved the following equivalent definition of the degree of freedom of \( e^{-V} \). It is the largest integer \( k \) such that:

- there exist \( z > 0 \) and linearly independent continuous bounded functions \( W_1, \ldots, W_k : S \to \mathbb{R} \) such that for every \( (\varepsilon_1, \ldots, \varepsilon_k) \in [-z, z]^k \), the function \( e^{-V}(1 + \sum_{i=1}^k \varepsilon_i W_i) \) remains a log-concave function.

A second observation is the following. Let \( k, z \) and \( W_1, \ldots, W_k \) satisfying the above condition. If the function \( V \) is affine on a convex \( C \subset D \) then every function \( W_i \) is also affine on \( C \).
Indeed, for all $i$ and for all $\varepsilon \in [-\alpha, \alpha]$, the function $V - \log(1 + \varepsilon W_i)$ is concave. Since $V$ is affine on $C$, it follows that $\log(1 + \varepsilon W_i)$ is convex on $C$ for all $\varepsilon \in [-\alpha, \alpha]$. By sending $\varepsilon$ to $0^+$ and $0^-$ we get that $W_i$ is affine on $C$.

Since the dimension of the vector space of affine functions defined on a $d$-dimensional compact convex set is equal to $d + 1$, consequently we get that if $V$ is affine on $D$ then the degree of freedom of $e^{-V}$ is less or equal than $d + 1$.

It is easy to see that the set of log-concave functions supported by a convex compact set $S$ is a positive cone which is not convex (the sum of two log-concave functions is in general not log-concave). But this notion of degree of freedom allows us to find some convex subset in this set around a log-concave function. The determination of the degree of freedom of a log-concave function $e^{-V}$ is related to the problem of finding convex functions $C$ such that $V + C$ and $V - C$ are convex functions. This question has been studied in detail in [11,5]. Although the two notions are not equivalent, we attempted to explain (at least when $V$ is affine) how they are connected. Proposition 1 states some general result about the minimum degree of freedom of log-concave functions. In Propositions 2 and 3, we explain how the situation changes drastically when the dimension of the support of the log-concave function is equal to one or greater than 2. This phenomenon is not surprising in view of the work of Johansen [11] and Bronshtein [5].

**Proposition 1.** Let $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function, whose domain $D$ is bounded. Denote by $d$ the dimension of $D$. Then the degree of freedom of $e^{-V}$ is greater or equal than $d + 1$.

If we assume moreover that $V$ is bounded and not affine on $D$ then the degree of freedom of $e^{-V}$ is greater or equal than $d + 2$.

An easy consequence of this proposition is that if $V$ is affine on a bounded $d$-dimensional convex set then the degree of freedom of $e^{-V}$ is $d + 1$. Proposition 1 is a direct consequence of the following lemma.

**Lemma 1.** Let $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function, such that its domain $D$ is bounded. Then for any affine function $W$, there exists $\alpha > 0$ such that for every $\varepsilon \in [-\alpha, \alpha]$ the function $(1 + \varepsilon W)e^{-V}$ is log-concave on $D$.

Moreover, if $V$ is bounded on $D$ then there exists $\beta > 0$ such that for every $(\varepsilon, \eta) \in [-\beta, \beta]^2$ the function $(1 + \varepsilon W + \eta V)e^{-V}$ is log-concave on $D$.

**Proof.** Let $\alpha = (2 \max_D |W|)^{-1}$. Then for every $\varepsilon \in [-\alpha, \alpha]$ we have $1 + \varepsilon W \geq 1/2 > 0$. Using the facts that $V$ is convex, log is concave and $1 + \varepsilon W$ is affine we get that $\log(1 + \varepsilon W) - V$ is concave.

For the proof of the second part, let $\beta = \min(1/2, (2 \max_D |W| + |V|)^{-1})$. Then for every $(\varepsilon, \eta) \in [-\beta, \beta]^2$ we have $1 + \varepsilon W + \eta V \geq 1/2 > 0$. Let $\rho = (1 + \varepsilon W + \eta V)e^{-V}$. If $\eta = 0$ we have already proved the concavity of $\log(\rho)$. Assume now that $\eta \neq 0$. We observe that the function $h_\eta$ defined by $h_\eta(t) = t - \log(1 + \eta t)$ is convex and increasing.
on $[1 - 1/|\eta|, 1/|\eta|]$. Since we have
\[
\log(\rho) = \log(1 + \varepsilon W + \eta V) - V = \frac{\varepsilon W}{\eta} - h_\eta \left( V + \frac{\varepsilon W}{\eta} \right),
\]
the concavity of $\log(\rho)$ follows from the convexity of $V$ and $h_\eta$. □

If the support of the function $e^{-V}$ is one-dimensional, the following proposition gives an exact characterization of the degree of freedom of $e^{-V}$ according to the shape of $V$.

**Proposition 2.** Let $a < b \in \mathbb{R}$, let $V : (a, b) \to \mathbb{R}$ be a convex function and let $k$ be a positive integer. The degree of freedom of $e^{-V}$ is $k + 1$ if and only if there exist $k$ (but not less than $k$) affine functions $\phi_1, \ldots, \phi_k$ such that $V = \max_{1 \leq i \leq k} \phi_i$. Moreover, if $V$ is not affine by parts with a finite number of parts then the degree of freedom of $e^{-V}$ is infinite.

**Proof.** We first prove that if there exist $k$ points $a < x_1 < \cdots < x_k < b$ such that $V$ is differentiable at $x_i$ and $V'(x_1) < \cdots < V'(x_k)$ then the degree of freedom of $e^{-V}$ is greater or equal than $k + 1$. Let us assume that $V'(x_1) \neq 0$ (if it is not the case then $V'(x_k) \neq 0$ and $x_k$ should be used instead of $x_1$ in the following construction). For $i = 1, \ldots, k$ we define the functions $W_i$ by

\[
W_i(x) = \begin{cases} 
V(x_1) + (x - x_1)V'(x_1) & \text{if } x \in (a, x_1], \\
V(x) & \text{if } x \in [x_1, x_i], \\
V(x_i) + (x - x_i)V'(x_i) & \text{if } x \in [x_i, b).
\end{cases}
\]

Let $W_0 = \chi_{(a,b)}$, $x_0 = a$ and $x_{k+1} = b$. Since $W_1$ is affine but not constant and $V'(x_i) \neq V'(x_j)$ for $i \neq j$, it is clear that the family $(W_i)_{0 \leq i \leq k}$ is linearly independent. Let $M_1 = \max_{[a,b]} |V(x)|$, $M_2 = \max(|V'(a)|, |V'(b)|)$ and $M = \max(1, M_1 + (b-a)M_2)$. Then for every $i = 0, \ldots, k$, $|W_i| \leq M$. Let $(\varepsilon_1, \ldots, \varepsilon_k) \in [-1, 1]^k$ and $Z = \sum_{i=1}^k \varepsilon_i W_i$. For every $i = 0, \ldots, k$, there exist $\lambda_i \in \mathbb{R}$ and an affine function $A_i$ such that

\[
Z = \lambda_i V + A_i \text{ on } [x_i, x_{i+1}] \text{ with } |\lambda_i V| \leq kM \text{ and } |A_i| \leq kM.
\]

Choosing $\varepsilon = \min(1/2, 1/(4kM))$, we get from Lemma 1 that for every $\varepsilon \in [-\varepsilon, \varepsilon]$, the function $(1 + \varepsilon Z)e^{-V}$ is log-concave on each interval $[x_i, x_{i+1}]$. Moreover, since the functions $V$ and $W_0, W_1, \ldots, W_k$ are differentiable at each $x_i$, the function $(1 + \varepsilon Z)e^{-V}$ is also differentiable at each $x_i$ hence it is log-concave on $(a, b)$. Therefore the degree of freedom of $e^{-V}$ is greater or equal than $k + 1$.

This proves that if $V$ is the maximum of $k$ (but not less than $k$) affine functions then the degree of freedom of $e^{-V}$ is greater or equal than $k + 1$. It also proves that if $V$ is not affine by parts with a finite number of parts, then the degree of freedom of $e^{-V}$ is infinite.
To finish the proof, let $V$ be the maximum of $k$ (but not less than $k$) affine functions. Let $x > 0$ and $W$ be a continuous function such that for every $\varepsilon \in [-x, x]$, the function $e^{-V}(1 + \varepsilon W)$ is log-concave on $(a, b)$. Then there exist $a = y_0 < y_1 < \cdots < y_{k-1} < y_k = b$ such that for every $i = 0, \ldots, k - 1$, $V$ is affine on $[y_i, y_{i+1}]$. Hence the function $W$ is affine on $[y_i, y_{i+1}]$. Since the linear space of continuous functions on $(a, b)$ which are affine on each interval $[y_i, y_{i+1}]$, for $i = 0, \ldots, k - 1$, has dimension $k + 1$, the degree of freedom of $e^{-V}$ is at most $k + 1$. □

As a consequence of the preceding proposition, for any $k \geq 1$, the set of convex functions $V$ on $(a, b)$ such that $e^{-V}$ has a degree of freedom less than $k + 1$ is included in the set of functions on $(a, b)$ which are affine by parts with less than $k$ slopes, hence it is a closed subset (with an empty interior) of the set of continuous functions with the topology of uniform approximation on any convex compact subset of $(a, b)$.

Contrasting with this remark, which is valid only when the dimension $d = 1$, the next proposition proves that when $d \geq 2$, there is a dense set of functions $V$ for which the degree of freedom of $e^{-V}$ is less than $d + 2$. Moreover, although the degree of freedom of every log-affine function on $\mathbb{R}^d$ is exactly $d + 1$, they are not the only one sharing this property (unlike in the case $d = 1$). This means that it is not possible in this case to identify simply the set of functions with $k$ degrees of freedom $(k \geq d + 1)$.

**Proposition 3.** Let $D$ be a convex bounded domain in $\mathbb{R}^d$ with $d \geq 2$.

(i) In the set of bounded convex functions $V : D \to \mathbb{R}$ endowed with the topology of uniform approximation on any convex compact subset of $D$, the subset of functions $V$, such that the degree of freedom of $e^{-V}$ is exactly $d + 2$, is dense.

(ii) There exist unbounded (and consequently not affine) convex functions $V : D \to \mathbb{R}$ such that the degree of freedom of $e^{-V}$ is exactly $d + 1$.

These examples come from the study of Johansen [11] (when $d = 2$) and Bronshtein [5] (when $d \geq 3$). We reproduce now their construction. For $d \geq 2$, let $D$ be a bounded convex domain in $\mathbb{R}^d$ and let affine functions $a_1, \ldots, a_q$ be defined on $D$. The function $V(x) = \max a_i(x)$ is convex and moreover the sets $P_i = \{x; f(x) = a_i(x)\}$ are polytopes (possibly empty). By eliminating superfluous affine functions, we can assume that the dimension of every set $P_i$ is equal to $d$. The function $V$ constructed in this manner is called a polyhedral function and the polytopes $P_i$ satisfy clearly $D \subset P_1 \cup \cdots \cup P_q$. Moreover $P_i \cap P_j$ is a face of each of the polytopes $P_i$ and $P_j$ (possibly empty).

Let $\mathcal{F}$ be the set of all polyhedral functions such that the following three conditions hold:

1. every face of the polytope $P_i$ which meets $D$ has a vertex in $D$;
2. any two vertices of any face of $P_i$ lying in $D$ can be joined in $D$ by edges of this face;
3. every vertex of $P_i$ lying in $D$ is of order $d + 1$, i.e. it belongs to $(d + 1)$ of the polytopes $P_j$.  

Under these assumptions on the family of polytopes $P_i$, Lemma 1.1 in [5] (which is an extension of a “combinatorial lemma” in [11]) states that if $h$ is a continuous function on $D$, affine on each $P_i$, and such that $h = 0$ on two of the polytopes $P_i$ having a common $(d - 1)$-face, then $h = 0$ on $D$.

**Proof of Proposition 3.** We will show that every $V \in \mathcal{F}$ satisfies the claimed assertion in the proposition. Let $W : D \to \mathbb{R}$ be a continuous function and assume that there is $\varepsilon > 0$ such that for every $\varepsilon \in [-\varepsilon, \varepsilon]$, the function $(1 + \varepsilon W)e^{-V}$ is log-concave. Since $V$ is affine on each $P_i$ then (by the second observation after the definition of the degree of freedom) $W$ has to be affine on $P_i$. Now, let $a$ and $b$ be two affine functions on $D$ such that $V - a = W - b = 0$ on $P_1$ and let $V_1 = V - a$ and $W_1 = W - b$. Let $P_k$ having a common face with $P_1$, $k \neq 1$. Two cases can occur. If $V_1 = 0$ on $P_k$ then $V_1 = 0$ on $P_1 \cup P_k$ therefore by Lemma 1.1 in [5] (see above), $V_1 = 0$ on $D$, which proves that $V$ is affine on $D$ and so is $W$. If $V_1 \neq 0$ on $P_k$, since $W_1 = V_1 = 0$ on a $(d - 1)$ dimensional face common to $P_1$ and $P_k$, there exists $\lambda \in \mathbb{R}$ such that $W_1 = \lambda V_1$ on $P_k$. Then $W_1 - \lambda V_1 = 0$ on $P_1 \cup P_k$ and again by Lemma 1.1 in [5], $W_1 - \lambda V_1 = 0$ on $D$, which shows that $W = \lambda (V - a) + b$ on $D$, where $a$ and $b$ are affine functions. This proves that when $V$ is not affine, the degree of freedom of $e^{-V}$ is exactly $d + 2$.

Moreover, by Theorem 2.2 in [5], we know that this set of particular convex functions $\mathcal{F}$ is dense in the set of bounded convex functions on $D$ for the topology of uniform convergence on any convex compact subset of $D$.

To adapt the construction to get unbounded convex functions, to begin with we choose infinitely many affine functions $a_i$ in such a way that $V$ is unbounded and assuming the same hypothesis on the associated convex polyhedra, we have the same result as Lemma 1.1 of Bronshtein (the proof can be done by induction on the number of polytopes $P_i$). Hence, the proof we have done before works similarly but since $W$ has to be bounded and $V$ is unbounded, we necessarily get that $\lambda = 0$ and $W$ is just an affine function. This proves that the degree of freedom of $e^{-V}$ is exactly $d + 1$. $\square$

**3. Generalized localization theorem**

We start by setting some terminology used in this section. Let $n$ be a positive integer, let $K$ be a compact convex set in $\mathbb{R}^n$ and $p \in \mathbb{N}$. Let $f_1, \ldots, f_p : K \to \mathbb{R}$ be upper semi-continuous functions, $f = (f_1, \ldots, f_p)$ and denote by $P_f$ the set of log-concave probabilities $\mu$ on $K$ satisfying the linear constraints $\int f_i d\mu \geq 0$ for every $i = 1, \ldots, p$. Without loss of generality we assume that the family of functions $(1, f_1, \ldots, f_p)$ is linearly independent (otherwise, there could be relations of dependence so that the system of inequalities could not have any solution, or if this system has non-trivial solution then we can forget about the useless constraints). The following lemma explains the relationship between the number of saturated constraints of an extreme point of $\text{conv}P_f$ and the degree of freedom of its density.
Lemma 2. Let \( v \) be an extreme point of \( \text{conv} \ P_f \), denote by \( G \) the affine subspace generated by its support, denote by \( V \) the convex function such that \( v \) has the density \( e^{-V} \) with respect to the Lebesgue measure on \( G \), and let \( k = \# \{ i \in \{ 1, \ldots, p \} : \int f_i \, dv = 0 \} \) be the number of saturated constraints. Then the degree of freedom of \( e^{-V} \) is less or equal than \( k + 1 \).

Proof. Without loss of generality, we may assume that \( \int f_i \, dv = 0 \) for \( 1 \leq i \leq k \) and \( \int f_i \, dv > 0 \) for \( k + 1 \leq i \leq p \). If the degree of freedom of \( e^{-V} \) is greater or equal than \( k + 2 \) then, by definition, there exist \( \alpha > 0 \) and \( k + 2 \) linearly independent continuous functions \( V_1, \ldots, V_{k+2} \) such that \( \forall (\varepsilon_1, \ldots, \varepsilon_{k+2}) \in [-\alpha, \alpha]^{k+2} \), the function \( e^{-V} + \sum_{i=1}^{k+2} \varepsilon_i V_i \) remains a log-concave function. Then the set

\[
Q = \left\{ \mu \text{ of density } e^{-V} + \sum_{i=1}^{k+2} \varepsilon_i V_i, \ (\varepsilon_1, \ldots, \varepsilon_{k+2}) \in [-\alpha, \alpha]^{k+2} \right\}
\]

is a \( k + 2 \)-dimensional cube centered at \( v \) in the set of log-concave measures. Hence, by a simple argument of linear algebra the following central section of \( Q \) by \( k + 1 \) hyperplanes and \( p-k \) halfspaces

\[
\left\{ \mu \in Q, \ \int d\mu = 1, \ \int f_i \, d\mu = 0 \text{ if } 1 \leq i \leq k, \ \int f_i \, d\mu \geq 0 \text{ if } k + 1 \leq i \leq p \right\},
\]

which is included in \( P_f \), contains some segment centered at \( v \). This contradicts the fact that \( v \) is extremal in \( \text{conv} P_f \). Therefore we conclude that the degree of freedom of \( e^{-V} \) is less than \( k + 1 \). \( \square \)

We can now state the main result which establishes some necessary conditions satisfied by an extreme point in \( \text{conv} P_f \).

Theorem 1. Let \( v \) be an extreme point of \( \text{conv} P_f \), denote by \( G \) the affine subspace generated by its support, set \( d = \dim G \), denote by \( V \) the convex function such that \( v \) has the density \( e^{-V} \) with respect to the Lebesgue measure on \( G \), and let \( k = \# \{ i \in \{ 1, \ldots, p \} : \int f_i \, dv = 0 \} \) be the number of saturated constraints, then

\[
k \geq d.
\]

Moreover,

1. if \( d = 1 \), then there exists \( k \) affine functions \( \phi_1, \ldots, \phi_k \) on \( \text{supp} \ V \) such that \( V = \max_{1 \leq i \leq k} \phi_i \)
2. if \( V \) is bounded and \( d = k \) then \( V \) is affine on its support,
3. if \( d = k = p \) or \( d = k = p - 1 \) then \( V \) is affine on its support.
Notice that in [8], we already proved that \( d \leq p \) and that when \( d = p \) (which implies \( d = k = p \)) then \( V \) is affine on its support.

Although the picture is still not complete (because we do not have any information on the shape of \( V \) when \( 2 \leq d \leq k - 1 \), for example \( d = 2 \) and \( k = p = 3 \)), observe that when \( n = 1 \) or \( p = 2 \), it provides some precise information. With the preceding notation, we have the following

**Corollary 1.** (A) For any number of constraints \( p \), when \( n = 1 \), if \( v \in \mathcal{E}(\text{ext}P_f) \), then either it is a Dirac measure or there exist \( k \) affine functions \( \phi_i \) such that \( V = \max_{1 \leq i \leq k} \phi_i \).

(B) For any dimension \( n \), when \( p = 2 \), if \( v \in \mathcal{E}(\text{ext}P_f) \), then either it is a Dirac measure,

or \( d = 1 \), \( k = 1 \) and \( V \) is affine,

or \( d = 1 \), \( k = 2 \) and there exist exactly two affine functions \( \phi_1, \phi_2 \) such that \( V = \max(\phi_1, \phi_2) \),

or \( d = 2 \), \( k = 2 \) and \( V \) is affine.

**Proof of Theorem 1.** From Proposition 1, the degree of freedom of \( e^{-V} \) is greater than \( d + 1 \) and from Lemma 2, it is less than \( k + 1 \). Therefore \( k \geq d \).

We shall now prove the three assertions claimed in the “moreover” part of the Theorem.

1. From Lemma 2, the degree of freedom of \( e^{-V} \) is less than \( k + 1 \). Hence from Proposition 2 there exist \( k \) affine functions \( \phi_i \) on \( \text{supp} V \) such that \( V = \max_{1 \leq i \leq k} \phi_i \).

2. From Proposition 1, we know that if \( V \) is not affine then the degree of freedom of \( e^{-V} \) is greater than \( d + 2 \) and from Lemma 2, it is less than \( d + 1 \). Hence \( V \) is necessarily affine.

3. Without any assumption on \( V \), it can be impossible to find a cube centered at \( v \) in the set of log-concave probabilities supported by \( K \) with the good dimension. We have indeed described in Proposition 3 examples of unbounded convex functions \( V \) such that the degree of freedom of \( e^{-V} \) is \( d + 1 \). Then we go back to our previous approach in [8] and use an argument based on Borsuk’s theorem. Assume that \( k = d = p \) or \( k = d = p - 1 \).

If \( k = d = p \) then for every \( i = 1, \ldots, d, \int f_i d\nu = 0 \) and we define \( f_{d+1} \) as the constant function equal to 1 (so that \( \int f_{d+1} d\nu = 1 > 0 \)). If \( k = d = p - 1 \) then for every \( i = 1, \ldots, d, \int f_i d\nu = 0 \) and \( \int f_{d+1} d\nu > 0 \).

Identify the affine subspace \( G \) generated by the support of \( V \) with \( \mathbb{R}^d \). Let \( e_{d+1} \) be an \( d \) dimensional vector in \( \mathbb{R}^d \), the graph of the convex function \( V \) is \( \{(x, V(x)), x \in \mathbb{R}^d \} \subset \mathbb{R}^{d+1} = \mathbb{R}^d \oplus \mathbb{R}_{d+1} \). We parametrize the halfspaces of \( \mathbb{R}^{d+1} \) by a sphere of dimension \( d + 1 \) as follows: let \( e_{d+2} \) a vector orthogonal to \( \mathbb{R}^{d+1} \) and for every \( u \in \mathbb{S}^{d+1} \), let \( H_u^+ = \{z; \langle z - e_{d+2}, u \rangle \geq 0 \} \cap \mathbb{R}^{d+1} \). Let \( H_u = H_u^+ \cap H_{-u}^+ \). When \( u_{d+1} \neq 0 \), \( H_u \) is the graph of an affine function \( A_u \) on \( G \) defined by: \( x_{d+1} = A_u(x_1, \ldots, x_d) = (u_{d+2} - \sum_{i=1}^d x_i u_i)/u_{d+1} \). When \( u_{d+1} = 0 \), we define \( A_u \) on the two halfspaces separated by \( H_u \) by, \( A_u(x) = +\infty \) if \( \sum_{i=1}^d x_i u_i > u_{d+2} \) and \( A_u(x) = -\infty \) if not.
For all \( u \in S^{d+1} \), we define a new measure \( \nu_u \) on the support of \( V \) by

\[
d
\nu_u = \begin{cases} 
(e^{-V} - e^{-A_v})_+ dm & \text{if } u_{d+2} > 0, \text{ or } (u_{d+2} = 0 \text{ and } u_{d+1} > 0), \\
\inf (e^{-V}, e^{-A_v}) dm & \text{if } u_{d+2} < 0, \text{ or } (u_{d+2} = 0 \text{ and } u_{d+1} \leq 0).
\end{cases}
\]

By the arithmetico-geometric inequality (see for example lemma 1 in [8]), \((e^{-V} - e^{-A_v})_+ \) is a log-concave function then for all \( u \in S^{d+1} \), \( \nu_u \) is a log-concave measure on \( G \). Also, by construction, for all \( u \in S^{d+1} \), we have \( \nu_u + \nu_{-u} = \nu \). Define \( \phi : S^{d+1} \rightarrow \mathbb{R}^{d+1} \) by

\[
\phi(u) = \left( \int f_1 d\nu_u, \ldots, \int f_{d+1} d\nu_u \right).
\]

It is easily checked that \( \phi \) is continuous on \( S^{d+1} \). Hence from Borsuk’s theorem, there exists \( v \in S^{d+1} \) such that \( \phi(v) = \phi(-v) \). Therefore, for all \( i = 1, \ldots, d \), \( \int f_i d\nu_v = \int f_i d\nu_{-v} = 0 \) and \( \int f_{d+1} d\nu_v = \int f_{d+1} d\nu_{-v} = \frac{1}{2} \int f_{d+1} d\nu > 0 \). From the last equality, \( \nu_v \) is not trivial i.e. it is different from 0 and from \( v \) (which means that the hyperplane \( H_v \) cuts properly the graph of \( V \)). It proves that \( \lambda = \int d\nu_v \) belongs to \( (0, 1) \) so taking \( v_1 = v/v_n \), and \( v_2 = v_{-v}/(1 - \lambda) \), we get that \( v = \lambda v_1 + (1 - \lambda)v_2 \) where \( v_1, v_2 \in P_f \). Since \( v \) is an extreme point of \( \text{conv} P_f \), we get that \( v = v_1 = v_2 \). This means that \( V \) is proportional to \( A_v \) and in that case, \( v_{d+1} \neq 0 \). Therefore \( V \) is an affine function on its support. \( \square \)

We start to explain how to use these informations to get dimension free functional inequalities for log-concave probabilities on \( \mathbb{R}^n \). For proving an inequality involving several integrals valid for every log-concave probabilities, our strategy is to define some linear constraints on the set \( \mathcal{L}(K) \) such that the inequality is equivalent to the fact that the supremum of an upper continuous convex functional on the set of log-concave probabilities satisfying these linear constraints is negative. The next theorem explains how to use the information about the extreme points of this set to study a constrained optimization problem. Therefore it remains to test the initial inequality on these extreme points. Since we have some information about the dimension of the support and the density of the corresponding log-concave probabilities, this principle can be understood as a generalized localization theorem.

**Theorem 2.** Under the same assumptions as in Theorem 1, if \( \Phi : \mathcal{P}(K) \rightarrow \mathbb{R} \) is a convex upper semi-continuous function then \( \sup \{ \Phi(\mu) ; \mu \in P_f \} \) is attained at a probability \( v \) such that \( v \) has the properties described in Theorem 1.

The proof is completely similar to the proof of Theorem 2 in [8]. We reproduce it here for completeness.

**Proof.** By Theorem 2.2 of [3], we know that the set of log-concave probabilities supported by \( K \) is \( w^* \)-compact. Since \( f_1, \ldots, f_p \) are upper semi-continuous, the condition
\[ \forall i \in \{1, \ldots, p\}, \int f_i \, d\mu \geq 0 \] is \( w^* \)-closed, therefore the set \( P_f \) is \( w^* \)-compact. By application of the Krein–Milman’s theorem, \( \sup \{ \Phi(\mu) : \mu \in P_f \} \) is achieved at a probability \( v \in \text{Ext}(\text{conv} w^* P_f) \subset \text{Ext}(\text{conv} P_f) \). The result follows by the description of the extreme points of \( \text{conv} P_f \) given in Theorem 1. \( \square \)

**Remark.** Using Theorem 2, we can write a result similar to the localization theorem of Lovász and Simonovits [13] with three integrals instead of two. Assume that there are three integrable u.s.c. functions \( f_1, f_2 \) and \( f_3 \) such that

\[
\int_{\mathbb{R}^n} f_1(x) \, dx > 0, \quad \int_{\mathbb{R}^n} f_2(x) \, dx > 0, \quad \int_{\mathbb{R}^n} f_3(x) \, dx > 0.
\]

Then either there exists \( x_0 \in \mathbb{R}^n \) such that for every \( i = 1, 2, 3, f_i(x_0) \geq 0 \) or there exist distinct points \( a, b \in \mathbb{R}^n \), a convex function \( \ell = \max(\ell_1, \ell_2) : [0, 1] \to \mathbb{R} \) maximum of two affine functions such that

\[
\int_0^1 f_i(ta + (1 - t)b)e^{-\ell(t)} \, dt \geq 0 \quad \text{for all } i \in \{1, 2, 3\},
\]

or there exist a 2 dimensional bounded convex set \( C \subset \mathbb{R}^2 \), a point \( a \in \mathbb{R}^n \) and vectors \( y, z \in \mathbb{R}^n \), and a linear function \( \ell : \text{span}(y, z) \to \mathbb{R} \) such that

\[
\int_{(s,t) \in C} f_i(a + sy + tz)e^{\ell(sy + tz)} \, ds \, dt \geq 0 \quad \text{for all } i \in \{1, 2, 3\}.
\]

Indeed by the assumptions about the \( f_i \)'s, we can find a Euclidean ball of radius \( R \) such that

\[
\int_{B(0,R)} f_1(x) \, dx \geq 0, \quad \int_{B(0,R)} f_2(x) \, dx \geq 0, \quad \int_{B(0,R)} f_3(x) \, dx \geq 0
\]

and we call \( P_f \) the set of log-concave probabilities \( \theta \) supported by \( B(0, R) \) such that \( \int f_1 \, d\theta \geq 0 \) and \( \int f_2 \, d\theta \geq 0 \). Define the function \( \Phi \) on the set of probabilities supported by \( B(0, R) \) by

\[
\Phi(\theta) = \int f_3 \, d\theta.
\]

Since the uniform probability measure on \( B(0, R) \), \( \mu \), is log-concave and belongs to \( P_f \) then

\[
\sup \{ \Phi(\theta) : \theta \in P_f \} \geq \Phi(\mu) = \int f_3 \, d\mu \geq 0.
\]
By Theorem 2, this supremum is achieved at a probability \( v \) which has the properties described in Theorem 1. Since we are in the case of two constraints, the properties are described in Corollary 1 part B and this gives in particular the announced result (in fact, Corollary 1 part B gives even more information according to the number of saturated constraints).

4. Application to a geometric problem

An hyperplane being given, a natural question on non-symmetric convex bodies is to compare the volume of the following three parallel sections (which coincide if the body is symmetric): the one passing through the center of gravity, the one separating the convex body into two parts of equal volume and the one of maximal volume. Such a study was started in [14] (see also [7] for a generalization). The aim of this part is to give a complete solution to this problem. More precisely, let \( K \) be a convex body (i.e. a compact convex set with no empty interior) in \( \mathbb{R}^n \), denote by \( g_K = \int_K x \, dx / \text{Vol}(K) \) its center of gravity. Let \( H \) be a fixed hyperplane passing through the origin in \( \mathbb{R}^n \) and define \( m_K \in \mathbb{R}^n \) such that the hyperplane \( H + m_K \) cuts \( K \) into two parts of equal volume. It was proved in [14] that

\[
\max_{x \in \mathbb{R}^n} \text{Vol}_{n-1}(K \cap (x + H)) \leq (1 + 1/n)^{n-1} \text{Vol}_{n-1}(K \cap (g_K + H))
\]

with equality when \( K \) is a convex cone with a basis parallel to \( H \). Using the methods developed in [1] it not difficult to see that

\[
\max_{x \in \mathbb{R}^n} \text{Vol}_{n-1}(K \cap (x + H)) \leq 2^{1 - 1/n} \text{Vol}_{n-1}(K \cap (m_K + H))
\]

and using the methods of [7] and [15], it can be shown that

\[
\text{Vol}_{n-1}(K \cap (m_K + H)) \leq 2^{1/n - 1} (1 + 1/n)^{n-1} \text{Vol}_{n-1}(K \cap (g_K + H)),
\]

with equality when \( K \) is a convex cone with a basis parallel to \( H \) (in both inequalities). Hence the last remaining question is to determine the quantity

\[
q = \sup \left\{ \frac{\text{Vol}_{n-1}(K \cap (g_K + H))}{\text{Vol}_{n-1}(K \cap (m_K + H))} \right\}, \tag{1}
\]

where the supremum is taken over every convex body \( K \) in \( \mathbb{R}^n \), every hyperplane \( H \) and every integer \( n \geq 2 \). From the preceding inequalities, \( q \leq 2 \). Since the convex cone is the extremal case in these inequalities, it is not easy to imagine a non-symmetric convex body such that \( \text{Vol}_{n-1}(K \cap (g_K + H)) \) is greater than \( \text{Vol}_{n-1}(K \cap (m_K + H)) \) and one could have conjectured that the extremal case in (1) would be the symmetric
convex body i.e. that $q = 1$. However, if we denote by $c$ the constant
\[ c = \max_{x>0} \frac{2}{1 + e^{-x}} + \sqrt{1 - (1 + x)e^{-x}}, \tag{2} \]
where a computer calculation gives $c = 1.0629955\ldots$, then Theorem 3 asserts that $q = c$. Notice that the maximum in (2) is reached at a unique positive real number $x_0$. Notice also that, on the way, we prove a more general result on sections by slabs instead of hyperplanes.

**Theorem 3.** Let $n$ be a positive integer, $H$ a hyperplane passing through the origin in $\mathbb{R}^n$. Let $K$ be a convex body in $\mathbb{R}^n$, let $g_K = \int_K x \, dx / \text{Vol}(K)$ be its center of gravity and let $m_K \in \mathbb{R}^n$ such that the hyperplane $H + m_K$ cuts $K$ into two parts of equal volume then
\[ \text{Vol}_{n-1}(K \cap (g_K + H)) \leq c \text{Vol}_{n-1}(K \cap (m_K + H)), \]
where $c$ is defined in (2). More generally, let $h > 0$ and let $S_h$ be the slab $S_h = \{x \in \mathbb{R}^n ; 0 \leq \langle x, u \rangle \leq h\}$ of width $h$ and orthogonal to $u$ where $u$ is defined such that $u \in S^{n-1}$, $H = u^\perp$ and $\langle g_K - m_K, u \rangle \geq 0$. Then
\[ \text{Vol}_n(K \cap (g_K + S_h)) \leq c \text{Vol}_n(K \cap (m_K + S_h)). \]
Moreover, there exists a sequence of convex bodies $(K_n)_{n \in \mathbb{N}}$ such that $K_n \subset \mathbb{R}^n$ and
\[ \lim_{n \to \infty} \frac{\text{Vol}_{n-1}(K_n \cap (g_{K_n} + H))}{\text{Vol}_{n-1}(K_n \cap (m_{K_n} + H))} = c, \tag{3} \]
where $K_n$ is defined as the union of a cone in the direction $u$ and a truncature of another cone in the direction $-u$, with the same basis parallel to $u^\perp = H$.

**Remark.** In fact, we can describe more precisely the sequence of convex bodies $(K_n)_{n \in \mathbb{N}}$ that proves the sharpness of our result. Without loss of generality, we can assume that $u = e_1$ and $H = \text{span}(e_2, \ldots, e_n)$. Let $a < 0$ and $C$ be a convex body in $H$ containing the origin. Let $\beta = -x_0/a$, $\alpha = \beta / \sqrt{1 - (1 + x_0)e^{-x_0}}$, where $x_0$ is the positive real number attaining the maximum in (2). Let $C^+_n = \text{conv}(C, \frac{n}{\beta} e_1)$, $C^-_n = \text{conv}(C, -\frac{n}{\beta} e_1) \cap \{x_1 \geq a\}$ and define $K_n$ by $K_n = C^-_n \cup C^+_n$. Using the equality case given in Theorem 4, it is easy to check that the sequence $(K_n)$ satisfies (3).

We derive this geometric inequality from a functional inequality valid for every log-concave function on $\mathbb{R}$.

**Theorem 4.** Let $f : \mathbb{R} \to \mathbb{R}^+$ be an integrable log-concave function. Let $g$ be defined by $g = \int_{\mathbb{R}} f(t) \, dt / \int_{\mathbb{R}} f(t) \, dt$ and let $m \in \mathbb{R}$ be such that $\int_{-\infty}^m f((t) \, dt = \int_m^{+\infty} f(t) \, dt$. We have
\[ f(g) \leq c f(m). \tag{4} \]
Moreover, assume that \( m < g \) then for every \( x, y \in \mathbb{R} \) such that \( m \leq x < y \leq g \) and for every \( h > 0 \)
\[
\int_{y}^{y+h} f(t) \, dt \leq c \int_{x}^{x+h} f(t) \, dt .
\]
(5)
These inequalities are optimal since we have \( f(g) = c f(m) \) for the functions \( f_a \) defined by \( f_a(t) = e^{ht} \chi_{(a,0]} + e^{-at} \chi_{[0, +\infty)} \) where \( a \) is an arbitrary negative real number, and
\[
\beta = -x_0/a \quad \text{and} \quad \alpha = \beta/\sqrt{1 - (1 + x_0)e^{-x_0}}.
\]

We first derive Theorem 3 from Theorem 4.

**Proof of Theorem 3.** Let \( f \) be the parallel section function defined by \( f(t) = \text{Vol}_{n-1} \left( (tu + u^\perp) \cap K \right) \) for every \( t \in \mathbb{R} \). It is well known, using the Brunn-Minkowski inequality, that \( f \) is a log-concave function on \( \mathbb{R} \). Moreover using Fubini, we have
\[
\langle g_K, u \rangle = g = \int_{\mathbb{R}} t f(t) \, dt / \int_{\mathbb{R}} f(t) \, dt,
\]
and if we define \( m \) by \( \int_{-\infty}^{m} f(t) \, dt = \int_{m}^{+\infty} f(t) \, dt \) then
\[
\text{Vol}_n(K \cap (mK + S_h)) = \int_{m}^{m+h} f(t) \, dt
\]
and if we define \( m \) by \( \int_{-\infty}^{m} f(t) \, dt = \int_{m}^{+\infty} f(t) \, dt \) then
\[
\text{Vol}_n(K \cap (mK + S_h)) = \int_{m}^{m+h} f(t) \, dt \quad \text{and} \quad \langle mK, u \rangle = m
\]
and then Theorem 3 follows easily from Theorem 4. \( \square \)

The purpose of the end of this section is to prove Theorem 4. We start by proving that it is easy to deduce (4) from (5). Indeed if \( g = m \), there is nothing to do since \( c \geq 1 \) and if \( g < m \), then replacing \( f(t) \) by \( \tilde{f}(t) = f(-t) \) we get that \( \tilde{g} > \tilde{m} \), \( \tilde{f}(\tilde{g}) = f(g) \) and \( \tilde{f}(\tilde{m}) = f(m) \) so that there is no loss of generality in assuming that \( m < g \). Now applying inequality (5) for \( x = m \), \( y = g \), and sending \( h \) to 0, we get inequality (4).

It remains to prove (5). We will divide its proof in two different parts. The first important one is to express our problem in term of a constrained optimization problem (with two constraints) on log-concave probabilities supported by a finite interval in \( \mathbb{R} \). Hence, using Theorem 1, we will manage to reduce the study of these inequalities to a particular class of densities of our log-concave probabilities. As it is now a classical constrained optimization problem, we will be able to analyze more easily the inequality that we will have to prove for such densities. For a function in this class, we will prove in the next Lemma that the functions described above in the Theorem are the optimal cases of the relevant inequality.
Lemma 3. Define by $A_2$ the set of log-concave functions $f : \mathbb{R} \to \mathbb{R}^+$ defined by
$$f(t) = \gamma e^{\beta(t-u)}[a,u] + \gamma e^{-\alpha(t-u)}[u,b]$$
for some $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and such that $a < \mu < g < u < b$ where \( g = \int_{\mu}^t f / \int f \) and \( \int_{-\infty}^{m} f = \int_{m}^{+\infty} f \).

For every function $f$ in $A_2$,
$$f(g) \leq c f(m)$$
and there is equality if $b = +\infty$, $\beta = -x_0/a$ and $\alpha = \beta/\sqrt{1-(1+x_0)e^{-x_0}}$.

Proof. To simplify the notations, we may and do assume that $g = 0$ and $\gamma = e^{\beta u}$ so that we have
$$f(t) = e^{\beta t}[a,u] + e^{-\alpha(t-u)+\beta u}[u,b].$$

The strategy is to show that we can find a new function $\tilde{f} \in A_2$ with
$$\tilde{f}(t) = e^{\beta t}[\tilde{\alpha},0] + e^{-\tilde{\alpha}t}[0,+\infty)$$
with $\tilde{\alpha} < 0$, $\tilde{\alpha} > 0$

such that $\tilde{\alpha} = 0$, $\tilde{\mu} < \mu$ and $f(g)/f(m) \leq \tilde{f}(\tilde{\mu})/\tilde{f}(\tilde{\mu})$.

Let $\tilde{\alpha} = \int_{0}^{\tilde{\mu}} f(t) dt$ and let
$$\tilde{f}(t) = e^{\beta t}[\tilde{\alpha},0] + e^{-\tilde{\alpha}t}[0,+\infty),$$
where $\tilde{\alpha}$ will be defined later. From the definition of $\tilde{\alpha}$, we have $\int_{0}^{+\infty} \tilde{f}(t) dt = 1/\tilde{\alpha} = \int_{0}^{+\infty} f(t) dt$. Hence the sign of the function $\tilde{f} - f$ has to change at some point $t_0 \in (0, +\infty)$. Using that $\log(\tilde{f})$ is affine on $(0, +\infty)$, we get
$$(t - t_0)(f(t) - \tilde{f}(t)) \leq 0 \quad \forall t \geq 0.$$ Integrating this inequality on $(0, +\infty)$, this gives
$$\int_{0}^{b} tf(t) dt < \int_{0}^{+\infty} t \tilde{f}(t) dt = 1/\tilde{\alpha}^2.$$ Moreover since $m < 0$, we have
$$\frac{1}{\tilde{\alpha}} = \int_{0}^{+\infty} \tilde{f}(t) dt = \int_{0}^{b} f(t) dt < \int_{a}^{0} f(t) dt = \int_{a}^{0} e^{\beta t} dt \leq \frac{1}{\beta}.$$
Now we define \( \rho(x) = \int_x^0 te^{\beta t} \, dt \), for \( x \leq 0 \). The function \( \rho \) is continuous, negative, increasing on \( (-\infty, 0) \) and \( \lim_{-\infty} \rho = -1/\beta^2 \). Moreover, from inequalities (6) and (7)

\[
\rho(a) = \int_a^0 te^{\beta t} \, dt = -\int_b^0 tf(t) \, dt > -\frac{1}{\tilde{a}^2} > -\frac{1}{\beta^2}.
\]

Then there exists \( \tilde{a} < a \) such that \( \rho(\tilde{a}) = -1/\tilde{a}^2 \). Hence \( \tilde{a} \) is chosen so that \( \int_{\mathbb{R}} t \tilde{f}(t) \, dt = 0 \), which gives \( \tilde{g} = g = 0 \). From the definition of \( \tilde{f} \) we have

\[
\int_{m}^{+\infty} \tilde{f}(t) \, dt = \int_{m}^{+\infty} f(t) \, dt = \int_{a}^{m} f(t) \, dt < \int_{\tilde{a}}^{m} \tilde{f}(t) \, dt.
\]

Hence \( \tilde{m} \leq m \), thus we get \( \tilde{f}(\tilde{g}) = 1 = f(g) \) and \( \tilde{f}(\tilde{m}) = e^{\beta \tilde{m}} < e^{\beta m} = f(m) \). Therefore \( \tilde{f} \in A_2 \) and the proof of \( f(g) \leq e f(m) \) reduces to the proof of \( \tilde{f}(g) \leq e \tilde{f}(m) \).

So we managed to reduce to the case of functions \( f \) of the form

\[
f(t) = e^{\beta t} \chi_{[a,0]} + e^{-\beta t} \chi_{(0,+\infty)} \quad \text{with } a < 0, \ \beta > 0
\]

and such that \( \int_{\mathbb{R}} tf(t) \, dt = 0 \) and \( \int_{-\infty}^{0} f(t) \, dt > \int_{0}^{+\infty} f(t) \, dt \). Replacing \( f \) by its expression this gives

\[
\frac{\beta^2}{\tilde{x}^2} = 1 - (1 - \beta a)e^{\beta a} \quad \text{and} \quad \frac{\beta}{\tilde{x}} < 1 - e^{\beta a}.
\]

Actually if the equality is satisfied, it immediately implies that the inequality is also satisfied. By definition of \( m \), we get

\[
f(m) = e^{\beta m} = \frac{1}{2} \left( 1 + e^{\beta a} + \frac{\beta}{\tilde{x}} \right) = \frac{1}{2} \left( 1 + e^{\beta a} + \sqrt{1 - (1 - \beta a)e^{\beta a}} \right).
\]

Since \( f(g) = f(0) = 1 \), we finally obtain our result,

\[
\frac{f(g)}{f(m)} \leq e = \max_{x > 0} 2/(1 + e^{-x} + \sqrt{1 - (1 + x)e^{-x}})
\]

with equality for the functions described in the Lemma. \( \square \)

**Proof of inequality (5) of Theorem 4.** For any log-concave probability \( \theta \) on \( \mathbb{R} \), we define \( g_\theta = \int_{\mathbb{R}} t \theta(t) \, dt \) and \( m_\theta \in \mathbb{R} \) by \( \theta((-\infty, m_\theta]) \geq 1/2 \) and \( \theta([m_\theta, +\infty)) \geq 1/2 \). Let \( f_0 \) be a fixed function satisfying the hypothesis of Theorem 4. Because of the homogeneity of the problem, we may assume, without loss of generality that \( \int_{\mathbb{R}} f_0(t) \, dt = 1 \).
Using continuity we may also assume that $\text{supp}(f_0)$ is a compact interval $K$ in $\mathbb{R}$. We denote by $\mu$ the log-concave probability on $\mathbb{R}$ whose density is $f_0$. Let $x, y \in \mathbb{R}$ such that $m_\mu \leq x < y \leq g_\mu$. We define two upper semi-continuous functions $f_1(t) = t - y$, $f_2 = \chi_{(-\infty,x]} - \chi_{(x, +\infty)}$ and denote by $P_{f_1, f_2}$ the set of log-concave probabilities $\theta$ supported by $K$ such that $\int f_1 d\theta \geq 0$ and $\int f_2 d\theta \geq 0$. This means that $\theta$ satisfies $g_\theta \geq y$ and $m_\theta \leq x$. In particular, $\mu \in P_{f_1, f_2}$ and no Dirac measure belongs to $P_{f_1, f_2}$.

Let $h > 0$ be fixed and define the function $\Phi$ on the set of probabilities supported by $K$ by

$$\Phi(\theta) = \theta([y, y + h]) - c \theta((x, x + h)).$$

The theorem asserts that $\Phi(\mu) \leq 0$. We will prove that

$$\Phi(\mu) \leq \sup_{\theta \in P_{f_1, f_2}} \Phi(\theta) \leq 0. \quad (8)$$

Since $\Phi$ is convex and upper semi-continuous, by Theorem 2, the supremum of $\Phi$ is attained at an extreme point $v$ of $\text{conv}(P_{f_1, f_2})$. Since no Dirac measure belongs to $P_{f_1, f_2}$, the measure $v$ is supported by a segment $[a, b] \subset \mathbb{R}$ and there exists a convex function $V$ on $(a, b)$ such that $dv = e^{-V} dx$ on $[a, b]$. Let $k = \# \{i \in \{1, 2\} : \int f_i dv = 0 \}$. From Theorem 1 (or Corollary 1 in the case $n = 1$), there are two cases: either $k = 1$ and $V$ is affine or $k = 2$ and $V$ is the maximum of two affine functions. Denote by $F$ the repartition function of $v$, i.e. $F(x) = v((-\infty, x]) = \int_{-\infty}^{x} f(t) dt$.

If $V$ is affine then $f$ is monotone. If $f$ is non-decreasing on $[a, b]$ then $F$ is convex on $[a, b]$. From Jensen’s inequality we get

$$F(g) = F\left(\int_{\mathbb{R}} tf(t) dt\right) \leq \int_{\mathbb{R}} F(t) f(t) dt = \frac{1}{2} [F^2]_{+\infty}^{+\infty} = \frac{1}{2} = F(m).$$

Since $F$ is increasing on $[a, b]$, this gives $g \leq m$, hence $v \notin P_{f_1, f_2}$. We get that necessarily $f$ is decreasing on $[a, b]$. But if $f$ is decreasing on $[m, +\infty)$, then $F$ is concave on $[m, +\infty)$. This implies that the function $z \mapsto F(z + h) - F(z)$ is non-increasing on $[m, +\infty)$. Since $x < y \leq g$ and $c > 1$, this gives

$$\Phi(v) = \int_{y}^{y+h} f(t) dt - c \int_{x}^{x+h} f(t) dt \leq (1 - c) \int_{x}^{x+h} f(t) dt < 0.$$  

Therefore inequality (8) is satisfied. This ends the case of monotone functions $f$.

Hence, in the following we may assume that $k = 2$ (which means that both constraints are saturated), $V$ is the maximum of two affine functions, $V$ is not monotone on $[a, b]$ and not decreasing on $[m, +\infty)$. Therefore there exist $\alpha > 0$, $\beta > 0$, $\gamma > 0$
and \( u \in (a, b) \) such that
\[
 f(t) = \gamma e^{\beta(t-u)} \mathcal{X}_{[a,u]} + \gamma e^{-\alpha(t-u)} \mathcal{X}_{(a,b]}.
\]
Moreover \( g_v = y > m_v = x \) and \( x < u \). In the following, we will simplify the notations and denote \( g = g_v = y \) and \( m = m_v = x \).

We start by proving that necessarily, \( f \in A_2 \), i.e. that \( g \leq u \) (the set \( A_2 \) is defined in Lemma 3).

Indeed, suppose that \( g > u \). Let
\[
 h(t) = f(g) \left( \frac{f(m)}{f(g)} \right)^{(t-g)/(m-g)} \quad \text{and} \quad \tilde{f} = \min(f, h)
\]
and denote by \( \tilde{v} \) the probability measure of density \( \tilde{f}/\tilde{f} \). It is easily seen that \( \tilde{v} \) is log-concave, \( \tilde{v} > g \) and \( m_\tilde{v} < m \), hence we have that \( \tilde{v} \in \mathcal{P}_K^{f_1, f_2} \). Moreover, by construction, \( \int_g^{g+h} \tilde{f} = \int_g^{g+h} f \) and \( \int_m^{m+h} \tilde{f} < \int_m^{m+h} f \) which proves that \( \Phi(\tilde{v}) > \Phi(v) \) and contradicts the extremality of \( \Phi(v) \). This proves that \( g \leq u \).

We are now able to prove that
\[
 \frac{v([g, g + h])}{f(g)} \leq \frac{v([m, m + h])}{f(m)}.
\]
(9)

First assume that \( g + h \leq b \). Then we have three cases:

1. If \( h \leq u - g \) we get
\[
 \frac{v([g, g + h])}{f(g)} = \int_g^{g+h} e^{\beta(t-g)} dt = \frac{e^{\beta h} - 1}{\beta} = \frac{v([m, m + h])}{f(m)},
\]
which is equality in (9).

2. If \( u - g \leq h \leq u - m \) then the calculation for \( v([m, m + h]) \) is the same, but since \( -\alpha < \beta \), we get
\[
 \frac{v([g, g + h])}{f(g)} = \int_g^{u} e^{\beta(t-g)} dt + \int_u^{g+h} e^{-\alpha(t-g)} dt
\]
\[
 \leq \int_g^{g+h} e^{\beta(t-g)} dt = \frac{v([m, m + h])}{f(m)}.
\]

3. If \( h \geq u - m \), then we first apply the case (2) to \( h = u - m \) to get
\[
 \frac{v([g, g + u - m])}{f(g)} \leq \frac{v([m, u])}{f(m)}.
\]
Since $f$ is decreasing on $[u, b]$, then $F$ is concave on $[u, b]$. Using that $g > m$ and $f(g) \geq f(m)$ this implies that

$$v([g + u - m, g + h]) f(g) \leq v([u, m + h]) f(g) \leq v([u, m + h]) f(m).$$

Summing the two inequalities gives (9).

In particular, we have proved that

$$\frac{v([g, b])}{f(g)} \leq \frac{v([m, m + b - g])}{f(m)}$$

therefore, if $g + h > b$, we get

$$\frac{v([g, g + h])}{f(g)} = \frac{v([g, b])}{f(g)} \leq \frac{v([m, m + b - g])}{f(m)} \leq \frac{v([m, m + h])}{f(m)}$$

and this concludes the proof of (9).

Since $f \in A^2$, we know by Lemma 3 that $f(g) \leq c f(m)$ which proves by (9) that $\Phi(v) \leq 0$ and this concludes the proof of (8). □

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References