Fields defined by locally nilpotent derivations and monomials

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Abstract

We prove a necessary and sufficient condition for certain fields defined by locally nilpotent derivations and monomials to be algebraically closed in a rational function field. This implies that a counterexample to the Fourteenth Problem of Hilbert in dimension four, which was recently given by the author, is obtained as the kernel of a derivation. It was previously unknown only in dimension four whether there exists such a counterexample.

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1. The Fourteenth Problem of Hilbert

The Fourteenth Problem of Hilbert asks the following. Let $K[x] = K[x_1, \ldots, x_m]$ be the polynomial ring in $m$ variables over a field $K$, and $K(x)$ its field of fractions. Suppose that $L$ is a subfield of $K(x)$ containing $K$. Then, is the $K$-subalgebra $L \cap K[x]$ of $K[x]$ finitely generated? In 1958, Nagata [15] gave the first counterexample for $m = 32$. Lower dimensional counterexamples were found by Roberts [18] for $m = 7$, Freudenburg [5] for

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It was previously unknown only for $m = 5$ (see also Mukai [14] and Steinberg [19]). Furthermore, the author recently gave counterexamples for $m = 4$ [11] and for $m = 3$ [12].

The counterexamples of Roberts, Freudenburg and Daigle were obtained by using locally nilpotent derivations of $K[x]$ as follows. A $K$-linear map $D : A \to A$ of a commutative $K$-algebra is called a derivation if $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. We say that $D$ is \textit{locally nilpotent} if, for each $a \in A$, there exists $r \geq 0$ such that $D^r(a) = 0$. For a $K$-subalgebra $B$ of $A$, we will consider the $K$-subalgebra

$$B^D = \{ b \in B \mid D(b) = 0 \}$$

of $B$. If $D$ is a derivation of $K[x]$, then $D$ extends uniquely to a derivation of $K(x)$, which we also denote by $D$. Since $K[x]^D = K(x)^D \cap K[x]$, the problem of finite generation of the kernel of a derivation of $K[x]$ is a special case of the Fourteenth Problem of Hilbert, and well studied (cf. [2,3,6–10,13,16,17]). The counterexamples of Roberts, Freudenburg and Daigle can be realized as $K(x)^D$ for locally nilpotent derivations $D$ of $K[x]$. Note that, if $D$ is a derivation of $K(x)$, then there exists a derivation $D'$ of $K[x]$ such that $K[x]^D = K[x]^{D'}$. Actually, for $h \in K(x) \setminus \{0\}$ with $hD(x_i) \in K[x]$ for each $i$, the derivation $D'$ defined by $D'(f) = hD(f)$ for each $f$ has this property. Hence, the problem of the finite generation of $K[x]^D$ for a derivation $D$ of $K[x]$ is the same as that for a derivation $D$ of $K(x)$.

On what follows, we will assume that the characteristic of $K$ is zero. Then, due to Zariski [21], the kernel of any derivation of $K[x]$ is finitely generated if $m \leq 3$ (see also [16]). On the other hand, Daigle and Freudenburg [1] showed that there exists a locally nilpotent derivation of $K[x]$ whose kernel is not finitely generated for each $m \geq 5$. It was previously unknown only for $m = 4$ whether there exists a derivation $D$ of $K[x]$ whose kernel $K[x]^D$ is not finitely generated, even when $D$ is not locally nilpotent. In the present paper, we will show that the counterexample for $m = 4$ in [11] can be realized as the kernel of a derivation, as a consequence of our main result. Thereby, the problem of finite generation of the kernel of a derivation of $K[x]$ is settled for all $m$.

The counterexamples for $m = 4$ in [11] and for $m = 3$ in [12] can be uniformly obtained by the following construction (see also [10]), although it was not clearly mentioned in these papers. Let $K[y] = K[y_1, \ldots, y_n]$ be the polynomial ring in $n$ variables over $K$, and $D$ a locally nilpotent derivation of $K[y]$. For $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we denote the monomial $x^a = x_1^{a_1} \cdots x_m^{a_m}$. Let $\Omega = (\omega_1, \ldots, \omega_n)$ be an element of $(\mathbb{Z}^m)^n$, and $\Phi_{\Omega} : K[y] \to K(x)$ the homomorphism defined by $y_i \mapsto x^{\omega_i}$ for each $i$. We will always assume that $D$ and $\Omega$ satisfy $K[y]^D \cap \ker \Phi_{\Omega} = \{0\}$. Then, define $K(D, \Omega)$ to be the field of fractions of $\Phi_{\Omega}(K[y]^D)$. The rank of $\Omega$ is defined as the dimension of the $\mathbb{R}$-vector space $\sum_{i=1}^n \mathbb{R}\omega_i$. Note that the rank of $\Omega$ is $n$ if and only if $\Phi_{\Omega}$ is injective. If this is the case, then $K[y]^D \cap \ker \Phi_{\Omega} = \{0\}$.

Using this construction, the main results of [11,12] can be restated as follows. Let $m = n = 4$, $D_1$ the locally nilpotent derivation of $K[y]$ defined by $D_1(y_i) = 1$ for each $i$, and $\Omega_1 = (\omega_1, \ldots, \omega_4) \in (\mathbb{Z}^4)^4$, where

$$\omega_1 = (-\omega_{1,1}, \omega_{1,2}, \omega_{1,3}, \omega_{1,4}), \quad \omega_2 = (\omega_{2,1}, -\omega_{2,2}, \omega_{2,3}, \omega_{2,4}), \quad \omega_3 = (\omega_{3,1}, \omega_{3,2}, -\omega_{3,3}, \omega_{3,4})$$
and \( \omega_4 = (0, 0, 0, \omega_{4,4}) \) such that \( \omega_{i,j} > 0, \omega_{i,4} \geq 0 \) for \( 1 \leq i, j \leq 3 \) and \( \omega_{4,4} > 0 \).

**Theorem 1.1** [11, Theorem 1.1]. If
\[
\frac{\omega_{1,1}}{\omega_{1,1} + \min\{\omega_{2,1}, \omega_{3,1}\}} + \frac{\omega_{2,2}}{\omega_{2,2} + \min\{\omega_{3,2}, \omega_{1,2}\}} + \frac{\omega_{3,3}}{\omega_{3,3} + \min\{\omega_{1,3}, \omega_{2,3}\}} < 1, \tag{1.2}
\]
then \( K(D_1, \Omega_1) \cap K[x] \) is not finitely generated. Moreover, \( K(D_1, \Omega_1) \cap K[x] \) is not contained in \( K[x]^{D'} \) for any nonzero locally nilpotent derivation \( D' \) of \( K[x] \).

We note that the rank of \( \Omega_1 \) is four if (1.2) holds (see [11, Section 2]). Hence, \( K[y]^{D_1} \cap \ker \Phi_{\Omega_1} = \{0\} \).

The counterexample for \( m = 3 \) in [12] is obtained as follows. Let \( m = 3, n = 4, D_2 \) the locally nilpotent derivation of \( K[y] \) defined by \( D_2(y_1) = D_2(y_2) = D_2(y_3) = 1, D_2(y_4) = y_1 \), and
\[
\Omega_2 = \left( (-\omega_{1,1}, \omega_{1,2}, 0), (\omega_{2,1}, -\omega_{2,2}, 0), (0, 0, 1), (-\omega_{1,1} + \omega_{2,1}, \omega_{1,2} - \omega_{2,2}, 0) \right),
\]
an element of \( (Z^3)^4 \), where \( \omega_{i,j} > 0 \) for each \( i, j \). Then, \( \ker \Phi_{\Omega_2} = (y_1 y_2 - y_4) K[y] \) and \( D(y_1 y_2 - y_4) = y_2 \). This implies that \( K[y]^{D_2} \cap \ker \Phi_{\Omega_2} = \{0\} \) by Lemma 2.3(i) below.

**Theorem 1.2** [12, Theorem 1.1]. If
\[
\frac{\omega_{1,1}}{\omega_{1,1} + \omega_{2,1}} + \frac{\omega_{2,2}}{\omega_{2,2} + \omega_{1,2}} < \frac{1}{2}, \tag{1.3}
\]
then \( K(D_2, \Omega_2) \cap K[x] \) is not finitely generated.

Although the field \( K(D, \Omega) \) is defined by using the kernel of a locally nilpotent derivation, it is not necessarily equal to \( K(x)^E \) for a derivation \( E \) of \( K[x] \). So, it is interesting to find a condition on \( D \) and \( \Omega \) under which the field \( K(D, \Omega) \) is equal to \( K(x)^E \) for a derivation \( E \) of \( K[x] \). The purpose of this paper is to give a necessary and sufficient condition on \( D \) and \( \Omega \) for \( K(D, \Omega) \) to be algebraically closed in \( K(x) \). By Theorem 1.3 below, our result implies a necessary and sufficient condition for \( K(D, \Omega) \) to be \( K(x)^E \) for a derivation \( E \) of \( K[x] \).

**Theorem 1.3** (Derksen [3], Nowicki [17], Suzuki [20]). Assume that the characteristic of \( K \) is zero. For an intermediate field \( K \subset L \subset K(x) \), there exists a derivation \( E \) of \( K[x] \) such that \( L = K(x)^E \) if and only if \( L \) is algebraically closed in \( K(x) \).

Here is our main result.

**Theorem 1.4.** Assume that the rank of \( \Omega \) is \( n \). Then, \( K(D, \Omega) \) is algebraically closed in \( K(x) \) if and only if \( \left( \sum_{i \in I_D} \mathbf{R} \omega_i \right) \cap \mathbb{Z}^n = \sum_{i \in I_D} \mathbf{Z} \omega_i \), where \( I_D = \{ i \mid D(y_i) = 0 \} \).
Now, consider the field $K(D_1, \Omega_1)$ defined above with $\Omega_1$ satisfying (1.2). By definition, $I_{D_1} = \emptyset$. Furthermore, the rank of $\Omega_1$ is four as mentioned. Hence, $K(D_1, \Omega_1)$ is algebraically closed in $K(x)$ by Theorem 1.4. By Theorem 1.3, there exists a derivation $E$ of $K[x]$ such that $K(x)^E = K(D_1, \Omega_1)$, and its kernel $K[x]^E = K(x)^E \cap K[x]$ is not finitely generated by Theorem 1.1. Thus, we have the following.

**Corollary 1.5.** There exists a derivation $E$ of $K[x]$ for $m = 4$ whose kernel $K[x]^E$ is not finitely generated over $K$.

We may also prove Corollary 1.5 without assuming Theorem 1.3 as follows. Let $E$ be a derivation of $K[x]$ such that $E(\Phi_{\Omega_1}(f)) = h\Phi_{\Omega_1}(D_1(f))$ for each $f \in K[y]$ for some $h \in K(x) \setminus \{0\}$. We show that $K(x)^E = K(D_1, \Omega_1)$. The condition implies that $K(D_1, \Omega_1) \subset K(x)^E$. Since the characteristic of $K$ is zero, the transcendence degree of $K(x)^E$ over $K$ is less than four. On the other hand, the transcendence degree of $K(D_1, \Omega_1)$ over $K$ is three, since that of $K[y]^{D_1}$ is three by Lemma 2.5 below and $\Phi_{\Omega_1}$ is injective. Hence, $K(x)^E$ is algebraic over $K(D_1, \Omega_1)$. Since $K(D_1, \Omega_1)$ is algebraically closed in $K(x)$ by Theorem 1.4, we have $K(D_1, \Omega_1) = K(x)^E$. In particular, $K[x]^E$ is not finitely generated.

For example, let $E$ be the derivation of $K[x]$ defined by

\[
E(x_1) = tx_1x_3^{t+1} + tx_1x_2^{t+1} + (1 - t)x_1^{t+2}, \\
E(x_2) = tx_2x_1^{t+1} + tx_2x_3^{t+1} + (1 - t)x_2^{t+2}, \\
E(x_3) = tx_3x_2^{t+1} + tx_3x_1^{t+1} + (1 - t)x_3^{t+2}, \\
E(x_4) = (2t^2 + t - 1)x_1^tx_2^tx_3^t , \tag{1.4}
\]

and $\Omega_1 = ((-1, t, t, 0), (t, -1, t, 0), (t, t, -1, 0), (0, 0, 0, 1))$, where $t \in \mathbb{Z}$ with $t \geq 3$. Then, by straightforward computation, we get

\[
E(\Phi_{\Omega_1}(f)) = (2t^2 + t - 1)x_1^tx_2^tx_3^t \Phi_{\Omega_1}(D_1(f)) \tag{1.5}
\]

for each $f \in K[y]$. Moreover, $\Omega_1$ satisfies (1.2). Hence, $K[x]^E$ is not finitely generated.

By the latter part of Theorem 1.1, $K(D_1, \Omega_1) \cap K[x]$ cannot be equal to the kernel $K[x]^E$ of any locally nilpotent derivation of $K[x]$ if $\Omega_1$ satisfies (1.2). For $m = 4$, the problem of finite generation of the kernel of a locally nilpotent derivation of $K[x]$ is still open (see [2] for a partial positive result).

If a derivation $\tilde{D}$ of $K[y]$ is not locally nilpotent, then the field $K(\tilde{D}, \Omega)$ of fractions of $\Phi_{\Omega}(K[y])$ is not always algebraically closed in $K(x)$ even if the rank of $\Omega$ is $n$ and $I_{\tilde{D}} = \emptyset$. Consider the case where $m = n = 2$, $\Omega = ((1, -1), (0, 2))$, and $\tilde{D}$ is the derivation of $K[y]$ defined by $\tilde{D}(y_1) = 1 - y_1$, $\tilde{D}(y_2) = 2y_2$. Then, the rank of $\Omega$ is two and $I_{\tilde{D}} = \emptyset$. Furthermore, $f = y_2 - 2y_1y_2 + y_1^2y_2$ is in $K[y]^{\tilde{D}}$, so $\Phi_{\Omega}(f) = (x_2 - x_1)^2$ is in $K(\tilde{D}, \Omega)$. On the other hand, $x_2 - x_1$ is not contained in $K(x_1x_2^{-1}, x_2^2)$. Since $K(x_1x_2^{-1}, x_2^2)$ is the field of fractions of $\Phi_{\Omega}(K[y])$, we have $K(\tilde{D}, \Omega) \subset K(x_1x_2^{-1}, x_2^2)$. Hence, $x_2 - x_1$ is not contained in $K(\tilde{D}, \Omega)$. Therefore, $K(\tilde{D}, \Omega)$ is not algebraically closed in $K(x)$. 
In the case where the rank of $\Omega$ is less than $n$, we have the following.

**Theorem 1.6.** Assume that the rank of $\Omega$ is less than $n$ and $K[y]^D \cap \ker \Phi_\Omega = \{0\}$. Then, $K(D, \Omega)$ is algebraically closed in $K(x)$ if and only if $(\sum_{i=1}^n R_{\omega_i}) \cap \mathbb{Z}^m = \sum_{i=1}^n \mathbb{Z}\omega_i$ and $D^2(u) = 0$ for some $u \in \ker \Phi_\Omega \setminus \{0\}$. If this is the case, then $K(D, \Omega) = K(x^{\omega_1}, \ldots, x^{\omega_n})$.

We will show Theorem 1.4 in Section 2 after proving some lemmas. Theorem 1.6 will be shown in Section 3.

### 2. Fibers of a morphism

First, we show some properties of the field $K(D, \Omega)$ for $D$ and $\Omega$.

Let $r$ be the rank of the $\mathbb{Z}$-module $\sum_{i=1}^n \mathbb{Z}\omega_i$. Then, there exist a $\mathbb{Z}$-basis $\{b_1, \ldots, b_m\}$ of $\mathbb{Z}^m$ and positive integers $v_1, \ldots, v_r$ such that $\sum_{i=1}^n \mathbb{Z}\omega_i = \sum_{i=1}^r v_i \mathbb{Z}b_i$. Let $e_1, \ldots, e_m$ be the coordinate unit vectors of $\mathbb{Z}^m$. For any $J \subset \{1, \ldots, n\}$, the condition $(\sum_{i \in J} R_{\omega_i}) \cap \mathbb{Z}^m = \sum_{i \in J} \mathbb{Z}\omega_i$ is equivalent to the condition $(\sum_{i \in J} R_{\omega_i}) \cap \mathbb{Z}^m = \sum_{i \in J} \mathbb{Z}\omega_i$. It follows that $\Phi_\Omega' = \psi \circ \Phi_\Omega$. In particular, $K[y]^D \cap \ker \Phi_\Omega' = \{0\}$, $\psi(K(D, \Omega)) = K(D, \Omega')$ and $\psi(K(x^{\omega_1}, \ldots, x^{\omega_n})) = K(x^{\omega_1'}, \ldots, x^{\omega_n'})$.

**Lemma 2.2.** The field $K(x^{\omega_1}, \ldots, x^{\omega_n})$ is algebraically closed in $K(x)$ if and only if $(\sum_{i=1}^n R_{\omega_i}) \cap \mathbb{Z}^m = \sum_{i=1}^n \mathbb{Z}\omega_i$.

**Proof.** By Lemma 2.1, we may assume that $\sum_{i=1}^n \mathbb{Z}\omega_i = \sum_{i=1}^r v_i \mathbb{Z}b_i$ for some positive integers $v_1, \ldots, v_r$, by replacing $\omega$ with $\omega'$ if necessary. Then, we have $K(x^{\omega_1}, \ldots, x^{\omega_n}) = K(x_1^{v_1}, \ldots, x_r^{v_r})$. Note that $(\sum_{i=1}^n R_{\omega_i}) \cap \mathbb{Z}^m = \sum_{i=1}^n \mathbb{Z}\omega_i$ if and only if $v_1 = \cdots = v_r = 1$. If this is the case, then $K(x)$ is the rational function field of $x_{r+1}, \ldots, x_m$ over $K(x^{\omega_1}, \ldots, x^{\omega_n})$. Hence, $K(x^{\omega_1}, \ldots, x^{\omega_n})$ is algebraically closed in $K(x)$. Conversely, if $v_i > 1$ for some $i$, then $x_i \not\in K(x_1^{v_1}, \ldots, x_r^{v_r})$ and $x_i^{v_i} \in K(x_1^{v_1}, \ldots, x_r^{v_r})$. Hence, $K(x^{\omega_1}, \ldots, x^{\omega_n})$ is not algebraically closed in $K(x)$.

The following fact is well known (see for example [4, Chapter 1.3]).

**Lemma 2.3.** Let $D$ be a locally nilpotent derivation of a $K$-domain $A$.

(i) If $D(ab) = 0$ for $a, b \in A \setminus \{0\}$, then $D(a), D(b) = 0$.

(ii) If $B$ is the field of fractions of $A$, then $B^D$ is equal to the field of fractions of $A^D$. 

In particular, if $y_1^{b_1} \cdots y_n^{b_n}$ belongs to $K(y)^D$ for $b_1, \ldots, b_n \in \mathbb{Z}$, then $b_i \neq 0$ implies that $i \in I_D$. Indeed, $y_1^{b_1} \cdots y_n^{b_n} = g_1/g_2$ for some $g_1, g_2 \in K[y]^D$ by Lemma 2.3(ii), and $y_i$ must be a factor of $g_1$ or $g_2$. Hence, $y_i$ is in $K[y]^D$ by Lemma 2.3(i).

Let $K[y]'$ be the localization of $K[y]$ by the prime ideal ker $\Phi_\Omega$. If the rank of $\Omega$ is $n$, then $K[y]'$ is equal to the field $K(y)$ of fractions of $K[y]$. The homomorphism $\Phi_\Omega$ can be extended to a homomorphism $\tilde{\Phi}_\Omega : K[y]' \to K(x)$ naturally.

**Lemma 2.4.** It follows that $K(y)^D \subset K[y]'$ and $\tilde{\Phi}_\Omega(K(y)^D) = K(D, \Omega)$.

**Proof.** Take any $f \in K(y)^D$. Then, there exist $g, h \in K[y]^D$ such that $f = g/h$ by Lemma 2.3(ii). By the assumption that ker $\Phi_\Omega \cap K[y]^D = \{0\}$, we have $h \notin \ker \Phi_\Omega$. Hence, $f$ is in $K[y]'$, and so $K(y)^D \subset K[y]'$. The latter part is readily verified. □

By Lemma 2.4, $K(D, \Omega)$ is isomorphic to $K(y)^D$, although $K(D, \Omega)$ itself is not equal to $K(x)^E$ for a derivation $E$ of $K[x]$ in general.

The rest of this section is devoted to the proof of Theorem 1.4. First, we show the only if part. Suppose to the contrary that there exists $x$ in $K(D, \Omega)$ such that $\Phi_\Omega(x) = 1$. Then, $I_D$ is in $K(D, \Omega)$. Since $K(D, \Omega) = \tilde{\Phi}_\Omega(K(y)^D)$ by Lemma 2.4, there exists $g \in K(y)^D$ such that $\tilde{\Phi}_\Omega(g) = x$. We show that $g$ is a monomial. Write $g = g_1/g_2$, where $g_1, g_2 \in K[y]$. Then, $x^d \Phi_\Omega(g_2) = \Phi_\Omega(g_1)$. Since $\Phi_\Omega(g_1)$ and $\Phi_\Omega(g_2)$ are Laurent polynomials in $x_1, \ldots, x_m$, there appears $\Phi_\Omega(g_i)$ a monomial $p_i$ for $i = 1, 2$ such that $x^d p_2 = p_1$. By the definition of $\Phi_\Omega$, we have $p_i = \Phi_\Omega(q_i)$ for some monomial $q_i$ appearing in $g_i$ for $i = 1, 2$. Then, $x^d = \Phi_\Omega(q_1/q_2)$. Since the rank of $\Omega$ is $n$, the map $\Phi_\Omega$ is injective. Thus, $g = q_1/q_2$, so $g$ is a monomial. Write $g = y_1^{b_1} \cdots y_n^{b_n}$, where $b_i \in \mathbb{Z}$ for each $i$. Then, $a = \sum_{i=1}^n b_i \omega_i$ by definition. As mentioned after Lemma 2.3, $b_i = 0$ if $i \notin I_D$. Hence, $a$ is in $\sum_{i \in I_D} \mathbb{Z} \omega_i$, a contradiction. Thus, the only if part of Theorem 1.4 is proved.

Now we will prove the converse. First, we show that we may restrict ourselves to the case where $K$ is algebraically closed. Let us denote by $\tilde{K}$ an algebraic closure of $K$, and by $\tilde{D}$ and by $\tilde{\Phi}_\Omega$ the $\tilde{K}$-linear map $\text{id}_{\tilde{K}} \otimes D : \tilde{K} \otimes K K[y] \to \tilde{K} \otimes K K[y]$ and the homomorphism $\text{id}_{\tilde{K}} \otimes \Phi_\Omega : \tilde{K} \otimes K K[y] \to \tilde{K} \otimes K K(x)$ of $\tilde{K}$-algebras, respectively. Note that $\tilde{D}$ is a locally nilpotent derivation of $\tilde{K} \otimes K K[y]$. Let $\tilde{K}(\tilde{D}, \Omega)$ be the field of fractions of $\tilde{\Phi}_\Omega((\tilde{K} \otimes K K[y])^D)$. Then, it follows that $\tilde{K}(\tilde{D}, \Omega) = \tilde{K} \otimes K K(D, \Omega)$. The field $\tilde{K} \otimes K K(D, \Omega)$ is algebraically closed in $\tilde{K} \otimes K K(x)$ if and only if $K(D, \Omega)$ is algebraically closed in $K(x)$. Thus, by replacing $K, D$ and $\Phi_\Omega$ with $\tilde{K}, \tilde{D}$ and $\tilde{\Phi}_\Omega$ if necessary, we may assume that $K$ is algebraically closed.

The following fact is well known (see for example [4, Chapter 1.3]).

**Lemma 2.5.** Let $D$ be a nonzero locally nilpotent derivation of a finitely generated $K$-domain $A$, and $B$ the field of fractions of $A$. If the transcendence degree of $B$ over $K$ is $d$, then that of $B^D$ is $d - 1$. If there exists $s \in A$ such that $D(s) = 1$, then $A$ is equal to the polynomial ring $A^D[s]$ in $s$ over $A^D$, and the $K$-algebra $A^D$ is finitely generated.
In case $D$ is zero, $K(D, \Omega) = K(x^{\omega_1}, \ldots, x^{\omega_n})$. Moreover, $(\sum_{i=1}^n R\omega_i) \cap \mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}\omega_i$ by assumption, since $I_D = \{1, \ldots, n\}$. Thus, the if part of Theorem 1.4 follows from Lemma 2.2 in this case.

Assume that $D$ is not zero. Then, there exists $w \in K[y]$ such that $D^{l-1}(w) \neq 0$ and $D^l(w) = 0$ for some $l \geq 2$. Put $h = D^{l-1}(w)$, $s = D^{l-2}(w)h^{-1}$ and $R = K[y][h^{-1}]$. Then, $D$ extends uniquely to a locally nilpotent derivation of $R$. By Lemma 2.5, $R$ is equal to the polynomial ring $R^D[s]$ in $s$ over $R^D$ and the $K$-algebra $R^D$ is finitely generated, since $D(s) = 1$. By Lemma 2.3(ii), the field of fractions of $R^D$ is equal to $K(y)^D$.

**Lemma 2.6.** Let $f$ be a prime element of $K[y]$ such that $D(f) \neq 0$. Then, $f$ is irreducible over $K(y)^D$, where we regard $f$ as a polynomial in $s$.

**Proof.** Let $A$ be the localization of $K[y]$ by the multiplicatively closed subset $K[y]^D \setminus \{0\}$. Then, $A = K[y]^D[s]$. Since $D(f) \neq 0$, the prime ideal $fK[y]$ of $K[y]$ does not intersect $K[y]^D \setminus \{0\}$ by Lemma 2.3(i). Hence, $fA$ is a prime ideal of $A$. Therefore, the polynomial $f$ in $s$ is irreducible over $K(y)^D$. 

Let $H_i'$ be the discriminant of $y_i$ for each $i$, and $H_{j,k}$ the resultant of $y_j$ and $y_k$ for each $j,k$, where we regard $y_i$, $y_j$ and $y_k$ as polynomials in $s$ over $R^D$. Then, define $H_i$ to be an element of $R^D$ obtained from $H_i'$ by multiplying a power of the leading coefficient of $y_i$ for each $i$. We show that $H = (\prod_i H_i)(\prod_{j,k} H_{j,k})$ is a nonzero element of $R^D$, where the first product is taken over $i$ with $i \notin I_D$ and the second product is taken over $j,k$ with $j,k \notin I_D$ and $j \neq k$. Clearly, $H$ is in $R^D$. By Lemma 2.6, the polynomial $y_i$ in $s$ is irreducible over $K(y)^D$ if $i \notin I_D$. Hence, $H_i$ is not zero, since $K$ is of characteristic zero. By a similar reason, if $H_{j,k}$ were zero, then $y_j/y_k$ would be in $K(y)^D$. Since $j \neq k$, it implies that $j,k \in I_D$, as mentioned after Lemma 2.3. This is a contradiction, so $H_{j,k}$ is not zero. Therefore, $H$ is not zero.

Let $S = K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, \Phi_\Omega(h)^{-1}]$. We study the fibers of the morphism $Spec S \to Spec R^D$ of affine schemes defined from $\tilde{\Phi}_\Omega|_{R^D} : R^D \to S$. The following is the key proposition.

**Proposition 2.7.** Let $P$ be a maximal ideal of $R^D$ which does not contain $H \prod_{i \in I_D} y_i$. Then, $\kappa_P \otimes_{R^D} S$ is an integral domain, where $\kappa_P = R^D / P$.

Proposition 2.7 implies that the fiber of $Spec S \to Spec R^D$ over each closed point contained in some nonempty open subset of $Spec R^D$ is integral.

Here, we recall a fact on algebraic geometry. Let $A$ and $B$ be affine $K$-domains such that $A \subset B$ and $A$ is not algebraically closed in $B$, i.e., the field $A'$ of fractions of $A$ does not contain an element of $B$ which is algebraic over $A'$. Then, the fiber of $Spec B \to Spec A$ over each closed point in some nonempty open subset of $Spec A$ is not connected. This is explained as follows. Assume that $b \in B$ is algebraic over $A'$ of degree $l \geq 2$. Since $K$ is of characteristic zero, the discriminant $d$ of the minimal polynomial of $b$ over $A'$ is a nonzero element of $A'$. Let $P$ be any maximal ideal of $A$ not containing $d_1d_2$, where $d_1, d_2 \in A$ such that $d = d_1/d_2$. Then, there exist $l$ distinct maximal ideals of $A[b]$ which lie over $P$, as $K$ is an algebraically closed field. This means that the fiber of $Spec A[b] \to Spec A$
over each closed point in a nonempty open subset of \( \text{Spec } A \) is not connected. On the other hand, \( \text{Spec } B \rightarrow \text{Spec } A \) is equal to the composite of the dominant morphisms \( \text{Spec } B \rightarrow \text{Spec } A[b] \rightarrow \text{Spec } A \), and \( \text{Spec } A[b] \) and \( \text{Spec } A \) have the same dimension. Hence, the fiber of \( \text{Spec } B \rightarrow \text{Spec } A \) over each closed point in some nonempty open subset of \( \text{Spec } A \) is not connected.

We may prove the if part of Theorem 1.4 as a consequence of Proposition 2.7 and the fact above. Suppose to the contrary that \( K(D, \Omega) \) is not algebraically closed in \( K(x) \). Then, there exists \( s \in S \setminus \{0\} \) such that \( \Phi_\Omega(R^D) \) is not algebraically closed in \( S[s^{-1}] \).

Hence, the fiber of \( \text{Spec } S[s^{-1}] \rightarrow \text{Spec } R^D \) over each closed point in a nonempty open subset of \( \text{Spec } A \) is not connected, as mentioned above. On the other hand, we know from Proposition 2.7 that the fiber of \( \text{Spec } S[s^{-1}] \rightarrow \text{Spec } R^D \) over each closed point in some nonempty open subset of \( \text{Spec } R^D \) is integral. This is a contradiction, and hence \( K(D, \Omega) \) is algebraically closed in \( K(x) \). Therefore, the proof of Theorem 1.4 is completed on the assumption that Proposition 2.7 is true.

To prove Proposition 2.7, we need a lemma. First, note that \( \kappa_P \) is equal to \( K \). Actually, the \( K \)-algebra \( R^D \) is finitely generated by Lemma 2.5, and \( K \) is algebraically closed by assumption. Since \( R = R^D[s] \), the \( \kappa_P \)-algebra \( \kappa_P \otimes_{R^D} R \) is identified with the polynomial ring \( \kappa_P[s] \) in \( s \) over \( \kappa_P \). Let \( \tilde{y}_i = \alpha_i \prod_{j=1}^{p_i} (s - \beta_{i,j}) \) be the image of \( y_i \) in \( \kappa_P[s] \) for each \( i \), where \( p_i \geq 0 \) and \( \alpha_i, \beta_{i,j} \in \kappa_P \) for each \( j \). Then, \( \alpha_i \neq 0 \), since \( y_i \notin P \) if \( i \in I_D \) and \( H_i \notin P \) otherwise. Moreover, \( p_i = 0 \) if and only if \( i \in I_D \).

Since the images of \( H_i \) and \( H_{j,k} \) in \( \kappa_P \) are not zero for any \( i \) and \( j, k \) with \( i, j, k \notin I_D \) and \( j \neq k \), we have \( \beta_{i,j} \neq \beta_{k,l} \) for any \( (i, j) \neq (k, l) \).

By Lemma 2.1, we may assume without loss of generality that \( \sum_{i=1}^{n} Z \omega_i = \sum_{i=1}^{n} v_i Z e_i \) for some positive integers \( v_1, \ldots, v_n \). Let \( (\tau_{i,j})_{i,j} \) be an element of \( GL_n(\mathbb{Z}) \) such that \( \sum_{j=1}^{n} \tau_{i,j} \omega_j = v_i e_i \) for each \( i \), and \( g_i \) a \( v_i \)th root of \( \prod_{j=1}^{n} \tilde{y}_j^{\tau_{i,j}} \) in an algebraic closure \( \kappa_P(s) \) of \( \kappa_P(s) \) for each \( i \).

**Lemma 2.8.** With the notation above, \( [\kappa_P(s)(g_1, \ldots, g_n) : \kappa_P(s)] = v_1 \cdots v_n \).

**Proof.** First, we show that, if \( g_1^{t_1} \cdots g_n^{t_n} \) is in \( \kappa_P(s) \) for \( (t_1, \ldots, t_n) \in \mathbb{Z}^n \), then \( t_i \) is in \( v_i \mathbb{Z} \) for each \( i \). Let \( u = v_1 \cdots v_n \), and \( \sqrt[s]{s - \beta_{j,k}} \) a \( v \)th root of \( s - \beta_{j,k} \) in \( \kappa_P(s) \) for each \( j, k \).

Then, \( g_i \) is equal to \( \prod_{j=1}^{n} \prod_{k=1}^{p_j} \sqrt[s]{s - \beta_{j,k}}^{t_{i,j} v_i/v_i} \) up to a multiplication of an element of \( \kappa_P \setminus \{0\} \). Hence, there exists \( \alpha \in \kappa_P \setminus \{0\} \) such that \( (g_1^{t_1} \cdots g_n^{t_n})^u = \alpha \prod_{i=1}^{n} \prod_{j=1}^{p_j} (s - \beta_{j,k})^{t_{i,j} v_i/v_i} = \alpha \prod_{j=1}^{n} \prod_{k=1}^{p_j} (s - \beta_{j,k})^{\sum_{i=1}^{n} t_{i,j} v_i/v_i} \). (2.1)
Recall that $\beta_{i,j} \neq \beta_{k,l}$ if $(i, j) \neq (k, l)$, and $p_j = 0$ if and only if $j \in I_D$. Since $g_1^{l_1} \cdots g_n^{l_n}$ is in $\kappa_P(s)$ by assumption, (2.1) implies that $\sum_{j=1}^n t_i \tau_{i,j} v_j / v_i$ is divisible by $v$ if $j \not\in I_D$. Hence, $\sum_{j=1}^n t_i \tau_{i,j} v_j / v_i$ is in $\mathbb{Z}$ if $j \not\in I_D$. Since $\sum_{j=1}^n \tau_{i,j} \omega_j = v_i e_i$ for each $i$, we have

$$\sum_{j \in I_D} \left( \sum_{i=1}^n \frac{t_i \tau_{i,j}}{v_i} \right) \omega_j = \sum_{i=1}^n \frac{t_i}{v_i} \left( \sum_{j=1}^n \tau_{i,j} \omega_j \right) - \sum_{j \not\in I_D} \left( \sum_{i=1}^n \frac{t_i \tau_{i,j}}{v_i} \right) \omega_j.$$

The left-hand side of the first equality of (2.2) is contained in $\sum_{j \in I_D} R \omega_j$, while the right-hand side of the second equality is contained in $\mathbb{Z}^m$. By the assumption that $(\sum_{j \in I_D} R \omega_j) \cap \mathbb{Z}^m = \sum_{j \in I_D} \mathbb{Z} \omega_j$, we have $\sum_{j \in I_D} (\sum_{i=1}^n \frac{t_i \tau_{i,j}}{v_i} \omega_j) \neq \sum_{j \in I_D} \mathbb{Z} \omega_j$. The linear independence of $\omega_1, \ldots, \omega_n$ implies that $\sum_{i=1}^n t_i \tau_{i,j} / v_i$ is in $\mathbb{Z}$ for each $j \in I_D$. Thus, $\sum_{i=1}^n (t_i / v_i) \tau_i$ is in $\mathbb{Z}$, where $\tau_i = (\tau_{i,1}, \ldots, \tau_{i,n})$ for each $i$. Since $\tau_1, \ldots, \tau_n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, each $t_i / v_i$ must be in $\mathbb{Z}$. Therefore, $t_i$ is in $v_i \mathbb{Z}$ for each $i$.

We set $L_l = \kappa_P(s)(g_1, \ldots, g_l)$ and show that $[L_l : L_{l-1}] = v_i$ for each $l = 1, \ldots, n$ by contradiction. Let $l$ be the minimal number where the assertion is false. Then, the polynomial $X^{v_l} - \prod_{j=1}^n \xi_j^{v_i}$ in $X$ is not irreducible over $L_{l-1}$. There exist $0 < p < v_l$ and $v_i$th roots $\xi_1, \ldots, \xi_p$ of unity such that $h(X) = \prod_{i=1}^p (X - \xi_i g_l)$ is in $L_{l-1}[X]$. Since $g_l^p = h(0) / \prod_{i=1}^p (-\xi_i)$ is in $L_{l-1}$ and $L_{l-1} = \kappa_P(s)[g_1, \ldots, g_{l-1}]$, we have

$$g_l^p = \sum_u \lambda_u g_1^{u_1} \cdots g_{l-1}^{u_{l-1}}$$

(2.3)

for some $\lambda_u \in \kappa_P(s)$ for each $u$, where the sum is taken over $u = (u_1, \ldots, u_{l-1})$ with $1 \leq u_i \leq v_i$ for each $i$. By the argument in the preceding paragraph, there exist $u, u'$ with $u \neq u'$ such that $\lambda_u, \lambda_{u'}$ are not zero. Without loss of generality, we may assume that the $(l - 1)$st components of $u$ and $u'$ are distinct. Then, (2.3) is written as

$$g_l^p = \sum_{i=1}^{v_i} \mu_i g_{l-1}^{i},$$

(2.4)

where $\mu_i \in L_{l-2}$ for each $i$, and $\mu_i, \mu_{i_2}$ are not zero for some $1 \leq i_1 < i_2 \leq v_i - 1$.

Since $g_{l-1}^{u_{l-1}}$ is in $L_{l-2}$, each element of the automorphism group $G$ of $L_{l-1}$ over $L_{l-2}$ sends $g_{l-1}^{u_{l-1}}$ to $\xi_{i_1}^{u_{i_1}} g_{l-1}^{i_1}$ for some $u$, where $\xi_i$ denotes a primitive $v_i$th root of unity for each $i$. By the minimality of $l$, we have $[L_{l-1} : L_{l-2}] = v_{l-1}$. So, the order of $G$ is $v_{l-1}$. Hence, there exists $\sigma \in G$ such that $\sigma(g_{l-1}) = \xi_{l-1} g_{l-1}$. Then, $\sigma$ extends to an automorphism $\tilde{\sigma}$ of $L_l$, which satisfies $\tilde{\sigma}(g_l) = \xi_l^q g_l$ for some $q$. By (2.4), we get

$$0 = \tilde{\sigma}(g_l^p) - \xi_l^{pq} g_l^p = \sum_{i=1}^{v_i} \mu_i (\xi_{l-1}^{i_1} - \xi_l^{pq}) g_{l-1}^{i_1}.$$
Since \([L_{l-1} : L_{l-2}] = v_{l-1}\), we have \(\mu_i(\xi^i_{l-1} - \zeta_{pq}^{i}) = 0\) for all \(i\). Since \(\mu_i, \mu_i \neq 0\), it follows that \(\xi^i_{l-1} - \zeta_{pq}^{i} = 0\) and \(\zeta_{l-2} - \zeta_{pq}^{i} = 0\), so \(\xi^i_{l-1} = \zeta_{l-2}^{i}\). This contradicts that \(\zeta_{l-1}\) is a primitive \(v_{l-1}\)st root of unity and \(1 \leq i_1 < i_2 \leq v_{l-1}\). \(\square\)

Now, let us prove Proposition 2.7. Note that \(A = \kappa P \otimes_R D R[(y_1 \cdots y_n)^{-1}]\) is a \(\kappa P\)-subalgebra of \(\kappa P(s)\) containing \(\tilde{y}_1^{\pm 1}, \ldots, \tilde{y}_n^{\pm 1}\). We show that \(\kappa P \otimes_R D S\) is isomorphic to the Laurent polynomial ring

\[
B = A[g_1, \ldots, g_n][x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]
\]

in \(x_{n+1}, \ldots, x_m\) over the \(\kappa P\)-subalgebra \(A[g_1, \ldots, g_n]\) of \(\kappa P(s)\). Clearly, this implies that \(\kappa P \otimes_R D S\) is an integral domain.

We set

\[
T = \tilde{\Phi}_\Omega(R[(y_1 \cdots y_n)^{-1}]) = K[x^{\pm 0_1}, \ldots, x^{\pm 0_n}, \Phi_\Omega(h)^{-1}]
\]

The homomorphism \(R^D \to S\) passes through \(R^D \to R[(y_1 \cdots y_n)^{-1}] \to T \to S\), where the second map is an isomorphism induced from \(\tilde{\Phi}_\Omega\). Hence, we have

\[
\kappa P \otimes_R D S = \kappa P \otimes_R D R[(y_1 \cdots y_n)^{-1}] \otimes_T S = A \otimes_T S.
\]

It follows that \(g_i^{-1} = g_i^{-v_i} \prod_{j=1}^n \tilde{y}_j^{v_i j} \) for each \(i\). So, \(g_i^{-1}\) is in \(B\), since \(\tilde{y}_1^{\pm 1}, \ldots, \tilde{y}_n^{\pm 1}\) are in \(A\). We define a homomorphism \(K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \to B\) by \(x_i \mapsto g_i\) for \(i = 1, \ldots, n\) and \(x_i \mapsto x_i\) for \(i = n+1, \ldots, m\). It sends \(\Phi_\Omega(h)\) to the image \(\tilde{h}\) of \(h\) in \(\kappa P(s)\). Since \(h\) is an invertible element of \(R^D\), we have \(\tilde{h} \in \kappa P \setminus \{0\}\). Hence, the homomorphism \(K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \to B\) extends to \(S \to B\). The composites \(T \to A \to B\) and \(T \to S \to B\) are the same. Thus, we obtain a homomorphism \(A \otimes_T S \to B\). It is surjective, and defined over \(C = A[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]\).

We show that the rank of the \(C\)-module \(A \otimes_T S\) is at most \(v_1 \cdots v_n\), while that of \(B\) is at least \(v_1 \cdots v_n\). It implies that \(A \otimes_T S \to B\) is an isomorphism. Then, we know by (2.6) that \(\kappa P \otimes_R D S\) is isomorphic to \(B\), and the proof will be completed.

The \(C\)-algebra \(A \otimes_T S\) is generated by \(1 \otimes x_1^{\pm 1}, \ldots, 1 \otimes x_m^{\pm 1}\), since \(1 \otimes \Phi_\Omega(h)^{-1} = \tilde{h}^{-1} \otimes 1\) is already in \(C\). Since \(x_i^{v_i} = \tilde{\Phi}_\Omega(\prod_{j=1}^n \tilde{y}_j^{v_i j})\), we have \(1 \otimes x_i^{v_i} = (\prod_{j=1}^n \tilde{y}_j^{v_i j}) \otimes 1 \in A \otimes 1\) for each \(i\). Hence, the \(C\)-module \(A \otimes_T S\) is generated by \(1 \otimes \prod_{i=1}^n x_i^{k_i}\) for \(1 \leq k_i \leq v_i\) for each \(i = 1, \ldots, n\), whose rank is at most \(v_1 \cdots v_n\).

By Lemma 2.8, the dimension of the \(\kappa P(s)\)-vector space \(\kappa P(s)(g_1, \ldots, g_n)\) is \(v_1 \cdots v_n\). Since \(\kappa P(s)(g_1, \ldots, g_n) = \kappa P(s)[g_1, \ldots, g_n]\) and \(A \subset \kappa P(s)\), the rank of the \(A\)-module \(A[g_1, \ldots, g_n]\) is at least \(v_1 \cdots v_n\). The rank of the \(C\)-module \(B\) is equal to the rank of the \(A\)-module \(A[g_1, \ldots, g_n]\). Hence, the rank of the \(C\)-module \(B\) is also at least \(v_1 \cdots v_n\).
3. The case where the rank of \( \Omega \) is less than \( n \)

In this section, we prove Theorem 1.6. Since the rank of \( \Omega \) is less than \( n \), the map \( \Phi_\Omega \) is not injective. Hence, the assumption \( K[y]^D \cap \ker \Phi_\Omega = \{0\} \) implies that \( D \) is not zero.

Lemma 3.1. With the notation as in Section 1, assume that the rank of \( \Omega \) is less than \( n \). Then, \( K(D, \Omega) \) is algebraically closed in \( K(x) \) if and only if the following hold:

(i) \( K(D, \Omega) = K(x_1, \ldots, x_n) \).
(ii) \( K(x_1, \ldots, x_n) \) is algebraically closed in \( K(x) \).

Proof. It suffices to show that the extension \( K(D, \Omega) \subset K(x_1, \ldots, x_n) \) is algebraic. Since the rank of \( \Omega \) is less than \( n \), the transcendence degree of \( K(x_1, \ldots, x_n) \) over \( K \) is less than \( n \). On the other hand, that of \( K(D, \Omega) \) is \( n - 1 \). Actually, \( K(D, \Omega) \) is isomorphic to \( K(y)^D \) by Lemma 2.4, and the transcendence degree of \( K(y)^D \) over \( K \) is \( n - 1 \) by Lemma 2.5. \( \square \)

The last assertion of Theorem 1.6 follows from Lemma 3.1. By Lemma 2.2, the condition (ii) is equivalent to \( \left( \sum_{i=1}^n R_{\omega_i} \right) \cap \mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}_{\omega_i} \). We show that the condition (i) is equivalent to the existence of \( u \in \ker \Phi_\Omega \setminus \{0\} \) such that \( D^2(u) = 0 \). Then, the proof of Theorem 1.6 will be completed.

First, assume that the condition (i) holds. There exists \( w \in K[y] \) such that \( D(w) \neq 0 \) and \( D^2(w) = 0 \). Actually, \( D^{l-1}(w') \neq 0 \) and \( D^l(w') = 0 \) for some \( w' \in K[y] \) and \( l \geq 2 \), since \( D \) is a nonzero locally nilpotent derivation of \( K[y] \). Then, \( w = D^{l-2}(w') \) has this property. By the condition (i) and Lemma 2.4, we have

\[
\Phi_\Omega(w) \in \Phi_\Omega(K[y]) = K(x_1, \ldots, x_n) \subset K(D, \Omega) = \Phi_\Omega(K(y)^D).
\]

Hence, there exists \( f \in K(y)^D \) such that \( \Phi_\Omega(w) = \Phi_\Omega(f) \). We may write \( f = g/h \) for some \( g, h \in K[y]^D \) by Lemma 2.3(ii). Put \( u = hw - g \). Then, \( u \neq 0 \) follows from \( D(w) \neq 0 \). Moreover, we have

\[
D^2(u) = hD^2(w) = 0, \quad \Phi_\Omega(u) = \Phi_\Omega(h(w - f)) = \Phi_\Omega(h)(\Phi_\Omega(w) - \Phi_\Omega(f)) = 0.
\]

Conversely, assume that there exists \( u \in \ker \Phi_\Omega \setminus \{0\} \) such that \( D^2(u) = 0 \). The assumption \( K[y]^D \cap \ker \Phi_\Omega = \{0\} \) implies \( D(u) \neq \ker \Phi_\Omega \). Let \( A = K[y][D(u)^{-1}] \). Since \( D(uD(u)^{-1}) = 1 \), we have \( A = A^D[uD(u)^{-1}] \) by Lemma 2.5. So, it follows that

\[
\Phi_\Omega(A) = \Phi_\Omega(A^D)[\Phi_\Omega(u)\Phi_\Omega(D(u))^{-1}] = \Phi_\Omega(A^D), \quad \text{(3.1)}
\]

since \( \Phi_\Omega(u) = 0 \). The field of fractions of \( \Phi_\Omega(A) \) is equal to \( K(x_1, \ldots, x_n) \), while that of \( \Phi_\Omega(A^D) \) is equal to \( K(D, \Omega) \). Hence, (i) follows from (3.1). We have thus proved Theorem 1.6.
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References