Irreducibility of Associated Matrices

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ABSTRACT

For an n by n matrix A, let $K(A)$ be the associated matrix corresponding to a permutation group (of degree m) and one of its characters. Let $D_r(A)$ be the coefficient of x^{m-r} in $K(A+xI)$. If A is reducible, then $D_r(A)$ is reducible. If A is irreducible and the character is identically one, then $D_1(A)$ is irreducible. If A is row stochastic and the character is identically one, then $D_r(A)$ is essentially row stochastic. Finally, the results motivate the definition of group induced digraphs.

1. INTRODUCTION

The primary purpose of this note is to extend the results of [2] to associated matrices based on groups and characters. To define these associated matrices we first consider the set $\Gamma_{m,n}$ of all functions from the first m positive integers to the first n. Let H be a permutation group of degree m, and let χ be a character of H of degree 1. (We shall point out in the sequel which results continue to hold for characters of degree greater than one.) The group H induces an equivalence relation on $\Gamma_{m,n}$ as follows: $\alpha\equiv\beta$ (mod H) if there is a $\sigma\in H$ such that $\alpha\sigma=\beta$. Let Δ be a system of distinct representatives for the equivalence classes mod H so chosen that $\alpha\in\Delta$ if and only if α is first, in lexicographic (dictionary) ordering, in its equivalence

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class. For each $\gamma\in\Gamma_{m,n}$ define $H\gamma=\{\sigma\in H:\gamma\sigma=\gamma\}$, i.e., the stabilizer subgroup of γ. Let

$$\bar{\Delta}=\left\{\gamma\in\Delta:\sum_{\sigma\in H_\gamma}\chi(\sigma)\neq 0\right\}.$$

If $B=(b_{ij})$ is an m by m matrix, define

$$d(B)=\sum_{\sigma\in H}\chi(\sigma)\prod_{t=1}^{m}b_{t\sigma(t)}.$$

For example, if $H=S_m$, the full symmetric group, and if $\chi=\epsilon$, the alternating character, then d is the determinant. If $H=S_m$ and $\chi\equiv 1$, d is the permanent.

Suppose, now, that $A=(a_{ij})$ is an n by n matrix. For $\alpha,\beta\in\Gamma_{m,n}$, define $A[\alpha|\beta]$ to be the m by m matrix, the (i,j) entry of which is the $(\alpha(i),\beta(j))$ entry of A. Finally, $K(A)$ is a matrix indexed by Δ. The (α,β) entry of this matrix is $[\nu(\alpha)\nu(\beta)]^{-1/2}d(A^t[\beta|\alpha])$, where $\nu(\alpha)$ is the order of H_α, and A^t is the transpose of A [4, p. 126]. For example, if $H=S_m$ and $\chi=\epsilon$, $K(A)$ is the mth compound of A. If $H=S_m$ and $\chi\equiv 1$, $K(A)$ is the mth induced power of A. If $H=\{\mathrm{id}\}$, $K(A)$ is the mth Kronecker power of A. The most important feature of associated matrices is that they are multiplicative, i.e., $K(A_1A_2)=K(A_1)K(A_2)$. Of course K depends on the quadruple (m,n,H,χ), a fact which, for the sake of simplicity, we suppress in the notation.

For an indeterminate x, the associated matrix $K(A+xI)$, I being the identity matrix, is a matrix polynomial in x of degree m:

$$K(A+xI)=\sum_{r=0}^{m}D_r(A)x^{m-r}.$$

The matrices $D_r(A)$, $1\leqslant r\leqslant m$, have variously been called *generalized associated matrices* [8] and *derivations* [5].

We shall investigate what influence the irreducibility or reducibility of A has on the irreducibility or reducibility of $D_r(A)$. Recall that the n by n matrix A is *weakly reducible* if there are n by n permutation matrices P and Q such that

$$PAQ=\begin{pmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{pmatrix},$$

where A_{11} and A_{22} are square matrices of order at least one. The matrix A is *reducible* if it can be brought to the above form with $Q=P^t$. A matrix which is not reducible is *irreducible*; one which is not weakly reducible is *fully irreducible*.

We shall also have occasion to discuss the directed graph $G(A)$ of the n by n matrix A: $G(A)=[N, E]$, where $N=\{1,2,\dots,n\}$ is the set of vertices and $(i,j)\in E$, the set of edges, if and only if $i\neq j$ (we ignore diagonal entries) and $a_{ij}\neq 0$. It is well known [3] that a square matrix of order $n\geqslant 2$ is irreducible if and only if $G(A)$ is strongly connected, i.e., if there is a path in $G(A)$ from every vertex into any other vertex.

2. PRELIMINARIES

We may think of a $\gamma\in\Gamma_{m,n}$ as a sequence of length m chosen from $N=\{1,2,\dots,n\}$, i.e., $\gamma=(\gamma(1),\gamma(2),\dots,\gamma(m))$. For each $t\in N$, let $m_t(\gamma)$ be the multiplicity of t in the sequence γ. (The next definition and lemma are valid for characters of degree greater than one.)

DEFINITION. The quadruple (m,n,H,χ) is *regular* if there is a pair $\alpha,\beta\in\bar{\Delta}$ and an integer $i\in N$ such that $m_i(\alpha)=0\neq m_i(\beta)$.

We will need the following technical observation about this definition.

LEMMA. *If (m,n,H,χ) is regular, then for all $j\in N$ there is a pair $\alpha,\beta\in\bar{\Delta}$ such that $m_j(\alpha)=0\neq m_j(\beta)$.*

Proof. Suppose $\gamma,\delta\in\Gamma_{m,n}$ are equivalent (mod H). Then there is a $\sigma\in H$ for which $\gamma\sigma=\delta$. If $\pi\in H_\gamma$, then $\delta\sigma^{-1}\pi\sigma=\gamma\pi\sigma=\gamma\sigma=\delta$, i.e., $\sigma^{-1}\pi\sigma\in H_\delta$. Similarly, if $\tau\in H_\delta$, then $\sigma\tau\sigma^{-1}\in H_\gamma$. In other words, if $\gamma\equiv\delta$ (mod H), then H_γ is conjugate to H_δ in H. Since characters are conjugacy class functions, it follows that

$$\sum_{\sigma\in H_\gamma}\chi(\sigma)\neq 0 \quad\Leftrightarrow\quad \sum_{\sigma\in H_\delta}\chi(\sigma)\neq 0. \tag{1}$$

Let π be a permutation of N. Then π is a permutation of $\Gamma_{m,n}$ by means of the action $\gamma\to\pi\gamma$, $\gamma\in\Gamma_{m,n}$. Suppose $\gamma,\delta\in\Gamma_{m,n}$. Notice that $\gamma\equiv\delta$ (mod H) if and only if $\pi\gamma\equiv\pi\delta$ (mod H). Now if $\gamma\in\bar{\Delta}$, $\pi\gamma$ need not be an element of $\bar{\Delta}$, but, since $H_{\pi\gamma}=H_\gamma$, it follows from (1) that there is a unique sequence $r(\pi\gamma)\in\bar{\Delta}$ such that $\pi\gamma\equiv r(\pi\gamma)$ (mod H).

Since (m,n,H,χ) is regular, there is a pair $\alpha,\beta\in\bar{\Delta}$ and an integer $i\in N$ such that $m_i(\alpha)=0\neq m_i(\beta)$. Suppose $j\in N$, $j\neq i$. Let π be some permutation of N such that $\pi(i)=j$. Then $m_j(\pi\alpha)=0\neq m_j(\pi\beta)$. Since $r(\pi\alpha)$ $[r(\pi\beta)]$ contains the same integers as $\pi\alpha$ $[\pi\beta]$, the result follows. ■

Further remarks about the definition: If $n>m$, then (m,n,H,χ) is always regular. This is because any increasing sequence of distinct integers belongs to $\bar{\Delta}$. Moreover, if $n_1>n_2$ and if (m,n_2,H,χ) is regular, then (m,n_1,H,χ) is regular. Finally, if $n>1$ and $\chi\equiv 1$, then (m,n,H,χ) is regular.

THEOREM 1. *Suppose (m,n,H,χ) is regular, Let A be an n by n matrix. If A is reducible, then $D_r(A)$ is reducible, $1\leqslant r\leqslant m$.*

Proof. Since A is reducible, there is a subset $M\subsetneqq N$, $\varnothing\neq M$, such that $a_{ij}=0$ whenever $i\in M$ and $j\in N\setminus M$. Let the cardinality of M be k. Without loss of generality, we may assume $M=\{1,2,\dots,k\}$. Let s be the smallest number of distinct integers that occur in any sequence of $\bar{\Delta}$. There are two cases.

Case 1: $k\geqslant s$. Let $\Omega=\Gamma_{m,k}\cap\bar{\Delta}$. Since $k<n$, $\bar{\Delta}\setminus\Omega\neq\varnothing$. Let $\alpha\in\Omega$ and $\beta\in\bar{\Delta}\setminus\Omega$. Since $\beta\notin\Omega$, there is an $r\in\{1,2,\dots,m\}$ such that $\beta(r)\in N\setminus M$. Therefore, the rth column of $A[\alpha|\beta]$ is zero. It follows from the definition that the α,β entry of $K(A)$ is zero $(A^t[\beta|\alpha]=A[\alpha|\beta]^t)$.

Case 2: $k<s$. Let $\mu_k(\gamma)=m_1(\gamma)+\cdots+m_k(\gamma)$, $\gamma\in\Gamma_{m,n}$. Let

$$t=\max_{\gamma\in\bar{\Delta}}\mu_k(\gamma).$$

Let $\Omega=\{\gamma\in\bar{\Delta}:\mu_k(\gamma)=t\}$. Then $\Omega\neq\varnothing$. Suppose $\bar{\Delta}=\Omega$. Since (m,n,H,χ) is regular, there exist $\omega,\nu\in\bar{\Delta}$ such that $m_k(\omega)=0\neq m_k(\nu)$. Since $s>k$, there is an integer $j>k$ such that $m_j(\omega)\neq 0$. Consider the sequence $\gamma\in\Gamma_{m,n}$ defined as follows: $\gamma(i)=\omega(i)$ if $\omega(i)\neq j$ and $\gamma(i)=k$ if $\omega(i)=j$. Let δ be that sequence of Δ which is equivalent (mod H) to γ. Since $H_\gamma=H_\omega$, it follows from (1) that $\delta\in\bar{\Delta}$. But this contradicts the definition of t, since $\mu_k(\delta)=\mu_k(\gamma)>\mu_k(\omega)$. Thus $\bar{\Delta}\setminus\Omega\neq\varnothing$.

Now, let $\alpha\in\Omega$ and $\beta\in\bar{\Delta}\setminus\Omega$. Consider $A[\alpha|\beta]$. Since $\mu_k(\beta)<t$, at least $m-t+1$ of the entries of β (multiplicities included) come from $N\setminus M$. Therefore, $A[\alpha|\beta]$ contains a t by $m-t+1$ submatrix consisting of zeros. By König's theorem, every diagonal product from $A[\alpha|\beta]$ is zero. It follows again that the α,β entry of $K(A)$ is zero.

Since $A+xI$ is reducible in the same manner as A, $K(A+xI)$ is reducible. It follows that $D_r(A)$ is reducible, $1\leqslant r\leqslant m$. The proof is complete. ■

REMARK. This theorem remains valid for higher degree characters. (For a definition of $K(A)$ in this case, see [9, Theorem 3].)

COROLLARY 1. *Suppose* (m, n, H, χ) *is regular. Let* A *be an* n *by* n *matrix. If* A *is weakly reducible, then* $K(A)$ *is weakly reducible.*

Proof. If A is weakly reducible, there is a permutation matrix P such that AP is reducible. Therefore $K(AP)=K(A)K(P)$ is reducible. It is proved in [6] that $K(P)$ is a generalized permutation, i.e., a diagonal matrix times a permutation matrix. It follows that $K(A)$ is weakly reducible. ∎

3. THE FIRST (ADDITIVE) DERIVATION

We begin with an explicit description of the matrix $D_1(A)$ valid when $\chi(\mathrm{id})=1$. Let $\alpha,\beta\in\overline{\Delta}$. Then, according to [8, Eq. (2.7)], the (α,β) entry of $D_1(A)$ is

$$\frac{1}{\sqrt{\nu(\alpha)\nu(\beta)}}\sum_{i=1}^{m}\sum_{\sigma\in H}\chi(\sigma)a_{\alpha\sigma(i)\beta(i)}\prod_{\substack{t=1\\t\neq i}}^{m}\delta_{\alpha\sigma(t)\beta(t)}.$$

We identify three cases:

Case 1: $\alpha=\beta$. For each $\sigma\in H$, either $\alpha\sigma=\alpha$ or $\alpha\sigma$ differs from α in at least two places. Therefore, only those $\sigma\in H_\alpha$ contribute nonzero terms to the summation. Since the restriction of χ to H_α is identically 1, we obtain in this case

$$[D_1(A)]_{\alpha\alpha}=\sum_{i=1}^{m}a_{\alpha(i)\alpha(i)}. \tag{2a}$$

Case 2: There is a $\tau\in H$ such that $\alpha\tau$ and β differ in exactly one place, say $m_p(\alpha)=m_p(\beta)+1$ and $m_q(\alpha)=m_q(\beta)-1$. Let $H=H_\alpha\pi_1\cup H_\alpha\pi_2\cup\cdots\cup H_\alpha\pi_r$, $r=[H:H_\alpha]=o(H)/\nu(\alpha)$, be the decomposition of H into right cosets afforded by H_α. Assume that $\pi_1,\pi_2,\dots,\pi_r$ are ordered so that $\alpha\pi_1,\alpha\pi_2,\dots,\alpha\pi_s$ each differ from β in exactly one place, and $\alpha\pi_{s+1},\dots,\alpha\pi_r$ each differ from β in at least two places. (It may happen that $s=r$.) A moment of reflection will establish that for $1 \leqslant i \leqslant s$, the entry which differs will always be a p in $\alpha\pi_i$ and a q in β. So, in this case,

$$[D_1(A)]_{\alpha\beta} = \frac{1}{\sqrt{\nu(\alpha)\nu(\beta)}} \sum_{j=1}^{s} \sum_{\sigma \in H_\alpha} \chi(\sigma\pi_j) a_{pq}$$

$$= \sqrt{\frac{\nu(\alpha)}{\nu(\beta)}}\, a_{pq} \sum_{j=1}^{s} \chi(\pi_j), \tag{2b}$$

since $\chi(\sigma\pi_j) = \chi(\sigma)\chi(\pi_j)$, and $\chi(\sigma) = 1$, $\sigma \in H_\alpha$.

Case 3: For all $\tau \in H$, $\alpha\tau$ differs from β in at least two places. In this case,

$$[D_1(A)]_{\alpha\beta} = 0. \tag{2c}$$

EXAMPLE. Take $m=4$, $H = \langle(1234)\rangle$, $\chi(1234) = -1$, $n=2$. Then $\bar{\Delta} = \{(1,1,1,2),(1,1,2,2),(1,2,1,2),(1,2,2,2)\}$. [In particular, (m,n,H,χ) is not regular.] If $A = (a_{ij})$ is a 2 by 2 matrix, then

$$D_1(A) = \begin{bmatrix} 3a_{11} + a_{22} & 0 & \sqrt{2}\, a_{12} & 0 \\ 0 & 2(a_{11}+a_{22}) & 0 & 0 \\ \sqrt{2}\, a_{21} & 0 & 2(a_{11}+a_{22}) & \sqrt{2}\, a_{12} \\ 0 & 0 & \sqrt{2}\, a_{21} & a_{11} + 3a_{22} \end{bmatrix}.$$

Notice that in this example $D_1(A)$ is reducible for every 2 by 2 matrix A. This cannot happen if χ is identically 1.

THEOREM 2. *Let H be a subgroup of S_m. Let χ be the principal (identically 1) character of H. Let A be an n by n matrix. Then A is reducible if and only if $D_1(A)$ is reducible.*

Proof. Suppose A is irreducible. Then $G(A)$ is strongly connected. Let $\alpha, \beta \in \Delta$ $(= \bar{\Delta})$, $\alpha \neq \beta$. Of all the $\tau \in H$, there will be (at least) one for which $\alpha\tau$ and β differ in the fewest places. The proof is by induction on this number. So, suppose τ is chosen so that $\alpha\sigma$ differs from β in at least as many places as $\alpha\tau$, $\sigma \in H$. Let $\gamma = \alpha\tau$ and assume $\gamma(r) \neq \beta(r)$. There is a path in

$G(A)$ from $\gamma(r)$ to $\beta(r)$, say $\gamma(r)=i_1\to i_2\to\cdots\to i_k=\beta(r)$, i.e., $i_1\neq i_2\neq\cdots\neq i_k$, and the (i_1,i_2), the $(i_2,i_3),\ldots,$ and the (i_{k-1},i_k) entries of A are all different from zero. For each $t=1,2,\ldots,k$, there is a unique sequence $\gamma_t\in\Delta$ such that $\gamma_t\equiv(\gamma(1),\gamma(2),\ldots,\gamma(r-1),i_t,\gamma(r+1),\ldots,\gamma(m))$ (mod H). By (2b), the (γ_t,γ_{t+1}) entry of $D_1(A)$ is a nonzero multiple of the (i_t,i_{t+1}) entry of A, $1\leqslant t<k$. Therefore, in the directed graph $G(D_1(A))$, there is a path from $\alpha=\gamma_1$ to γ_k. Now, γ_k is equivalent to a sequence which differs from β in one fewer place than γ. By the induction assumption, there is a directed path from γ_k to β and hence one from α to β. It follows that $G(D_1(A))$ is strongly connected.

The converse is a special case of Theorem 1. ■

REMARK. One of us (Fiedler [2]) has shown for $H=S_m$, $\chi=\epsilon$, A irreducible n by n with $n\geqslant m$, that $D_1(A)$ is irreducible. It would be interesting to know if the assumption $n\geqslant m$ [or perhaps (m,n,H,χ) regular] eliminates examples of the type given above, i.e., if $m\leqslant n$, is $D_1(A)$ irreducible for every irreducible n by n matrix A, and every subgroup H of S_m and character χ of degree one?

4. STOCHASTIC MATRICES

In [7], M. Marcus and M. Newman investigated conditions on A which made $K(A)$ doubly stochastic for the case $\chi\equiv 1$. In general, they found that A must be an rth root of unity times a permutation.

THEOREM 3. *Let H be a subgroup of S_m. Let χ be the principal character of H. Let n be a positive integer. Let $\Delta\subseteq\Gamma_{m,n}$ be the corresponding system of distinct representatives for the equivalence classes* (mod H). *For each $\gamma\in\Delta$, let $d_\gamma=[H:H_\gamma]^{1/2}$. Let Y be the diagonal matrix indexed by Δ and with d_γ in the (γ,γ) position. If A is an n by n row stochastic matrix, then*

$$\binom{m}{r}^{-1}Y^{-1}D_r(A)Y$$

is row stochastic.

Proof. Let V be a real (or complex) inner product space of dimension n. Let $\mathfrak{B}=\{e_1,e_2,\ldots,e_n\}$ be an ordered, orthonormal basis of V. Let T be the linear operator on V whose matrix representation, with respect to $\mathfrak{B}$, is A. Then $Tu=u$, where $u=e_1+e_2+\cdots+e_n$.

Let $\otimes^m V$ be the mth tensor power of V, and write $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ for the decomposable tensor product of the indicated vectors. To each $\sigma \in S_m$, there corresponds a linear operator $P(\sigma)$ on $\otimes^m V$ such that

$$P(\sigma^{-1}) v_1 \otimes \cdots \otimes v_m = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

Define

$$\Theta_H = \frac{1}{o(H)} \sum_{\sigma \in H} P(\sigma).$$

Then Θ_H is an orthogonal projection (with respect to the induced inner product of $\otimes^m V$) onto its range $V_1(G)$. Define $v_1 * v_2 * \cdots * v_m = \Theta_H(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$, and write $e_\gamma^* = e_{\gamma(1)} * \cdots * e_{\gamma(m)}$. Then it is well known [4] that $\mathcal{F} = \{d_\gamma e_\gamma^* : \gamma \in \Delta\}$ is an orthonormal basis of $V_1(G)$. Define $D_r(T)$ on $V_1(G)$ by

$$D_r(T) e_\gamma^* = \sum_{\omega \in Q_{r,m}} e_{\gamma(1)} * \cdots * Te_{\gamma\omega(1)} * \cdots * Te_{\gamma\omega(r)} * \cdots * e_{\gamma(m)},$$

$\gamma \in \Delta$, and linear extension, where $Q_{r,m}$ is the set of *strictly increasing* functions from the first r to the first m positive integers. Then it turns out that

$$D_r(T) v_1 * \cdots * v_m = \sum_{\omega \in Q_{r,m}} v_1 * \cdots * Tv_{\omega(1)} * \cdots * Tv_{\omega(r)} * \cdots * v_m$$

for all $v_1, v_2, \ldots, v_m \in V$. Finally, the matrix representation of $D_r(T)$ with respect to F is $D_r(A)$ [8].

We next observe that

$$\begin{aligned} u * u * \cdots * u &= \left(\sum_{i=1}^n e_i \right) * \left(\sum_{i=1}^n e_i \right) * \cdots * \left(\sum_{i=1}^n e_i \right) \\ &= \sum_{\gamma \in \Gamma_{m,n}} e_\gamma^* \\ &= \sum_{\gamma \in \Delta} \frac{1}{\nu(\gamma)} \sum_{\sigma \in H} e_{\gamma\sigma}^* \\ &= \sum_{\gamma \in \Delta} [H : H_\gamma] e_\gamma^*, \end{aligned}$$

since $e^*_{\gamma\sigma}=e^*_\gamma$ for all $\sigma\in H$. If S is the linear operator on $V_1(G)$ defined by $Se^*_\gamma=d_\gamma e^*_\gamma$, then the matrix representation of S with respect to $\mathcal{F}$ is Y. Finally, observe

$$S^{-1}D_r(T)S\Big(\sum_{\gamma\in\Delta} d_\gamma e^*_\gamma\Big)=S^{-1}D_r(T)\Big(\sum_{\gamma\in\Delta}[H:H_\gamma]e^*_\gamma\Big)$$

$$=S^{-1}D_r(T)(u*u*\cdots*u)$$

$$=S^{-1}\Big(\sum_{\omega\in Q_{r,m}} u*\cdots*Tu*\cdots*Tu*\cdots*u\Big)$$

$$=S^{-1}\Big[\binom{m}{r}(u*u*\cdots*u)\Big]$$

$$=\binom{m}{r}\Big(\sum_{\gamma\in\Delta} d_\gamma e^*_\gamma\Big),$$

i.e., $S^{-1}D_r(T)S$ applied to the sum of the basis elements is $\binom{m}{r}$ times the sum of the basis elements. In other words, the row sums of $Y^{-1}D_r(A)Y$ are all equal to $\binom{m}{r}$. The result now follows from the definitions. ■

We remark that

$$\binom{m}{r}^{-1}YD_r(A)Y^{-1}$$

is column stochastic for all column stochastic A.

5. A CLASS OF INDUCED DIGRAPHS

Let $G=[V,E]$ be a finite directed graph without multiple edges, with vertex set $V=\{1,2,\ldots,n\}$, and with edge set E. Let m be a positive integer, and let H be a subgroup of S_m. Let $\Delta\subseteq\Gamma_{m,n}$ be the corresponding system of distinct representations for the equivalence classes (mod H). The *H-induced graph* G_H of G is the graph $[\Delta, E_H]$, where $(\alpha,\beta)\in E_H$ if and only if there is a $\tau\in H$ such that $\alpha\tau$ and β differ in exactly one place, say with a p in $\alpha\tau$ and a q in β, and $(p,q)\in E$.

THEOREM 4. *Let H be a subgroup of S_m. Let χ be the principal character of H. If A is an n by n matrix, then $G(D_1(A))=(G(A))_H$.*

Proof. This follows immediately from (2b), (2c), and the definitions. ■

THEOREM 5. *Let G be a finite directed graph without multiple edges. Let H be a subgroup of S_m. Then G is strongly connected if and only if G_H is strongly connected.*

Proof. In view of Theorem 4, this is but a restatement of Theorem 2. ■

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