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Weak Levi-Civita Connection for the Damped Metric on the Riemannian Path Space and Vanishing of Ricci Tensor in Adapted Differential Geometry

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DEDICATED TO OUR PROFESSOR PAUL MALLIAVIN WITH OUR GRATITUDE AND ADMIRATION

We shall establish in the context of adapted differential geometry on the path space $\mathbf{P}_{m_o}(M)$ a Weitzenböck formula which generalizes that in (A. B. Cruzeiro and P. Malliavin, *J. Funct. Anal.* **177** (2000), 219–253), without hypothesis on the Ricci tensor. The renormalized Ricci tensor will be vanished. The connection introduced in (A. B. Cruzeiro and S. Fang, 1997, *J. Funct. Anal.* **143**, 400–414) will play a central role. © 2001 Academic Press

Let M be a compact Riemannian manifold (of dimension d). We shall be interested in the following space of paths

(0.1) $$\mathbf{P}_{m_o}(M) = \{\gamma\colon [0, 1] \to M \text{ continuous}\,;\, \gamma(0) = m_o\},$$

where $m_o \in M$ is a fixed point. $\mathbf{P}_{m_o}(M)$ is an infinite dimensional manifold, which has a differential structure via Malliavin calculus and a structure of Itô filtration. We refer to [CM1] for the foundations of the differential geometry on $\mathbf{P}_{m_o}(M)$.

Take on M the Levi-Civita connection. We have the following Itô stochastic parallel transport

(0.2) $$\mathbf{t}^{\gamma}_{0 \leftarrow \tau}\colon T_{\gamma(\tau)}(M) \to T_{m_o}(M).$$

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By fixing an orthonormal basis of $T_{m_o}(M)$, we shall identify $T_{m_o}(M)$ with $\mathbb{R}^d$. A vector field Z on $\mathbf{P}_{m_o}(M)$ is given by the data $Z(\gamma, \tau) \in T_{\gamma(\tau)}(M)$ such that $z \in P_o(\mathbb{R}^d)$ where $z(\gamma, \tau) = \mathbf{t}^{\gamma}_{0 \leftarrow \tau}(Z(\gamma, \tau))$. We shall call Z a $\mathbb{H}$-vector field on $\mathbf{P}_{m_o}(M)$ if $z(\gamma, \cdot) \in \mathbb{H}$, where

$$\mathbb{H} = \left\{ h \in P_o(\mathbb{R}^d);\ |h|_H^2 = \int_0^1 |\dot{h}(\tau)|^2 \, d\tau < +\infty \right\}. \tag{0.3}$$

We define a differential 1-form Θ on $\mathbf{P}_{m_o}(M)$ with values in $P_o(\mathbb{R}^d)$ by

$$\Theta(Z) = z. \tag{0.4}$$

This last formula emphasizes the underlying structure of parallelized manifold of $\mathbf{P}_{m_o}(M)$. So the Hilbertian norm $|\cdot|_H$ on $\mathbb{H}$ induces a Riemannian metric on $\mathbf{P}_{m_o}(M)$ by

$$\langle Z_1, Z_2 \rangle_\gamma = \langle \Theta(Z_1),\ \Theta(Z_2) \rangle_H. \tag{0.5}$$

The Levi-Civita covariant derivative on $\mathbf{P}_{m_o}(M)$ has been computed in [CM1], but it does not preserve the $\mathbb{H}$-vector fields, nor the Itô filtration, due to the phenomenon of Lie bracket. To overcome this difficulty, the new concept of Markovian connection has been introduced in [CM1]: it preserves the category of adapted $\mathbb{H}$-vector fields on $\mathbf{P}_{m_o}(M)$. The price to pay is that the associated torsion tensor is not free nor skew-symmetric in the sense of Driver [Dr1].

A challenging problem on the space $\mathbf{P}_{m_o}(M)$ is the establishment of a *good* Weitzenböck formula, which will give some geometrical informations by doing the analysis. Severals works have been done (see [CM1,2], [CFM], [Fa2]). In [CFM], the authors addressed themselves the following questions:

(i) Does there a Markovian connection ∇ (which preserves the adapted $\mathbb{H}$-vector fields) exist such that for all smooth adapted vector fields h and k,

$$\mathbb{E}(D_{T(h,k)} F) = 0, \tag{0.6}$$

where $T(h, k) = \nabla_h k - \nabla_k h - [h, k]$ is the associated torsion?

(ii) With respect to what metric on $\mathbf{P}_{m_o}(M)$, is it compatible ?

By the structure equation established in [CM1], we know that $T(h, k)$ is a tangent process. If $T(h, k)$ were a $\mathbb{H}$-vector field, the relation (0.6) would imply that $T(h, k) = 0$ by Itô energy identity. For these reasons, such a connection will be called the weak Levi-Civita connection. In the case where $\mathrm{Ric}_M = 0$, the Markovian connection introduced in [CM1] is a weak

Levi-Civita connection with the metric defined in (0.5). In this situation, a *good* Weitzenböck formula has been established in [CM2].

The purpose of this work is to establish a *good* Weitzenböck formula without the hypothesis on Ric_M. The concept of the weak Levi-Civita connection will play a key role. A shorter version, but from another point of view, has been done in the note [CF2].

1. GENERAL FRAMEWORK

Let M be a compact Riemannian manifold, endowed with its Levi-Civita connection. Let $A_1, \ldots, A_d$ be the canonical horizontal vector fields on the orthonormal frame bundle $O(M)$. Consider the Stratonovich s.d.e:

$$dr_x(\tau) = \sum_{i=1}^{d} A_i(r_x(\tau)) \circ dx^i(\tau), \qquad r_x(0) = r_o \in \pi^{-1}(m_o),$$

where $\pi: O(M) \to M$ is the canonical projection and x is the standard Brownian motion on $\mathbb{R}^d$. Denote by X the classical Wiener space of the Brownian trajectories in $\mathbb{R}^d$. Define:

$$\gamma_x(\tau) = \pi(r_x(\tau)).$$

It is well known that the map $I: (X, \mu) \to (\mathbf{P}_{m_o}(M), \nu)$ defined by $x \to I(x) = \gamma_x$ is a measurable isomorphism, where μ and ν are the Wiener measures on X and $\mathbf{P}_{m_o}(M)$ respectively.

Following [CM1], a semi-martingale $d\xi_x(t) = a_x(t)\, dx(t) + b_x(t)\, dt$ on $\mathbb{R}^d$ is called a tangent process if $t \to (a_x(t), b_x(t))$ is an adapted process taking values in $so(d) \times \mathbb{R}^d$ such that

$$\mathbb{E}(e^{c \int_0^1 |b_x(s)|^2\, ds}) < +\infty \quad \text{for some} \quad c > 0. \tag{1.1}$$

Consider now $Z_\xi(t) = r_x(t)\, \xi_x(t)$. Then Z_ξ is a vector field along the Brownian curves γ_x on M. We shall consider ξ as a vector field on $\mathbf{P}_{m_o}(M)$ throughout Z_ξ. Denote

$$\Gamma_\xi(t) = \int_0^t \Omega_s(\xi_x(s), \circ\, dx_s), \tag{1.2}$$

where $\Omega_s = \Omega_{r_x(s)}$ is the curvature tensor on M read in the frame $r_x(s)$. Define

$$d\xi^*_x(t) = d\xi_x(t) + \Gamma_\xi(t) \circ dx(t). \tag{1.3}$$

$\xi_x^*(t)$ is again a tangent process over $\mathbb{R}^d$. If $d\xi_x^*(t) = q_x(t)\,dx(t) + \dot{h}(t)\,dt$, then

$$q_x(t) = a_x(t) + \Gamma_\xi(t) \qquad \dot{h}(t) = b_x(t) + \tfrac{1}{2}\,\mathrm{Ric}_t\,\xi(t), \tag{1.4}$$

where $\mathrm{Ric}_t = \mathrm{Ric}_{r_x(t)}$. Define (see [FM]):

$$\psi_\varepsilon^\xi(x, t) = \int_0^t e^{\varepsilon\, q_x(s)}\, dx_s + \varepsilon h_x(t) \qquad \text{and} \qquad \sigma_\varepsilon^\xi = I \circ \psi_\varepsilon^\xi \circ I^{-1}. \tag{1.5}$$

Definition 1.1. Let $F \in \bigcap_{p>1} L^p(\mathbf{P}_{m_o}(M))$. We shall say that F is derivable with respect to the tangent process ξ, if

$$D_\xi F(\gamma) = \left\{ \frac{d}{d\varepsilon}\, F(\sigma_\varepsilon^\xi) \right\}_{\varepsilon=0}$$

exists in all L^p.

Remark 1.2. In fact, ξ^* is just the pull back of Z_ξ by the Itô map I on the Wiener space X (see [Dr1], [FM], [CM1]).

By Girsanov's theorem, we have:

Proposition 1.3.

$$\mathbb{E}(D_\xi F) = \mathbb{E}\left(F \int_0^1 \langle \dot{h}(s), dx(s) \rangle \right). \tag{1.6}$$

Now we shall consider a class $\mathscr{C}$ of test functions F on $\mathbf{P}_{m_o}(M)$, which are cylindrical

$$F(\gamma) = f(\gamma(\tau_1), \ldots, \gamma(\tau_k))$$

for some $0 \leqslant \tau_1 < \cdots < \tau_k \leqslant 1$ and $f \in C^\infty(M^k)$. For $F \in \mathscr{C}$, we have (see [Dr1], [FM], [CM1])

$$(D_\xi F) = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f})\, \xi^\alpha(\tau_i), \tag{1.7}$$

where $\tilde{f}(r_1, \ldots, r_k) = f(\pi(r_1), \ldots, \pi(r_k))$, and $\partial^i_{A_\alpha}$ denotes the partial derivative with respect to the ith component.

COROLLARY 1.4. *Let $d\xi_x(t) = a_x(t)\,dx(t) + b_x(t)\,dt$ be a tangent process. Suppose that*

$$b_x(t) + \tfrac{1}{2}\,\mathrm{Ric}_t\left(\int_0^t b_x(s)\,ds\right) + \tfrac{1}{2}\,\mathrm{Ric}_t\left(\int_0^t a_x(s)\,dx(s)\right) = 0. \tag{1.8}$$

Then for all $F \in \mathscr{C}$, we have $\mathbb{E}(D_\xi F) = 0$.

Proof. It follows from (1.4) and (1.6). ∎

2. WEAK LEVI-CIVITA CONNECTION

We shall denote by χ_p the space of $\mathbb{H}$-vector fields Z on $\mathbf{P}_{m_o}(M)$ such that

$$\mathbb{E}(|z|_H^p) = \mathbb{E}\left[\left(\int_0^1 |\dot{z}(t)|^2\,dt\right)^{p/2}\right] < +\infty,$$

where $z = \Theta(Z)$ is defined in (0.4). In what follows, we shall identify Z with z throughout the parallelism Θ. We shall say that z is an adapted vector field if $z \in \chi_2$ and $x \to z(\gamma_x, t)$ is adapted to the filtration $\mathscr{F}_t = \sigma(x(s);\, s \leqslant t)$. Denote χ_p^a the space of adapted vector fields in χ_p.

DEFINITION 2.1. Let $z \in \chi_p$. We define $\hat{z} \in \chi_p$ by

$$\dot{\hat{z}}(t) = \dot{z}(t) + \tfrac{1}{2}\,\mathrm{Ric}_t\, z(t). \tag{2.1}$$

It is obvious that the map $z \to \hat{z}$ preserves the class of adapted vector fields. Let $Q_{t,s}$ be the solution of the following resolvant equation

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\,\mathrm{Ric}_t\, Q_{t,s}, \qquad Q_{s,s} = I, \quad t > s. \tag{2.2}$$

Then the inverse map of $z \to \hat{z}$ is given by

$$\tilde{z}(t) = \int_0^t Q_{t,s}\dot{z}(s)\,ds \tag{2.3}$$

or in the form

$$\dot{\tilde{z}}(t) = \dot{z}(t) - \tfrac{1}{2}\,\mathrm{Ric}_t \int_0^t Q_{t,s}\dot{z}(s)\,ds. \tag{2.4}$$

Remark that the map $z \to \tilde{z}$ preserves also the class of adapted vector fields on $\mathbf{P}_{m_o}(M)$. Let ξ be a tangent process, and $z \in \bigcap_{p>1} \chi_p$. We shall say that z is derivable with respect to ξ if

$$D_\xi z = \left\{ \frac{d}{d\varepsilon} z(\sigma_\varepsilon^\xi) \right\}_{\varepsilon=0} \quad \text{exists in all } L^p(\mathbf{P}_{m_o}(M), \mathbb{H}). \tag{2.5}$$

We shall say that z is in $\mathscr{C}^1$ if for any tangent process ξ, $D_\xi z$ exists in all L^p; z is in $\mathscr{C}^2$ if for any adapted vector field $h \in \mathscr{C}^1$, $D_h z$ is in $\mathscr{C}^1$; similarly we define the class $\mathscr{C}^k$ of vector fields. We shall say that z is smooth if $z \in \mathscr{C}^k$ for any $k \geqslant 1$. Remark that if z is adapted, then $D_\xi z$ is also adapted. Following [CM1], the covariant derivative $\nabla^o_\xi Z$ with respect to the Markovian connection ∇^o, throughout the parallelism Θ, is defined by

$$\overset{\cdot}{\widetilde{\nabla^o_\xi z}} = \overset{\cdot}{\widetilde{D_\xi z}} - \Gamma_\xi \dot{z}. \tag{2.6}$$

From this expression, we see that ∇^o_ξ preserves the class of adapted vector fields. For further properties of the Markovian connection, we refer to [CM1,2] and [Fa2]. Now following [CF1], we shall introduce another Markovian connection, which takes into account the Ricci effect.

DEFINITION 2.2. We define the covariant derivative $\nabla_\xi z$ by

$$\widehat{\nabla_\xi z} = \nabla^o_\xi \hat{z}. \tag{2.7}$$

It is clear that the covariant derivative ∇_ξ preserves also the class of adapted vector fields, but ∇ is not compatible with the metric induced by $|\cdot|_H$. With what metric is it compatible ?

DEFINITION 2.3. We define

$$(z_1 \mid z_2)_\gamma = \int_0^1 \langle \dot{\hat{z}}_1(\gamma, t), \dot{\hat{z}}_2(\gamma, t) \rangle_{\mathbb{R}^d} \, dt. \tag{2.8}$$

In what follows, we shall denote by $\mathbb{H}_\gamma = \mathbb{H}$ endowed with the inner product defined in (2.8). Therefore it is more convenient to consider the $\mathbb{H}$-vector fields as the measurable sections of a Hilbertian fiber over $\mathbf{P}_{m_o}(M)$. The constant vector fields are not natural in our situation.

THEOREM 2.4. *Let ξ be a tangent process, z_1, z_2 be two derivable vector fields with respect to ξ, then*

$$D_\xi (z_1 \mid z_2) = (\nabla_\xi z_1 \mid z_2) + (z_1 \mid \nabla_\xi z_2). \tag{2.9}$$

Proof. The relation (2.9) was proved in [CF1, p. 406]. For the convenience of readers, we include the proof. ∇^o is compatible with $|\cdot|_H$, then

$$D_\xi (z_1 \mid z_2) = D_\xi \langle \hat{z}_1, \hat{z}_2 \rangle_H = \langle \nabla^o_\xi \hat{z}_1, \hat{z}_2 \rangle_H + \langle \hat{z}_1, \nabla^o_\xi \hat{z}_2 \rangle_H$$
$$= \langle \widehat{\nabla_\xi z_1}, \hat{z}_2 \rangle_H + \langle \hat{z}_1, \widehat{\nabla_\xi z_2} \rangle_H = (\nabla_\xi z_1 \mid z_2) + (z_1 \mid \nabla_\xi z_2). \quad \blacksquare$$

Let z be an adapted vector field. Denote $\delta(z) = \int_0^1 \langle \dot{\hat{z}}(\tau), dx(\tau) \rangle$. The following commutation formula will play an important role in our approach to the Weitzenböck formula (see [FF] for loop groups case).

THEOREM 2.5 ([CF1, p. 406]). *Let $h \in \bigcap_{p>1} \chi^a_p$. Assume that z is derivable with respect to h. Then*

$$D_h\, \delta(z) = \delta(\nabla_h z) + (h \mid z). \tag{2.10}$$

Let z_1 and z_2 be two smooth adapted vector fields on $\mathbf{P}_{m_o}(M)$. The torsion $T(z_1, z_2)$ between z_1, z_2 is defined by

$$T(z_1, z_2) = \nabla_{z_1} z_2 - \nabla_{z_2} z_1 - [z_1, z_2]. \tag{2.11}$$

The explicit expression for $[z_1, z_2]$ was first discovered in [CM1] when z_1 and z_2 are constant vector fields. It was generalized in [Dr2] for $z_1, z_2 \in \mathscr{C}^1$. $[z_1, z_2]$ is a tangent process; therefore $T(z_1, z_2)$ is a tangent process.

THEOREM 2.6. *We have for all $F \in \mathscr{C}$,*

$$\mathbb{E}(D_{T(z_1, z_2)} F) = 0. \tag{2.12}$$

Proof. By the formula of integration by parts (1.6),

$$\mathbb{E}(D_{z_1} D_{z_2} F) = \mathbb{E}(D_{z_2} F\, \delta(z_1)) = \mathbb{E}(-D_{z_2}\, \delta(z_1)\, F + F\, \delta(z_1)\, \delta(z_2)).$$

According to (2.10),

$$\mathbb{E}(-D_{z_2}\, \delta(z_1)\, F) = -\mathbb{E}(\delta(\nabla_{z_2} z_1)\, F) - \mathbb{E}(F\, (z_1 \mid z_2)).$$

Therefore

$$\mathbb{E}(D_{[z_1, z_2]} F) = \mathbb{E}(D_{z_1} D_{z_2} F) - \mathbb{E}(D_{z_2} D_{z_1} F) = \mathbb{E}(\delta(\nabla_{z_1} z_2 - \nabla_{z_2} z_1)\, F)$$

and the result follows. $\blacksquare$

Remark 2.7. (i) By Proposition 1.3, equality (2.12) remains true for any function which is two times derivable in the sense of Definition 1.1.

(ii) Let $h, k \in \mathbb{H}$. $T(h, k)$ has the following expression (see [Fa1, p. 257–258]):

$$T(h,k)(t) = \int_0^t \Omega_s(h,k)\, dx_s - \tfrac{1}{2} \int_0^t Q_{t,s}\, \mathrm{Ric}_s \left(\int_0^s \Omega(h,k)\, dx \right) ds. \tag{2.13}$$

Put $d\,T(h,k)(t) = a_x(t)\, dx(t) + b_x(t)\, dt$. Then $a(t) = \Omega_t(h(t), k(t))$ and $b_x(t)$ satisfies the relation, due to (2.3)

$$b_x(t) + \tfrac{1}{2} \mathrm{Ric}_t \int_0^t b_x(s)\, ds = -\tfrac{1}{2} \mathrm{Ric}_t \left(\int_0^t \Omega(h,k)\, dx \right).$$

It follows that condition (1.8) is satisfied. Therefore by Corollary 1.4, we obtain

$$\mathbb{E}(D_{T(h,k)} F) = 0.$$

3. ORNSTEIN–UHLENBECK OPERATORS ON $\mathbf{P}_{m_o}(M)$

Let $F \in \mathscr{C}$. Define the gradient $\nabla F(\gamma) \in \mathbb{H}_\gamma$ by

$$(D_h F)(\gamma) = (\nabla F \mid h)_\gamma \qquad \text{for all} \quad h \in \mathbb{H}. \tag{3.1}$$

The damped gradient ∇F introduced in [FM] satisfies the relation

$$D_h F = \langle \nabla F, \hat{h} \rangle_H \qquad \text{for all} \quad h \in \mathbb{H}. \tag{3.2}$$

It follows that

$$\widehat{\nabla F} = \nabla F. \tag{3.3}$$

Consider now the Dirichlet form on $\mathbf{P}_{m_o}(M)$:

$$\mathscr{E}(F, G) = \mathbb{E}((\nabla F \mid \nabla G)) \quad F, G \in \mathscr{C}. \tag{3.4}$$

THEOREM 3.1. *The generator $\mathscr{L}$ of $\mathscr{E}$ is the Norris-O.U. operator* [No].

Proof. By relation (3.3), we have:

$$\mathscr{E}(F, G) = \mathbb{E}(\langle \nabla F, \nabla G \rangle_H) = \mathbb{E}(\mathscr{L} F\, G). \quad \blacksquare$$

THEOREM 3.2. *We have for $F \in \mathscr{C}$,*

$$\mathscr{L}F = -\sum_{\alpha=1}^{d} \int_0^1 \tilde{D}^2_{\tau,\alpha} F \, d\tau + \int_0^1 \langle \tilde{D}_{\tau,\alpha} F, \circ\, dx(\tau) \rangle, \tag{3.5}$$

where $\tilde{D}_{\tau,\alpha}F$ is defined by $D_h F = \sum_\alpha \int_0^1 \langle \tilde{D}_{\tau,\alpha} F, \dot{\hat{h}}^\alpha(\tau) \rangle_{\mathbb{R}^d} \, d\tau$.

Proof. Let $c_n \in \mathbb{H}(\mathbb{R})$ such that $\{\dot{c}_n; n \geqslant 1\}$ is an orthonormal basis of $L^2([0,1])$. Let $\{\varepsilon_1, ..., \varepsilon_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Define

$$h_{n,\alpha}(\gamma, t) = \int_0^t (Q_{t,s} \varepsilon_\alpha) \, \dot{c}_n(s) \, ds. \tag{3.6}$$

Then $\{h_{n,\alpha}; n \geqslant 1, \alpha = 1, ..., d\}$ is an orthonormal basis of H_γ. let $F \in \mathscr{C}$. Then

$$\mathscr{L}F = \sum_{n,\alpha} (-D_{h_{n,\alpha}} D_{h_{n,\alpha}} F + \delta(h_{n,\alpha}) \, D_{h_{n,\alpha}} F) \qquad \text{in} \quad L^2. \tag{3.7}$$

By [Og], the following equality holds in $L^2(X)$ that

$$\sum_{n,\alpha} \delta(h_{n,\alpha}) \, D_{h_{n,\alpha}} F = \sum_{n,\alpha} \left(\int_0^1 \langle \dot{\hat{h}}_{n,\alpha}, dx \rangle \right) \left(\int_0^1 \langle \tilde{D}_{\tau,\alpha} F, \dot{\hat{h}}_{n,\alpha} \rangle \, d\tau \right)$$
$$= \sum_\alpha \int_0^1 \langle \tilde{D}_{\tau,\alpha} F, \circ\, dx^\alpha(\tau) \rangle,$$

where the stochastic integral is the Stratonovich anticipative integral. On the other hand,

$$D^2_{h_{n,\alpha}} F = \int_0^1 \int_0^1 \tilde{D}_{s,\alpha} \tilde{D}_{\tau,\alpha} F \dot{c}_n(\tau) \, \dot{c}_n(s) \, d\tau \, ds.$$

Define the operator $A : L^2([0,1]) \to L^2([0,1])$ by

$$(Au)(s) = \int_0^1 \tilde{D}_{s,\alpha} \, \tilde{D}_{\tau,\alpha} F \, u(\tau) \, d\tau.$$

A is of trace class. Therefore

$$\sum_n D^2_{h_{n,\alpha}} F = \sum_n \langle A\dot{c}_n, \dot{c}_n \rangle_{L^2} = \int_0^1 \tilde{D}^2_{\tau,\alpha} F \, d\tau.$$

Using expression (3.7), we obtain (3.5). ∎

Now we shall define the Ornstein-Uhlenbeck operator $\mathcal{L}_1$ on $\mathbb{H}$-vector fields on $\mathbf{P}_{m_o}(M)$, with respect to the connection ∇ defined in (2.7). Let z be a smooth $\mathbb{H}$-vector field. Assume that there exists $(\nabla z)(\gamma) \in H_\gamma \otimes H_\gamma$ such that

(i) $\mathbb{E}(|\nabla z|^p_{H_\gamma \otimes H_\gamma}) < +\infty$ for all $p > 1$;

(ii) $(\nabla_h z \mid k)_\gamma = (\nabla z \mid h \otimes k)_{H_\gamma \otimes H_\gamma}$ for all $h, k \in \mathbb{H}$.

We shall denote by $D^1(\chi)$ the space of all such vector fields on $\mathbf{P}_{m_o}(M)$. Consider the following Dirichlet form on $D^1(\chi)$:

$$\mathcal{E}_1(z_1, z_2) = \mathbb{E}((\nabla z_1 \mid \nabla z_2)_{H_\gamma \otimes H_\gamma}). \tag{3.8}$$

Let $\{h_{n,\alpha}; n \geqslant 1, \varepsilon = 1, ..., d\}$ be the vector fields defined in (3.6).

Proposition 3.3. *The generator $\mathcal{L}_1$ of $\mathcal{E}_1$ is given by*

$$\mathcal{L}_1 z = \sum_{n,\alpha} (-\nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}} z + \delta(h_{n,\alpha}) \nabla_{h_{n,\alpha}} z). \tag{3.9}$$

We shall say that $z \in Dom(\mathcal{L}_1)$ if the series in (3.9) converges in L^2.

Proof. Let $h = h_{n,\alpha}$. Applying the formula of integration by parts (1.6) to the identity

$$D_h(z_1 \mid z_2) = (\nabla_h z_1 \mid z_2) + (z_1 \mid \nabla_h z_2),$$

we obtain

$$\mathbb{E}((\nabla_h z_1 \mid z_2)) = E(\delta(h)(z_1 \mid z_2)) - \mathbb{E}((z_1 \mid \nabla_h z_2)).$$

It follows that the dual operator ∇_h^* of ∇_h has the expression

$$\nabla_h^* z = -\nabla_h z + \delta(h)\, z.$$

Therefore

$$\mathcal{E}_1(z_1, z_2) = \sum_{n,\alpha} \mathbb{E}((\nabla_{h_{n,\alpha}} z_1 \mid \nabla_{h_{n,\alpha}} z_2))$$

$$= \sum_{n,\alpha} \mathbb{E}((-\nabla_{h_{n,\alpha}} \nabla_{h_{n,\alpha}} z_1 + \delta(h_{n,\alpha}) \nabla_{h_{n,\alpha}} z_1 \mid z_2)).$$

If $z_1 \in \text{Dom}(\mathcal{L}_1)$, we obtain $\mathcal{E}_1(z_1, z_2) = \mathbb{E}((\mathcal{L}_1 z_1 \mid z_2))$. ▮

Proposition 3.4. *Let $F \in \mathcal{C}$. Then $\nabla F \in Dom(\mathcal{L}_1)$*

Proof. Denote by $Q^*_{t,s}$ the transposed matrix of $Q_{t,s}$. Then by (1.7) and (2.3), we have:

$$D_h F = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f})\, h^\alpha(\tau_i)$$
$$= \sum_i \sum_{\alpha,\beta} (\partial^i_{A_\alpha} \tilde{f}) \cdot \int_0^1 Q^{\alpha\beta}_{\tau_i,s}\, \dot{\hat{h}}^\beta(s)\, \mathbf{1}_{(s<\tau_i)}\, ds$$
$$= \int_0^1 \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f}) \cdot \langle Q^*_{\tau_i,s}(\mathbf{1}_{(s<\tau_i)}\varepsilon_\alpha), \dot{\hat{h}}(s)\rangle\, ds.$$

It follows that

$$\dot{\widehat{\nabla F}}(s) = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f}) \cdot Q^*_{\tau_i,s}(\mathbf{1}_{(s<\tau_i)}\varepsilon_\alpha).$$

According to (2.3), we obtain :

$$\nabla F(\tau) = \sum_{i,\alpha} (\partial^i_{A_\alpha} \tilde{f}) \cdot \left(\int_0^{\tau\wedge\tau_i} Q_{\tau,s} Q^*_{\tau_i,s}\, \varepsilon_\alpha\, ds \right). \tag{3.10}$$

From this expression, by a straightforward computation, we derive the result. ∎

Let $z \in D^1(\chi)$. Define $(\nabla_{s,\alpha} z)(\gamma,\tau)$ by

$$(\nabla_h z)(\gamma,\tau) = \int_0^1 \langle (\nabla_{s,\alpha} z)(\gamma,\tau), \dot{\hat{h}}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds. \tag{3.11}$$

THEOREM 3.5. *Assume that z is adapted, then almost surely*

$$(\nabla_{s,\alpha} z)(\gamma,\tau) = 0 \qquad \textit{for} \quad s > \tau. \tag{3.12}$$

Proof. Fix $\tau_o \in [0,1]$. Let h be an adapted $\mathbb{H}$-vector field such that $h(\gamma,s) = 0$ for $s \leqslant \tau_o$. Then (see [CM1])

$$(\nabla^o_h z)(\gamma,\tau) = 0 \qquad \text{for} \quad \tau \leqslant \tau_o.$$

Therefore for $\tau \leqslant \tau_o$,

$$(\nabla_h z)(\gamma,\tau) = \int_0^\tau Q_{\tau,s} \dot{\widehat{\nabla_h z}}(\gamma,s)\, ds = \int_0^\tau Q_{\tau,s} (\dot{\widehat{\nabla^o_h z}})(\gamma,s)\, ds = 0.$$

Remark that $\tilde{h}(\gamma, s)=0$ for $s \leqslant \tau_o$. Hence,

$$0=(\nabla_{\tilde{h}} z)(\gamma, \tau)=\sum_\alpha \int_\tau^1 \langle(\nabla_{s, \alpha} z)(\gamma, \tau_o), \dot{h}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds.$$

$L^2([\tau_o, 1])$ being separable, il follows that almost surely, for any $\dot{h} \in L^2([\tau_o, 1])$,

$$\sum_\alpha \int_{\tau_o}^1 \langle(\nabla_{s, \alpha} z)(\gamma, \tau), \dot{h}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds=0, \qquad \tau \leqslant \tau_o.$$

Let D be a countable dense subset of $[0,1]$. For any τ_o, denote

$$W_{\tau_o}=\left\{\gamma; \text{ for any } \dot{h} \in L^2([\tau_o, 1]), \int_{\tau_o}^1 \langle(\nabla_{s, \alpha} z)(\gamma, \tau), \dot{h}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds=0, \tau \leqslant \tau_o\right\}.$$

Let $\gamma \in \bigcap_{\tau_o \in D} W_{\tau_o}$. Let $\tau \in\,]0,1[$. There exists a decreasing sequence $\tau_n \in D$, which converges to τ. Let $\dot{h} \in L^2([\tau, 1])$. Then $\dot{h} \in L^2([\tau_n, 1])$. Therefore

$$\int_{\tau_n}^1 \langle(\nabla_{s, \alpha} z)(\gamma, \tau), \dot{h}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds=0.$$

Letting $n \rightarrow+\infty$, we obtain $\int_\tau^1 \langle(\nabla_{s, \alpha} z)(\gamma, \tau), \dot{h}^\alpha(s)\rangle_{\mathbb{R}^d}\, ds=0$. It follows that

$$(\nabla_{s, \alpha} z)(\gamma, \tau)=0 \qquad \text{for} \quad s>\tau. \quad \blacksquare$$

4. WEITZENBÖCK FORMULA

We shall establish the following main result

THEOREM 4.1. *We have for $F \in \mathscr{C}$ and any adapted vector field z,*

$$\mathbb{E}(D_z \mathscr{L} F)=\mathbb{E}((\mathscr{L}_1(\nabla F) \mid z))+\mathbb{E}((\nabla F \mid z)). \tag{4.1}$$

Proof. Let k be a smooth adapted vector field. For simplicity, denote $h=h_{n, \alpha}$. Define

$$I_1(h)=\mathbb{E}(-D_k D_h D_h F+D_k(\delta(h)\, D_h F)). \tag{4.2}$$

We have

$$D_h D_h F=D_h(\nabla F \mid h)=(\nabla^2 F \mid h \otimes h)_{H_\gamma \otimes H_\gamma}+(\nabla F \mid \nabla_h h)$$

and

$$(4.3)\quad (D_k D_h D_h F)(\gamma) = (\nabla^3 F \mid k \otimes h \otimes h)_{H_\gamma^{\otimes 3}} + (\nabla^2 F \mid \nabla_k (h \otimes h))_{H_\gamma \otimes H_\gamma} + (\nabla^2 F \mid k \otimes \nabla_h h)_{H_\gamma \otimes H_\gamma} + (\nabla F \mid \nabla_k \nabla_h h).$$

On the other hand, using (2.10),

$$\begin{aligned} D_k(\delta(h)\, D_h F) &= D_k\, \delta(h)\, D_h F + \delta(h)\, D_k(\nabla F \mid h) \\ &= \delta(\nabla_k h)(\nabla F \mid h) + (k \mid h)(\nabla F \mid h) \\ &\quad + \delta(h)(\nabla^2 F \mid k \otimes h)_{H_\gamma \otimes H_\gamma} + \delta(h)(\nabla F \mid \nabla_k h). \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \mathbb{E}(D_k(\delta(h)\, D_h F)) &= \mathbb{E}((\nabla^2 F \mid \nabla_k h \otimes h) + (\nabla F \mid \nabla_{\nabla_k h} h) + (k \mid h)(\nabla F \mid h) \\ &\quad + (\nabla^3 F \mid h \otimes k \otimes h) + (\nabla^2 F \mid \nabla_h (k \otimes h)) \\ &\quad + (\nabla^2 F \mid h \otimes \nabla_k h) + (\nabla F \mid \nabla_h \nabla_k h)). \end{aligned}$$

Combining with (4.3), we obtain

$$(4.4)\quad \begin{aligned} I_1(h) &= \mathbb{E}((\nabla^3 F \mid h \otimes k \otimes h) - (\nabla^3 F \mid k \otimes h \otimes h) \\ &\quad + (\nabla F \mid (\nabla_h \nabla_k - \nabla_k \nabla_h)\, h) + (\nabla^2 F \mid \nabla_h k \otimes h) \\ &\quad + (\nabla F \mid \nabla_{\nabla_k h} h) + (k \mid h)(\nabla F \mid h)). \end{aligned}$$

Now define

$$I_2(h) = \mathbb{E}(-(\nabla_h \nabla_h (\nabla F) \mid k) + \delta(h)(\nabla_h (\nabla F) \mid k)).$$

We have

$$(4.5)\quad (\nabla_h \nabla_h (\nabla F) \mid k) = (\nabla^3 F \mid h \otimes h \otimes k) + (\nabla^2 F \mid \nabla_h h \otimes k).$$

and

$$\mathbb{E}(\delta(h)(\nabla_h \nabla F \mid k)) = \mathbb{E}((\nabla^3 F \mid h \otimes h \otimes k) + (\nabla^2 F \mid \nabla_h (h \otimes k))).$$

It follows that

$$(4.6)\quad I_2(h) = \mathbb{E}((\nabla^2 F \mid h \otimes \nabla_h k)).$$

In order to simplify the quantity $I(h) = I_1(h) - I_2(h)$, we need the following algebraic lemmas.

LEMMA 4.2. *We have*

$$(\nabla^2 F \mid h \otimes k) - (\nabla^2 F \mid k \otimes h) = -D_{T(h,k)} F. \tag{4.7}$$

Proof. A direct computation gives the result. ∎

LEMMA 4.3.

$$\begin{aligned}(\nabla^3 F \mid h \otimes k \otimes h) &- (\nabla^3 F \mid k \otimes h \otimes h)\\ &= (\nabla F \mid R^P(k, h)\, h) - (\nabla_{T(h,k)}(\nabla F) \mid h),\end{aligned} \tag{4.8}$$

where

$$R^P(k, h)\, h = (\nabla_k \nabla_h - \nabla_h \nabla_k - \nabla_{[k,h]})\, h. \tag{4.9}$$

Proof. Using (4.5),

$$(\nabla^3 F \mid h \otimes k \otimes h) = (\nabla_h \nabla_k (\nabla F) \mid h) - (\nabla^2 F \mid \nabla_h k \otimes h).$$

and

$$(\nabla^3 F \mid k \otimes h \otimes h) = (\nabla_k \nabla_h (\nabla F) \mid h) - (\nabla^2 F \mid \nabla_k h \otimes h).$$

Therefore

$$\begin{aligned}(\nabla^3 F \mid h \otimes k \otimes h) &- (\nabla^3 F \mid k \otimes h \otimes h)\\ &= ((\nabla_h \nabla_k - \nabla_k \nabla_h)(\nabla F) \mid h) - (\nabla^2 F \mid (\nabla_h k - \nabla_k h) \otimes h)\\ &= (R^P(h, k)\, \nabla F \mid h) - (\nabla_{T(h,k)}(\nabla F) \mid h).\end{aligned}$$

To conclude (4.8), it is sufficient to remark that the operator $R^P(k, h)$ is skew-symmetric, although the covariant derivative ∇_h is not skew-symmetric. More precisely, we have:

$$(R^P(h, k)\, z_1 \mid z_2) = -(z_1 \mid R^P(h, k)\, z_2). \quad ∎ \tag{4.10}$$

Now using (4.4) (4.6), (4.7) and (4.8), we obtain

$$\begin{aligned}I(h) &= I_1(h) - I_2(h)\\ &= \mathbb{E}((\nabla F \mid R^P(k, h)\, h) - (\nabla_{T(h,k)} \nabla F \mid h)\\ &\quad + (\nabla F \mid (\nabla_h \nabla_k - \nabla_k \nabla_h)\, h) - D_{T(\nabla_h k, h)} F + (\nabla F \mid \nabla_{\nabla_k h} h))\\ &\quad + \mathbb{E}((k \mid h)(\nabla F \mid h)).\end{aligned}$$

We have $R^P(k, h)\,h + (\nabla_h \nabla_k - \nabla_k \nabla_h)\,h = -\nabla_{[k,h]} h$ and by Theorem (2.4),

$$(\nabla_{T(h,k)} \nabla F \mid h) = D_{T(h,k)} D_h F - (\nabla F \mid \nabla_{T(h,k)} h).$$

Therefore using (2.12), we obtain

$$\text{(4.11)} \quad \begin{aligned} I(h) &= \mathbb{E}((\nabla F \mid \nabla_{[h,k]} h + \nabla_{T(h,k)} h + \nabla_{\nabla_k h} h)) + \mathbb{E}((k \mid h)(\nabla F \mid h)) \\ &= \mathbb{E}((\nabla F \mid \nabla_{\nabla_h k} h)) + \mathbb{E}((k \mid h)(\nabla F \mid h)) \end{aligned}$$

Now using (3.7),

$$\begin{aligned} \mathbb{E}(D_k \mathscr{L} F) &= \mathbb{E}(\mathscr{L} F\, \delta(k)) \\ &= \sum_{n,\alpha} \mathbb{E}((-D_{h_{n,\alpha}} D_{h_{n,\alpha}} F + \delta(h_{n,\alpha})\, D_{h_{n,\alpha}} F)\, \delta(k)) = I_1(h_{n,\alpha}). \end{aligned}$$

According to Proposition 3.4, we have, for all $F \in \mathscr{C}$,

$$\text{(4.12)} \quad \begin{aligned} \mathbb{E}(D_k \mathscr{L} F) - \mathbb{E}((\mathscr{L}_1(\nabla F) \mid k)) &= \sum_{n,\alpha} I(h_{n,\alpha}) \\ &= \sum_{n,\alpha} \mathbb{E}((\nabla F \mid \nabla_{\nabla_{h_{n,\alpha}} k} h_{n,\alpha})) + \mathbb{E}((\nabla F \mid k)). \end{aligned}$$

Therefore, to conclude (4.1), it is sufficient to establish the following result.

Proposition 4.5. *Let $z \in \chi_p$ for some $p > 2$. Then*

$$\text{(4.13)} \qquad \sum_{n,\alpha} (z \mid \nabla_{\nabla_{h_{n,\alpha}} k} h_{n,\alpha}) = 0 \qquad \textit{in} \quad L^1.$$

Proof. Using definition (2.7),

$$(\nabla_{h_{n,\alpha}} k \mid h_{m,\beta}) = \int_0^1 \langle \dot{\overbrace{\nabla^o_{h_{n,\alpha}} \hat{k}}}\,(\tau), \varepsilon_\beta \rangle\, \dot{c}_m(\tau)\, d\tau.$$

Put $\dot{\overbrace{\nabla^o_h z}}\,(\tau) = \sum_{\beta=1}^d \int_0^1 \nabla^o_{s,\beta} z(\tau)\, \dot{h}^\gamma(s)\, ds$. If z is adapted, $\nabla^o_{s,\beta} z(\tau) = 0$ for $s > \tau$. By (2.4),

$$\dot{h}_{n,\alpha}(s) = \dot{c}_n(s)\, \varepsilon_\alpha - \tfrac{1}{2} \int_0^s \mathrm{Ric}_s\, Q_{s,\xi}\, \varepsilon_\alpha\, \dot{c}_n(\xi)\, d\xi.$$

We have

$$\begin{aligned}
&\overbrace{\nabla^o_{h_{n,\alpha}} \hat{k}}^{\cdot}(\tau) \\
&\quad = \sum_\beta \int_0^1 \nabla^o_{s,\beta} \hat{k}(\tau)\, \dot{h}^\gamma_{n,\alpha}(s)\, ds \\
&\quad = \int_0^1 \nabla^o_{s,\alpha} \hat{k}(\tau)\, \dot{c}_n(s)\, ds - \tfrac{1}{2} \sum_\beta \int_0^1 \nabla^o_{s,\beta} \hat{k}(\tau) \left(\int_0^s \langle \mathrm{Ric}_s\, Q_{s,\xi}\, \varepsilon_\alpha, \varepsilon_\beta \rangle\, \dot{c}_n(\xi)\, d\xi \right) ds \\
&\quad = \int_0^1 \nabla^o_{s,\alpha} \hat{k}(\tau)\, \dot{c}_n(s)\, ds - \tfrac{1}{2} \sum_\beta \int_0^1 \dot{c}_n(\xi) \left(\int_\xi^\tau \nabla^o_{s,\beta} \hat{k}(\tau)\, \langle \mathrm{Ric}_s\, Q_{s,\xi}\, \varepsilon_\alpha, \varepsilon_\beta \rangle\, ds \right) d\xi.
\end{aligned}$$

Denote

$$A_\alpha(s,\tau) = \mathbf{1}_{(s<\tau)} \left(\nabla^o_{s,\alpha} \hat{k}(\tau) - \tfrac{1}{2} \sum_\beta \int_s^\tau \nabla^o_{\xi,\beta} \hat{k}(\tau)\, \langle \mathrm{Ric}_\xi\, Q_{\xi,s} \varepsilon_\alpha, \varepsilon_\beta \rangle\, d\xi \right). \tag{4.13}$$

Then

$$\overbrace{\nabla^o_{h_{n,\alpha}} \hat{k}}^{\cdot}(\tau) = \int_0^1 A_\alpha(s,\tau)\, \dot{c}_n(s)\, ds.$$

Therefore

$$(\nabla_{h_{n,\alpha}} k \mid h_{m,\beta}) = \int_0^1 \int_0^1 \langle A_\alpha(s,\tau), \varepsilon_\beta \rangle\, \dot{c}_m(\tau)\, \dot{c}_n(s)\, ds\, d\tau. \tag{4.14}$$

On the other hand,

$$(z \mid \nabla_{h_{m,\beta}} h_{n,\alpha}) = \int_0^1 \langle \dot{\hat{z}}(\tau), \overbrace{\nabla^o_{h_{m,\beta}} \hat{h}_{n,\alpha}}^{\cdot}(\tau) \rangle\, d\tau.$$

We have

$$\overbrace{\nabla^o_{h_{m,\beta}} \hat{h}_{n,\alpha}}^{\cdot}(\tau) = -\Gamma_{h_{m,\beta}}(\tau)\, \varepsilon_\alpha\, \dot{c}_n(\tau).$$

Denote

$$B_{\alpha,\beta}(\xi,\tau) = -\mathbf{1}_{(\xi<\tau)} \int_\xi^\tau \Omega_s(Q_{s,\xi}\, \varepsilon_\beta, \circ\, dx_s)\, \varepsilon_\alpha. \tag{4.15}$$

Then $\Gamma_{h_{m,\beta}}(\tau) = -\int_0^\tau \dot{c}_m(\xi)\, B_{\alpha,\beta}(\xi,\tau)\, d\xi$. Therefore

$$(z \mid \nabla_{h_{m,\beta}} h_{n,\alpha}) = \int_0^1 \int_0^1 \langle \dot{\hat{z}}(s), B_{\alpha,\beta}(\tau,s)\rangle\, \dot{c}_n(s)\, \dot{c}_m(\tau)\, ds\, d\tau. \tag{4.16}$$

By (4.13) and (4.15), we see that

$$\mathbb{E}\left(\int_0^1 \int_0^1 |\langle A_\alpha(s,\tau), \varepsilon_\beta\rangle|^2\, ds\, d\tau\right) < +\infty,$$

and

$$\mathbb{E}\left(\int_0^1 \int_0^1 |\langle \dot{\hat{z}}(s), B_{\alpha,\beta}(\tau,s)\rangle|^2\, ds\, d\tau\right) < +\infty.$$

Then by (4.14) and (4.16), the series

$$\sum_{n,m} (\nabla_{h_{n,\alpha}} k \mid h_{m,\beta})(z \mid \nabla_{h_{m,\beta}} h_{n,\alpha})$$

converges absolutely in L^1. It follows that

$$\sum_{n,\alpha} (z \mid \nabla_{\nabla_{h_{n,\alpha}} k} h_{n,\alpha}) = \sum_{\alpha,\beta} \sum_{n,m} (\nabla_{h_{n,\alpha}} k \mid h_{m,\beta})(z \mid \nabla_{h_{m,\beta}} h_{n,\alpha})$$
$$= \sum_{\alpha,\beta} \int_0^1 \int_0^1 \langle A_\alpha(s,\tau), \varepsilon_\beta\rangle \langle \dot{\hat{z}}(s), B_{\alpha,\beta}(\tau,s)\rangle\, ds\, d\tau = 0. \quad \blacksquare$$

As a consequence of Theorem 4.1, we obtain the following Stroock commutation formula (see [Ma, p. 169] for flat case).

COROLLARY 4.6. *Let z be an adapted vector field in* $\mathrm{Dom}(\mathscr{L}_1)$, *then*

$$\mathscr{L}\delta(z) = \delta(\mathscr{L}_1 z) + \delta(z). \tag{4.17}$$

Proof. By (3.9) and (3.12), we see that $\mathscr{L}_1 z$ is adapted. Now by duality from (4.1), we obtain the result. $\blacksquare$

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