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Existence of symmetric solutions for a class of BVP with integral boundary conditions

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Abstract

In this paper, we study the symmetric solutions of second-order BVP with integral boundary conditions. By using a generalized Leggett-Williams fixed point theorem and some other techniques, we obtain sufficient conditions for the existence of symmetric positive solutions for the system. Meanwhile, an example is devoted to demonstrate our results in the end.

Keywords: integral boundary conditions; Green's function; boundary value problem; fixed point theorem

1 Introduction

With the development of science and technology, boundary value problems have acquired more attention in these years. Many methods are used to solve this kind of problems, such as fixed point theorems, coincidence degree theory, iterative method with upper and lower solutions, etc. Readers can see [1–13] for details.

However, the past researches focus on the existence of positive solution, periodic solution, uniqueness, etc.

For example, Boucherif [14] considered the following BVP:

$$\begin{cases} u''(t) + f(u(t)) = 0, \quad t \in (0,1), \\ u(0) - au'(0) = \int_0^1 g_0(s)u(s)\,ds, \\ u(1) - bu'(1) = \int_0^1 g_1(s)u(s)\,ds. \end{cases}$$

By applying the Krasnoselskii fixed point theorem, they obtained the existence of positive solutions of the system.

After Hayasida [15] drew some interesting results about symmetric positive solutions for a kind of BVP, several papers concerned about the subject with the same method; see [16–19]. Inspired by them, we consider a similar problem for the following system:

$$\begin{cases} u''(t) + f(u(t)) = 0, \quad t \in I, \\ u(0) - au'(0) = \int_0^1 \varrho_0(s)u(s)\,ds, \\ u(1) - bu'(1) = \int_0^1 \varrho_1(s)u(s)\,ds, \end{cases} \tag{1}$$

where $f:\mathbb{R}\to[0,+\infty)$ is continuous, $I=[0,1]$, $\varrho_0>0$ and $\varrho_1>0$ are continuous, and a and b are real parameters.

Abdulkadir [20] studied the case $a=b=0$, $\varrho_0,\varrho_1\equiv 0$ and drew some conclusions by using a generalization of the Leggett-Williams fixed point theorem. Obviously, the general case is much complicated. In this paper, we generalize the corresponding results.

The structure of this paper is as follows. In Section 2 and Section 3, we introduce the Leggett-Williams fixed point theorem, some definitions, and some lemmas. In particular, we deduce system (4) and some properties of the Green function, which will be used to prove the main results. Section 4 is devoted to developing the main results, which will be stated in detail. Finally, an example is included to display the main results.

2 Preliminaries

In this section, we give some definitions and the fixed point theorem that will be used in this paper.

Definition 1 Let E be a real Banach space. A nonempty, closed, and convex set $P\subset E$ is a cone if the following two conditions are satisfied:

(i) if $x\in P$ and $\mu\geq 0$, then $\mu x\in P$;
(ii) if $x\in P$ and $-x\in P$, then $x=0$.

Every cone $P\subset E$ induces the ordering in E given by $x_1\leq x_2$ if and only if $x_2-x_1\in P$.

Definition 2 A map α is called a nonnegative continuous convex functional on a cone P in a real Banach space E if $\alpha:P\to[0,+\infty)$ is continuous and

$$\alpha\bigl(\lambda x_1+(1-\lambda)x_2\bigr)\leq\lambda\alpha(x_1)+(1-\lambda)\alpha(x_2)$$

for all $x_1,x_2\in P$ and $0\leq\lambda\leq 1$. Likewise, we know the map β is a nonnegative continuous concave functional on a cone P in a real Banach space E if $\beta:P\to[0,+\infty)$ is continuous and

$$\beta\bigl(\lambda x_1+(1-\lambda)x_2\bigr)\geq\lambda\beta(x_1)+(1-\lambda)\beta(x_2)$$

for all $x_1,x_2\in P$ and $0\leq\lambda\leq 1$.

We denote $E=C(I)$, $I=[0,1]$, with the maximum norm, and for all $0<\widetilde{t}\leq\frac{1}{2}$, we define the cone $P\subset E$ by

$$\begin{aligned}P=\Bigl\{&u\in E:u(t)\text{ is concave, symmetric, and nonnegative-valued on }I,\\&\text{and }\min_{t\in[\widetilde{t},1-\widetilde{t}]}u(t)\geq 2\widetilde{t}\|u\|\Bigr\}.\end{aligned}$$

Theorem 1 (Leggett-Williams fixed-point theorem [21]) *Let $P\subset E$ be a cone in a real Banach space E. Let $l>0$ and $N>0$, let β and χ be nonnegative continuous concave functionals on P, and let ζ, α, and ρ be nonnegative continuous convex functionals on P with*

$$\beta(u)\leq\alpha(u),\qquad\|u\|\leq N\zeta(u),$$

for all $u \in \overline{\mathbb{P}(\zeta,l)}$. *Suppose that* $Y : \overline{\mathbb{P}(\zeta,l)} \to \overline{\mathbb{P}(\zeta,l)}$ *is a completely continuous operator and that there exist numbers* $h > 0$, $d > 0$, $p, q > 0$ *with* $d < p$ *such that*:

$$\begin{aligned}
&u \in \mathbb{P}(\zeta,\rho,\beta,p,q,l): \quad \beta(u) > p \neq \emptyset \quad \text{and} \quad \beta(Fu) > p \quad \text{for } u \in \mathbb{P}(\zeta,\rho,\beta,p,q,l);\\
&u \in \mathbb{Q}(\zeta,\alpha,\chi,h,d,l): \quad \alpha(u) < d \neq \emptyset \quad \text{and} \quad \alpha(Fu) < d \quad \text{for } u \in \mathbb{Q}(\zeta,\alpha,\chi,h,d,l);\\
&\alpha(Fu) > p \quad \text{for } u \in \mathbb{P}(\zeta,\beta,p,l) \text{ with } \rho(Fu) > q;\\
&\beta(Fu) < d \quad \text{for } u \in \mathbb{Q}(\zeta,\alpha,d,l) \text{ with } \chi(Fu) < h.
\end{aligned}$$

Then there exist at least three fixed points $u_1, u_2, u_3 \in \overline{\mathbb{P}(\zeta,l)}$ *such that*

$$\alpha(u_1) < d, \qquad p < \beta(u_2), \quad \text{and} \quad d < \alpha(u_3) \quad \text{with } \beta(u_3) < p.$$

Thereinto, some sets are as follows:

$$\begin{aligned}
&\mathbb{P}(\zeta,l) = \{u \in P : \zeta(u) < l\},\\
&\mathbb{P}(\zeta,\beta,p,l) = \{u \in P : p \leq \beta(u), \zeta(u) < l\},\\
&\mathbb{P}(\zeta,\rho,\beta,p,q,l) = \{u \in P : p \leq \beta(u), \rho(u) \leq q, \zeta(u) < l\},\\
&\mathbb{Q}(\zeta,\alpha,d,l) = \{u \in P : \alpha(u) \leq d, \zeta(u) < l\},\\
&\mathbb{Q}(\zeta,\alpha,\chi,h,d,l) = \{u \in P : h \leq \chi(u), \alpha(u) \leq d, \zeta(u) < l\}.
\end{aligned}$$

Many other functionals on the cone P are defined by

$$\begin{aligned}
&\beta(u) = \min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u(t) = u(t_0),\\
&\chi(u) = \min_{t\in[\frac{1}{\omega},\frac{\omega-1}{\omega}]} u(t) = u\left(\frac{1}{\omega}\right),\\
&\alpha(u) = \max_{t\in[\frac{1}{\omega},\frac{\omega-1}{\omega}]} u(t) = u\left(\frac{1}{2}\right),\\
&\rho(u) = \max_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u(t) = u(t_1),\\
&\zeta(u) = \max_{t\in[0,\widetilde{t}]\cup[1-\widetilde{t},1]} u(t) = u(\widetilde{t}),
\end{aligned}$$

where t_0, t_1, and ω are nonnegative numbers such that

$$0 < t_0 < t_1 \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{\omega} \leq t_1.$$

It is clear that, for all $u \in P$,

$$\beta(u) = u(t_0) \leq u\left(\frac{1}{2}\right) = \alpha(u), \tag{2}$$

$$\|u\| = u\left(\frac{1}{2}\right) \leq \frac{1}{2\widetilde{t}} u(\widetilde{t}) = \frac{1}{2\widetilde{t}} \zeta(u). \tag{3}$$

Throughout the paper, we suppose that the following two conditions hold.

(H_0) $a > 1$, $-1 < b < 0$;

(H_1) $\varrho_0 > 0$ and $\varrho_1 > 0$ are continuous on I, and the supplementary function $\varphi(t,s)$, defined by

$$\varphi(t,s) = \frac{a+t}{1+a-b}\varrho_1(s) - \frac{b-1+t}{1+a-b}\varrho_0(s), \quad t,s \in I,$$

satisfies

$$0 \le m := \min_{t,s\in I} \varphi(t,s) \le M := \max_{t,s\in I} \varphi(t,s) < 1.$$

3 Some lemmas

In order to get the main results, we consider the following linear system:

$$\begin{cases} -u''(t) = h(t), & t \in [0,1], \\ u(0) - au'(0) = \int_0^1 \varrho_0(s)\sigma_0(s)\,ds, \\ u(1) - bu'(1) = \int_0^1 \varrho_1(s)\sigma_1(s)\,ds. \end{cases} \tag{4}$$

Lemma 1 *Assume that h, σ_0, and σ_1 are continuous functionals. If condition* (H_0) *is satisfied, then problem* (4) *has a unique solution given by*

$$u(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{a+t}{1+a-b}\int_0^1 \varrho_1(s)\sigma_1(s)\,ds - \frac{b-1+t}{1+a-b}\int_0^1 \varrho_0(s)\sigma_0(s)\,ds, \tag{5}$$

where $G(t,s)$ is given by

$$G(t,s) = \begin{cases} \frac{(s+a)(1-b-t)}{1+a-b}, & 0 \le s < t < 1, \\ \frac{(1-s-b)(a+t)}{1+a-b}, & 0 \le t < s \le 1. \end{cases} \tag{6}$$

Here $G(t,s)$ is Green's function of (4), which has the following different properties:

$$\int_0^1 G(t,s)\,ds = \frac{-(1+a-b)t^2 + (1-2b)t + a(1-2b)}{2(1+a-b)}, \quad 0 \le t \le 1,$$

$$\int_0^{\frac{1}{\omega}} G\left(\frac{1}{2},s\right) ds = \frac{(1-2b)(1+2a\omega)}{4\omega^2(1+a-b)}, \quad \omega > 2,$$

$$\int_{\frac{1}{\omega}}^{\frac{1}{2}} G\left(\frac{1}{2},s\right) ds = \frac{(1-2b)(\omega-2)[(4a+1)\omega+2]}{16\omega^2(1+a-b)}, \quad \omega > 2,$$

$$\int_{t_0}^{t_1} G(t_1,s)\,ds + \int_{1-t_1}^{1-t_0} G(t,s)\,ds = \frac{(t_1-t_0)(1-2b)(a+t)}{2(1+a-b)}, \quad 0 < t_0 < t_1 < \frac{1}{2},$$

$$\min_{\omega\in[0,1]} \frac{G(t_0,\omega)}{G(t_1,\omega)} = 1, \quad 0 < t_0 < t_1 < \frac{1}{2}, \qquad \max_{\omega\in[0,1]} \frac{G(\frac{1}{2},\omega)}{G(t,\omega)} = 1, \quad 0 < t \le \frac{1}{2}.$$

Define the linear operator $B : C(I) \to C(I)$ by

$$(Bu)(t) = \int_0^1 \varphi(t,s)u(s)\,ds. \tag{7}$$

Lemma 2 *If* (H_0) *and* (H_1) *are satisfied, then the operator* B *has the following properties*:
(i) B *is a bounded linear operator*, $B(P) \subset P$;
(ii) $(I - B)$ *is invertible*;
(iii) $\|(I - B)^{-1}\| \leq \frac{1}{1-M}$.

Proof
(i) For all $k_1, k_2 \in \mathbb{R}$ and $u_1(t), u_2(t) \in C(I)$,

$$\begin{aligned} &B\bigl(k_1 u_1(t) + k_2 u_2(t)\bigr) \\ &\quad = \int_0^1 \phi(t,s)\bigl[k_1 u_1(t) + k_2 u_2(t)\bigr]\, ds \\ &\quad = k_1 (Bu_1)(t) + k_2 (Bu_2)(t). \end{aligned}$$

By using (H_1) and $\varphi(t,s) \leq M$ we have that

$$\bigl|(Bu)(t)\bigr| \leq M\|u\|.$$

For $u \in P$, we have $u(s) \geq 0$, $s \in [0,1]$. Since $\varphi(t,s) \geq m > 0$, we can obtain the following inequalities:

$$\begin{aligned} &(Bu)(t) \geq 0; \qquad (Bu)(t) = (Bu)(1-t) \quad \left(0 < t < \frac{1}{2}\right); \\ &(Bu)''(t) = -u(t) < 0; \qquad (Bu)(\widetilde{t}) \leq 2\widetilde{t}Bu\left(\frac{1}{2}\right). \end{aligned}$$

Then, $B(P) \subset P$.

(ii) To prove that $(I - B)$ is invertible, we just need to show that 1 is not an eigenvalue of B.

Since $M < 1$, $\|Bu\| \leq M\|u\|$, and thus $\sup_{u\neq 0} \frac{\|Bu\|}{\|B\|} \leq M < 1$.

Besides, if we suppose that 1 is an eigenvalue of B, then there is $u \in C(I)$ such that $Bu = u$. Moreover, we can obtain that $\frac{\|Bu\|}{\|B\|} = 1$. So $\|B\| \geq 1$. Thus, this assumption is false, so that 1 is not an eigenvalue of B and, equivalently, $(I - B)$ is invertible.

(iii) To obtain the expression for $(I - B)^{-1}$, we make use of the theory of Fredholm integral equations.

For each $t \in I$, $u(t) = (I - B)^{-1}x(t) \Leftrightarrow u(t) = x(t) + (Bu)(t)$. We obtain

$$u(t) = x(t) + \int_0^1 \varphi(t,s)u(s)\, ds. \tag{8}$$

By using successive substitutions in (8), the condition $M < 1$ implies that 1 is not an eigenvalue of the kernel $\varphi(t,s)$. So (8) has a unique continuous solution $u(t)$ for every continuous function $x(t)$.

By successive substitutions in (8) we get

$$u(t) = x(t) + \int_0^1 K(t,s)x(s)\, ds, \tag{9}$$

where the kernel $K(t,s)$ is given by

$$K(t,s) = \sum_{j=1}^{\infty} \varphi_j(t,s). \tag{10}$$

Here $\varphi_j(t,s) = \int_0^1 \varphi(t,\tau)\varphi_{j-1}(\tau,s)\,ds$, $j = 2,3,\ldots$, and $\varphi_1(t,s) = \varphi(t,s)$.

Because $|\varphi(t,s)| \le M < 1$, the series on the right in (10) is convergent. It can be easily verified that $K(t,s) \le \frac{M}{1-M}$. So we can get

$$(I-B)^{-1}x(t) = x(t) + \int_0^1 K(t,s)x(s)\,ds. \tag{11}$$

Therefore,

$$\begin{aligned}\bigl|(I-B)^{-1}x(t)\bigr| &\le \bigl|x(t)\bigr| + \frac{M}{1-M}\int_0^1 x(s)\,ds \\ &\le \|x\|\left(1 + \frac{M}{1-M}\right) \\ &= \frac{1}{1-M}\|x\|,\end{aligned}$$

so that

$$\frac{\|(I-B)^{-1}x\|}{\|x\|} \le \frac{1}{1-M}.$$

Thus,

$$\bigl\|(I-B)^{-1}\bigr\| \le \frac{1}{1-M}.$$

The proof of the lemma is over. □

Remark 1 Since $\varphi(t,s) \ge m$ for each $(t,s) \in I^2$, we analogously have $K(t,s) \ge \frac{m}{1-m}$.

Lemma 3 *A function $u(t) \in P$ is a solution of* (1) *if and only if*

$$u(t) = \int_0^1 G(t,s)f\bigl(u(s)\bigr)\,ds + \int_0^1 \varphi(t,s)u(s)\,ds, \quad \textit{for } t \in [0,1]. \tag{12}$$

Define the nonlinear operator $T : C(I) \to C(I)$ by

$$(Tu)(t) = \int_0^1 G(t,s)f\bigl(u(s)\bigr)\,ds. \tag{13}$$

From (12) we obtain another form of $u(t)$,

$$u(t) = (Tu)(t) + (Bu)(t). \tag{14}$$

By Lemma 2, (14) is equivalent to

$$u(t) = (I-B)^{-1}(Tu)(t).$$

Hence, we construct a composite operator Φ:

$$\Phi := (I - B)^{-1} T. \tag{15}$$

Lemma 4 *If $u(t)$ of* (12) *is a cone in P and Φ is as in* (15), *then $\Phi : P \to P$.*

Proof By the above lemmas we easily get

$$(\Phi u)(t) = \int_0^1 G(t,s) f\big(u(s)\big)\,ds + \int_0^1 K(t,s) \int_0^1 G(s,\tau) f\big(u(\tau)\big)\,d\tau\,ds.$$

First, we will verify the conditions of cone P,

(i) if $u \in p$, then, by some properties of $G(t,s)$, $\Phi u(t) \geq 0$,

(ii) $(\Phi u)''(t) = -f(u(s)) \leq 0$, $0 < t < 1$, that is, $\Phi u(t)$ is concave.

(iii) $\Phi u(\widetilde{t}) \geq 2\widetilde{t}\Phi u(\frac{1}{2})$, $\Phi u(t) = \Phi u(1-t)$, $0 < t < \frac{1}{2}$, that is, $\Phi u(t)$ is symmetric.

This means that $\Phi u(t) \in P$, and so $\Phi : P \to P$. □

4 Main results

To show our main result, we first assume that the following condition is established:

$$\begin{aligned}
(H_2)\text{(i)}\quad & f(\epsilon) \leq \frac{8[pa\omega^2 - l(1+2a\omega)](1+a-b)(1-M)}{a(1-2b)(\omega-2)[(4a+1)\omega+2]}, \quad \frac{2p}{l} < \epsilon < p; \\
\text{(ii)}\quad & f(\epsilon) > \frac{2q(1-m)(1+a-b)}{(a+1)(t_1-t_0)(1-2b)}, \quad q < \epsilon < \frac{qt_1}{t_0}; \\
\text{(iii)}\quad & f(\epsilon) \leq \frac{8l(1-M)(1+a-b)^2}{[2(a+b)+1]^2 - 4ab(2a-2b+5)}, \quad 0 < \epsilon < \frac{l}{2\widetilde{t}}.
\end{aligned}$$

Theorem 2 *Suppose that f satisfies (H_2). Let real constants $p > 0$, $q > 0$, and $l > 0$ be such that $0 < p < q \leq \frac{lt_0}{t_1}$. Then system* (1) *has three symmetric positive solutions $u_1(t)$, $u_2(t)$, and $u_3(t)$ satisfying*

$$\max_{t\in[0,\widetilde{t}]\cup[1-\widetilde{t},1]} u_i(t) \leq l, \quad i = 1,2,3,$$

$$\min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u_1(t) > q, \qquad \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]} u_2(t) < p,$$

$$\min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u_3(t) < q, \quad \textit{with} \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]} u_3(t) > p.$$

Proof The following five steps are used for verifying the conditions of Theorem 1.

(i) For all $u \in P$, from (2) and (3) we obtain $\beta(u) \leq \alpha(u)$, $\|u\| \leq \frac{1}{2\widetilde{t}}\zeta(u)$.

If $u \in \overline{\mathbb{P}(\zeta, l)}$, then $\|u\| \leq \frac{1}{2\widetilde{t}}\zeta(u) < \frac{l}{2\widetilde{t}}$, and from assumption (iii) of (H_2) we get

$$\begin{aligned}
\zeta(\Phi u) &= \max_{t\in[0,\widetilde{t}]\cup[1-\widetilde{t},1]} \left\{ \int_0^1 G(t,s) f\big(u(s)\big)\,ds + \int_0^1 K(t,s) \int_0^1 G(s,\tau) f\big(u(\tau)\big)\,d\tau\,ds \right\} \\
&\leq \left(1 + \frac{M}{1-M}\right) \int_0^1 G(\widetilde{t},s) f\big(u(s)\big)\,ds \\
&\leq \frac{1}{1-M} \frac{8l(1-M)(1+a-b)^2}{[2(a+b)+1]^2 - 4ab(2a-2b+5)} \int_0^1 G(\widetilde{t},s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{1-M}\frac{2l(1-M)(1+a-b)}{-(1+a-b)\widetilde{t}^2+(1-2b)\widetilde{t}+a(1-2b)}\int_0^1 G(\widetilde{t},s)\,ds \\
&= l.
\end{aligned}
$$

Thus, $\Phi:\overline{\mathbb{P}(\zeta,l)}\to\overline{\mathbb{P}(\zeta,l)}$, and we immediately get that

$$
\begin{aligned}
&\left\{u\in\mathbb{P}\left(\zeta,\rho,\beta,q,\frac{qt_1}{t_0},l\right):\beta(u)>q\right\}\neq\emptyset \quad \text{and} \\
&\left\{u\in\mathbb{Q}\left(\zeta,\alpha,\chi,\frac{2p}{l},p,l\right):\alpha(u)<p\right\}\neq\emptyset.
\end{aligned}
$$

(ii) If $u\in\mathbb{Q}(\zeta,\alpha,p,l)$ with $\chi(\Phi u)<\frac{2p}{l}$, then we obtain

$$
\begin{aligned}
\alpha(\Phi u) &= \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]}\left\{\int_0^1 G(t,s)f\big(u(s)\big)\,ds+\int_0^1 K(t,s)\int_0^1 G(s,\tau)f\big(u(\tau)\big)\,d\tau\,ds\right\} \\
&\le \left(1+\frac{M}{1-M}\right)\int_0^1 G\left(\frac{1}{2},s\right)f\big(u(s)\big)\,ds \\
&= \frac{1}{1-M}\int_0^1 \frac{G(\frac{1}{2},s)}{G(\frac{1}{\omega},s)}G\left(\frac{1}{\omega},s\right)f\big(u(s)\big)\,ds \\
&\le \frac{1}{1-M}\int_0^1 G\left(\frac{1}{\omega},s\right)f\big(u(s)\big)\,ds \\
&\le \frac{1-m}{1-M}\chi(\Phi u) \\
&< \frac{1-m}{1-M}\frac{2p}{l} \\
&< p.
\end{aligned}
$$

(iii) If $u\in\mathbb{Q}(\zeta,\alpha,\chi,\frac{2p}{l},p,l)$, then from assumption (i) and (iii) of (H_2) we get

$$
\begin{aligned}
\alpha(\Phi u) &= \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]}\left\{\int_0^1 G(t,s)f\big(u(s)\big)\,ds+\int_0^1 K(t,s)\int_0^1 G(s,\tau)f\big(u(\tau)\big)\,d\tau\,ds\right\} \\
&\le \left(1+\frac{M}{1-M}\right)\int_0^1 G\left(\frac{1}{2},s\right)f\big(u(s)\big)\,ds \\
&= \frac{1}{1-M}\int_0^1 G\left(\frac{1}{2},s\right)f\big(u(s)\big)\,ds \\
&= \frac{2}{1-M}\left\{\int_0^{\frac{1}{\omega}} G\left(\frac{1}{2},s\right)f\big(u(s)\big)\,ds+\int_{\frac{1}{\omega}}^{\frac{1}{2}} G\left(\frac{1}{2},s\right)f\big(u(s)\big)\,ds\right\} \\
&\le \frac{2}{1-M}\left\{\frac{8l(1-M)(1+a-b)^2}{[2(a+b)+1]^2-4ab(2a-2b+5)}\int_0^{\frac{1}{\omega}} G\left(\frac{1}{2},s\right)ds\right. \\
&\quad \left.+\frac{8[pa\omega^2-l(1+2a\omega)](1+a-b)(1-M)}{a(1-2b)(\omega-2)[(4a+1)\omega+2]}\int_{\frac{1}{\omega}}^{\frac{1}{2}} G\left(\frac{1}{2},s\right)ds\right\} \\
&\le \frac{2}{1-M}\left\{\frac{2l(1-M)(1+a-b)}{a(1-2b)}\int_0^{\frac{1}{\omega}} G\left(\frac{1}{2},s\right)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{8[pa\omega^2 - l(1+2a\omega)](1+a-b)(1-M)}{a(1-2b)(\omega-2)[(4a+1)\omega+2]} \times \int_{\frac{1}{\omega}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) ds\biggr\} \\
&= p.
\end{aligned}
$$

(iv) If $u \in \mathbb{Q}(\zeta, \beta, q, l)$ with $\rho(\Phi u) > \frac{qt_1}{t_0}$, then we get

$$
\begin{aligned}
\beta(\Phi u) &= \min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} \biggl\{ \int_0^1 G(t,s) f\bigl(u(s)\bigr)\,ds \\
&\quad + \int_0^1 K(t,s) \int_0^1 G(s,\tau) f\bigl(u(\tau)\bigr)\,d\tau\,ds \biggr\} \\
&\geq \left(1 + \frac{m}{1-m}\right) \int_0^1 G(t_0,s) f\bigl(u(s)\bigr)\,ds \\
&= \frac{1}{1-m} \int_0^1 \frac{G(t_0,s)}{G(t_1,s)} G(t_1,s) f\bigl(u(s)\bigr)\,ds \\
&\geq \frac{1}{1-m} \int_0^1 G(t_1,s) f\bigl(u(s)\bigr)\,ds \\
&\geq \frac{1-M}{1-m} \rho(\Phi u) \\
&> \frac{1-M}{1-m} \frac{qt_1}{t_0} \\
&> q.
\end{aligned}
$$

(v) If $u \in \mathbb{Q}(\zeta, \rho, \beta, q, \frac{qt_1}{t_0}, l)$, then from assumption (ii) of (H_2) we get

$$
\begin{aligned}
\beta(\Phi u) &= \min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} \biggl\{ \int_0^1 G(t,s) f\bigl(u(s)\bigr)\,ds \\
&\quad + \int_0^1 K(t,s) \int_0^1 G(s,\tau) f\bigl(u(\tau)\bigr)\,d\tau\,ds \biggr\} \\
&\geq \left(1 + \frac{m}{1-m}\right) \int_0^1 G(t_0,s) f\bigl(u(s)\bigr)\,ds \\
&= \frac{1}{1-m} \int_0^1 G(t_0,s) f\bigl(u(s)\bigr)\,ds \\
&> \frac{1}{1-m} \left\{ \int_{t_0}^{t_1} G(t_1,s) f\bigl(u(s)\bigr)\,ds + \int_{1-t_1}^{1-t_0} G(t_0,s) f\bigl(u(s)\bigr)\,ds \right\} \\
&\geq \frac{1}{1-m} \cdot \frac{2q(1-m)(1+a-b)}{(a+t_0)(t_1-t_0)(1-3b)} \left\{ \int_{t_0}^{t_1} G(t_0,s)\,ds + \int_{1-t_1}^{1-t_0} G(t_0,s)\,ds \right\} \\
&= q. \qquad \square
\end{aligned}
$$

5 Example

In the section, we provide a simple example illustrating the application of our main results.

Example 1 Let $a = \frac{3}{2}$, $b = -\frac{1}{2}$, $\varrho_0 = \varrho_1 = \frac{1}{3}$, and $f(u(t)) = u(t)$, Then (1) turns to the following equation:

$$\begin{cases} -u''(t) = u(t), \quad t \in I, \\ u(0) - \frac{3}{2}u'(0) = \frac{1}{3}\int_0^1 u(s)\,ds, \\ u(1) + \frac{1}{2}u'(1) = \frac{1}{3}\int_0^1 u(s)\,ds. \end{cases} \tag{16}$$

Because of the main result in Section 4, we have the following result.

Corollary 1 *Choose $p = \frac{1}{3}$, $q = 2$, $l = 3$, $t_0 < t_1$, and $\omega > 2$ such that $0 < p < q \leq \frac{lt_0}{t_1}$. Then the boundary value problem* (16) *has three symmetric positive solutions $u_1(t)$, $u_2(t)$ and $u_3(t)$ satisfying*

$$\max_{t\in[0,\widetilde{t}]\cup[1-\widetilde{t},1]} u_i(t) \leq 3, \quad i = 1,2,3,$$

$$\min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u_1(t) > 2, \qquad \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]} u_2(t) < \frac{1}{3},$$

$$\min_{t\in[t_0,t_1]\cup[1-t_1,1-t_0]} u_3(t) < 2, \quad \textit{with} \max_{t\in[\frac{1}{\omega},1-\frac{1}{\omega}]} u_3(t) > \frac{1}{3}.$$

Proof Now, we only verify conditions of (H_2) in Theorem 2.

(i) For $\frac{2}{9} < u(t) < \frac{1}{3}$,

$$f\bigl(u(t)\bigr) \leq \frac{8(\omega^2 - 3\omega - 6)}{21(\omega^2 - \frac{12}{7}\omega - \frac{4}{7})}, \quad \text{and, obviously,} \quad \frac{1}{3} \leq \frac{8(\omega^2 - 3\omega - 6)}{21(\omega^2 - \frac{12}{7}\omega - \frac{4}{7})}.$$

(ii) For $2 < u(t) < \frac{2t_1}{t_0}$,

$$f\bigl(u(t)\bigr) > \frac{8}{5(t_1 - t_0)}, \quad \text{and} \quad 2 > \frac{8}{5(t_1 - t_0)}.$$

(iii) For $0 < u(t) < \frac{3}{2\widetilde{t}}$,

$$f\bigl(u(t)\bigr) \leq 4, \quad \text{and} \quad \frac{3}{2\widetilde{t}} \leq 4.$$

We can complete the proof according Theorem 2. □

Competing interests
The authors have no any competing interests.

Authors' contributions
Zhiying Tong wrote the first draft of a paper, revising and editing. Wei Ding is in charge of the topic, the method, revising and editing.

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