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Limit theorems of Hilbert valued semimartingales and Hilbert valued martingale measures ☆

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Abstract

In this paper, we study tight criteria of càdlàg Hilbert valued processes and prove the tightness of Hilbert valued square integrable martingales and Hilbert valued semimartingales by using their characteristics. These extend appropriate results of Jacod and Shiryaev (1987). We also discuss the property of Hilbert valued martingale measure and introduce the concept of convergence of martingale measures in distribution. The sufficient and necessary conditions are provided for strongly orthogonal martingale measures with independent increments. The conditions are given for convergence of martingale measures.

Keywords: Hilbert valued semimartingale; Limit theorem; Martingale measures; The Skorokhod topology; Tightness

0. Introduction

The tight criteria of càdlàg Hilbert valued processes has been discussed by Joffe and Métivier (1986) and Métivier and Nakao (1987). But it is difficult to apply this tight criteria to prove the tightness of càdlàg Hilbert valued semimartingales by using their characteristics as Jacod and Shiryaev (1987). This problem will be solved in this paper. Another purpose of this paper is to study the weak convergence of integrable Hilbert valued martingale measures in distribution which is a sort of organic combination of weak convergence of vector random measures (Thang, 1991) and weak convergence of Hilbert valued martingales in distribution.

In Section 1, we will review the property of Hilbert valued semimartingale and define characteristics of Hilbert valued semimartingale and study the principal property of Hilbert valued semimartingales. In Section 2, we will study martingale measures and the relationship of martingale measure with independent increments and its characteristics. In Section 3, we will discuss the property of Skorokhod space $\boldsymbol{D}(\boldsymbol{H})$ which is the space of all càdlàg function: $\mathbf{R}_+ \to \boldsymbol{H}$, where $\boldsymbol{H}$ is a real separable Hilbert

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space. We obtain the sufficient and necessary conditions for subsets of $\boldsymbol{D}(\boldsymbol{H})$ being relatively compact (Theorem 3.2). It extends Theorem VI-3.21 of Jocad and Shiryaev to infinite dimension space. As Jocad and Shiryaev (1987), we will give tightness of sequences of càdlàg Hilbert valued locally square integrable martingales (Theorem 3.6). In Theorem 3.7, we will give the sufficient and necessary conditions for tightness of càdlàg Hilbert valued semimartingales which is the extension of Theorem VI-4.18 of Jacod and Shiryaev on Hilbert space. In Section 4, we will discuss the convergence of Hilbert valued semimartingales as Jacod and Shiryaev: (i) convergence of semimartingales with independent increments; (ii) convergence to a semimartingale with independent increments. These extend the appropriate results of Jacod and Shiryaev. In the end, we will define and study the convergence of integrable Hilbert valued martingale measures in distribution. The general theorems are given for convergence of martingale measures in distribution. As the discussion in convergence of semimartingales, we also study the following cases: (i) convergence of martingale measures with independent increments; (ii) convergence to a martingale measure with independent increments.

1. Preliminaries

Let $\boldsymbol{H}$ be a real separable Hilbert space with scalar product $x \cdot y$ and norm $\|\cdot\|$. In this paper, stochastic processes with values in $\boldsymbol{H}$ are studied.

Let $\{e_n\}_{n\geqslant 1}$ be an orthonormal basis of $\boldsymbol{H}$. Put $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H} = \{y : y = \sum_{i,j}\lambda_{ij}e_i \otimes e_j, \|y\|_1 = \sum_{i,j} \mid \lambda_{ij} \mid < \infty\}$, then $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$ is a Banach space with the norm $\|\cdot\|_1$. It is said to be the nuclear space of $\boldsymbol{H}$, which is included in the Hilbert–Schmidt tensor product $\boldsymbol{H}\hat{\otimes}_2\boldsymbol{H} = \{y : y = \sum_{i,j}\lambda_{ij}e_i\otimes e_j, \sum_{i,j}\lambda_{ij}^2 < \infty\}$. The space $\boldsymbol{H}\hat{\otimes}_2\boldsymbol{H}$ is a Hilbert space with $\{e_i \otimes e_j\}_{i,j\geqslant 1}$ as an orthonormal basis and norm $\|y\|_2 = (\sum_{i,j}\lambda_{ij}^2)^{1/2}$. The injection from $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$ into $\boldsymbol{H}\hat{\otimes}_2\boldsymbol{H}$ is continuous. We assume once and for all the stochastic basis $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is given and $\mathscr{F}_\infty = \vee_{t\geqslant 0}\mathscr{F}_t$.

Let M be a $\boldsymbol{H}$-valued square integrable martingale, then $\|M\|^2$ is a non-negative submartingale. By Doob–Meyer decomposition theorem, there exists a predictable, increasing process, denoted by $\langle M\rangle(\langle M\rangle_0 = 0)$ such that $\|M\|^2 - \langle M\rangle$ is a real martingale. Also there exists a finite variation process, denoted by $[M]$, which is uniquely defined up to P-equality with the following properties:

(i) $\|M\|^2 - [M]$ is a martingale;

(ii) $\langle M\rangle$ is the dual predictable projection of $[M]$;

(iii) $[M]_t = \langle M^c\rangle_t + \sum_{s\leqslant t}\|\Delta M_s\|^2$, for all $t \geqslant 0$, where M^c is the continuous martingale part of M.

Let M be a $\boldsymbol{H}$-valued square integrable martingale. For every pair stopping times $S, T (S \leqslant T)$, put

$$\alpha_M\left(]S,T]\right) = E\left(\|M_T\|^2 - \|M_S\|^2\right) = E\left(\|M_T - M_S\|^2\right),$$

$$\mu_M\left(]S,T]\right) = E\left(M_T^{\otimes_2} - M_S^{\otimes_2}\right) = E\left((M_T - M_S)^{\otimes_2}\right),$$

we have $\alpha_M = Tr\mu_m$. As Theorem 15.8 of Métivier (1982), we may use Radon–Nikodym theorem for $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued measure to obtain the existence of a $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued predictable process Q_M, for every predictable G

$$\mu_M(G) = \int_G Q_M \mathrm{d}\alpha_M.$$

Moreover, Q_M takes values in the set of positive symmetric elements of $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$ and

$$TrQ_M(\omega, s) = \|Q_M(\omega, s)\|_1 = 1, \quad \alpha_M\text{-a.s.}$$

The process $\langle\langle M\rangle\rangle_t = \int_0^t Q_M \mathrm{d}\langle M\rangle$ is predictable with finite variation, admits μ_M as its Doleans measure, and $M^{\otimes 2} - \langle\langle M\rangle\rangle$ is a $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued martingale.

Also there exists a $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued càdlàg process, which is uniquely defined up to P-equality, denoted by $[\![M]\!]$ and called the tensor quadratic variation of M with the following properties:

(i) $M^{\otimes 2} - [\![M]\!]$ is a $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued martingale,

(ii) $[\![M]\!] = \langle\langle M^c\rangle\rangle + \sum_{s\leqslant\cdot}(\Delta M_s)^{\otimes 2} = [\![M^c]\!] + \sum_{s\leqslant\cdot}(\Delta M_s)^{\otimes 2}$ P-a.s.,

the series on the right hand side are absolutely convergent in $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$ for all $t\geqslant 0$.

Lemma 1.1. *If a semimartingale X satisfies $\|\Delta X\|\leqslant a$, it is a special semimartingale and its canonical decomposition $X = X_0 + M + A$ satisfies $\|\Delta A\|\leqslant a$ and $\|\Delta M\|\leqslant 2a$.*

Definition 1.2. A map $h : \boldsymbol{H} \to \boldsymbol{H}$ is called truncation if it is bounded, continuous, and that exist $b > 0$ and $c > 0$ such that $h(x) = x$ when $\|x\|\leqslant b$ and $h(x) = 0$ when $\|x\| > c$. We denote by $\mathscr{C}$ the class of all truncation functions.

Let X be a $\boldsymbol{H}$-valued semimartingale on $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. For $h \in \mathscr{C}$, $\Delta X - h(\Delta X) \neq 0$ only if $\|\Delta X\| > b$ for some $b > 0$ and the following formulae

$$\check{X}(h) = \sum_{s\leqslant\cdot}[\Delta X_s - h(\Delta X_s)], \quad X(h) = X - \check{X}(h) \tag{1.1}$$

define a $\boldsymbol{H}$-valued right continuous process $\check{X}(h)$ with finite variation and a $\boldsymbol{H}$-valued semimartingale $X(h)$. Since $\Delta X(h) = h(\Delta X)$ is bounded, by Lemma 1.1, $X(h)$ is a special and we consider its canonical decomposition

$$X(h) = X_0 + M(h) + B(h).$$

where $M(h)$ is a local martingale and $B(h)$ is a locally integrable, predictable process with finite variation.

Definition 1.3. Let $h \in \mathscr{C}$ be fixed. We call characteristics of X (or characteristics associated with h if there is an ambiguity on h) the triplet (B, C, ν) consisting of:

(i) B is a $\boldsymbol{H}$-valued predictably finite variation process $B = B(h)$;

(ii) C is a $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued continuous process and $C_t - C_s$ takes values in the set of positive symmetric element of $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$ for every $s < t$, namely $C = \langle\langle X^c\rangle\rangle$, where X^c is the continuous martingale part of X;

(iii) ν is a predictable random measure on $\mathscr{B}_+ \times \mathscr{B}(\boldsymbol{H})$, namely the compensator of the random measure μ^X associated to the jumps of X:

$$\mu^X(\mathrm{d}t, \mathrm{d}x) = \sum_{s>0} I_{\{\Delta X_s \neq 0\}} \varepsilon_{(s, \Delta X_s)}(\mathrm{d}t, \mathrm{d}x)$$

where $\varepsilon_a(\mathrm{d}x)$ is Dirac measure.

We see that C and ν do not depend on the choice of the function h, while $B = B(h)$ does.

Since $\Delta X(h)$ is bounded and so does $\Delta M(h)$, $M(h)$ is a locally square integrable martingale. This implies that $\langle\langle M(h) \rangle\rangle$ exists. We denote by $\tilde{C} = \langle\langle M(h) \rangle\rangle$, which is called modified second characteristics of X (associated to h).

Theorem 1.4. *Let X be a $\boldsymbol{H}$-valued semimartingale with $X_0 = 0$. Then it is a process with independent increments if and only if there is a version (B, C, ν) of its characteristics that is deterministic.*

A $\boldsymbol{H}$-valued semimartingale X is called locally square integrable if it is a special semimartingale whose canonical decomposition $X = X_0 + N + A$ satisfies that N is a locally square integrable martingale.

As the proof of Proposition II-2.29 in Jacod and Shiryaev (1987), we have the following:

Proposition 1.5. *Let X be a semimartingale with characteristics (B, C, ν) relative to the truncation h.*

(a) *X is a special semimartingale if and only if $(\|x\|^2 \wedge \|x\|) \cdot \nu$ is locally integrable. In this case, the canonical decomposition $X = X_0 + N + A$ satisfies*

$$A = B + (x - h(x)) \cdot \nu, \quad \Delta A_t = \int_{\boldsymbol{H}} x\nu(\{t\} \times \mathrm{d}x). \tag{1.2}$$

(b) X is a locally square integrable semimartingale if and only if $\|x\|^2 \cdot \nu$ is locally integrable. In this case, the canonical decomposition $X = X_0 + N + A$ satisfies (1.2) and

$$\begin{aligned} \langle\langle N \rangle\rangle &= C + x^{\otimes_2} \cdot \nu - \sum_{s \leqslant \cdot} \left[\int_{\boldsymbol{H}} x\nu(\{s\} \times \mathrm{d}x) \right]^{\otimes_2} \\ &= C + x^{\otimes_2} \cdot \nu - \sum_{s \leqslant \cdot} (\Delta A_s)^{\otimes_2}. \end{aligned} \tag{1.3}$$

2. Definition and basic properties of Hilbert valued martingale measures

Let $\boldsymbol{E}$ be a Lusin space, $\mathscr{B}(\boldsymbol{E})$ be the Borel σ-field on $\boldsymbol{E}$ and $\mathscr{M}(\boldsymbol{E})$ be the linear space formed by all $\boldsymbol{H}$-valued measures with finite variation on $\mathscr{B}(\boldsymbol{E})$. We consider a $\boldsymbol{H}$-valued set function $U(\omega, A)$ defined on $\Omega \times \mathscr{A}$, where $\mathscr{A}$ is a subring of $\mathscr{B}(\boldsymbol{E})$

which satisfies:

$$\|U(A)\|_2^2 = E[\|U(A)\|^2] < \infty, \quad \forall A \in \mathscr{A},$$

$$A \cap B = \emptyset \ \Rightarrow \ U(A) + U(B) = U(A \cup B) \quad \text{a.s.} \quad \forall A, B \in \mathscr{A}.$$

We will say that the map U is σ-finite when there exists an increasing sequence $\{E_n\}_{n \geq 1}$ of $\boldsymbol{E}$ such that:

(1) $\cup_n E_n = \boldsymbol{E}$,

(2) $\forall n, \mathscr{E}_n = \mathscr{E}|_{E_n} \subseteq \mathscr{A}$,

(3) $\sup\{\|U(A)\|_2 : A \in \mathscr{E}_n\} < \infty$.

The set function U will be said countable additive if for each n, for each sequence $\{A_j\}_{j \geq 1}$ of $\mathscr{E}_n$ decreasing to $\emptyset$, $\|U(A_j)\|_2$ tends to 0. Then it is easy to extend U by $U(A) = \lim_n U(A \cap E_n)$ on every set of $\mathscr{E}$ such that the limit exists in $L^2_{\boldsymbol{H}}(\Omega, \mathscr{F}, P)$. A set function which satisfies all these properties is called a σ-finite $L^2_{\boldsymbol{H}}$-valued measure.

Definition 2.1. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space satisfying the "usual conditions".

(1) $\{M_t(A), t \geq 0, A \in \mathscr{A}\}$ is said to be a $\boldsymbol{H}$-valued $\mathscr{F}_t$-martingale measure if:
  (i) $M_0(A) = 0$ for all $A \in \mathscr{A}$,
  (ii) $\{M_t(A)\}_{t \geq 0}$ is a $\mathscr{F}_t$-martingale for all $A \in \mathscr{A}$,
  (iii) $M_t(\cdot) : \mathscr{A} \to L^2_{\boldsymbol{H}}$ is a $\boldsymbol{H}$-valued σ-finite measure for all $t > 0$.

(2) A $\boldsymbol{H}$-valued martingale measure M is said to be orthogonal if, for any two disjoint sets $A, B \in \mathscr{A}$, $\langle\langle M(A), M(B) \rangle\rangle = 0$.

(3) A $\boldsymbol{H}$-valued martingale measure M is said to be strongly orthogonal if, for any two disjoint sets $A, B \in \mathscr{A}$, $[\![M(A), M(B)]\!] = 0$.

Definition 2.2. If M is a $\boldsymbol{H}$-valued martingale measure and if, for all $A \in \mathscr{A}$, the map $t \to M_t(A)$ is continuous, we will say that M is continuous. If the map $t \to M_t(A)$ is càdlàg, we will say that M is càdlàg.

Definition 2.3. Let M and N be two $\boldsymbol{H}$-valued $\mathscr{F}_t$-martingale measures on Lusin spaces $\boldsymbol{E}$ and $\boldsymbol{E}'$, respectively. If they satisfy: for all $A \in \mathscr{A}$ and $B \in \mathscr{A}', M(A) \otimes N(B)$ is a $\boldsymbol{H} \hat{\otimes}_1 \boldsymbol{H}$-valued $\mathscr{F}_t$-martingale, then we will say that M and N are orthogonal.

It is clear that we can associate each set $A \in \mathscr{A}$ with a predictable process Q_A which takes values in the set of positive symmetric elements of $\boldsymbol{H} \hat{\otimes}_1 \boldsymbol{H}$ and the increasing process $\langle M(A) \rangle$, such that $\langle\langle M(A) \rangle\rangle = \int_0^{\cdot} Q_A \mathrm{d}\langle M(A) \rangle$. The processes can be regularized to be a $\boldsymbol{H} \hat{\otimes}_1 \boldsymbol{H}$-valued measure on $\mathscr{B}_+ \times \mathscr{A}$ in the following case.

Theorem 2.4. (a) *If M is an orthogonal $\mathscr{F}_t$ martingale measure, there exist $\mathscr{F}_t$ predictable, σ-finite positive random measure $\nu(\mathrm{d}s, \mathrm{d}x)$ on $\boldsymbol{R}_+ \times \boldsymbol{E}$ and positive symmetric $\boldsymbol{H} \hat{\otimes}_1 \boldsymbol{H}$-valued process $Q(s,x)$, $\mathscr{P} \times \mathscr{A}$-measurable, such that for all $A \in \mathscr{A}$, the processes $(\nu([0,t] \times A))_{t \geq 0}$ and $\int_0^{\cdot} \int_E Q(s,x) \nu(\mathrm{d}s, \mathrm{d}x)$ are predictable, and satisfy*

$$\nu([0,t] \times A) = \langle M(A) \rangle_t, \quad \int_0^t \int_A Q(s,x) \nu(\mathrm{d}s, \mathrm{d}x) = \langle\langle M(A) \rangle\rangle_t, \quad P\text{-a.s.}$$

for all $t > 0$ and $A \in \mathscr{A}$, where $\mathscr{P}$ is the predictable σ-field. We denote $\langle M \rangle$ by ν and $\langle\langle M \rangle\rangle$ by $\bar{\nu}$. It is clear that $\nu = Tr\bar{\nu}$.

(b) *If M is a strongly orthogonal $\mathscr{F}_t$-martingale measure, there exist random σ-finite positive measure $\mu(ds, dx)$ and σ-finite positive symmetric $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued measure $\bar{\mu}(ds, dx)$, $\mathscr{F}_t$-optional, such that for all $A \in \mathscr{A}$, the processes $(\mu([0,t] \times A))_{t \geqslant 0}$ and $(\bar{\mu}([0,t] \times A))_{t \geqslant 0}$ are optional, and satisfy*

$$\mu([0,t] \times A) = [M(A)]_t, \quad \bar{\mu}([0,t] \times A) = [\![M(A)]\!]_t, \quad P\text{-a.s.}$$

for all $t > 0$ and $A \in \mathscr{A}$. Moreover, we have that ν and $\bar{\nu}$ are predictable dual projection of μ and $\bar{\mu}$, respectively. We denote by $[M]$ by μ and $[\![M]\!]$ by $\bar{\mu}$.

Proof. It is the same as the proof of Theorem 2.5 of Walsh (1986). □

Definition 2.5. Let M be an orthogonal martingale measure with $\langle M \rangle = \nu$. M is said to be integrable if $E\nu(\boldsymbol{R}_+ \times \boldsymbol{E}) < \infty$. M is said to be locally integrable if there exist a sequence of stopping times $T_n \uparrow \infty$ and a sequence of compact subsets $\{K_n\}_{n \geqslant 1}$ which exhausts $\boldsymbol{E}$ such that $E\nu([0, T_n] \times K_n) < \infty$ for all $n \geqslant 1$.

In this paper, we only consider the following martingale measure M: for all $t > 0$, $M(\{t\} \times dx)$ is a random $\boldsymbol{H}$-valued measure on $\mathscr{B}(\boldsymbol{E})$. Put $M(\{t\} \times A) = M_t(A) - M_{t-}(A)$, for all $A \in \mathscr{B}(\boldsymbol{E})$. $M(\{t\} \times dx)$ is called the jump of M at time t. Put

$$\alpha(dt, dy) = \sum_{s>0} I_{\{M(\{s\} \times dx) \neq 0\}} \varepsilon_{(s, M(\{s\} \times dx))}(dt, dy).$$

We will say that α is the random measure associated to the jumps of M. In the following, we will assume that α is an integer-valued random measure and has dual predictable projection, denoted by β. It is easy to compute for all $f \in C_b(\boldsymbol{R}_+ \times \boldsymbol{E}), g \in C_f(\boldsymbol{H})$ (which is the set of g which is continuous on $\boldsymbol{H}$ and there exists $a > 0$ such that $g(x) = 0$),

$$\int_0^{\cdot} \int_{\boldsymbol{H}} g(x) \gamma(ds, dx) = \int_0^{\cdot} \int_{\mathscr{M}(\boldsymbol{E})} g\left(\int_{\boldsymbol{E}} f(s,x) y(dx)\right) \alpha(ds, dy),$$
$$\int_0^{\cdot} \int_{\boldsymbol{H}} g(x) \lambda(ds, dx) = \int_0^{\cdot} \int_{\mathscr{M}(\boldsymbol{E})} g\left(\int_{\boldsymbol{E}} f(s,x) y(dx)\right) \beta(ds, dy), \tag{2.1}$$

where γ is the jump measure of $X = \int_0^{\cdot} \int_{\boldsymbol{E}} f(s,x) M(ds, dx)$ and λ is the dual predictable projection of γ.

We will say that $(\bar{\nu}, \beta)$ is the characteristics of martingale measure M.

In the following, we will study the properties of martingale measures with independent increments.

Definition 2.6. Let M be an $\mathscr{F}_t$-martingale measure.

(a) M is said to be with independent increments (in short MMII), if for all $0 \leqslant s \leqslant t$, the random measure $M_t - M_s$ is independent from σ-field $\mathscr{F}_s$.

(b) A time $t \geqslant 0$ is called a fixed time of discontinuity for M if $P(M(\{t\} \times dx) \neq 0) > 0$.

Theorem 2.7. *Let M be an orthogonal martingale measure. Then M is MMII if and only if there is a version $(\bar{\nu},\beta)$ of its characteristics that is deterministic.*

Proof. (*Necessity*) Let M be an orthogonal martingale measure with independent increments. For every $f \in C_b(\boldsymbol{R}_+ \times \boldsymbol{E})$, $X = \int_0^{\cdot} \int_E f(s,x)M(\mathrm{d}s,\mathrm{d}x)$ is a square integrable $\boldsymbol{H}$-valued martingale with independent increments. Hence, by Theorem 1.4, we have

$$\langle\langle X\rangle\rangle = \int_0^{\cdot} \int_E f^2(s,x)Q(s,x)\nu(\mathrm{d}s,\mathrm{d}x), \tag{2.2}$$

is positive symmetric $\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}$-valued processes and deterministic. Under the basis $(e_i \otimes e_j)_{i,j\geqslant 1}$, we obtain that the real processes $\int_0^{\cdot} \int_E f^2(s,x)Q_{ij}(s,x)\nu(\mathrm{d}s,\mathrm{d}x) i,j \geqslant 1$ are deterministic. The arbitrariness of f yields that $Q_{ij}(s,x)\nu(\mathrm{d}s,\mathrm{d}x)$ is deterministic. Hence $Q(s,x)\nu(\mathrm{d}s,\mathrm{d}x)$ is deterministic. By (2.1), we deduce that β is deterministic.

(*Sufficiency*) Suppose that $\langle\langle M\rangle\rangle$ and β are deterministic. From (2.1) and (2.2), we have the characteristics of $X = \int_0^{\cdot} \int_E f(s,x)M(\mathrm{d}s,\mathrm{d}x)$ are deterministic for all $f \in C_b(\boldsymbol{R}_+ \times \boldsymbol{E})$. Hence X is $\boldsymbol{H}$-valued martingale with independent increments. This implies that M is a martingale measure with independent increments. □

Corollary 2.8. *Let M be an orthogonal, continuous martingale measure. Then M is MMII if and only if $\langle\langle M\rangle\rangle$ is deterministic.*

Corollary 2.9. *Let $\boldsymbol{E} = \{a_1,\ldots,a_n\}$ and let $m^1,\ldots,m^n$ be n orthogonal square integrable, continuous $\boldsymbol{H}$-valued martingales. Put $M_t(A) = \sum_{i=1}^n m_t^i \delta_{a_i}(A)$. Then M is MMII if and only if $(m^1,\ldots,m^n)$ is a $\boldsymbol{H}^n$-valued martingale with independent increments.*

3. Tightness of a sequence of Hilbert valued semimartingales

In this section, we will lay down the last cornerstone that is needed to derive functional limit theorem for Hilbert valued processes and study tight conditions of Hilbert valued semimartingales. These extend appropriate results of Jacod and Shiryaev (1987) and Joffe and Métivier (1986) and Thang (1991).

Let $\{e_n\}_{n\geqslant 1}$ be an orthonormal basis of $\boldsymbol{H}$. For any $x \in \boldsymbol{H}$, put $x = \sum_{k=1}^{\infty} x_k e_k$, if Π_n maps $\boldsymbol{H}$ onto the finite dimensional space $\mathbf{R}^n$ of vectors $(x_1,\ldots,x_n)$

$$x \longmapsto (x_1,\ldots,x_n), \tag{3.1}$$

then there is a continuous mapping V_n of $\mathbf{R}^n$ into $\boldsymbol{H}$, where

$$V_n(x_1,\ldots,x_n) = \sum_{k=1}^{n} x_k e_k \tag{3.2}$$

and clearly $\|x - V_n \circ \Pi_n x\| \to 0$ when $n \to \infty$, for all $x \in \boldsymbol{H}$.

Definition 3.1. (a) We denote by $\boldsymbol{D}(\boldsymbol{H})$ the space of all càdlàg function $\alpha : \mathbf{R}_+ \to \boldsymbol{H}$ (it is called the Skorokhod space).

(b) If $\alpha \in \boldsymbol{D}(\boldsymbol{H})$, we denote by $\alpha(t)$ the value of α at time t and by $\alpha(t-)$ its left-hand limit at time t (with $\alpha(0-) = \alpha(0)$ by convention), and $\Delta\alpha(t) = \alpha(t) - \alpha(t-)$.

For every $\alpha \in \boldsymbol{D}(\boldsymbol{H})$, we define

$$w(\alpha; I) = \sup_{s,t \in I} \|\alpha(s) - \alpha(t)\|$$

where I is an interval of $\mathbf{R}_+$;

$$w_N(\alpha, \theta) = \sup\{w(\alpha; [t, t+\theta]) : 0 \leqslant t \leqslant t + \theta \leqslant N\}, \quad \theta > 0, \ N > 0,$$

$$w'_N(\alpha, \theta) = \inf\{\max_{i \leqslant r} w(\alpha; [t_{i-1}, t_i)) : 0 = t_0 < t_1 < \cdots < t_r = N, \\ \inf_{i<r}(t_i - t_{i-1}) \geqslant \theta\}.$$

It is easy to prove the following theorem.

Theorem 3.2. (a) *There is a metrizable topology on $\boldsymbol{D}(\boldsymbol{H})$, called the Skorokhod topology, for which this space is Polish, and which is characterized as follows: a sequence $\{\alpha_n\}_{n \geqslant 1}$ converges to α if and only if there is a sequence $\{\lambda_n\}_{n \geqslant 1} \subset \Lambda$ such that*

$$\|\lambda_n - I\|_\infty = \sup_s |\lambda(s) - s| \to 0 \quad n \to \infty,$$

$$\sup_{s \leqslant N} \|\alpha_n \circ \lambda_n(s) - \lambda(s)\| \to 0 \quad n \to \infty, \forall N > 0.$$

(b) *A subset A of $\boldsymbol{D}(\boldsymbol{H})$ is relatively compact under the Skorokhod topology if and only if the following conditions hold*:

(i) $\sup_{\alpha \in A} \sup_{s \leqslant N} \|\alpha(s)\| < \infty$ *for all* $N > 0$;

(ii) $\lim_{\theta \to 0} \sup_{\alpha \in A} w'_N(\alpha, \theta) = 0$ *for all* $N > 0$;

(iii) *For every $\varepsilon > 0, N > 0$, there exists $n \in \mathbb{N}$, such that*

$$\sup_{\alpha \in A} \sup_{s \leqslant N} \|\alpha(s) - V_n \circ \Pi_n \alpha(s)\| \leqslant \varepsilon.$$

Where Λ is the set of all continuous function $\lambda : \mathbf{R}_+ \to \mathbf{R}_+$ that are strictly increasing, with $\lambda(0) = 0$ and $\lambda(t) \to \infty$ as $t \to \infty$.

Let X be a $\boldsymbol{H}$-valued càdlàg process, defined on a triple $(\Omega, \mathscr{F}, P)$. Then it may be considered as a random variable taking its values in the Polish space $\boldsymbol{D}(\boldsymbol{H})$ equipped with Skorokhod topology.

Theorem 3.3. *Let X^n be $\boldsymbol{H}$-valued càdlàg process which is defined on some space $(\Omega^n, \mathscr{F}^n, P^n)$ for $n \geqslant 1$. The sequence $\{X^n\}_{n \geqslant 1}$ is tight if and only if*

(i) *for all $N > 0, \varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that*

$$n \geqslant n_0 \quad \Longrightarrow \quad P^n\left(\sup_{s \leqslant N} \|X_s^n\| > K\right) \leqslant \varepsilon;$$

(ii) *for all $N > 0, \varepsilon > 0, \eta > 0$, there are $n_0 \in \mathbb{N}$ and $\theta > 0$ such that*

$$n \geqslant n_0 \quad \Longrightarrow \quad P^n(w'_N(X^n, \theta) \geqslant \eta) \leqslant \varepsilon;$$

(iii) *for all* $N > 0, \varepsilon > 0, \eta > 0$, *there are* $n_0, m \in \mathbb{N}$ *such that*

$$n \geqslant n_0 \quad \Longrightarrow \quad P^n\left(\sup_{s\leqslant N} \|X_s^n - V_m \circ \Pi_m X_s^n\| \geqslant \eta\right) \leqslant \varepsilon$$

Proof. It is the same as the proof of Theorem VI-3.21 of Jacod and Shiryaev. □

Definition 3.4. A sequence $\{X^n\}_{n\geqslant 1}$ of processes is called C-tight if it is tight and if all limit points of the sequence $\{\mathscr{L}(X^n)\}_{n\geqslant 1}$ are law of continuous processes.

Lemma 3.5. *Suppose that for all* $n, q \in \mathbb{N}$, *we have a decomposition*

$$X^n = U^{nq} + V^{nq} + W^{nq}$$

with (i) *the sequences* $\{U^{nq}\}_{n\geqslant 1}$ *are tight;*

(ii) *the sequences* $\{V^{nq}\}_{n\geqslant 1}$ *are tight and there is a sequence* $\{a_q\}_{q\geqslant 1}$ *of real numbers with* $\lim_{q\to\infty} a_q = 0$, $\lim_{n\to\infty} P^n\left(\sup_{s\leqslant N}\left\|\Delta V_s^{nq}\right\| > a_q\right) = 0$ *for all* $N > 0$;

(iii) *for all* $N > 0$, $\varepsilon > 0$, $\lim_{q\to\infty} \limsup_{n\to\infty} P^n\left(\sup_{s\leqslant N}\left\|W_s^{nq}\right\| > \varepsilon\right) = 0$.

Proof. That $\{X^n\}$ satisfies condition 3.3(i) is trivial. By inequality

$$\begin{aligned}\|X^n - V_m \circ \Pi_m X^n\| \leqslant\; & \|U^{nq} - V_m \circ \Pi_m U^{nq}\| \\ & + \|V^{nq} - V_m \circ \Pi_m V^{nq}\| + 2\,\|W^{nq}\|\end{aligned}$$

and the conditions (i)–(iii), we know that $\{X^n\}_{n\geqslant 1}$ satisfies condition 3.3(iii). As in the proof of Lemma VI-3.32 of Jacod and Shiryaev, we get that $\{X^n\}_{n\geqslant 1}$ meets the condition 3.3(iii). □

Let X and Y be two increasing processes defined on the same stochastic basis. We say that X strongly majorizes Y, and we write $Y \prec X$, if the process $X - Y$ is itself increasing.

The following theorem is the extension of theorem VI-4.13 of Jacod and Shiryaev on Hilbert space.

Theorem 3.6. *We suppose that* $X^n - X_0^n$ *is a locally square integrable martingale on* $\mathscr{B}^n = (\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, P^n)$ *for each n. Then for the sequence* $\{X^n\}_{n\geqslant 1}$ *to be tight, it is sufficient that:*

(i) *the sequence* $\{X_0^n\}_{n\geqslant 1}$ *is tight (in* $\boldsymbol{H}$*);*

(ii) *the sequence* $\{\langle\langle X^n\rangle\rangle\}_{n\geqslant 1}$ *is C-tight (in* $\mathbf{D}(\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H})$*).*

Proof. Put $U^{nq} = V_q \circ \Pi_q(X^n), V^{nq} = 0$ and $W^{nq} = X - U^{nq}$. We have $X = U^{nq} + V^{nq} + W^{nq}$. By the Lenglart's inequality and the hypotheses, we get Lemma 3.5(iii) by using Theorem 3.3 for $\langle\langle W^{nq}\rangle\rangle$. Lemma VI-3.32 of Jacod and Shiryaev implies the tightness of $\{U^{nq}\}_{n\geqslant 1}$. Hence, $\{X^n\}_{n\geqslant 1}$ is tight by Lemma 3.5. □

Theorem 3.7. *Let $\{X^n\}_{n\geq 1}$ be a sequence of $\boldsymbol{H}$-valued semimartingales. For the sequence $\{X^n\}_{n\geq 1}$ to be tight it suffices that*

(i) *the sequence $\{X_0^n\}_{n\geq 1}$ is tight (in $\boldsymbol{H}$);*

(ii) *for all $N > 0, \varepsilon > 0$,*

$$\lim_{a\to\infty} \limsup_{n\to\infty} P^n(\nu^n([0,N] \times \{\|x\| > a\}) > \varepsilon) = 0. \tag{3.3}$$

(iii) *for all $N > 0, \varepsilon > 0, \eta > 0$ and $p \in \mathbb{N}$, there exist $n_0, m \in \mathbb{N}$ with*

$$n \geq n_0 \implies P^n\left(g_p \circ (I - V_m \circ \Pi_m) \cdot \nu_N^n \geq \eta\right) \leq \varepsilon. \tag{3.4}$$

(iv) *each one of the following sequences of processes is C-tight:*

(1) $\{B^n\}_{n\geq 1}$,

(2) $\left\{\tilde{C}^n\right\}_{n\geq 1}$,

(3) $\{g_p \cdot \nu^n\}_{n\geq 1}$ *for all $p \in \mathbb{N}$. where $g_p(x) = (p\|x\| - 1)^+ \wedge 1$, I is the identical transformation on $\boldsymbol{H}$.*

Moreover, (i)ϰ(iii) *are also necessary for tightness of $\{X^n\}_{n\geq 1}$.*

Proof. (a) Let U^{nq}, V^{nq} and W^{nq} be the same as in the proof of Theorem 3.6. From (1.1), we get $X = X(h) + \check{X}(h)$ for some truncation h. By using Theorem VI-4.8 of Jacod and Shiryaev, we deduce that $\{U^{nq}\}_{n\geq 1}$ is tight for all $q \geq 1$.

From Theorem 3.3 and Lenglart's inequality, the hypotheses imply, for any $\delta > 0$ and $N > 0$,

$$\lim_{q\to\infty} \limsup_{n\to\infty} P^n\left\{\sup_{t\leq N} \|X(h)_t - V_q \circ \Pi_q X(h)_t\| > \delta\right\} = 0 \tag{3.5}$$

and by the definition and (3.4), we obtain

$$\begin{aligned} &\lim_{q\to\infty} \limsup_{n\to\infty} P^n\left\{\sup_{t\leq N} \|\check{X}(h)_t - V_q \circ \Pi_q \check{X}(h)_t\| > \delta\right\} \\ &\leq \lim_{q\to\infty} \limsup_{n\to\infty} P^n\left\{\sup_{t\leq N} \|\Delta(X_t - V_q \circ \Pi_q X_t\| \geq a\right\} \qquad (3.6) \\ &\leq \lim_{q\to\infty} \limsup_{n\to\infty} P^n\{\nu^n([0,N] \times \{\|x - V_q \circ \Pi_q x\| \geq a\}) > \varepsilon\} = 0 \end{aligned}$$

for all $\varepsilon > 0$ and some $a > 0$. (3.5) and (3.6) yield that $\{W^{nq}\}_{n\geq 1}$ satisfies the condition 3.5(iii). Hence $\{X^n\}_{n\geq 1}$ is tight by Lemma 3.5.

(b) Conversely, the proof of the conditions (i) and (ii) is the same as in the proof of Theorem VI-4.18 of Jacod and Shiryaev.

Let

$$A_t^n = \sum_{s\leq t} I_{\{\|\Delta X_s^n - V_m \circ \Pi_m(\Delta X_s^n)\| \geq 1/p\}},$$

$$\tilde{A}_t^n = \nu^n([0,N] \times \{x - V_m \circ \Pi_m x : \|x - V_n \circ \Pi_m x\| \geq 1/p\}).$$

Then $\tilde{A}^n$ is the compensator of A^n on $\mathscr{B}^n$. So A^n is L-dominated by $\tilde{A}^n$. Since

$$g_p \circ (I - V_m \circ \Pi_m) \cdot \mu_t^n = \sum_{s \leqslant t} (p \|\Delta X_s^n - V_m \circ \Pi_m(\Delta X_s^n)\| - 1)^+ \wedge 1 \prec \tilde{A}^n \quad (3.7)$$

and

$$\sup_{s \leqslant t} \|\Delta X_s^n - V_m \circ \Pi_m(\Delta X_s^n)\| \leqslant 2 \sup_{s \leqslant t} \|X_s^n - V_m \circ \Pi_m X_s^n\|,$$

the tightness of $\{X^n\}_{n \geqslant 1}$ implies that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P^n \left(\tilde{A}_t^n > \varepsilon \right) = 0 \quad (3.8)$$

for all $\varepsilon > 0$ by 3.3(iii). From (3.7) and (3.8), we obtain

$$\lim_{m \to \infty} \limsup_{n \to \infty} \left(g_p \circ (I - V_m \circ \Pi_m) \cdot \mu_t^n > \varepsilon \right) = 0$$

for all $\varepsilon > 0$. Hence $g_p(I - V_m \circ \Pi_m) \cdot v^n$ meets the condition (iii) because it is L-dominated by $g_p \circ (I - V_m \circ \Pi_m) \cdot \mu_t^n$. □

4. Convergence to semimartingale with independent increments

In this section and Section 5, the setting is as follows: for every $n \geqslant 1$, we consider a stochastic basis $\mathscr{B}^n = (\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, P^n)$, E^n denotes the expectation with respect to P^n. All sets, variables, processes, martingale measures, with the superscript n are defined on $\mathscr{B}^n$, and the limit process, martingale measure are defined on stochastic basis $\mathscr{B} = (\Omega, \mathscr{F}, \mathscr{F}_t, P)$, usually without mentioning.

As the proof of Lemma VII-3.20 of Jacod and Shiryaev, we have the following lemma.

Lemma 4.1. *Let X^n and X be locally square integrable $\boldsymbol{H}$-valued semimartingales with independent increments and suppose X has no fixed time of discontinuity. Suppose that $X^n = N^n + A^n$ and $X = N + A$ are canonical decomposition. If $X^n \xrightarrow{\mathscr{L}} X$, we have $g \cdot v^n \to g \cdot v$ under Skorokhod topology in $\boldsymbol{D}(\boldsymbol{R})$ for all g being continuous, bounded function on $\boldsymbol{H}$ which is 0 around 0 and has a limit at infinity.*

Lemma 4.2. *Assume that $X^n \xrightarrow{\mathscr{L}} X$ and that for each $t > 0$, the sequence of random variables $\{\sup_{s \leqslant t} \|X_s^n\|\}_{n \geqslant 1}$ is uniformly integrable. Then if $\beta_n(t) = E^n X_t^n$ and $\beta(t) = EX_t$, we have $\beta_n \to \beta$ under Skorokhod topology in $\boldsymbol{D}(\boldsymbol{H})$.*

Proof. Put $\beta_n^k(t) = E^n(\Pi_k X_t^n), \beta^k(t) = E(\Pi_k X_t), k \geqslant 1, n \geqslant 1$. For all $k \geqslant 1, X^n \xrightarrow{\mathscr{L}} X$ implies $\beta_n^k \to \beta^k$ in $\boldsymbol{D}(\boldsymbol{R}^k)$ by Lemma VII-3.8 of Jacod and Shiryaev. Hence β is the only possible limit point of the sequence $\{\beta^n\}_{n \geqslant 1}$ in $\boldsymbol{D}(\boldsymbol{H})$. Therefore it remains to prove that the sequence $\{\beta^n\}_{n \geqslant 1}$ is relatively compact. In view of Theorem 3.2(b), it is enough to prove that the set $A = \{\beta^n\}_{n \geqslant 1}$ meets the conditions 3.2(b) (i)–(iii).

For any $N > 0, \varepsilon > 0$ and $\delta > 0$, by Theorem 3.3, $X^n \xrightarrow{\mathscr{L}} X$ yields that there exist n_0, $m \in \mathbb{N}$ such that

$$P^n\left(\sup_{s\leqslant N} \|X_s^n - V_m \circ \Pi_m X_s^n\| > \delta\right) < \varepsilon, \quad n \geqslant n_0. \tag{4.1}$$

Since

$$\sup_{n\geqslant 1}\sup_{s\leqslant N} \|\beta_n(s)\| \leqslant \sup_{n\geqslant 1} E^n\left(\sup_{s\leqslant N}\|X_s^n\|\right) < \infty \tag{4.2}$$

and

$$\sup_{s\leqslant N}\|X_s^n - V_m \circ \Pi_m X_s^n\| \leqslant 2 \sup_{s\leqslant N}\|X_s^n\|, \tag{4.3}$$

we have that $\{\beta_n\}_{n\geqslant 1}$ meets the condition 3.2(b)(i) by (4.2) and $\left\{\sup_{s\leqslant N}\|X_s^n - V_m \circ \Pi_m X_s^n\|\right\}_{n\geqslant 1}$ is uniformly integrable from $\left\{\sup_{s\geqslant N}\|X_s^n\|\right\}_{n\geqslant 1}$ uniformly integrable. Hence

$$\sup_{n\geqslant 1}\sup_{s\leqslant N}\|\beta_n(s) - V_m \circ \Pi_m \beta_n(s)\| \leqslant \sup_{n\geqslant 1} E^n(\sup_{s\leqslant N}\|X_s^n - V_m \circ \Pi_m X_s^n\|)$$

implies $\{\beta_n\}_{n\geqslant 1}$ meets the condition 3.2(b)(iii).

Finally, by $\beta_n^m \to \beta^m$ $(n \to \infty)$ in $\boldsymbol{D}(\boldsymbol{R}^m)$ and $\{\beta_n\}_{n\geqslant 1}$ meeting the condition 3.2(b)(iii), we easily deduce that $\{\beta_n\}_{n\geqslant 1}$ meets the condition 3.2(b)(ii). □

Theorem 4.3. *Let X^n and X be the same as in Lemma 4.1. If $X^n \xrightarrow{\mathscr{L}} X$, $\|\Delta N^n\| \leqslant a$ for all $n \geqslant 1$, and $\sup_n \langle N^n\rangle_t < \infty$ for all $t > 0$, then we have the following:*

(i) *$A^n \longrightarrow A$ under Skorokhod topology in $\boldsymbol{D}(\boldsymbol{H})$;*

(ii) *$\langle\langle N^n\rangle\rangle \longrightarrow \langle\langle N\rangle\rangle$ under Skorokhod topology in $\boldsymbol{D}\left(\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H}\right)$;*

(iii) *$g \cdot v_t^n \longrightarrow g \cdot v_t$ for all $t > 0$, $g \in C_0^+(\boldsymbol{H})$, where $C_0^+(\boldsymbol{H})$ is the set of all continuous, bounded function $g \geqslant 0$ on $\boldsymbol{H}$ satisfying that there are $a > 0$, $b > 0$ $(a < b)$ such that $g(x) = 0$ for $\|x\| \leqslant a, \|x\| > b$.*

Proof. We suppose that $X^n \xrightarrow{\mathscr{L}} X$. By Lemma 4.1, we have the condition (iii). Since $\sup_n Var(\langle\langle N^n\rangle\rangle)_t < \infty$ for all $t > 0$, by using Lemma VII-3.34 of Jacod and Shiryaev, we obtain

$$\sup_n E\left(\sup_{s\leqslant t}\|N_s^n\|^4\right) \leqslant \sup_n\left\{K_1 a^2\left[E(\langle N_t^n\rangle_t^2)\right]^{1/2} + K_2 E(\langle N^n\rangle_t^2)\right\} < \infty,$$

where K_1, K_2 are constants, and the sequence $\{\sup_{s\leqslant t}\|N_s^n\|^p\}_{n\geqslant 1}$ is uniformly integrable if $p < 4$. As in the proof of Theorem VII-3.13 of Jacod and Shiryaev, we deduce that the sequence $\left\{\sup_{s\leqslant t}\|X_s^n\|\right\}_{n\geqslant 1}$ is uniformly integrable.

Note $A_t^n = E^n X_t^n, A_t = EX_t, X^n \xrightarrow{\mathscr{L}} X$, and X^n and X are $\boldsymbol{H}$-valued semimartingales with independent increments, Lemma 4.2 yields that $A^n \to A$ under the Skorokhod topology in $\boldsymbol{D}(\boldsymbol{H})$, that is, $\{X^n\}$ meets (i).

Finally, since X has no fixed time of discontinuity and A is continuous, $X^n \xrightarrow{\mathscr{L}} X$ and $A^n \xrightarrow{s.k} A$ yield $N^n \xrightarrow{\mathscr{L}} N$. This implies $(N^n)^{\otimes_2} \xrightarrow{\mathscr{L}} N^{\otimes_2}$. Hence we have $\langle\langle N^n\rangle\rangle \to \langle\langle N\rangle\rangle$

under Skorokhod topology in $\boldsymbol{D}(\boldsymbol{H}\hat{\otimes}_1\boldsymbol{H})$ by Lemma 4.2. That is, $\{X^n\}$ meets the condition (ii). □

Theorem 4.4. *Let X^n and X be locally square integrable $\boldsymbol{H}$-valued semimartingales and let X be with independent increments and without fixed time of discontinuity. Suppose that $\{X^n\}$ satisfies*

$$\lim_{a\to\infty}\ \limsup_{n\to\infty} P^n\left(\|x\|^2 I_{\{\|x\|>a\}}\cdot v_t^n > \eta\right) = 0, \quad \forall t\geqslant 0,\ \eta > 0. \tag{4.4}$$

Then we have $X^n \xrightarrow{\mathscr{L}} X$ if the following conditions hold.

(i) $\sup_{s\leqslant t}\|A_s^n - A_s\| \xrightarrow{P} 0$ *for all* $t\geqslant 0$;

(ii) $\sup_{s\leqslant t}\|\langle\langle N^n\rangle\rangle_s - \langle\langle N\rangle\rangle_s\|_1 \xrightarrow{P} 0$ *for all* $t\geqslant 0$;

(iii) $g\cdot v_t^n \xrightarrow{P} g\cdot v_t$ *for all* $t > 0$, $g\in C_0^+(\boldsymbol{H})$.

Proof. Since X has no fixed time of discontinuity, A and $\langle\langle N\rangle\rangle$ are continuous by (1.2) and (1.3). The hypotheses (i) and (ii) yield that $\{A^n\}$ and $\{\langle\langle N^n\rangle\rangle\}_{n\geqslant 1}$ are C-tight. $g\cdot v$ is increasing, continuous function on $\boldsymbol{R}_+$ for every $g\in C_0^+(\boldsymbol{H})$, the hypothesis (iii) implies $g\cdot v^n \xrightarrow{\mathscr{L}} g\cdot v$. Hence $\{g\cdot v^n\}_{n\geqslant 1}$ is C-tight. So $\{X^n\}_{n\geqslant 1}$ meets the condition 3.7(iv). Because of $X_0^n = 0$, it is clear that 3.7(i) is met. For all $N > 0, \varepsilon > 0$, there is $a\in\boldsymbol{Q}_+$ such that $g_a\cdot v_N \leqslant v([0,N]\times\{\|x\| > 1/a\})\leqslant\varepsilon$, we have by the condition (iii)

$$P^n(v^n([0,N]\times\{\|x\| > 2/a\}) > 2\varepsilon) \leqslant P^n(\|g_a\cdot v_N^n - g_a\cdot v_N\| > \varepsilon)\to 0.$$

That is, 3.7(ii) is met. For every $g\in C_0^+(\boldsymbol{H})$, there are $a > 0, b > 0$ such that $g(x) = 0$ for $\|x\|\leqslant a$ and $g\leqslant b$, then

$$\begin{aligned} &g\circ(I - V_m\circ\Pi_m)\cdot v_N \\ &\quad\leqslant b v([0,N]\times\{x - V_m\circ\Pi_m x : \|x - V_m\circ\Pi_m x\|\geqslant a\}) \longrightarrow 0. \quad m\to\infty. \end{aligned}$$

Hence

$$\begin{aligned} &P^n(g\circ(I - V_m\circ\Pi_m)\cdot v_N^n > 2\varepsilon) \\ &\quad\leqslant P^n(\|g\circ(I - V_m\circ\Pi_m)\cdot v_N^n - g\circ(I - V_m\circ\Pi_m)\cdot v_N\| > \varepsilon) \\ &\qquad + P^n(g\circ(I - V_m\circ\Pi_m)\cdot v_N > \varepsilon) \end{aligned}$$

implies that the condition 3.7(iii) is met by the condition (iii). Therefore $\{X^n\}_{n\geqslant 1}$ is tight by Theorem 3.7.

From Theorem VIII-2.18 of Jacod and Shiryaev, we obtain that $\Pi_m X^n \xrightarrow{\mathscr{L}} \Pi_m X$ on $[0,T]$ for all $m\geqslant 1$ and $T > 0$. Hence X is the only possible limit for the sequence $\{X^n\}$. This means $X^n \xrightarrow{\mathscr{L}} X$. □

Corollary 4.5. *Let X^n and X be the same as in Lemma 4.1. Assume that*

$$\lim_{a\to\infty}\ \limsup_{n\to\infty} \|x\|^2 I_{\{\|x\|>a\}}\cdot v_t^n = 0, \quad \forall t > 0 \tag{4.5}$$

Then we have the conditions 4.3(i)–(iii).

5. Convergence of integrable H-valued martingale measures

In this section, we only study integrable H-valued martingale measures. For simplicity, we still call them martingale measures.

Definition 5.1. Let M^n and M be martingale measures. We say that M^n converges to M in distribution and write $M^n \xrightarrow{L} M$ if for every $f \in C_b(\mathbf{R}_+ \times \boldsymbol{E})$,

$$\int_0^\cdot \int_{\boldsymbol{E}} f(s,x) M^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_{\boldsymbol{E}} f(s,x) M(\mathrm{d}s,\mathrm{d}x).$$

Theorem 5.2. *Let M^n and M be orthogonal martingale measures, $\langle M^n \rangle = \nu^n$ and $\langle M \rangle = \nu$.*

(i) *Suppose*

$$\lim_{a\to\infty} \limsup_{n\to\infty} P^n(\nu^n([0,N] \times \boldsymbol{E}) > a) = 0, \quad \forall N > 0, \tag{5.1}$$

$$\left(\int_0^\cdot \int_{\boldsymbol{E}} I_{A_1} M^n(\mathrm{d}s,\mathrm{d}x), \ldots, \int_0^\cdot \int_{\boldsymbol{E}} I_{A_k} M^n(\mathrm{d}s,\mathrm{d}x)\right) \xrightarrow{\mathscr{L}} \left(\int_0^\cdot \int_{\boldsymbol{E}} I_{A_1} M(\mathrm{d}s,\mathrm{d}x), \ldots, \int_0^\cdot \int_{\boldsymbol{E}} I_{A_k} M(\mathrm{d}s,\mathrm{d}x)\right) \tag{5.2}$$

for all sequence $\{A_1,\ldots,A_k\}$ of ν-continuous disjoint sets.

Then we have $M^n \xrightarrow{L} M$.

(ii) *Let $M^n \xrightarrow{L} M$ and*

$$\lim_{m\to\infty} \limsup_{n\to\infty} P^n(\nu^n(A_m) > \varepsilon) = 0, \quad \forall \varepsilon > 0 \tag{5.3}$$

for all sequence $\{A_m\}_{m\geq 1}$ of closed ν-continuous sets such that $\lim_{m\to\infty} \nu(A_m) = 0$, we have

$$\int_0^\cdot \int_{\boldsymbol{E}} I_A(s,x) M^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_{\boldsymbol{E}} I_A(s,x) M(\mathrm{d}s,\mathrm{d}x).$$

for all ν-continuous set $A \subset \mathbf{R}_+ \times \boldsymbol{E}$.

Proof. It is the same as the proof of Theorem 2.4 of Xie (1994). □

Remark. This theorem is an extension of Theorem 3.2 of Thang.

Corollary 5.3. *Let M^n and M be strongly orthogonal martingale measures with independent increments and M be continuous. If (5.2) is replaced by the condition that for all ν-continuous set A*

$$\int_0^\cdot \int_{\boldsymbol{E}} I_A(s,x) M^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_{\boldsymbol{E}} I_A(s,x) M(ds,dx), \tag{5.4}$$

the conclusion of Theorem 5.2(i) remains true.

Proof. Let $\{A_i\}_{i\leqslant k}$ be a sequence of ν-continuous disjoint sets. From (5.4) we have

$$X^{ni}=\int_0^{\cdot}\int_E I_{A_i}M^n(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^{\cdot}\int_E I_{A_i}M(\mathrm{d}s,\mathrm{d}x)=X^i,\quad 1\leqslant i\leqslant k. \tag{5.5}$$

Since M is continuous martingale measures with independent increments, we know that X^i $(i\leqslant k)$ are continuous square integrable independent $\boldsymbol{H}$-valued martingales. (5.5) implies that $\{X^{ni}\}_{n\geqslant 1}$ is C-tight for all $i\leqslant k$. Hence, $\{X^n=(X^{n1},\ldots,X^{nk})\}_{n\geqslant 1}$ is C-tight. Suppose Y is limit point of $\{X^n\}$. By Skorokhod's theorem, there exists a subsequence $\{X^{n_j}\}$ of $\{X^n\}$ such that $X^{n_j}\longrightarrow Y$, a.s. under Skorokhod topology on $\boldsymbol{D}(\boldsymbol{H}^k)$. Since M^n and M are strongly orthogonal martingale measures with independent increments, we have that $X^{n,1},\ldots,X^{n,k}$ and $X^1,\ldots,X^k$ are independent, respectively. $X^{n_j i}\longrightarrow X^i$, a.s. implies $\mathscr{L}(Y)=\mathscr{L}(X)$. This implies that (5.2) holds. □

Let $\nu^n,\nu\in\mathscr{M}(\boldsymbol{E})$ be random measures on $\mathscr{B}(\boldsymbol{E})$, we say that ν^n converges to ν in distribution and write $\nu^n\xrightarrow{\mathscr{L}}\nu$ if for any $f\in C_K(\boldsymbol{E})$, $\int_E f(x)\nu^n(\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_E f(x)\nu(\mathrm{d}x)$.

Theorem 5.4. *Let M^n and M be $\boldsymbol{H}$-valued orthogonal martingale measures. $\langle\langle M^n\rangle\rangle=\bar{\nu}^n$, $\langle\langle M\rangle\rangle=\bar{\nu}$, and let β^n and β be the dual predictable projections of random measures associated to the jumps of M^n and M, respectively. Suppose that M is strongly orthogonal and has no fixed time of discontinuity and MMII, and for all $t>0$, $\delta>0$, $f\in C_b(\mathbf{R}_+\times\boldsymbol{E})$,*

$$\lim_{a\to\infty}\limsup_{n\to\infty}P^n\left\{\int_0^t\int_{\mathscr{M}(E)}\left\|\int_E f(s,x)y(\mathrm{d}x)\right\|^2 I_{\left\{\left\|\int_E f(s,x)y(\mathrm{d}x)\right\|>a\right\}}\beta^n(\mathrm{d}s,\mathrm{d}y)>\delta\right\}=0. \tag{5.6}$$

Then we have $M^n\xrightarrow{L}M$ if the following conditions hold:

(i) $\bar{\nu}^n\xrightarrow{\mathscr{L}}\bar{\nu}$;

(ii) *for all $f\in C_b(\mathbb{R}_+\times\boldsymbol{E}), g\in C_0^+(\boldsymbol{H}), t>0$,*

$$\int_0^t\int_{\mathscr{M}(E)}g\left(\int_E f(s,x)y(\mathrm{d}x)\right)\beta^n(\mathrm{d}s,\mathrm{d}y)\xrightarrow{P}\int_0^t\int_{\mathscr{M}(E)}g\left(\int_E f(s,x)y(\mathrm{d}x)\right)\beta(\mathrm{d}s,\mathrm{d}y).$$

Proof. Since M is MMII, we know that $\bar{\nu}$ and β are deterministic by Theorem 2.7. For $f\in C_b(\mathbf{R}_+\times\boldsymbol{E})$, we put

$$X^n=\int_0^{\cdot}\int_E f(s,x)M^n(\mathrm{d}s,\mathrm{d}x),\quad X=\int_0^{\cdot}\int_E f(s,x)M(\mathrm{d}s,\mathrm{d}x). \tag{5.7}$$

Then X^n and X are $\boldsymbol{H}$-valued square integrable martingales, and X has no fixed time of discontinuity and with independent increments. Let λ^n and λ be the dual predictable projections of the random measures associated to the jumps of X^n and X, respectively. For all $g \in C_0^+(\boldsymbol{H})$, we have $g \cdot \lambda_t^n \xrightarrow{P} g \cdot \lambda_t$ for all $t > 0$ by the condition (ii) and (2.1). The condition (i) yields

$$\langle\langle X^n \rangle\rangle = \int_0^{\cdot} \int_E f^2(s,x)\bar{v}^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^{\cdot} \int_E f^2(s,x)\bar{v}(\mathrm{d}s,\mathrm{d}x) = \langle\langle X \rangle\rangle.$$

(5.6) implies that X^n satisfies (4.4). Hence $X^n \xrightarrow{\mathscr{L}} X$ by using Theorem 4.4. The arbitrariness of f implies $M^n \xrightarrow{L} M$. □

Corollary 5.5. *Let M^n and M be strongly orthogonal MMII and let M have no fixed time of discontinuity. Let $\bar{v}^n$, $\bar{v}$, β^n and β be the same as in Theorem 5.4. Suppose*

$$\lim_{a\to\infty} \limsup_{n\to\infty} \int_0^t \int_{\mathscr{M}(E)} \left\| \int_E f(s,x)y(\mathrm{d}x) \right\|^2 I_{\left\{\left\|\int_E f(s,x)y(\mathrm{d}x)\right\| > a\right\}} \beta^n(\mathrm{d}s,\mathrm{d}y) = 0, \tag{5.8}$$

for all $t > 0, f \in C_b(\mathbb{R}_+ \times \boldsymbol{E})$. we have $M^n \xrightarrow{L} M$ if the following conditions hold,

(i) $\bar{v}^n \xrightarrow{\mathscr{L}} \bar{v}$;

(ii) *for all $t > 0, g \in C_0^+(\boldsymbol{H}), f \in C_b(\mathbf{R}_+ \times \boldsymbol{E})$,*

$$\int_0^t \int_{\mathscr{M}(E)} g\left(\int_E f(s,x)y(\mathrm{d}x)\right) \beta^n(\mathrm{d}s,\mathrm{d}y) \longrightarrow \int_0^t \int_{\mathscr{M}(E)} g\left(\int_E f(s,x)y(\mathrm{d}x)\right) \beta(\mathrm{d}s,\mathrm{d}x).$$

Proof. The proof is exactly as in Theorem 5.4, one has only to replace Theorem 4.4 by Corollary 4.5. □

Theorem 5.6. *Let M^n and M be the same as in above theorem. Suppose $| M^n(\{s\} \times \boldsymbol{E}) | \leqslant b, s \geqslant 0, n \geqslant 1, b$ is a constant. Then $M^n \xrightarrow{L} M$ if and only if the condition 5.5(i) and (ii) hold.*

Proof. We only prove necessity. Suppose $M^n \xrightarrow{L} M$. For any $f \in C_b(\mathbf{R}_+ \times \boldsymbol{E})$, let X^n and X be the same as in the proof of Theorem 5.4. Then X^n and X are square integrable $\boldsymbol{H}$-valued martingales with independent increments and X has no fixed time of discontinuity. From the hypotheses, there is $a > 0$ such that $\|\Delta X^n\| \leqslant a$ for $n \geqslant 1$ and $\sup_n \langle X^n \rangle_t < \infty$ for all $t > 0$. By Theorem 4.3, we have

(a) $\langle\langle X^n \rangle\rangle \longrightarrow \langle\langle X \rangle\rangle$ for Skorokhod topology in $\boldsymbol{D}\left(\boldsymbol{H}\hat{\otimes}_1 \boldsymbol{H}\right)$;

(b) $g \cdot \lambda_t^n \longrightarrow g \cdot \lambda_t$ for all $t > 0, g \in C_0^+(\boldsymbol{H})$.

This means that $\int_0^{\cdot} \int_E f^2(s,x)\bar{v}^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{s.k} \int_0^{\cdot} \int_E f^2(s,x)\bar{v}(\mathrm{d}s,\mathrm{d}x)$ and

$$\int_0^t \int_{\mathscr{M}(E)} g\left(\int_E f(s,x)y(\mathrm{d}x)\right) \beta^n(\mathrm{d}s,\mathrm{d}y) \longrightarrow \int_0^t \int_{\mathscr{M}(E)} g\left(\int_E f(s,x)y(\mathrm{d}x)\right) \beta(\mathrm{d}s,\mathrm{d}y).$$

By f and g being arbitrary, we deduce that 5.5(i) and (ii) hold. □

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